Neutrosophic Sets and Systems, Vol. 53, 2023
University of New Mexico

The Neutrosophic Limits

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Abstract: the purpose of this article is to study the neutrosophic limits, where the methods of neutrosophic factorization and neutrosophic rationalization were applied, useful theorems have been proven for facilitating the calculation of the neutrosophic limits. Also, the definition of a positive neutrosophic number was presented, and the necessary condition to find the square root of the neutrosophic number, in addition to studying some special limits and neutrosophic trigonometric limits. Where detailed examples were given to clarify each case.

Keywords: the neutrosophic limits; neutrosophic trigonometric limits; indeterminacy; method of neutrosophic factorization.

1. Introduction

As an alternative to the existing logics, Smarandache proposed the neutrosophic logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3-14]. He presented the definition of the standard form of neutrosophic real number [2-4], studying the concept of the Neutrosophic probability [3-6], the Neutrosophic statistics [4][7], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus [1-9]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [10]. Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side represented with triangular neutrosophic numbers [11]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [12-13]. Y. Alhasan studied the concepts of neutrosophic complex numbers and the general exponential form of a neutrosophic complex [8][15]. On the other hand, M. Abdel-Basset presented study in the science of neutrosophic about an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number [16]. H. Khalid, F. Smarandache and A. Essa have presented a study on a neutrosophic binomial factorial theorem with their Refrains [5].

Paper consists of 5 sections. In 1st section, provides an introduction, in which neutrosophic science review has given. In 2nd section, some definitions, examples of neutrosophic real number and new theorems in neutrosophic limits are discussed. The 3rd section frames methods of neutrosophic factorization, neutrosophic rationalization, and neutrosophic trigonometric limits. The 4th section introduces the definition of a positive neutrosophic number, and the necessary condition to find the square root of the neutrosophic number, in addition to studying some special limits and neutrosophic trigonometric limits. In 5th section, a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic real number [4]

Suppose that w is a neutrosophic real number, then it takes the following standard form: $w = a + bI$ where a, b are real coefficients, and I represent indeterminacy, such $0.\mathrm{I} = 0$ and $\mathrm{I}^n = \mathrm{I}$, for all positive integers n.

2.2. Division of neutrosophic real numbers [4]

Suppose that w_1, w_2 are two neutrosophic numbers, where:

$$w_1 = a_1 + b_1 I \ , \qquad w_2 = a_2 + b_2 I$$

To find $(a_1 + b_1 I) \div (a_2 + b_2 I)$, we can write:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} \equiv x + yI$$

where x and y are real unknowns.

$$a_1 + b_1 I \equiv (a_2 + b_2 I)(x + yI)$$

$$a_1 + b_1 I \equiv a_2 x + (b_2 x + a_2 y + b_2 y)I$$

by identifying the coefficients, we get

$$a_1 = a_2 x$$

$$b_1 = b_2 x + (a_2 + b_{2)} y$$

We obtain unique one solution only, provided that:

$$\begin{vmatrix} a_2 & 0 \\ b_2 & a_2 + b_2 \end{vmatrix} \neq 0 \quad \Rightarrow \quad a_2(a_2 + b_2) \neq 0$$

Hence: $a_2 \neq 0$ and $a_2 \neq -b_2$ are the conditions for the division of two neutrosophic real numbers to exist.

Then:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)}.I$$

2.3 New theorems in neutrosophic limits [5]

Theorem2.3.1 (Binomial Factorial Theorem)

$$\lim_{x \to \infty} \left(I + \frac{1}{x}\right)^x = Ie$$

where I is the literal indeterminacy, $e = 2.7182828$.

Corollary 2.3.1

$$\lim_{x\to 0}(I+x)^{\frac{1}{x}} = Ie$$

Corollary 2.3.2

$$\lim_{x\to\infty}\left(I+\frac{k}{x}\right)^{x} = Ie^{k}$$

where $k > 0$ & $k \neq 0$, I is the literal indeterminacy.

Corollary 2.3.3

$$\lim_{x\to 0}\left(I+\frac{x}{k}\right)^{\frac{1}{x}} = \sqrt[k]{Ie}$$

where $k \neq 0$ & $k > 0$.

Theorem2.3.2

$$\lim_{x\to 0}\frac{(lna)[Ia^{x}-I]}{xlna+lnI} = \frac{lna}{1+lnI}$$

where $a > 0, a \neq 0$.

Corollary 2.3.4

$$\lim_{x\to 0}\frac{Ia^{kx}-I}{x+\frac{lnI}{lna^{k}}} = \frac{k\ lna}{1+lnI}$$

Corollary 2.3.5

$$\lim_{x\to 0}\frac{Ia^{x}-I}{x+lnI} = \frac{1}{1+lnI}$$

Corollary 2.3.6

$$\lim_{x\to 0}\frac{Ia^{kx}-I}{x+\frac{lnI}{k}} = \frac{k}{1+lnI}$$

Theorem2.3.3

$$\lim_{x\to 0}\frac{ln(I+kx)}{x} = k(1+lnI)$$

Theorem2.3.4

Prove that, for any two real numbers a, b

$$\lim_{x\to 0}\frac{Ia^{x}-I}{Ib^{x}-I} = 1$$

where $a, b > 0$ & $a, b \neq 1$.

3. Method of neutrosophic factorization

Suppose $\frac{f(x,I)}{g(x,I)}$ is rational neutrosophic function, if $f(x,I), g(x,I)$ contains some common factors, then we can cancel out the common factors from the numerator and denominator and then put $x = a + bI$
where a , b are real coefficients, and I represent indeterminacy.

Example 3.1
Evaluate:

$$\lim_{x\to 2-3I}\frac{x-2+3I}{x^2-4+3I}$$

Solution:

$$\lim_{x\to 2-3I}\frac{x-2+3I}{x^2-4+3I}=\frac{0}{0}$$

Method1:

$$x^2-4+3I=(x-2+3I)(x+2-3I)$$

$$\lim_{x\to 2-3I}\frac{x-2+3I}{x^2-4+3I}=\lim_{x\to 2-3I}\frac{x-2+3I}{(x-2+3I)(x+2-3I)}$$

$$\lim_{x\to 2-3I}\frac{1}{x+2-3I}=\frac{1}{4-6I}=\frac{1}{4}-\frac{3}{4}I$$

Method2:
by using L'Hôpital's rule

$$\Rightarrow \quad \lim_{x\to 2-3I}\frac{x-2+3I}{x^2-4+3I}=\lim_{x\to 2-3I}\frac{1}{2x}$$

$$=\frac{1}{2(2-3I)}=\frac{1}{4-6I}=\frac{1}{4}-\frac{3}{4}I$$

3.1 The method of neutrosophic rationalization

Example 3.1.1
Evaluate:

$$\lim_{x\to 0+0I}\frac{\sqrt{1-(2+4I)x}-\sqrt{1+(2+4I)x}}{(1+3I)x}$$

Solution:

$$\lim_{x\to 0+0I}\frac{\sqrt{1-(2+4I)x}-\sqrt{1+(2+4I)x}}{(1+3I)x}=\frac{0}{0}$$

Method1:

$$\Rightarrow \quad \lim_{x\to 0+0I}\frac{\sqrt{1-(2+4I)x}-\sqrt{1+(2+4I)x}}{(1+3I)x}$$

$$= \lim_{x\to 0+0I} \frac{\left(\sqrt{1-(2+4I)x}-\sqrt{1+(2+4I)x}\right)\left(\sqrt{1-(2+4I)x}+\sqrt{1+(2+4I)x}\right)}{(1+3I)x\left(\sqrt{1-(2+4I)x}+\sqrt{1+(2+4I)x}\right)}$$

$$= \lim_{x\to 0+0I} \frac{1-(2+4I)x-(1+(2+4I)x)}{(1+3I)x\left(\sqrt{1-(2+4I)x}+\sqrt{1+(2+4I)x}\right)}$$

$$= \lim_{x\to 0+0I} \frac{-(4+8I)x}{(1+3I)x\left(\sqrt{1-(2+4I)x}+\sqrt{1+(2+4I)x}\right)}$$

$$= \lim_{x\to 0+0I} \frac{-4-8I}{(1+3I)\left(\sqrt{1-(2+4I)x}+\sqrt{1+(2+4I)x}\right)}$$

$$= \frac{-2-4I}{1+3I} = -2+\frac{1}{2}I$$

Method2:
by using L'Hôpital's rule

$$\Rightarrow \quad \lim_{x\to 0+0I} \frac{\sqrt{1-(2+4I)x}-\sqrt{1+(2+4I)x}}{(1+3I)x}$$

$$= \lim_{x\to 0+0I} \frac{\frac{-(2+4I)}{2\sqrt{1-(2+4I)x}}-\frac{(2+4I)}{2\sqrt{1+(2+4I)x}}}{1+3I}$$

$$= \frac{\frac{-(2+4I)}{2\sqrt{1-0}}-\frac{(2+4I)}{2\sqrt{1+0}}}{1+3I} = \frac{-1-2I-1-2I}{1+3I}$$

$$= \frac{-2-4I}{1+3I} = -2+\frac{1}{2}I$$

Example 3.1.2
Evaluate:

$$\lim_{x\to 5-I} \frac{1-\sqrt{x-4+I}}{x-5+I}$$

Solution:

$$\lim_{x\to 5-I} \frac{1-\sqrt{x-4+I}}{x-5+I} = \frac{0}{0}$$

Method1:

$$\Rightarrow \quad \lim_{x\to 5-I} \frac{\left(1-\sqrt{x-4+I}\right)\left(1+\sqrt{x-4+I}\right)}{(x-5+I)\left(1+\sqrt{x-4+I}\right)}$$

$$= \lim_{x\to 5-I} \frac{1-(x-4+I)}{(x-5+I)\left(1+\sqrt{x-4+I}\right)}$$

$$= \lim_{x \to 5-I} \frac{-x+5-I}{(x-5+I)\left(1+\sqrt{x-4+I}\right)}$$

$$= \lim_{x \to 5-I} \frac{-(x-5+I)}{(x-5+I)\left(1+\sqrt{x-4+I}\right)}$$

$$= \lim_{x \to 5-I} \frac{-1}{1+\sqrt{x-4+I}} = \frac{-1}{2}$$

Method2:
by using L'Hôpital's rule

$$\Rightarrow \quad \lim_{x \to 5-I} \frac{1-\sqrt{x-4+I}}{x-5+I}$$

$$= \lim_{x \to 5-I} \frac{\frac{-1}{2\sqrt{x-4+I}}}{1} = \lim_{x \to 5-I} \frac{-1}{2\sqrt{x-4+I}} = \frac{-1}{2}$$

Example 3.1.3

Evaluate:

$$\lim_{x \to a+bI} \frac{\sqrt{a+bI+2x}-\sqrt{3x}}{\sqrt{3a+3bI+x}-2\sqrt{x}}$$

Solution:

$$\lim_{x \to a+bI} \frac{\sqrt{a+bI+2x}-\sqrt{3x}}{\sqrt{3a+3bI+x}-2\sqrt{x}} = \frac{0}{0}$$

$$\Rightarrow \quad \lim_{x \to a+bI} \frac{\sqrt{a+bI+2x}-\sqrt{3x}}{\sqrt{3a+3bI+x}-2\sqrt{x}}$$

$$= \lim_{x \to a+bI} \frac{\left(\sqrt{a+bI+2x}-\sqrt{3x}\right)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}{\left(\sqrt{3a+3bI+x}-2\sqrt{x}\right)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI} \frac{a+bI+2x-3x}{\left(\sqrt{3a+3bI+x}-2\sqrt{x}\right)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI} \frac{a+bI-x}{\left(\sqrt{3a+3bI+x}-2\sqrt{x}\right)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)} = \frac{0}{0}$$

$$\Rightarrow \quad \lim_{x \to a+bI} \frac{a+bI-x}{\left(\sqrt{3a+3bI+x}-2\sqrt{x}\right)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)} \frac{\sqrt{3a+3bI+x}+2\sqrt{x}}{\sqrt{3a+3bI+x}+2\sqrt{x}}$$

$$= \lim_{x \to a+bI} \frac{(a+bI-x)\left(\sqrt{3a+3bI+x}+2\sqrt{x}\right)}{(3a+3bI+x-4x)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x\to a+bI} \frac{(a+bI-x)\left(\sqrt{3a+3bI+x}+2\sqrt{x}\right)}{3(a+bI-x)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x\to a+bI} \frac{\sqrt{3a+3bI+x}+2\sqrt{x}}{3\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)} = \frac{\sqrt{3a+3bI+a+bI}+2\sqrt{a+bI}}{3\left(\sqrt{a+bI+2(a+bI)}+\sqrt{3(a+bI)}\right)}$$

$$= \frac{\sqrt{4(a+bI)}+2\sqrt{a+bI}}{3\left(\sqrt{3(a+bI)}+\sqrt{3(a+bI)}\right)} = \frac{4\sqrt{a+bI}}{6\sqrt{3(a+bI)}}$$

$$= \frac{2\sqrt{a+bI}}{3\sqrt{3}\sqrt{a+bI}} = \frac{2}{3\sqrt{3}}$$

3.2 Neutrosophic trigonometric limits

1) $\lim_{x\to 0+0I} sin(a+bI)\,x = 0$

2) $\lim_{x\to 0+0I} cos(a+bI)\,x = 0$

3) $\lim_{x\to 0+0I} \frac{sin(a+bI)x}{x} = a+bI$

Proof (3):

Put $(a+bI)x = y \quad \Longrightarrow \quad x = \frac{1}{a+bI}y$

When $x \longrightarrow 0+0I$ then: $y \longrightarrow 0+0I$

$$\Longrightarrow \quad \lim_{x\to 0+0I} \frac{sin(a+bI)\,x}{x} = \lim_{y\to 0+0I} \frac{siny}{\frac{1}{a+bI}y} = (a+bI) \lim_{y\to 0+0I} \frac{siny}{y} = a+bI$$

4) $\lim_{x\to 0+0I} \frac{x}{sin(a+bI)x} = \frac{1}{a+bI} = \frac{1}{a} - \frac{b}{a(a+b)}I$

Where $a\,,b$ are real coefficients, $a_2 \neq 0$ $and\; a_2 \neq -b_2$, I represent indeterminacy.

Proof (4):

Put $(a+bI)x = y \quad \Longrightarrow \quad x = \frac{1}{a+bI}y$

When $x \longrightarrow 0+0I$ then: $y \longrightarrow 0+0I$

$$\Longrightarrow \quad \lim_{x\to 0+0I} \frac{x}{sin(a+bI)\,x} = \lim_{y\to 0+0I} \frac{\frac{1}{a+bI}y}{siny}$$

$$= \frac{1}{a+bI} \lim_{y\to 0+0I} \frac{siny}{y}$$

$$= \frac{1}{a+bI} = \frac{1}{a} - \frac{b}{a(a+b)}I$$

5) $\lim_{x\to 0+0I} \frac{tan(a+bI)x}{x} = a+bI$

6) $\lim_{x\to 0+0I} \frac{x}{tan(a+bI)x} = \frac{1}{a+bI} = \frac{1}{a} - \frac{b}{a(a+b)}I$

Where a, b are real coefficients, $a_2 \neq 0$ $and\ a_2 \neq -b_2$, I represent indeterminacy.

We can prove 5 and 6 the same method in 3, 4.

Example 3.2.1

1) $\lim_{x\to 0+0I} \frac{sin(5+4I)x}{(6-7I)x} = \frac{5+4I}{6-7I} \lim_{x\to 0+0I} \frac{sin(5+4I)x}{(5+4I)x} = \frac{5+4I}{6-7I} = \frac{5}{6} - \frac{59}{6}I$

2) $\lim_{x\to 0+0I} \frac{x}{sin(1+2I)x} = \frac{1}{1+2I} \lim_{x\to 0+0I} \frac{(1+2I)x}{sin(1+2I)x} = \frac{1}{1+2I} = 1 - \frac{2}{3}I$

3) $\lim_{x\to 0+0I} \frac{sin(3+4I)x}{tan(2-8I)x} = \lim_{x\to 0+0I} \frac{\frac{sin(3+4I)x}{x}}{\frac{tan(2-8I)x}{x}} = \frac{\lim_{x\to 0+0I} \frac{sin(3+4I)x}{x}}{\lim_{x\to 0+0I} \frac{tan(2-8I)x}{x}} = \frac{3+4I}{2-8I} = \frac{3}{2} - \frac{8}{3}I$

4) $\lim_{x\to 0} \frac{1-cos(10+4I)x}{x^2} = \lim_{x\to 0+0I} \frac{2sin^2(5+2I)x}{x^2} = 2 \lim_{x\to 0+0I} \left(\frac{sin(5+2I)x}{x}\right)^2 = 2(5+2I)^2 = 50 + 28I$

5) $\lim_{x\to 0+0I} \frac{(3-5I)x - sin(2+1I)\,x}{(1-4I)x} = \lim_{x\to 0+0I} \left(\frac{(3-5I)x}{(1-4I)x} - \frac{sin(2+1I)\,x}{(1-4I)x}\right)$

$= \lim_{x\to 0+0I} \left(\frac{3-5I}{1-4I} - \frac{sin(2+1I)x}{(1-4I)x}\right) = \frac{3-5I}{1-4I} - \left(\frac{2+1I}{1-4I}\right) = 1 + \frac{2}{3}I$

4.

Deffinition4.1

Let the neutrosophic number $a + bI$, then $a + bI$ is positive neutrosophic number If it fulfills one of the following conditions:

1) $a > 0, b > 0\ and\ I > 0$
2) $a > 0, b < 0\ and\ I < 0$
3) $a < 0\ and \begin{cases} b > 0\ and\ I > \frac{a}{b} \\ b < 0\ and\ I < \frac{a}{b} \end{cases}$

Example 4.1

1) $5 + 3I,\ I > 0\ then: 5 + 3I > 0$
2) $1 - 3I,\ I < 0\ then: 1 - 3I > 0$
3) $-7 + 3I,\ I > \frac{7}{3}\ then: -7 + 3I > 0$
4) $-4 - I,\ I < -4\ then: -4 - I > 0$

Deffinition4.2

Let the neutrosophic real number $a + bI$, then $a + bI$ has a square root if it fulfills the following condition:

$$a \geq 0 \; and \; a + b \geq 0$$

Where:

$$\sqrt{a + bI} = \sqrt{a} + \left(-\sqrt{a} + \sqrt{a + b}\right)I$$

Or: $= \sqrt{a} - \left(\sqrt{a} + \sqrt{a + b}\right)I$

Or: $= -\sqrt{a} + \left(\sqrt{a} + \sqrt{a + b}\right)I$

Or: $= -\sqrt{a} + \left(\sqrt{a} - \sqrt{a + b}\right)I$

For fulfills the real square root condition $\sqrt{a} \; and \; \sqrt{a + b}$, the neutrosophic number $a + bI$ must fulfills the following condition:

$$a \geq 0 \; and \; a + b \geq 0$$

Example 4.2

1) $4 - 3I$, has a square root
2) $1 - 3I$, has not a square root because:

   $$a + b = 1 - 3 = -2 < 0$$

3) $-4 + 2I$, has not a square root because:

   $$a = -4 < 0 \; and \; a + b = -6 < 0$$

4.1 Some Special Limits

1) $\lim_{x \to 0+0I} e^{(a+bI)x} = 1$

2) $\lim_{x \to 0+0I} \dfrac{e^{(a+bI)x} - 1}{x} = a + bI$

Proof (2):

Put $(a + bI)x = y \quad \Longrightarrow \quad x = \frac{1}{a+bI} y$

When $x \longrightarrow 0 + 0I$ then: $y \longrightarrow 0 + 0I$

$$\Longrightarrow \quad \lim_{x \to 0+0I} \frac{e^{(a+bI)x} - 1}{x} = \lim_{y \to 0+0I} \frac{e^{y} - 1}{\frac{1}{a + bI} y}$$

$$= (a + bI) \lim_{y \to 0+0I} \frac{e^{y} - 1}{y} = (a + bI)(1) = a + bI$$

3) $\lim_{x\to 0+0I} \frac{ln(1+(a+bI)x)}{x} = a+bI$

Proof (3):

Put $(a+bI)x = y \quad \Rightarrow \quad x = \frac{1}{a+bI}y$

When $x \longrightarrow 0+0I$ then: $y \longrightarrow 0+0I$

$$\Rightarrow \quad \lim_{x\to 0+0I} \frac{ln(1+(a+bI)x)}{x} = \lim_{y\to 0+0I} \frac{ln(1+y)}{\frac{1}{a+bI}y}$$

$$= (a+bI) \lim_{y\to 0+0I} \frac{ln(1+y)}{y} = (a+bI)(1) = a+bI$$

4) $\lim_{x\to 0+0I} \frac{(a+bI)^x-1}{x} = ln(a+bI) \quad ; \quad a+bI > 0$

Proof (4):

Put $(a+bI)^x - 1 = y \quad \Rightarrow \quad (a+bI)^x = y+1$

$$ln(a+bI)^x = ln(1+y)$$

$$x\, ln(a+bI) = ln(1+y)$$

$$x = \frac{1}{ln(a+bI)} ln(1+y)$$

When $x \longrightarrow 0+0I$ then: $y \longrightarrow 0+0I$

$$\Rightarrow \quad \lim_{x\to 0+0I} \frac{(a+bI)^x-1}{x} = \lim_{x\to 0+0I} \frac{y}{\frac{1}{ln(a+bI)} ln(1+y)}$$

$$= ln(a+bI) \lim_{x\to 0+0I} \frac{y}{ln(1+y)} = ln(a+bI)\,(1) = ln(a+bI)$$

Corollary 4.1:

$$\lim_{x\to 0+0I} \frac{(a+bI)^x-1}{(c+dI)^x-1} = \frac{ln(a+bI)}{ln(c+dI)} \quad ; \quad a+bI > 0 \;\; and \;\; c+dI > 0$$

Proof:

$$\lim_{x\to 0+0I} \frac{\frac{(a+bI)^x-1}{x}}{\frac{(c+dI)^x-1}{x}} = \frac{\lim_{x\to 0+0I} \frac{(a+bI)^x-1}{x}}{\lim_{x\to 0+0I} \frac{(c+dI)^x-1}{x}} = \frac{ln(a+bI)}{ln(c+dI)}$$

Example 4.1.1

1) $\lim_{x\to 0+0I} e^{(4+5I)x} = 1$

2) $\lim_{x\to 0+0I} \frac{e^{(1+3I)x}-1}{x} = 1+3I$

3) $\lim\limits_{x\to 0+0I} \frac{(5+4I)^x - 1}{x} = ln(5+4I) \quad ; \ I > 0$

4) $\lim\limits_{x\to 0+0I} \frac{(9-4I)^x - 1}{e^{(1+3I)x} - 1} \quad ; \ I < 0$

$$\lim_{x\to 0+0I} \frac{(9-4I)^x - 1}{e^{(7+I)x} - 1} = \lim_{x\to 0+0I} \frac{\frac{(9-4I)^x - 1}{x}}{\frac{e^{(7+I)x} - 1}{x}}$$

$$= \frac{\lim\limits_{x\to 0+0I} \frac{(9-4I)^x - 1}{x}}{\lim\limits_{x\to 0+0I} \frac{e^{(7+I)x} - 1}{x}} = \frac{ln(9-4I)}{7+I} = \left(\frac{1}{7} - \frac{1}{56}I\right) ln(9-4I)$$

5) $\lim\limits_{x\to 0+0I} \frac{(5-4I)^x - 1}{(6-2I)^x - 1} = \frac{ln(5-4I)}{ln(6-2I)} \quad ; \quad I < 0$

6) $\lim\limits_{x\to 0+0I} \frac{(-7+3I)^x - 1}{x} = ln(-7+3I) \ ; \ -7+3I, \quad I > \frac{7}{3}$

7) $\lim\limits_{x\to 0+0I} \frac{ln(1+(3+3I)x)}{x} = 3+3I$

8) $\lim\limits_{x\to 0+0I} \frac{e^{(-4+2I)x} - 1}{\sin(1+2I)\,x} = \lim\limits_{x\to 0+0I} \frac{\frac{e^{(-4+2I)x} - 1}{x}}{\frac{\sin(1+2I)\,x}{x}}$

$$= \lim_{x\to 0+0I} \frac{\frac{e^{(-4+2I)x} - 1}{x}}{\frac{sin(1+2I)x}{x}} = \frac{\lim\limits_{x\to 0+0I} \frac{e^{(-4+2I)x} - 1}{x}}{\lim\limits_{x\to 0+0I} \frac{sin(1+2I)x}{x}} = \frac{-4+2I}{1+2I} = -4 + \frac{10}{3}I$$

10) $\lim\limits_{x\to 0+0I} \frac{(3+I)^x - (5+2I)^x}{x} \quad ; \ I > 0$

$$\lim_{x\to 0+0I} \frac{(3+I)^x - (5+2I)^x}{x} = \lim_{x\to 0+0I} \frac{(3+I)^x - (5+2I)^x - 1 + 1}{x}$$

$$= \lim_{x\to 0+0I} \frac{(3+I)^x - 1 - ((5+2I)^x - 1)}{x} = \lim_{x\to 0+0I} \frac{(3+I)^x - 1}{x} - \lim_{x\to 0+0I} \frac{(5+2I)^x - 1}{x}$$

$$= ln(3+I) - \ln(5+2I) = \ln\left(\frac{3+I}{5+2I}\right) = \ln\left(\frac{3}{5} - \frac{1}{35}I\right)$$

Theorem 4.1:

$$\lim_{x\to 0+0I} \frac{ln\big(I + (a+bI)x\big)}{x} = (a+bI)\,(1+lnI)$$

Proof:

$$\lim_{x\to 0+0I}\frac{ln\big(I+(a+bI)x\big)}{x}=\lim_{x\to 0+0I}\frac{ln\big(I+(a+bI)x\big)-lnI+lnI}{x}$$

$$Let \quad y=ln(I+(a+bI)x)-lnI \quad \Rightarrow \quad y+lnI=ln(I+(a+bI)x)$$

$$e^{y+lnI}=e^{ln(I+(a+bI)x)} \quad \Rightarrow \quad e^{y}e^{lnI}=I+(a+bI)x$$

$$Ie^{y}-I=(a+bI)x \quad \Rightarrow \quad x=\frac{Ie^{y}-I}{a+bI}$$

$$y\longrightarrow 0+0I \quad as \quad x\longrightarrow 0+0I$$

$$\Rightarrow \qquad \lim_{x\to 0+0I}\frac{ln\big(I+(a+bI)x\big)}{x}=\lim_{y\to 0+0I}\frac{y+lnI}{\frac{Ie^{y}-I}{a+bI}}$$

$$=\lim_{y\to 0+0I}\frac{a+bI}{\frac{Ie^{y}-I}{y+lnI}}=\frac{a+bI}{\lim_{y\to 0+0I}\left(\frac{Ie^{y}-I}{y+lnI}\right)}$$

$$=\frac{a+bI}{\frac{1}{1+lnI}}=(a+bI)(1+lnI)$$

Example 4.3

1) $\lim_{x\to 0+0I}\frac{ln(I+(3+5I)x)}{x}=(3+5I)\,(1+lnI)$

2) $\lim_{x\to 0+0I}\frac{ln(1+(1+2I)x)}{ln(I+(6+4I)x)}=\lim_{x\to 0+0I}\frac{\frac{ln(1+(1+2I)x)}{x}}{\frac{ln(I+(6+4I)x)}{x}}=\frac{\lim_{x\to 0+0I}\frac{ln(1+(1+2I)x)}{x}}{\lim_{x\to 0+0I}\frac{ln(I+(6+4I)x)}{x}}$

$$=\frac{1+2I}{(6+4I)(1+lnI)}=\left(\frac{1}{6}+\frac{2}{15}\right)\frac{1}{1+lnI}$$

Theorem 4.2:

$$\lim_{x\to 0+0I}\frac{ln(a+bI)\,[I(a+bI)^{x}-I]}{x\ln(a+bI)-lnI}=\frac{ln(a+bI)}{1+lnI} \quad ; \; a+bI>0$$

Proof:

Let $y=I(a+bI)^{x}-I \quad \Rightarrow \quad y+I=I(a+bI)^{x}$

$y\longrightarrow 0+0I \;\; as \;\; x\longrightarrow 0+0I$

$\Rightarrow \quad ln(y+I)=lnI+xln(a+bI) \quad \Rightarrow \quad x=\frac{ln(y+I)-lnI}{ln(a+bI)}$

Then:

$$\lim_{x\to 0+0I}\frac{ln(a+bI)\,[I(a+bI)^x-I]}{x\,ln(a+bI)-lnI}=\lim_{x\to 0+0I}\frac{I(a+bI)^x-I}{x-\dfrac{lnI}{ln(a+bI)}}=\lim_{y\to 0+0I}\frac{y}{\dfrac{\ln(y+I)-lnI}{ln(a+bI)}-\dfrac{lnI}{ln(a+bI)}}$$

$$\lim_{y\to 0+0I}\frac{ln(a+bI)}{\dfrac{ln(y+I)}{y}}=\frac{ln(a+bI)}{\lim\limits_{y\to 0+0I}\dfrac{ln(y+I)}{y}}=\frac{ln(a+bI)}{1+lnI}$$

Theorem 4.3:

$$\lim_{x\to 0+0I}\frac{I(a+bI)^x-I}{I(c+dI)^x-I}=1\ \ ;\quad a+bI>0\ \ and\quad c+dI>0$$

Proof:

$$\lim_{x\to 0+0I}\frac{I(a+bI)^x-I}{I(c+dI)^x-I}=\lim_{x\to 0+0I}\frac{\dfrac{ln(a+bI)[I(a+bI)^x-I]}{x\,ln(a+bI)-lnI}}{\dfrac{ln(c+dI)[I(c+dI)^x-I]}{xln(c+dI)-lnI}}.\frac{\dfrac{x\,ln(a+bI)-lnI}{ln(a+bI)}}{\dfrac{xln(c+dI)-lnI}{ln(c+dI)}}$$

$$=\frac{\lim\limits_{x\to 0+0I}\dfrac{ln(a+bI)[I(a+bI)^x-I]}{x\,ln(a+bI)-lnI}}{\lim\limits_{x\to 0+0I}\dfrac{ln(c+dI)[I(c+dI)^x-I]}{xln(c+dI)-lnI}}.\frac{\lim\limits_{x\to 0+0I}(x\,ln(a+bI)-lnI)}{\lim\limits_{x\to 0+0I}(xln(c+dI)-lnI)}.\frac{ln(c+dI)}{ln(a+bI)}$$

$$=\frac{\dfrac{ln(a+bI)}{1+lnI}}{\dfrac{ln(c+dI)}{1+lnI}}.\frac{lnI}{lnI}.\frac{ln(c+dI)}{ln(a+bI)}=1$$

Example 4.4

$$\lim_{x\to 0+0I}\frac{ln(5+3I)\,[I(5+3I)^x-I]}{x\ln(5+3I)-lnI}=\frac{ln(5+3I)}{1+lnI}\ ;\ I>0$$

Corollary 4.2:

$$\lim_{x\to\infty}\left[I+\frac{a}{x-b}\right]^x=I^{a+b}e^a\ ;\ a+b>0$$

Note: if $a+b=0$ then $I^{a+b}=I^0$ and if $a+b<0$ then $I^{a+b}=\frac{1}{I^{-(a+b)}}=\frac{1}{I}$ that is from forms of the indeterminate forms in neutrosophic calculus.

Proof:

$$y=\frac{a}{x-b}\quad\Rightarrow\ xy-by=a\quad\Rightarrow\ x=\frac{a}{y}+b$$

$$y\longrightarrow 0\ \ as\ \ x\longrightarrow\infty$$

$$\Rightarrow\ \lim_{x\to\infty}\left[I+\frac{a}{x-b}\right]^x=\lim_{y\to 0}[I+y]^{\frac{a}{y}+b}=\left(\lim_{y\to 0}[I+y]^{\frac{1}{y}}\right)^a.\lim_{y\to 0}[I+y]^b$$

$$=(Ie)^a.I^b=I^{a+b}e^a$$

Corollary 4.3:

$$\lim_{x\to\infty}\left[I+\frac{a}{x-b}\right]^{kx} = I^{k(a+b)}e^{ka} \ ;\ a+b>0 \ \&\ k\neq 0$$

Proof:

$$y=\frac{a}{x-b} \quad\Rightarrow\quad xy-by=a \quad\Rightarrow\quad x=\frac{a}{y}+b$$

$$y\longrightarrow 0 \ \ as \ \ x\longrightarrow\infty$$

$$\Rightarrow \lim_{x\to\infty}\left[I+\frac{a}{x-b}\right]^{x} = \lim_{y\to 0}[I+y]^{k\left(\frac{a}{y}+b\right)} = \left(\lim_{y\to 0}[I+y]^{\frac{1}{y}}\right)^{ka}.\lim_{y\to 0}[I+y]^{kb}$$

$$=(Ie)^{ka}.I^{kb} = I^{k(a+b)}e^{ka}$$

Example 4.5

1) $\lim_{x\to\infty}\left[I+\frac{5}{x-4}\right]^{x}$

$$y=\frac{5}{x-4} \quad\Rightarrow\quad x=\frac{5}{y}+4$$

$$y\longrightarrow 0 \ \ as \ \ x\longrightarrow\infty$$

$$\Rightarrow \lim_{x\to\infty}\left[I+\frac{5}{x-4}\right]^{x} = \lim_{y\to 0}[I+y]^{\frac{5}{y}+4} = \left(\lim_{y\to 0}[I+y]^{\frac{1}{y}}\right)^{5}.\lim_{y\to 0}[I+y]^{4} = (Ie)^{5}.I^{4} = I^{9}e^{5} = Ie^{5}$$

2) $\lim_{x\to\infty}\left[I+\frac{1}{x-2}\right]^{\frac{x}{2}}$

$$y=\frac{1}{x-2} \quad\Rightarrow\quad x=\frac{1}{y}+2$$

$$y\longrightarrow 0 \ \ as \ \ x\longrightarrow\infty$$

$$\Rightarrow \lim_{x\to\infty}\left[I+\frac{1}{x-2}\right]^{\frac{x}{2}} = \lim_{y\to 0}[I+y]^{\frac{1}{2}\left(\frac{1}{y}+2\right)} = \left(\lim_{y\to 0}[I+y]^{\frac{1}{y}}\right)^{\frac{1}{2}}.\lim_{y\to 0}[I+y]$$

$$=(Ie)^{\frac{1}{2}}.I = \sqrt{Ie}\,I = \sqrt{I}\sqrt{e}\,I = \pm I\sqrt{e}$$

Where: $\sqrt{I}=\pm I$

5. Conclusions

Limits are one of the important principles of calculus. It is concerned with the study of derivation by studying the basic concepts of infinitesimal quantities. This led us to study the neutrosophic limits. Where the methods of neutrosophic factorization and neutrosophic rationalization were applied, in addition to introduce definition of the positive neutrosophic number, and the necessary condition to find the square root of the neutrosophic number. Also, studying some special limits and neutrosophic trigonometric limits. This paper is considered an introduction of the neutrosophic calculus.

Acknowledgments: This publication was supported by the Deanship of Scientific Research at Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.

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Received: Sep 10, 2022. Accepted: Dec 20, 2022