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# A Study of NeutroAlgebra and AntiAlgebra of Ideals in a Factor Ring

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**Abstract**. If $I$ is an ideal in a ring $R$ and $M$ is the collection of all nontrivial ideals in the factor ring $R/I$, we find in this paper conditions under which $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ are NeutroAlgebras and AntiAlgebras where $\oplus$, $\otimes$ and $\cap$ are the usual sum, product and intersection of ideals in $R/I$.

**Keywords:** ClassicalAlgebra; PartialAlgebra; NeutroAlgebra; AntiAlgebra; NeutrosubAlgebra; sum of ideals, product of ideals; intersection of ideals.

## 1. **Introduction and Preliminaries**

In this section, we provide brief introduction to the concepts of NeutroAlgebraic structure and AntiAlgebraic structure. For completeness, basic definitions and results that will be used later in the paper are provided.

The concept of NeutroAlgebraic Structure was introduced by Smarandache in [16]. In [14], Smarandache introduced NeutroAlgebra as a generalization of Partial Algebra. Using the methods of NeutroSophication and AntiSophication, Smarandache in [15] presented and studied NeutroAlgebraic Structures and AntiAlgebraic Structures respectively. Since the presentation of seminal papers [ [16], [14] and [15]] by Smarandache, many Neutrosophic Researchers have further studied and published papers on NeutroAlgebraic and AntiAlgebraic Structures as well as NeutroAlgebraic and AntiAlgebraic

Hyper Structures. For full details, see [ [1], [2], [3], [4], [5], [6], [7], [9], [10] and [12]]. Kandasamy et.al. in [11] studied NeutroAlgebra of ideals in a ring under the usual sum and product of ideals. They proved that the set of nontrivial ideals in the ring $\mathbb{Z}$ is a NeutroAlgebra under the usual sum of ideals and not a NeutroAlgebra under the usual product of ideals. They also proved that the set of nontrivial ideals in the ring $\mathbb{Z}_n$ is a NeutroAlgebra under the usual sum and product of ideals. They equally showed that the set of nontrivial ideals in polynomial rings $\mathbb{Z}[x]$, $\mathbb{Q}[x]$ and $\mathbb{R}[x]$ are NeutroAlgebras under the usual sum of ideals and not NeutroAlgebras under the product of ideals. They finally showed that the set of nontrivial ideals under the usual product of ideals in the polynomial ring $\mathbb{Z}_n[x]$ is a NeutroAlgebra. The aim of the present paper is to extend the work done in [11] by studying NeutroAlgebra and AntiAlgebra of ideals in a factor ring.

**Definition 1.1.** (a) (i) A ClassicalOperation is an operation that is well defined for all the set's elements.
(ii) A NeutroOperation is an operation that is partially well defined, partially indeterminate, and partially outer defined on the given set.
(iii) An AntiOperation is an operation that is outer defined for all set's elements.

(b) (i) A ClassicalLaw/Axiom defined on a nonempty set is a law/axiom that is totally true for all the set's elements.
(ii) A NeutroLaw/Axiom defined on a nonempty set is a law/axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0, 1]$, with $(T, I, F) = (1, 0, 0)$ that represents the ClassicalAxiom/Law, and $(T, I, F) = (0, 0, 1)$ that represents the AntiAxiom.
(iii) An AntiLaw/Axiom defined on a nonempty set is a law/axiom that is false for all the set's elements.

(c) (i) A PartialOperation on a set is an operation that is well defined for some elements of the set and undefined for all the other elements of the set.
(ii) A PartialAlgebra is an algebra that has at least one PartialOperation, and all its other axioms are classical.

**Definition 1.2.** (a) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom and no AntiOperation or AntiAxiom.
(b) An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

(c) When a NeutroAlgebra has no NeutroAxiom, then it coincides with the PartialAlgebra.

**Theorem 1.3.** *[14] The NeutroAlgebra is a generalization of PartialAlgebra.*

**Theorem 1.4.** *[12] Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let $S$ be a finite or infinite subset of $\mathbb{U}$. If $n$ classical operations (laws and axioms) are defined on $S$ where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.*

**Example 1.5.** (i) Let $X = \mathbb{Z}^+$ and let $f : X \times X \to \mathbb{N}$ be a function defined $\forall x, y \in X$ by $f(x, y) = \sqrt{xy}$ . Then $(X, f)$ is a PartialAlgebra with respect to the ClassicalAxiom of commutativity.

(ii) Let $X = \{1, 2, 3\} \subseteq \mathbb{Z}_4$ and let $*$ be a binary operation defined in the Cayley table below.

| $*$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 2 | 0 | 2 |
| 3 | 3 | 2 | 1 |

Then $(X, *)$ is not a PartialAlgebra since $2 * 2$ is outer defined. However, $(X, *)$ is a NeutroAlgebra.

(iii) $(\mathbb{N}, \div)$ is not a PartialAlgebra eventhough $\div$ is a PartialOperation over $\mathbb{N}$. Axioms of commutativity and associativity are NeutroAxioms and not ClassicalAxioms.

(iv) $(\mathbb{Z}, \div)$ is a NeutroAlgebra.

(v) Let $X = \mathbb{Z} - \{0\}$ and let $f : X \times X \to X$ be a function defined $\forall x, y \in X$ by $f(x, y) = e^{xy}$. Then $(X, f)$ is an AntiAlgebra.

**Definition 1.6.** Let $I$ and $J$ be two ideals in a ring $R$.

(i) The sum of $I$ and $J$ denoted by $I + J$ is defined by

$$I + J = \{x + y : x \in I, y \in J\}.$$

(ii) The product of $I$ and $J$ denoted by $I \times J$ is defined by

$$I \times J = \{xy : x \in I, y \in J\}.$$

(iii) The intersection of $I$ and $J$ denoted by $I \cap J$ is defined by

$$I \cap J = \{x : x \in J \text{ and } x \in J\}.$$

**Lemma 1.7.** *If $I =< m >$ and $J =< n >$ are ideals in a ring $R$, then:*

(i) $I + J = \langle GCD(m, n)\rangle$.

(ii) $IJ =< mn >$.

(iii) $I \cap J = \langle LCM[m, n]\rangle$.

**Theorem 1.8.** *[11] Let $\mathbb{Z}$ be the ring of integers and let $J$ be the collection of all nontrivial ideals in $\mathbb{Z}$. Then $(J, +)$ is an infinite NeutroAlgebra.*

**Example 1.9.** Let $I =< 2 >, J =< 3 >, K =< 4 >, L =< 5 >, M =< 6 >, N =< 7 >$ be ideals in $\mathbb{Z}$. If $X = \{I, K, M\}$ and $Y = \{J, L, N\}$, then:

(i) $(X, +)$ is a ClassicalAlgebra,

(ii) $(Y, +)$ is a NeutroAlgebra.

(iii) $(X, \cap)$ is a NeutrolAlgebra,

(iv) $(Y, \cap)$ is a NeutrolAlgebra,

**Definition 1.10.** Let $N$ be a NeutroAlgebra and let $M$ be a nonempty subset of $N$. $M$ is said to be a NeutrosubAlgebra of $N$ if $M$ is also a NeutroAlgebra under the same operation(s) inherited from $N$.

**Theorem 1.11.** *[11] Let $\mathbb{Z}$ be the ring of integers. Let $J$ be the collection of nontrivial ideals in $\mathbb{Z}$ generated by singleton element $n \in \mathbb{Z} - \{1\}$ and let $S$ be the collection of ideals in $\mathbb{Z}$ generated by the primes $p \in \mathbb{Z} - \{1\}$. Then:*

(i) $(J, +)$ *is a NeutroAlgebra which is not a PartialAlgebra.*

(ii) $(J, \times)$ *is not a NeutroAlgebra.*

(iii) $(S, +)$ *is a NeutrosubAlgebra.*

(iv) $(S, \times)$ *is not a NeutrosubAlgebra, in fact, it is an AntiAlgebra.*

**Theorem 1.12.** *[11] Let $R = \mathbb{Z}_n$ be the ring of integers modulo $n$ where $n$ is a composite such that $6 \leq n < \infty$. Let $B$ be the collection of nontrivial ideals in $R$. Then:*

(i) $(B, +)$ *is a NeutroAlgebra which is neither a PartialAlgebra nor an AntiAlgebra.*

(ii) $(B, \times)$ *is a NeutroAlgebra which is neither a PartialAlgebra nor an AntiAlgebra.*

**Theorem 1.13.** *[11] Let $S = R[x]$ be a polynomial ring where $R = \mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{Z}$ or $\mathbb{Z}_p$ with $p$ a prime. Let $B$ be the collection of all proper ideals in $S$. Then*

(i) $(B, +)$ *is a NeutroAlgebra.*

(ii) $(B, \times)$ *is not a NeutroAlgebra.*

**Theorem 1.14.** *[11] Let $S = \mathbb{Z}_n[x]$ be a polynomial ring where $n$ is a composite. Let $B$ be the collection of all proper ideals in $S$. Then*

(i) $(B, +)$ *is a NeutroAlgebra.*

(ii) $(B, \times)$ *is a NeutroAlgebra.*

## 2. **Main Results**

In this section, we are going to study NeutroAlgebra and AntiAlgebra of ideals in a factor ring. If $I$ is an ideal in a ring $R$ and $M$ is the collection of all nontrivial ideals in the factor ring $R/I$, we want to find conditions under which $(M,\oplus)$, $(M,\otimes)$ and $(M,\cap)$ are NeutroAlgebras and AntiAlgebras where $\oplus$, $\otimes$ and $\cap$ are the usual sum, product and intersection of ideals in $R/I$.

**Theorem 2.1.** *Let $I$ be an ideal in a ring $R$. Then each ideal in $R/I$ is of the form $J/I$ where $J$ is an ideal in $R$ containing $I$.*

**Example 2.2.** Let $R = \mathbb{Z}$ be the ring of integers and let $I =< 24 >$ be an ideal in $\mathbb{Z}$ generated by 24. By Theorem 2.1, $M_1 =< 2 > /I, M_2 =< 4 > /I, M_3 =< 6 > /I, M_4 =< 8 > /I$ are nontrivial ideals in the factor ring $R/I$. If $M = \{M_1, M_2, M_3, M_4\}$, and $\oplus$ is the binary operation of addition of ideals in $M$, then we can generate the following Cayley table:

| $\oplus$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $M_1$ | $M_1$ | $M_1$ | $M_1$ | $M_1$ |
| $M_2$ | $M_1$ | $M_2$ | $M_1$ | $M_1$ |
| $M_3$ | $M_1$ | $M_1$ | $M_3$ | $M_1$ |
| $M_4$ | $M_1$ | $M_2$ | $M_1$ | $M_4$ |

It is clear from the table that $\oplus$ is a ClassicalOperation and therefore, $(M,\oplus)$ is a ClassicalAlgebra and not a NeutroAlgebra.

**Theorem 2.3.** *Let $I =< m >$ be an ideal in $R = \mathbb{Z}$ and let $J =< n >$ be an ideal in $\mathbb{Z}$ containing $I$ where $m \in 2\mathbb{Z}$ with $m \geq 8$ and $n \in 2\mathbb{Z}$ with $n \geq 2$. If $M$ is the collection of all nontrivial ideals in the factor ring $R/I$ of the form $J/I$ and $\oplus$ is the binary operation of addition of ideals in $M$, then:*

(i) *$\oplus$ is a ClassicalOperation.*
(ii) *$(M,\oplus)$ is a ClassicalAlgebra and not a NeutroAlgebra.*

*Proof.* (i) Suppose that $A, B \in M$ are arbitrary. Then $A \oplus B$ is nontrivial and $A \oplus B \in M$ $\forall A, B \in M$. Hence, $\oplus$ is a ClassicalOperation.
(ii) Since $\oplus$ is a ClassicalOperation over $M$, it follows that $(M,\oplus)$ is a ClassicalAlgebra and not a NeutroAlgebra. □

**Example 2.4.** Let $M = \{M_1, M_2, M_3, M_4\}$ be as defined in Example 2.2. If $\otimes$ is the binary operation of multiplication of ideals in $M$, then we can generate the following

Cayley table:

| $\otimes$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $M_1$ | $M_2$ | $M_4$ | outer defined | outer defined |
| $M_2$ | $M_4$ | outer defined | outer defined | outer defined |
| $M_3$ | outer defined | outer defined | outer defined | outer defined |
| $M_4$ | outer defined | outer defined | outer defined | outer defined |

It is clear from the table that $\otimes$ is a NeutroOperation and therefore, $(M, \otimes)$ is a NeutroAlgebra.

**Theorem 2.5.** *Let $I =< m >$ be an ideal in $R = \mathbb{Z}$ and let $J =< n >$ be an ideal in $\mathbb{Z}$ containing $I$ where $m \in 2\mathbb{Z}$ with $m \geq 8$ and $n \in 2\mathbb{Z}$ with $n \geq 2$. If $M$ is the collection of all nontrivial ideals in the factor ring $R/I$ of the form $J/I$ and $\otimes$ is the binary operation of multiplication of ideals in $M$, then:*

(i) $\otimes$ *is a NeutroOperation.*
(ii) $(M, \otimes)$ *is a NeutrolAlgebra.*

*Proof.* (i) Without any loss of generality, there exists at least one duplet $(A, A) \in M$ and at least one duplet $(A, B) \in M$ such that $A \otimes A \in M$ and $A \otimes B \in M$ with the degree of truth (T) and there exists at least one duplet $(C, D) \in M$ such that $C \otimes D \notin M$ with the degree of falsehood (F). Hence, $\otimes$ is a NeutroOperation.
(ii) Since $\otimes$ is a NeutroOperation over $M$, it follows that $(M, \otimes)$ is a NeutroAlgebra. □

**Example 2.6.** Let $X = \{M_1, M_2\}$ and $Y = \{M_3, M_4\}$ be subsets of $M$ where $M$ is the NeutroAlgebra of Example 2.4. Consider the following Cayley tables:

| $\otimes$ | $M_1$ | $M_2$ |
|---|---|---|
| $M_1$ | outer defined | outer defined |
| $M_2$ | outer defined | outer defined |

| $\otimes$ | $M_3$ | $M_4$ |
|---|---|---|
| $M_3$ | outer defined | outer defined |
| $M_4$ | outer defined | outer defined |

It is clear from the tables that both $(X, \otimes)$ and $(Y, \otimes)$ are AntisubAlgebras of $M$.

**Remark 2.7.** Every NeutroAlgebra $(M, \otimes)$ of Theorem 2.5 has at least one AntisubAlgebra.

**Example 2.8.** Let $M = \{M_1, M_2, M_3, M_4\}$ be as defined in Example 2.2. If $\cap$ is the binary operation of intersection of ideals in $M$, then we can generate the following

Cayley table:

| $\cap$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $M_1$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
| $M_2$ | $M_2$ | $M_2$ | outer defined | $M_4$ |
| $M_3$ | $M_3$ | outer defined | $M_3$ | outer defined |
| $M_4$ | $M_4$ | $M_4$ | outer defined | $M_4$ |

It is clear from the table that $\cap$ is a NeutroOperation and therefore, $(M, \otimes)$ is a NeutroAlgebra.

**Theorem 2.9.** *Let $I =< m >$ be an ideal in $R = \mathbb{Z}$ and let $J =< n >$ be an ideal in $\mathbb{Z}$ containing $I$ where $m \in 2\mathbb{Z}$ with $m \geq 8$ and $n \in 2\mathbb{Z}$ with $n \geq 2$. If $M$ is the collection of all nontrivial ideals in the factor ring $R/I$ of the form $J/I$ and $\cap$ is the binary operation of intersection of ideals in $M$, then:*

(i) *$\cap$ is a NeutroOperation.*
(ii) *$(M, \cap)$ is a NeutroAlgebra.*

*Proof.* (i) Let $A =< a > /I \in M$ be arbitrary with $a \in 2\mathbb{Z}$. Then $A \cap A = \langle \text{LCM}[a, a] \rangle /I =< a > /I \in M$. This shows that there exists at least a duplet $(A, A) \in M$ with 100% degree of truth (T). Without any loss of generality, there exists at least a duplet $(B, C) \in M$ such that $B \cap C \in M$ with degree of truth (T) and there exists a duplet $(D, E) \in M$ such that $D \cap E \in M$ with degree of falsehood $(F)$. These show that $\cap$ is a NeutroOperation.
(ii) Since $\cap$ is a NeutroOperation, it follows that $(M, \cap)$ is a NeutroAlgebra. □

**Example 2.10.** Let $X = \{M_1, M_2\}$ and $Y = \{M_3, M_4\}$ be subsets of $M$ where $M$ is the NeutroAlgebra of Example 2.8. Consider the following Cayley tables:

| $\cap$ | $M_1$ | $M_2$ |
|---|---|---|
| $M_1$ | $M_1$ | $M_2$ |
| $M_2$ | $M_2$ | $M_2$ |

| $\cap$ | $M_3$ | $M_4$ |
|---|---|---|
| $M_3$ | $M_3$ | outer defined |
| $M_4$ | outer defined | $M_4$ |

It is clear from the tables that $(X, \cap)$ is a ClassicalsubAlgebra of $(M, \cap)$ while $(Y, \cap)$ is a NeutrosubAlgebra of $(M, \cap)$.

**Remark 2.11.** Every NeutroAlgebra $(M, \cap)$ of Theorem 2.9 has at least one ClassicalsubAlgebra and at least one NeutrosubAlgebra.

**Example 2.12.** Let $R = \mathbb{Z}$ be the ring of integers and let $I =< 1155 >$ be an ideal in $\mathbb{Z}$ generated by 1155. By Theorem 2.1, $M_1 =< 3 > /I, M_2 =< 5 > /I, M_3 =< 7 > /I, M_4 =< 11 > /I$ are nontrivial ideals in the factor ring $R/I$. If $M = \{M_1, M_2, M_3, M_4\}$, and $\oplus$ is the binary operation of addition of ideals in $M$, then we can generate the following Cayley table:

| $\oplus$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $M_1$ | $M_1$ | outer defined | outer defined | outer defined |
| $M_2$ | outer defined | $M_2$ | outer defined | outer defined |
| $M_3$ | outer defined | outer defined | $M_3$ | outer defined |
| $M_4$ | outer defined | outer defined | outer defined | $M_4$ |

It is clear from the table that $\oplus$ is a NeutroOperation and therefore, $(M, \oplus)$ is a NeutroAlgebra.

**Example 2.13.** Let $X = \{M_1, M_2\}$ and $Y = \{M_3, M_4\}$ be subsets of $M$ where $M$ is the NeutroAlgebra of Example 2.12. Consider the following Cayley tables:

| $\oplus$ | $M_1$ | $M_2$ |
|---|---|---|
| $M_1$ | $M_1$ | outer defined |
| $M_2$ | outer defined | $M_2$ |

| $\oplus$ | $M_3$ | $M_4$ |
|---|---|---|
| $M_3$ | $M_3$ | outer defined |
| $M_4$ | outer defined | $M_4$ |

It is clear from the tables that both $(X, \oplus)$ and $(Y, \oplus)$ are NeutrosubAlgebras of $(M, \oplus)$.

**Theorem 2.14.** *Let $I =< p >$ be an ideal in $R = \mathbb{Z}$ and let $J =< q >$ be an ideal in $\mathbb{Z}$ containing $I$ where $p$ and $q$ are distinct prime numbers different from 1. If $M$ is the collection of all nontrivial ideals in the factor ring $R/I$ of the form $J/I$ and $\oplus$ is the binary operation of addition of ideals in $M$, then:*

(i) $\oplus$ *is a NeutroOperation.*
(ii) $(M, \oplus)$ *is a NeutrolAlgebra.*

*Proof.* (i) Let $A =< a >$ and $B =< b >$ be arbitrary elements of $M$ with $a$ and $b$ distinct primes different from 1. Then $A \oplus A = \langle \text{GCD}(a, a) \rangle /I =< a > /I \in M$. Also, $A \oplus B = \langle \text{GCD}(a, b) \rangle /I =< 1 > /I = R/I \notin M$. These show that there exists at least one duplet $(A, A) \in M$ such that $A \oplus A \in M$ with the degree of truth (T) and there exists at least one duplet $(A, B) \in M$ such that $A \oplus B \notin M$ with the degree of falsehood (F). Hence, $\oplus$ is a NeutroOperation.
(ii) Since $\oplus$ is a NeutroOperation over $M$, it follows that $(M, \oplus)$ is a NeutroAlgebra. □

**Remark 2.15.** Every NeutroAlgebra $(M, \oplus)$ of Theorem 2.14 has at least one NeutrosubAlgebra.

**Example 2.16.** Let $M = \{M_1, M_2, M_3, M_4\}$ be as defined in Example 2.12. If $\otimes$ is the binary operation of multiplication of ideals in $M$, then we can generate the following Cayley table:

| $\otimes$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $M_1$ | outer defined | outer defined | outer defined | outer defined |
| $M_2$ | outer defined | outer defined | outer defined | outer defined |
| $M_3$ | outer defined | outer defined | outer defined | outer defined |
| $M_4$ | outer defined | outer defined | outer defined | outer defined |

It is clear from the table that $\otimes$ is an AntiOperation and therefore, $(M, \otimes)$ is an AntiAlgebra.

**Theorem 2.17.** *Let $I =< p >$ be an ideal in $R = \mathbb{Z}$ and let $J =< q >$ be an ideal in $\mathbb{Z}$ containing $I$ where $p$ and $q$ are distinct prime numbers different from 1. If $M$ is the collection of all nontrivial ideals in the factor ring $R/I$ of the form $J/I$ and $\otimes$ is the binary operation of multiplication of ideals in $M$, then:*

(i) $\otimes$ *is an AntiOperation.*
(ii) $(M, \otimes)$ *is an AntiAlgebra.*

*Proof.* (i) Let $A =< a >$ and $B =< b >$ be arbitrary elements of $M$ with $a$ and $b$ distinct primes different from 1. Then $A \otimes A =< aa > /I \notin M$. This shows that $\forall A \in M$, the duplet $(A, A) \notin M$ with the degree of falsehood (F). Also, $A \otimes B =< ab > /I \notin M$. This shows that $\forall A, B \in M$, the duplet $(A, B) \notin M$ with the degree of falsehood (F). Hence, $\otimes$ is an AntiOperation.
(ii) Since $\otimes$ is an AntiOperation over $M$, it follows that $(M, \otimes)$ is an AntiAlgebra. □

**Remark 2.18.** All subAlgebras of AntiAlgebra $(M, \otimes)$ of Theorem 2.17 are all AntisubAlgebras.

**Example 2.19.** Let $M = \{M_1, M_2, M_3, M_4\}$ be as defined in Example 2.12. If $\cap$ is the binary operation of intersection of ideals in $M$, then we can generate the following

Cayley table:

| $\cap$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $M_1$ | $M_1$ | outer defined | outer defined | outer defined |
| $M_2$ | outer defined | $M_2$ | outer defined | outer defined |
| $M_3$ | outer defined | outer defined | $M_3$ | outer defined |
| $M_4$ | outer defined | outer defined | outer defined | $M_4$ |

It is clear from the table that $\cap$ is a NeutroOperation and therefore, $(M, \cap)$ is a NeutroAlgebra.

**Theorem 2.20.** *Let $I =< p >$ be an ideal in $R = \mathbb{Z}$ and let $J =< q >$ be an ideal in $\mathbb{Z}$ containing $I$ where $p$ and $q$ are distinct prime numbers different from 1. If $M$ is the collection of all nontrivial ideals in the factor ring $R/I$ of the form $J/I$ and $\cap$ is the binary operation of intersection of ideals in $M$, then:*

(i) *$\cap$ is a NeutroOperation.*

(ii) *$(M, \cap)$ is a NeutroAlgebra.*

*Proof.* (i) Let $A =< a >$ and $B =< b >$ be arbitrary elements of $M$ with $a$ and $b$ distinct primes different from 1. Then $A \cap A = \langle \text{LCM}[a, a] \rangle /I =< a > /I \in M$. This shows that $\forall A \in M$, the duplet $(A, A) \in M$ with 100% degree of truth (T). Also, $A \cap B = \langle \text{LCM}[a, b] \rangle /I \notin M$. This shows that for $A \neq B$, there exists at least a duplet $(A, B) \notin M$ with the degree of falsehood (F). Hence, $\cap$ is a NeutroOperation.
(ii) Since $\cap$ is a NeutroOperation over $M$, it follows that $(M, \cap)$ is a NeutroAlgebra. □

**Example 2.21.** Let $R = \mathbb{Z}_{12}$ be the ring of integers modulo 12 and let $I =< 6 >$ be an ideal in $R$ generated by 6. By Theorem 2.1, $M_1 =< 2 > / < 6 >, M_2 =< 3 > / < 6 >$ are nontrivial ideals in the factor ring $R/I$. Let $M = \{M_1, M_2\}$ and let $\oplus$, $\otimes$ and $\cap$ be the binary operations of addition, multiplication and intersection of ideals in $M$ respectively. Consider the following Cayley tables:

| $\oplus$ | $M_1$ | $M_2$ |
|---|---|---|
| $M_1$ | $M_1$ | outer defined |
| $M_2$ | outer defined | $M_2$ |

| $\otimes$ | $M_1$ | $M_2$ |
|---|---|---|
| $M_1$ | outer defined | outer defined |
| $M_2$ | outer defined | $M_2$ |

| $\cap$ | $M_1$ | $M_2$ |
|---|---|---|
| $M_1$ | $M_1$ | outer defined |
| $M_2$ | outer defined | $M_2$ |

It is clear from the tables that $\oplus$, $\otimes$ and $\cap$ are NeutroOperations and thus, $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ are NeutroAlgebras.

**Example 2.22.** Let $R = \mathbb{Z}_{24}$ be the ring of integers modulo 24 and let $I =< 12 >$ be an ideal in $R$ generated by 12. By Theorem 2.1, $M_1 =< 2 > / < 12 >, M_2 =< 3 > / < 12 >, M_3 =< 4 > / < 12 >, M_4 =< 6 > / < 12 >$ are nontrivial ideals in the factor ring $R/I$. Let $M = \{M_1, M_2, M_3, M_4\}$ and let $\oplus$, $\otimes$ and $\cap$ be the binary operations of addition, multiplication and intersection of ideals in $M$ respectively. Consider the following Cayley tables:

| $\oplus$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $M_1$ | $M_1$ | outer defined | $M_1$ | $M_1$ |
| $M_2$ | outer defined | $M_2$ | outer defined | $M_2$ |
| $M_3$ | $M_1$ | outer defined | $M_3$ | $M_1$ |
| $M_4$ | $M_1$ | $M_2$ | $M_1$ | $M_4$ |

| $\otimes$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $M_1$ | $M_3$ | $M_4$ | outer defined | outer defined |
| $M_2$ | $M_4$ | outer defined | outer defined | outer defined |
| $M_3$ | outer defined | outer defined | outer defined | outer defined |
| $M_4$ | outer defined | outer defined | outer defined | outer defined |

| $\cap$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $M_1$ | $M_1$ | $M_4$ | $M_3$ | $M_4$ |
| $M_2$ | $M_4$ | $M_2$ | outer defined | $M_4$ |
| $M_3$ | $M_3$ | outer defined | $M_3$ | outer defined |
| $M_4$ | $M_4$ | $M_4$ | outer defined | $M_4$ |

It is clear from the tables that $\oplus$, $\otimes$ and $\cap$ are NeutroOperations and thus, $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ are NeutroAlgebras.

**Theorem 2.23.** *Let $R = \mathbb{Z}_n$ be the ring of integers modulo $n$ where $n$ is a composite such that $12 \leq n < \infty$, let $I =< p >$ be an ideal in $R$ and let $J =< q >$ be an ideal in $R$ containing $I$ where $p, q \notin \{0, 1\}$. If $M$ is the collection of all nontrivial ideals in the factor ring $R/I$ of the form $J/I$, and $\oplus$, $\otimes$ and $\cap$ are respectively the binary operations of addition, multiplication and intersection of ideals in $M$, then:*

(i) $(M, \oplus)$ *is a NeutroAlgebra.*
(ii) $(M, \otimes)$ *is a NeutroAlgebra.*
(iii) $(M, \cap)$ *is a NeutroAlgebra.*

*Proof.* Similar to the proofs of Theorems 2.14 and 2.20 and so omitted. □

**Example 2.24.** Let $R = \mathbb{Z}[x]$ be the ring of polynomials in $\mathbb{Z}$ and let $I =< x^2 + 1 >$ be an ideal in $R$ generated by $x^2 + 1$. By Theorem 2.1, $J =< x^3 + x^2 + x + 1 > /I, K =< x^4 + x^2 > /I$ are nontrivial ideals in the factor ring $R/I$. Let $M = \{J, K\}$ and let $\oplus$, $\otimes$ and $\cap$ be the binary operations of addition, multiplication and intersection of ideals in $M$ respectively. Consider the following Cayley tables:

| $\oplus$ | $J$ | $K$ |
|---|---|---|
| $J$ | $J$ | outer defined |
| $K$ | outer defined | $K$ |

| $\otimes$ | $J$ | $K$ |
|---|---|---|
| $J$ | inner defined | inner defined |
| $K$ | inner defined | inner defined |

| $\cap$ | $J$ | $K$ |
|---|---|---|
| $J$ | inner defined | inner defined |
| $K$ | inner defined | inner defined |

It can be seen from the tables that $(M, \oplus)$ is a NeutroAlgebra whereas $(M, \otimes)$ and $(M, \cap)$ are not NeutroAlgebras but ClassicalAlgebras.

**Theorem 2.25.** *Let $I$ be an ideal in the polynomial ring $R = \mathbb{Z}[x]$ or $\mathbb{Q}[x]$ or $\mathbb{R}[x]$ or $\mathbb{Z}_p[x]$ where $p$ is a prime number and let $J$ be an ideal in $R$ containing $I$. If $M$ is the collection of all nontrivial ideals in the factor ring $R/I$ of the form $J/I$ and $\oplus$, $\otimes$ and $\cap$ are the binary operations of addition, multiplication and intersection of ideals in $M$ respectively. then:*

(i) $(M, \oplus)$ *is a NeutroAlgebra.*
(ii) $(M, \otimes)$ *is a ClassicalAlgebra.*
(iii) $(M, \cap)$ *is a ClassicalAlgebra.*

**Theorem 2.26.** *Let $I$ be an ideal in the polynomial ring $R = \mathbb{Z}_n[x]$ where $n$ is a composite and let $J$ be an ideal in $R$ containing $I$. If $M$ is the collection of all nontrivial ideals in the factor ring $R/I$ of the form $J/I$ and $\oplus$, $\otimes$ and $\cap$ are the binary operations of addition, multiplication and intersection of ideals in $M$ respectively. then:*

(i) $(M, \oplus)$ *is a NeutroAlgebra.*
(ii) $(M, \otimes)$ *is a NeutroAlgebra.*
(iii) $(M, \cap)$ *is a NeutroAlgebra.*

**Example 2.27.** Let $R = \mathbb{Z}_{10}[x]$ be the ring of polynomials in $\mathbb{Z}_{10}$ and let $I =< x + 1 >$ be an ideal in $R$ generated by $x + 1$. By Theorem 2.1, $J =< 2x^2 - 2) > /I, K =< 5x^2 + 5x > /I$ are nontrivial ideals in the factor ring $R/I$. Let $M = \{J, K\}$ and let $\oplus$,

$\otimes$ and $\cap$ be the binary operations of addition, multiplication and intersection of ideals in $M$ respectively. Consider the following Cayley tables:

| $\oplus$ | $J$ | $K$ |
|---|---|---|
| $J$ | $J$ | outer defined |
| $K$ | outer defined | $K$ |

| $\oplus$ | $J$ | $K$ |
|---|---|---|
| $J$ | $J$ | outer defined |
| $K$ | outer defined | $K$ |

| $\oplus$ | $J$ | $K$ |
|---|---|---|
| $J$ | $J$ | outer defined |
| $K$ | outer defined | $K$ |

It can be seen from the tables that $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ are NeutroAlgebras.

## 3. Conclusion

In this paper, we have extended the work done by Kandasamy et al. in [11]. If $I$ is an ideal in a ring $R$ and $M$ is the collection of all nontrivial ideals in the factor ring $R/I$, we have provided conditions under which $(M, \oplus)$, $(M, \otimes)$ and $(M, \cap)$ can be NeutroAlgebras and AntiAlgebras where $\oplus$, $\otimes$ and $\cap$ are the usual sum, product and intersection of ideals in $R/I$. Several examples were provided to illustrate the conditions.

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