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Multi-Valued Multi-Polar Neutrosophic Sets with an application in Multi-Criteria Decision-Making

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Abstract. This research directs to obtain optimum fuzzy soft constants through Bonferroni mean and TOPSIS with the initial data represented in terms of multi-valued m-polar neutrosophic soft set. Multi-valued m-polar neutrosophic soft set is defined in this paper, which is the generalization of m-polar neutrosophic soft set, obtained by combining it with multi-valued neutrosophic soft set. Optimum fuzzy soft constants play a fundamental role for the construction of the system of differential equations which helps to observe the experts' future attitudes. Sometimes experts feel a requirement to rethink their choices or decisions due to the observation of others' choice especially when others choose different alternatives. After the individual decisions of experts, an analysis of experts' attitudes is produced by using phase portraits and line graphs of the system of differential equations. This analysis can also be provided by using system of differential equations with fuzzy initial conditions. To find the multi-valued m-polar neutrosophic Bonferroni mean, some basic operations on the elements of the defined set are introduced. An illustrative example is given where a system of two differential equations is developed for attitude analysis of two persons with independent variable t.

Keywords: Multi valued neutrosophic set; Multi polar neutrosophic set; Bonferroni mean; Fuzzy soft differential equations.

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1. **Introduction**

Fuzzy sets have been conveniently utilized to deal with a plethora of problems regarding to uncertainties since when it was introduced by Zadeh [10]. It allocates each element of a set with a membership degree in the real standard $[0, 1]$. Intuitionistic fuzzy set (IFS) was introduced by Atanassov [11] which generalizes the concept of fuzzy set and handles some complicated fuzzy information in multi-criteria decision making (MCDM). IFS determines the membership and non-membership degrees for each element of a set. The concept of IFS was extended by Atanassov and Gargov [12] to interval-valued intuitionistic fuzzy set (IVIFS) which was applied for MCDM methods by several authors [13–18]. Despite of a number of research achievements on IFS, there is a need of indeterminate information. Smarandache [1] proposed an indeterminacy membership function which leads to the neutrosophic set (NS). NS generalizes the fuzzy set and IFS. Hesitant fuzzy set (HFS) was defined by Torra [20] which is identified by a function h_A on a universe U that returns a subset of $[0, 1]$. Many extensions of fuzzy set were further extended by combining with hesitant fuzzy set to Interval-valued hesitant fuzzy set (IVHFS) [25], hesitant fuzzy soft set (HFSS) [21], Interval-valued hesitant fuzzy soft set (IVHFSS) [22], dual hesitant fuzzy set (DHFS) [23], dual hesitant fuzzy soft set (DHFSS) [24], interval-valued dual hesitant fuzzy set (IVDHFS) [26] and some others. All these extended representations of hesitant fuzzy set have a substantial amount of research work for MCDM [27–29, 49]. Single valued neutrosophic set (SVNS) is an NS for which membership function, indeterminacy function and falsity function assign a single value from the interval $[0, 1]$ for each element of a set [2,45]. Interval-valued neutrosophic set (IVNS) involves the functions (membership, indeterminacy, non-membership) assigning the intervals from the interval $[0, 1]$ for each element [43]. NS has remarkably contributed in MCDM [44,47], and recently in TOPSIS [46]. Sometimes, decision makers hesitate to assign a single value to membership, non-membership or indeterminacy functions. They may suggest two or more values to these functions. HFS, IVHFS, DHFS and multi-valued neutrosophic set (MVNS) [3] facilitates those problems.

Bipolar fuzzy set [50] is an extension of a fuzzy set whose membership degree ranges from -1 to 1, It represents the double-sided uncertainties (e.g. positive-negative, yes-no, gains-loses,bright-dim, effect-side effect, etc.). These two sides are reciprocally related. Some bipolar representations with their applications have been done by different authors [30–33]. Chen *et al.* [34] presented a multi-polar fuzzy set which is an abstraction of a bipolar fuzzy set. They also explained some real world problems involving multi-agent, multi-attribute, multi-object and multi-index information. Deli *et al.* [5] defined multi-polar neutrosophic soft set and Saeed *et al.* [9] presented some operations on this set.

Bonferroni mean (BM) and geometric Bonferroni mean (WBM) are the aggregation operators which generalize arithmetic mean and geometric mean respectively [35]. BM and WBM represent the interrelationships between the arguments of individuals and have some properties discussed by Yager [36], Xu and Yager [37] and Xia *et al.* [39]. Multi-valued neutrosophic Bonferroni mean (MVNBM) was defined by Liu *et al.* [3] and some of its applications in multiple attribute group decision-making are also presented. Hesitant fuzzy Bonferroni mean (HFBM) was defined by Zhu *et al.* [38] which facilitates to calculate BM for hesitant fuzzy elements.

Beg *et al.* [41] utilized HFBM to analyze the human attitude by developing fuzzy soft differential equations. This investigation along with some others [42, 48] guides us to think about the changes in attitudes or experts' interpersonal influences after the decisions.

In this paper multi-valued m-polar neutrosophic set (MVmNS) is defined by combining the multi-valued neutrosophic set (MVNS) and m-polar neutrosophic set (mNS). Then operational laws are defined for its elements which lead to formulate multi-valued m-polar neutrosophic Bonferroni mean (MVmNBM) and multi-valued m-polar neutrosophic weighted Bonferroni mean (MVmNWBM) operators which are the extension of multi-valued neutrosophic Bonferroni mean (MVNBM) and multi-valued neutrosophic weighted Bonferroni mean (MVNWBM) operators [3] respectively. Then by utilizing the score values of MVmNWBM and coefficients of relative closeness obtained through TOPSIS for each alternative, a system of fuzzy soft differential equations is constructed to observe the change in experts' attitudes. Another contribution of this research work is the utilization of system of differential equations with fuzzy initial conditions.

2. **Preliminaries**

2.1. *Neutrosophic Set*

Neutrosophy is a branch of Philosophy and a basis of neutrosophic set. Neutrosophy considers a unit "A" in relation to "anti-A" and "neither A nor anti-A". Smarandache presented the neutrosophic set with some applications [1].

Definition 2.1. [2] Let Z be a universal set. A single valued neutrosophic set (SVNS) X is defined as:

$X = \{z, (T_X(z), I_X(z), F_X(z)) : z \in Z\}$,

where, $T_X(z)$, $I_X(z)$ and $F_X(z)$ are three real values in $[0, 1]$, denoting the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of the element $z \in Z$ to the set X respectively, satisfying

$0 \leq T_X(z) + I_X(z) + F_X(z) \leq 3$

for all $z \in Z$.

Definition 2.2. [3] Let Z be a universal set. An MVNS X is defined as:

$X = \{z, (\widetilde{T}_X(z), \widetilde{I}_X(z), \widetilde{F}_X(z)) : z \in Z\}$,

where, $\widetilde{T}_X(z)$, $\widetilde{I}_X(z)$ and $\widetilde{F}_X(z)$ are three collections of discrete real values in $[0, 1]$, denoting the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of the element $z \in Z$ to the set X respectively, satisfying

$0 \leq \gamma, \mu, \varphi \leq 1$, $0 \leq \gamma^+ + \mu^+ + \varphi^+ \leq 3$ and $\gamma \in \widetilde{T}_X(z)$, $\gamma^+ \in \sup \widetilde{T}_X(z)$, $\mu \in \widetilde{I}_X(z)$, $\mu^+ \in \sup \widetilde{I}_X(z)$, $\varphi \in \widetilde{F}_X(z)$, $\varphi^+ \in \sup \widetilde{F}_X(z)$.

An element $\widetilde{n}$ of an MVNS X can have the following expression:
$\widetilde{n} = \left(\widetilde{T}_X(z), \widetilde{I}_X(z), \widetilde{F}_X(z)\right)$ for some $z \in Z$, where
$\widetilde{T}_X(z) = \{\gamma, \gamma \in [0,1]\}$,
$\widetilde{I}_X(z) = \{\mu, \mu \in [0,1]\}$,
$\widetilde{F}_X(z) = \{\varphi, \varphi \in [0,1]\}$.

Definition 2.3. [3] Let $\widetilde{n}_1 = \left(\widetilde{T}_1, \widetilde{I}_1, \widetilde{F}_1\right)$ and $\widetilde{n}_2 = \left(\widetilde{T}_2, \widetilde{I}_2, \widetilde{F}_2\right)$ be two elements of an MVNS, then their operational laws are defined as follows:

(1) $\widetilde{n}_1 \oplus \widetilde{n}_2 = \left(\widetilde{T}_1 \oplus \widetilde{T}_2, \widetilde{I}_1 \otimes \widetilde{I}_2, \widetilde{F}_1 \otimes \widetilde{F}_2\right)$

$$= \bigcup_{\substack{\gamma_1 \in \widetilde{T}_1, \mu_1 \in \widetilde{I}_1, \varphi_1 \in \widetilde{F}_1 \\ \gamma_2 \in \widetilde{T}_2, \mu_2 \in \widetilde{I}_2, \varphi_2 \in \widetilde{F}_2}} (\gamma_1 + \gamma_2 - \gamma_1\gamma_2, \mu_1\mu_2, \varphi_1\varphi_2)$$

(2) $\widetilde{n}_1 \otimes \widetilde{n}_2 = \left(\widetilde{T}_1 \otimes \widetilde{T}_2, \widetilde{I}_1 \oplus \widetilde{I}_2, \widetilde{F}_1 \oplus \widetilde{F}_2\right)$

$$= \bigcup_{\substack{\gamma_1 \in \widetilde{T}_1, \mu_1 \in \widetilde{I}_1, \varphi_1 \in \widetilde{F}_1 \\ \gamma_2 \in \widetilde{T}_2, \mu_2 \in \widetilde{I}_2, \varphi_2 \in \widetilde{F}_2}} (\gamma_1\gamma_2, \mu_1 + \mu_2 - \mu_1\mu_2, , \varphi_1 + \varphi_2 - \varphi_1\varphi_2)$$

(3) $k\ \widetilde{n}_1 = \bigcup_{\gamma_1 \in \widetilde{T}_1, \mu_1 \in \widetilde{I}_1, \varphi_1 \in \widetilde{F}_1} \left(1 - (1-\gamma_1)^k, \mu_1^k, \varphi_1^k\right), k > 0$

(4) $\widetilde{n}_1^k = \bigcup_{\gamma_1 \in \widetilde{T}_1, \mu_1 \in \widetilde{I}_1, \varphi_1 \in \widetilde{F}_1} \left(\gamma_1^k, 1 - (1-\mu_1)^k, 1 - (1-\varphi_1)^k\right), k > 0$

For many real world problems (e.g. ordering results of a journal, ordering results of an institute and inclusion degrees), multipolar information exists. The notion of m-polar fuzzy set was put forward to deal with those problems where m is an arbitrary ordinal number [34].

2.2. *Multi-Polar Neutrosophic Set*

Definition 2.4. [9] An m-polar neutrosophic set (mNS) on a universal set Z is a mapping

$X = \{(s_1 \circ T_X(z), s_2 \circ T_X(z), ..., s_m \circ T_X(z)), (s_1 \circ I_X(z), s_2 \circ I_X(z), ..., s_m \circ I_X(z)), (s_1 \circ F_X(z), s_2 \circ F_X(z), ..., s_m \circ F_X(z))\} : Z \longrightarrow ([0,1]^m, [0,1]^m, [0,1]^m)$

where ith mapping is defined as
$s_i \circ T_X : Z \longrightarrow [0,1]$
$s_i \circ I_X : Z \longrightarrow [0,1]$
$s_i \circ F_X : Z \longrightarrow [0,1]$
and $0 \leq s_i \circ T_X(z) + s_i \circ I_X(z) + s_i \circ F_X(z) \leq 3$
for all $i = 1, 2, ..., m$ and $z \in Z$.

Example 2.5. Let $Z = \{z_1, z_2, z_3\}$ be a universal set. Then

$$X = \left\{ \begin{array}{l} ((0.4, 0.6, 0.7), (0.1, 0.2, 0.3), (0.3, 0.5, 0.6))/z_1 \\ ((0.2, 0.4, 0.5), (0.6, 0.7, 0.8), (0.7, 0.8, 0.9))/z_2 \\ ((0.2, 0.5, 0.6), (0.3, 0.4, 0.6), (0.4, 0.6, 0.8))/z_3 \end{array} \right\}$$

represents an 3-polar neutrosophic set (3NS).

2.3. *Neutrosophic Soft Set*

Let Z be a universal set and E be the set of attributes of elements in Z. Take X to be a subset of E.

Definition 2.6. [4] An neutrosophic soft set (NSS) (ω, X) over Z is a mapping from X to $P(Z)$ and is defined as

$\Omega_X = (\omega, X) = \{(e, \omega_X(e)) : e \in E, \omega_X(e) \in P(Z)\}$

where $P(Z)$ denotes the collection of all neutrosophic subsets of Z,

$\omega_X(e) = \{z, T_X(e)(z), I_X(e)(z), F_X(e)(z) : z \in Z\}$

and each of $T_X(e)(z)$, $I_X(e)(z)$ and $F_X(e)(z)$ is a mapping from Z to interval $[0, 1]$ with

$0 \leq T_X(e)(z) + I_X(e)(z) + F_X(e)(z) \leq 3$

for all $e \in E$ and $z \in Z$.

Definition 2.7. [5] An m-polar neutrosophic soft set (mNSS) (ω, X) over Z is a mapping from X to $P(Z)$ and is defined as

$\Omega_X = (\omega, X) = \{(e, \omega_X(e)) : e \in E, \omega_X(e) \in P(Z)\}$

where $P(Z)$ denotes collection of all neutrosophic subsets of Z,

$\omega_X(e) = \{z, s_i \circ T_X(e)(z), s_i \circ I_X(e)(z), s_i \circ F_X(e)(z) : z \in Z\}$

and

$0 \leq s_i \circ T_X(e)(z) + s_i \circ I_X(e)(z) + s_i \circ F_X(e)(z) \leq 3$

for all $i = 1, 2, ..., m$; $e \in E, z \in Z$.

2.4. *Bonferroni mean operator*

Definition 2.8. [37] Let l, m be two natural numbers and $x_i \geq 0$ where $i \in \{1, 2, ..., n\}$ then Bonferroni mean $B^{l,m}$is defined as follow:

$$B^{l,m} = \left(\frac{1}{n\,(n-1)} \sum_{i,j=1, i \neq j}^{n} x_i^l x_j^m \right)^{\frac{1}{l+m}}.$$

Definition 2.9. [37] Let l, m be two natural numbers and $x_i \geq 0$ $(i = 1, 2, ..., n)$ and $W = [w_i \geq 0]^T$ be the weight vector of $[x_i]$ with the condition $\sum_{i=1}^{n} w_i = 1$, then weighted Bonferroni mean (WBM) is defined as follow:

$$WBM^{l,m} = \left(\frac{1}{n\,(n-1)} \sum_{i,j=1, i\neq j}^{n} (w_i x_i)^l \, (w_j x_j)^m \right)^{\frac{1}{l+m}}.$$

2.5. *Fuzzy Differential Equations*

2.5.1. *Fuzzy Numbers and Fuzzy Functions*

Definition 2.10. [6] A fuzzy number x is defined by a pair $x = (\underline{x}, \overline{x})$ of functions $\underline{x}, \overline{x} : [0,1] \longrightarrow R$, satisfying the three conditions:

(1) $\underline{x}(\alpha)$ is a bounded, monotonically increasing left-continuous function for all $\alpha \in (0,1]$ and right-continous for $\alpha = 0$,
(2) $\overline{x}(\alpha)$ is a bounded, monotonically decreasing left-continuous function for all $\alpha \in (0,1]$ and right-continous for $\alpha = 0$,
(3) For all $\alpha \in (0,1]$ we have: $\underline{x} \leq \overline{x}$.

For every $x = (\underline{x}, \overline{x})$, $y = (\underline{y}, \overline{y})$ and $k > 0$, $\alpha \in (0,1]$, we define addition and multiplication as follows:

- $\underline{(x+y)}(\alpha) = \underline{x}(\alpha) + \underline{y}(\alpha)$,
- $\overline{(x+y)}(\alpha) = \overline{x}(\alpha) + \overline{y}(\alpha)$,
- $(\underline{kx})(\alpha) = k\underline{x}(\alpha)$,
- $(\overline{kx})(\alpha) = k\overline{x}(\alpha)$.

With this definition of addition and multiplication, the collection of all fuzzy numbers is denoted by E^1. For $0 < \alpha \leq 1$, we define $\alpha-$cuts of fuzzy number u with $[x]^\alpha = \{u \in R \mid x(u) \geq \alpha\}$ and for $\alpha = 0$, the support of x is defined as $[x]^0 = \{u \in R \mid x(u) > 0\}$.

Definition 2.11. [6] Let $x = (\underline{x}, \overline{x})$ and $y = (\underline{y}, \overline{y})$ be two arbitrary numbers, then distance between x and y is defined as follows:

$$d(x,y) = \sup_{\alpha \in [0,1]} \{\max[\mid \underline{x(\alpha)} - \underline{y(\alpha)} \mid, \mid \overline{x(\alpha)} - \overline{y(\alpha)} \mid]\}.$$

Definition 2.12. [6] A fuzzy function $g : R^1 \to E^1$ is said to be continuous if for an arbitrary fixed $t_\circ \in R^1$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that:

$$|t - t_\circ| < \delta \Rightarrow d(g(t), g(\hat{t})) < \varepsilon,$$

then g is said to be continuous.

Definition 2.13. [6] Let $x, y \in E^1$. If there exists $z \in E^1$ such that $x = y + z$ then z is called the H-difference of x, y and it is denoted by $x - y$.

Definition 2.14. [6] A function $g : (c, d) \to E^1$ is said to be H-differentiable at $t_\circ \in (c, d)$ if for a small $h > 0$, there exist the H-differences $g(t_\circ) - g(t_\circ - h)$, $g(t_\circ + h) - g(t_\circ)$ and the element $g'(t_\circ) \in E^1$ such that:

$$0 = \lim_{h \to 0^+} d\left(\frac{g(t_\circ) - g(t_\circ - h)}{h}, g'(t_\circ)\right) = \lim_{h \to 0^+} d\left(\frac{g(t_\circ + h) - g(t_\circ)}{h}, g'(t_\circ)\right),$$

then $g'(t_\circ)$ is called the fuzzy derivative of g at $t_\circ$.

Definition 2.15. [7] The triangular fuzzy numbers are common and are denoted by $x = (\alpha, c, \beta)$ and defined by:

$$x = \begin{cases} \frac{u-\alpha}{c-\alpha}, & \text{if } \alpha \le u \le c, \\ \frac{\beta-u}{\beta-c}, & \text{if } c \le u \le \beta, \\ 0, & \text{otherwise.} \end{cases}$$

2.5.2. *First Order Fuzzy Differential Equations*

A first order fuzzy differential equation is written in the following form:

$$x'(t) = g(t, x(t))$$

where $g(t, x)$ is a fuzzy function of the crisp variable t and the fuzzy variable x and x is a fuzzy function of t. Here x' is the fuzzy derivative of x. Consider the initial value problem

$$x'(t) = g(t, x(t)), \qquad x(t_\circ) = x_\circ, \tag{1}$$

a mapping $x : R^1 \to E^1$ is a solution to the problem (1) if and only if it is continuous and satisfies the integral equation

$$x(t) = x_\circ + \int_{t_\circ}^{t} g(s, x(s))ds$$

for all $t \in R^1$ [8]. Moreover, sufficient conditions for the existence of a unique solution to Eq (1) are:

- f is continuous,
- A Lipschitz condition $d(g(t, x), g(t, y)) \le Ld(x, y)$ satisfied for some $L > 0$.

To obtain the solution of Eq (1), it can be replaced by the following equivalent system:

$$\begin{aligned} \underline{x}'(t) &= \underline{g}(t, x(t)), & \underline{x}(t_\circ) &= \underline{x_\circ}, \\ \overline{x}'(t) &= \overline{g}(t, x(t)), & \overline{x}(t_\circ) &= \overline{x_\circ}. \end{aligned}$$

For example, to solve

$$\tfrac{dx}{dt} = t^2 x, \quad x(0) = (0, \tfrac{1}{2}, 1),$$

it is replaced by

$$\begin{aligned} \tfrac{d\underline{x}}{dt} &= t^2 \underline{x}, \quad \underline{x}(0) = (0, \tfrac{1}{2}), \\ \tfrac{d\overline{x}}{dt} &= t^2 \overline{x}, \quad \overline{x}(0) = (\tfrac{1}{2}, 1), \end{aligned}$$

$\underline{x}(0) = (0, \frac{1}{2})$ and $\overline{x}(0) = (\frac{1}{2}, 1)$ are replaced by parametric forms $\underline{x}(0) = 2\alpha$ and $\overline{x}(0) = 2(1-\alpha)$, respectively where $\alpha \in [0, 1]$. And the solution is:

$$x = (2\alpha e^{\frac{t^3}{3}}, 2(1-\alpha)e^{\frac{t^3}{3}}), \quad \alpha \in [0, 1].$$

2.6. *Human attitude analysis after a decision*

The constants which provide the base to rank the alternatives, can also provide a support for further process of rethinking after a decision. These constants were utilized in fuzzy soft differential equations [41] and in developing the influence matrix which can play a vital role in influence model and doubly extended TOPSIS [42, 48].

2.6.1. *Human attitude analysis based on fuzzy soft differential equations*

A system of linear fuzzy soft differential equations is developed by Beg et al. [41].

$$\begin{aligned} \tfrac{dP_1}{dt} &= a^1_{P_1}P_1 + a^1_{P_2}P_2, \\ \tfrac{dP_2}{dt} &= a^2_{P_1}P_1 + a^2_{P_2}P_2 \end{aligned} \tag{2}$$

where P_1 and P_2 are the variables representing the attitude of two persons after taking a decision at time t, $\frac{dP_1}{dt}$ and $\frac{dP_2}{dt}$ represent the change in persons' attitudes after some time due to that decision and $a^i_{P_j}$ $(i, j = 1, 2)$ are optimum fuzzy soft constants (OFSCs) taken as signed fuzzy numbers denoting the influence on ith person of his internal feelings and jth person's feelings. Positive sign is assigned to $a^i_{P_j}$ when the attitude of jth person for ith person is supportive, otherwise a negative sign is assigned to it. Stability of system (2) depends upon eigen values of the matrix $\begin{bmatrix} a^1_{P_1} & a^1_{P_2} \\ a^2_{P_1} & a^2_{P_2} \end{bmatrix}$

3. **Multi valued Multi-Polar Neutrosophic Set**

Multi-valued multi-polar neutrosophic set (MVmNS) is a generalization and composition of MVNS and mNS.

Definition 3.1. Let Z be a non empty set. An MVmNS X is a mapping defined as

$$X : Z \to \begin{pmatrix} m \text{ sets of discrete values in } [0,1], \\ m \text{ sets of discrete values in } [0,1], \\ m \text{ sets of discrete values in } [0,1] \end{pmatrix}$$

$$X(z) = \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z), s_2 \circ \widetilde{T}_X(z), ..., s_m \circ \widetilde{T}_X(z)\right), \\ \left(s_1 \circ \widetilde{I}_X(z), s_2 \circ \widetilde{I}_X(z), ..., s_m \circ \widetilde{I}_X(z)\right), \\ \left(s_1 \circ \widetilde{F}_X(z), s_2 \circ \widetilde{F}_X(z), ..., s_m \circ \widetilde{F}_X(z)\right) \end{pmatrix}$$

where $s_i \circ \widetilde{T}_X(z)$, $s_i \circ \widetilde{I}_X(z)$ and $s_i \circ \widetilde{F}_X(z)$ $(i = 1, 2, ..., m)$ are the collections of discrete real values γ_i, μ_i and φ_i denoting the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of the element $z \in Z$ to the set X respectively with $0 \leq \gamma_i,\ \mu_i,\ \varphi_i \leq 1$, $0 \leq \gamma_i^+ + \mu_i^+ + \varphi_i^+ \leq 3$, $\gamma_i^+ \in \sup\left(s_i \circ \widetilde{T}_X(z)\right)$, $\mu_i^+ \in \sup\left(s_i \circ \widetilde{I}_X(z)\right)$, $\varphi_i^+ \in \sup\left(s_i \circ \widetilde{F}_X(z)\right)$.

3.1. *Multi valued Multi-Polar Neutrosophic Soft Set*

Definition 3.2. Let Z be a universal set and E be a set of parameters with $X \subseteq E$. Define $\omega : X \to P(Z)$, where $P(Z)$ is the collection of all MVmN subsets of the set Z. Then (ω, X) is said to be an multi-valued m-polar neutrosophic soft set (MVmNSS) over Z which is represented as $\Omega_X = (\omega, X) = \{e, \omega_X(e) : e \in E, \omega_X(e) \in P(Z)\}$ and $\omega_X(e)$ is an MVmNS over Z.

3.2. *Operations on MVmNS*

$$\text{Let } X(z_1) = \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_1), s_2 \circ \widetilde{I}_X(z_1), ..., s_m \circ \widetilde{I}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right) \end{pmatrix}$$

$$= \left(\left(\cup\{\gamma_1^{(z_1)}\}, ..., \cup\{\gamma_m^{(z_1)}\}\right), \left(\cup\{\mu_1^{(z_1)}\}, ..., \cup\{\mu_m^{(z_1)}\}\right), \left(\cup\{\varphi_1^{(z_1)}\}, ..., \cup\{\varphi_m^{(z_1)}\}\right)\right)$$

$$\text{and } X(z_2) = \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_2), s_2 \circ \widetilde{T}_X(z_2), ..., s_m \circ \widetilde{T}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_2), s_2 \circ \widetilde{I}_X(z_2), ..., s_m \circ \widetilde{I}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_2), s_2 \circ \widetilde{F}_X(z_2), ..., s_m \circ \widetilde{F}_X(z_2)\right) \end{pmatrix}$$

$$= \left(\left(\cup\{\gamma_1^{(z_2)}\}, ..., \cup\{\gamma_m^{(z_2)}\}\right), \left(\cup\{\mu_1^{(z_2)}\}, ..., \cup\{\mu_m^{(z_2)}\}\right), \left(\cup\{\varphi_1^{(z_2)}\}, ..., \cup\{\varphi_m^{(z_2)}\}\right)\right)$$

be two elements of an MVmNS. Then their operational laws are defined as

$$(1)\ \ (X(z_1))^c = \begin{pmatrix} \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{I}'_X(z_1), s_2 \circ \widetilde{I}'_X(z_1), ..., s_m \circ \widetilde{I}'_X(z_1)\right), \\ \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right) \end{pmatrix}$$

$$= \left(\left(\cup\{\varphi_1^{(z_1)}\}, ..., \cup\{\varphi_m^{(z_1)}\}\right), \left(\cup\{1-\mu_1^{(z_1)}\}, ..., \cup\{1-\mu_m^{(z_1)}\}\right), \left(\cup\{\gamma_1^{(z_1)}\}, ..., \cup\{\gamma_m^{(z_1)}\}\right)\right).$$

(2) $X(z_1) \oplus X(z_2)$

$$= \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_1), s_2 \circ \widetilde{I}_X(z_1), ..., s_m \circ \widetilde{I}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right) \end{pmatrix} \oplus \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_2), s_2 \circ \widetilde{T}_X(z_2), ..., s_m \circ \widetilde{T}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_2), s_2 \circ \widetilde{I}_X(z_2), ..., s_m \circ \widetilde{I}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_2), s_2 \circ \widetilde{F}_X(z_2), ..., s_m \circ \widetilde{F}_X(z_2)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{T}_X(z_2), s_2 \circ \widetilde{T}_X(z_2), ..., s_m \circ \widetilde{T}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_1), s_2 \circ \widetilde{I}_X(z_1), ..., s_m \circ \widetilde{I}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{I}_X(z_2), s_2 \circ \widetilde{I}_X(z_2), ..., s_m \circ \widetilde{I}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{F}_X(z_2), s_2 \circ \widetilde{F}_X(z_2), ..., s_m \circ \widetilde{F}_X(z_2)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\left(s_1 \circ \widetilde{T}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{T}_X(z_2)\right), ..., \left(s_m \circ \widetilde{T}_X(z_1)\right) \oplus \left(s_m \circ \widetilde{T}_X(z_2)\right)\right), \\ \left(\left(s_1 \circ \widetilde{I}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{I}_X(z_2)\right), ..., \left(s_m \circ \widetilde{I}_X(z_1)\right) \otimes \left(s_m \circ \widetilde{I}_X(z_2)\right)\right), \\ \left(\left(s_1 \circ \widetilde{F}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{F}_X(z_2)\right), ..., \left(s_m \circ \widetilde{F}_X(z_1)\right) \otimes \left(s_m \circ \widetilde{F}_X(z_2)\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\cup\{\gamma_1^{(z_1)} + \gamma_1^{(z_2)} - \gamma_1^{(z_1)}\gamma_1^{(z_2)}\}, ..., \cup\{\gamma_m^{(z_1)} + \gamma_m^{(z_2)} - \gamma_m^{(z_1)}\gamma_m^{(z_2)}\}\right), \\ \left(\cup\{\mu_1^{(z_1)}\mu_1^{(z_2)}\}, ..., \cup\{\mu_m^{(z_1)}\mu_m^{(z_2)}\}\right), \\ \left(\cup\{\varphi_1^{(z_1)}\varphi_1^{(z_2)}\}, ..., \cup\{\varphi_m^{(z_1)}\varphi_m^{(z_2)}\}\right) \end{pmatrix}.$$

(3) $X(z_1) \otimes X(z_2)$

$$= \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_1), s_2 \circ \widetilde{I}_X(z_1), ..., s_m \circ \widetilde{I}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right) \end{pmatrix} \otimes \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_2), s_2 \circ \widetilde{T}_X(z_2), ..., s_m \circ \widetilde{T}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_2), s_2 \circ \widetilde{I}_X(z_2), ..., s_m \circ \widetilde{I}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_2), s_2 \circ \widetilde{F}_X(z_2), ..., s_m \circ \widetilde{F}_X(z_2)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{T}_X(z_2), s_2 \circ \widetilde{T}_X(z_2), ..., s_m \circ \widetilde{T}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_1), s_2 \circ \widetilde{I}_X(z_1), ..., s_m \circ \widetilde{I}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{I}_X(z_2), s_2 \circ \widetilde{I}_X(z_2), ..., s_m \circ \widetilde{I}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{F}_X(z_2), s_2 \circ \widetilde{F}_X(z_2), ..., s_m \circ \widetilde{F}_X(z_2)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\left(s_1 \circ \widetilde{T}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{T}_X(z_2)\right), ..., \left(s_m \circ \widetilde{T}_X(z_1)\right) \otimes \left(s_m \circ \widetilde{T}_X(z_2)\right)\right), \\ \left(\left(s_1 \circ \widetilde{I}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{I}_X(z_2)\right), ..., \left(s_m \circ \widetilde{I}_X(z_1)\right) \oplus \left(s_m \circ \widetilde{I}_X(z_2)\right)\right), \\ \left(\left(s_1 \circ \widetilde{F}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{F}_X(z_2)\right), ..., \left(s_m \circ \widetilde{F}_X(z_1)\right) \oplus \left(s_m \circ \widetilde{F}_X(z_2)\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\cup\{\gamma_1^{(z_1)}\gamma_1^{(z_2)}\}, ..., \cup\{\gamma_m^{(z_1)}\gamma_m^{(z_2)}\}\right), \\ \left(\cup\{\mu_1^{(z_1)} + \mu_1^{(z_2)} - \mu_1^{(z_1)}\mu_1^{(z_2)}\}, ..., \cup\{\mu_m^{(z_1)} + \mu_m^{(z_2)} - \mu_m^{(z_1)}\mu_m^{(z_2)}\}\right), \\ \left(\cup\{\varphi_1^{(z_1)} + \varphi_1^{(z_2)} - \varphi_1^{(z_1)}\varphi_1^{(z_2)}\}, ..., \cup\{\varphi_m^{(z_1)} + \varphi_m^{(z_2)} - \varphi_m^{(z_1)}\varphi_m^{(z_2)}\}\right) \end{pmatrix}.$$

$$(4)\ \ kX(z_1) = \begin{pmatrix} \left(\cup\{1 - \left(1 - \gamma_1^{(z_1)}\right)^k\}, ..., \cup\{1 - \left(1 - \gamma_m^{(z_1)}\right)^k\}\right), \\ \left(\cup\{\left(\mu_1^{(z_1)}\right)^k\}, ..., \cup\{\left(\mu_m^{(z_1)}\right)^k\}\right), \\ \left(\cup\{\left(\varphi_1^{(z_1)}\right)^k\}, ..., \cup\{\left(\varphi_m^{(z_1)}\right)^k\}\right) \end{pmatrix}.$$

$$(5)\ \left(X\left(z_1\right)\right)^k=\left(\begin{array}{c}\left(\cup\{\left(\gamma_1^{(z_1)}\right)^k\},...,\cup\{\left(\gamma_m^{(z_1)}\right)^k\}\right),\\ \left(\cup\{1-\left(1-\mu_1^{(z_1)}\right)^k\},...,\cup\{1-\left(1-\mu_m^{(z_1)}\right)^k\}\right),\\ \left(\cup\{1-\left(1-\varphi_1^{(z_1)}\right)^k\},...,\cup\{1-\left(1-\varphi_m^{(z_1)}\right)^k\}\right)\end{array}\right).$$

Example 3.3. Let

$x_1=((\{0.3\},\{0.4,0.5\},\{0.5,0.6\}),(\{0.4,0.5\},\{0.2\},\{0.7\}),(\{0.3\},\{0.5\},\{0.8\}))$ and

$x_2=((\{0.1\},\{0.3\},\{0.6\}),(\{0.2,0.4\},\{0.6\},\{0.7,0.8\}),(\{0.5\},\{0.6\},\{0.7,0.8\}))$

be two elements of an MV3NS. Then

$(x_1)^c=((\{0.3\},\{0.5\},\{0.8\}),(\{0.6,0.5\},\{0.8\},\{0.3\}),(\{0.3\},\{0.4,0.5\},\{0.5,0.6\}))$,

$$x_1\oplus x_2=\left(\begin{array}{c}(\{0.37\},\{0.58,0.65\},\{0.8,0.84\}),\\(\{0.08,0.2\},\{0.12\},\{0.49,0.56\}),\\(\{0.15\},\{0.30\},\{0.56,0.64\})\end{array}\right),$$

$$x_1\otimes x_2=\left(\begin{array}{c}(\{0.03\},\{0.12,0.15\},\{0.3,0.36\}),\\(\{0.52,0.7\},\{0.68\},\{0.91,0.94\}),\\(\{0.65\},\{0.8\},\{0.94,0.96\})\end{array}\right),$$

$$2x_1=\left(\begin{array}{c}(\{0.51\},\{0.64,0.75\},\{0.75,0.84\}),\\(\{0.16,0.25\},\{0.04\},\{0.49\}),(\{0.09\},\{0.25\},\{0.64\})\end{array}\right),$$

$$(x_1)^2=\left(\begin{array}{c}(\{0.09\},\{0.16,0.25\},\{0.25,0.36\}),\\(\{0.64,0.75\},\{0.36\},\{0.91\}),(\{0.51\},\{0.75\},\{0.96\})\end{array}\right).$$

Definition 3.4. Score function $s(X(z))$ and accuracy function $a(X(z))$ of an element $X(z)$ of an MVmNS is defined as follows:

$$s(X(z))=\frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{l_{iT}l_{iI}l_{iF}}\sum\left(\frac{\gamma_i-\mu_i-\varphi_i}{3}\right)\right),$$
$$a(X(z))=\frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{l_{iT}l_{iI}l_{iF}}\sum\left(\frac{\gamma_i+\mu_i+\varphi_i}{3}\right)\right),$$

where $\gamma_i\in s_i\circ\widetilde{T}_X(z)$, $\mu_i\in s_i\circ\widetilde{I}_X(z)$, $\varphi_i\in s_i\circ\widetilde{F}_X(z)$ and l_{iT},l_{iI},l_{iF} are the number of elements in $s_i\circ\widetilde{T}_X(z)$, $s_i\circ\widetilde{I}_X(z)$ and $s_i\circ\widetilde{F}_X(z)$ respectively.

It can be observed that, the score function and accuracy function satisfy the following properties:

(1) For an element $X(z)$ of an MVmNS,
$$-\frac{2}{3}\le s(X(z))\le\frac{1}{3}.$$

(2) For an element $X(z)$ of an MVmNS,
$$0\le a(X(z))\le 1.$$

Definition 3.5. Two elements $X(z_1)$ and $X(z_2)$ of an MVmNS are compared as:

- if $s\left(X\left(z_1\right)\right)>s\left(X\left(z_2\right)\right)$, then $X\left(z_1\right)>X\left(z_2\right)$,
- if $s\left(X\left(z_1\right)\right)=s\left(X\left(z_2\right)\right)$ and
  - if $a\left(X\left(z_1\right)\right)>a\left(X\left(z_2\right)\right)$, then $X(z_1)>X\left(z_2\right)$,
  - if $a\left(X\left(z_1\right)\right)=a\left(X\left(z_2\right)\right)$, then $X(z_1)=X\left(z_2\right)$.

Definition 3.6. Let $X(z_i)$, $(i=1,2,...,n)$ be the elements of an MVmNS. Then for two natural numbers p,q, MVmNBM operator is defined as

$$\mathrm{MV}m\mathrm{NBM}^{p,q}\left(X\left(z_1\right),X\left(z_2\right),...,X\left(z_n\right)\right)=\left(\frac{1}{n(n-1)}\left(\underset{i,j=1,j\neq i}{\overset{n}{\oplus}}\left(\left(X(z_i)\right)^p\otimes\left(X\left(z_j\right)\right)^q\right)\right)\right)^{\frac{1}{p+q}}$$

Theorem 3.7. *Let $X(z_i)$, $(i=1,2,...,n)$ be n elements of an MVmNS, then MVmNBM operator can be expressed as:*

$$MVmNBM^{p,q}\left(X\left(z_1\right),X\left(z_2\right),...,X\left(z_n\right)\right)=$$

$$\left(\begin{array}{c}\left(\cup\left\{\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\gamma_1^{(z_i)}\right)^p\left(\gamma_1^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\},...,\cup\left\{\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\gamma_m^{(z_i)}\right)^p\left(\gamma_m^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}\right),\\ \left(\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\mu_1^{(z_i)}\right)^p\left(1-\mu_1^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\},...,\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\mu_m^{(z_i)}\right)^p\left(1-\mu_m^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}\right),\\ \left(\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\varphi_1^{(z_i)}\right)^p\left(1-\varphi_1^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\},...,\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\varphi_m^{(z_i)}\right)^p\left(1-\varphi_m^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}\right)\end{array}\right)$$

Proof. Let

$$X(z_i)=\left(\left(\cup\{\gamma_1^{(z_i)}\},...,\cup\{\gamma_m^{(z_i)}\}\right),\left(\cup\{\mu_1^{(z_i)}\},...,\cup\{\mu_m^{(z_i)}\}\right),\left(\cup\{\varphi_1^{(z_i)}\},...,\cup\{\varphi_m^{(z_i)}\}\right)\right)$$

and $X\left(z_j\right)=\left(\left(\cup\{\gamma_1^{(z_j)}\},...,\cup\{\gamma_m^{(z_j)}\}\right),\left(\cup\{\mu_1^{(z_j)}\},...,\cup\{\mu_m^{(z_j)}\}\right),\left(\cup\{\varphi_1^{(z_j)}\},...,\cup\{\varphi_m^{(z_j)}\}\right)\right)$

$$\left(X\left(z_i\right)\right)^p=\left(\begin{array}{c}\left(\cup\{\left(\gamma_1^{(z_i)}\right)^p\},...,\cup\{\left(\gamma_m^{(z_i)}\right)^p\}\right),\\ \left(\cup\{1-\left(1-\mu_1^{(z_i)}\right)^p\},...,\cup\{1-\left(1-\mu_m^{(z_i)}\right)^p\}\right),\\ \left(\cup\{1-\left(1-\varphi_1^{(z_i)}\right)^p\},...,\cup\{1-\left(1-\varphi_m^{(z_i)}\right)^p\}\right)\end{array}\right)$$

$$\left(X\left(z_j\right)\right)^q=\left(\begin{array}{c}\left(\cup\{\left(\gamma_1^{(z_j)}\right)^q\},...,\cup\{\left(\gamma_m^{(z_j)}\right)^q\}\right),\\ \left(\cup\{1-\left(1-\mu_1^{(z_j)}\right)^q\},...,\cup\{1-\left(1-\mu_m^{(z_j)}\right)^q\}\right),\\ \left(\cup\{1-\left(1-\varphi_1^{(z_j)}\right)^q\},...,\cup\{1-\left(1-\varphi_m^{(z_j)}\right)^q\}\right)\end{array}\right)$$

$$\left(X\left(z_i\right)\right)^p\otimes\left(X\left(z_j\right)\right)^q=$$

$$\left(\begin{array}{c}\left(\cup\left\{\left(\gamma_1^{(z_i)}\right)^p\left(\gamma_1^{(z_j)}\right)^q\right\},...,\cup\left\{\left(\gamma_m^{(z_i)}\right)^p\left(\gamma_m^{(z_j)}\right)^q\right\}\right),\\ \left(\cup\left\{1-\left(1-\mu_1^{(z_i)}\right)^p\left(1-\mu_1^{(z_j)}\right)^q\right\},...,\cup\left\{1-\left(1-\mu_m^{(z_i)}\right)^p\left(1-\mu_m^{(z_j)}\right)^q\right\}\right),\\ \left(\cup\left\{1-\left(1-\varphi_1^{(z_i)}\right)^p\left(1-\varphi_1^{(z_j)}\right)^q\right\},...,\cup\left\{1-\left(1-\varphi_m^{(z_i)}\right)^p\left(1-\varphi_m^{(z_j)}\right)^q\right\}\right)\end{array}\right)$$

$$\underset{i,j=1,j\neq i}{\overset{n}{\oplus}}\left((X(z_i))^p\otimes(X(z_j))^q\right)=$$

$$\left(\begin{array}{l}\left(\cup\left\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\gamma_1^{(z_i)}\right)^p\left(\gamma_1^{(z_j)}\right)^q\right)\right\},...,\cup\left\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\gamma_m^{(z_i)}\right)^p\left(\gamma_m^{(z_j)}\right)^q\right)\right\}\right),\\ \left(\cup\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\mu_1^{(z_i)}\right)^p\left(1-\mu_1^{(z_j)}\right)^q\right)\right\},...,\cup\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\mu_m^{(z_i)}\right)^p\left(1-\mu_m^{(z_j)}\right)^q\right)\right\}\right),\\ \left(\cup\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\varphi_1^{(z_i)}\right)^p\left(1-\varphi_1^{(z_j)}\right)^q\right)\right\},...,\cup\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\varphi_m^{(z_i)}\right)^p\left(1-\varphi_m^{(z_j)}\right)^q\right)\right\}\right)\end{array}\right)$$

Finally, the required result is obtained by using operations 4 and 5, presented in section 3.2.

□

Definition 3.8. Let $X(z_i)$, $(i = 1, 2, ..., n)$ be the elements of an MVmNS with weight vector $W = (w_1, w_2, ..., w_n)^T$ satisfying $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. Then for two natural numbers p, q, MVmNWBM operator is defined as

$$MVmNWBM^{p,q}\left(X\left(z_1\right),X\left(z_2\right),...,X\left(z_n\right)\right)=\left(\frac{1}{n(n-1)}\left(\underset{i,j=1,j\neq i}{\overset{n}{\oplus}}\left((w_iX(z_i))^p\otimes(w_jX\left(z_j\right))^q\right)\right)\right)^{\frac{1}{p+q}}$$

Theorem 3.9. *Let $X(z_i)$, $(i = 1, 2, ..., n)$ be n elements of an MVmNS with weight vector $W = (w_1, w_2, ..., w_n)^T$ satisfying $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, then MVmNWBM operator can be expressed as:*

$$MVmNWBM^{p,q}\left(X\left(z_1\right),X\left(z_2\right),...,X\left(z_n\right)\right)=$$

$$\left(\begin{array}{l}\left(\begin{array}{l}\cup\left\{\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\gamma_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(1-\gamma_1^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\},...,\\ \cup\left\{\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\gamma_m^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(1-\gamma_m^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}\end{array}\right),\\ \left(\begin{array}{l}\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(\mu_1^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\},...,\\ \cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_m^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(\mu_m^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}\end{array}\right),\\ \left(\begin{array}{l}\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\varphi_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(\varphi_1^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\},...,\\ \cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\varphi_m^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(\varphi_m^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}\end{array}\right)\end{array}\right)$$

Proof. This result can be obtained similarly as the previous one. □

MVmNBM and MVmNWBM operators satisfy the following properties:

- Permutation
- Monotonicity
- Boundedness

Theorem 3.10. ***(Permutation)*** *Let $X(z_i)$, $(i = 1, 2, ..., n)$ be a set of n elements of an MVmNS. If $\acute{X}(z_i)$, $(i = 1, 2, ..., n)$ is a permutation of $X(z_i)$, $(i = 1, 2, ..., n)$, then*

$$MVmNWBM(X(z_1), X(z_2), ..., X(z_n)) = MVmNWBM\left(\acute{X}(z_1), \acute{X}(z_2), ..., \acute{X}(z_n)\right)$$

Proof. Following result can be obtained by definition

$$\left(\frac{1}{n(n-1)}\left(\bigoplus_{i,j=1,j\neq i}^{n}((w_i X(z_1))^p \otimes (w_j X(z_2))^q)\right)\right)^{\frac{1}{p+q}} = \left(\frac{1}{n(n-1)}\left(\bigoplus_{i,j=1,j\neq i}^{n}\left(\left(w_i \acute{X}(z_1)\right)^p \otimes \left(w_j \acute{X}(z_2)\right)^q\right)\right)\right)^{\frac{1}{p+q}}.$$

Hence the result. □

Theorem 3.11. ***(Monotonicity)*** *Let $X(z_i)$, $(i = 1, 2, ..., n)$ and $\acute{X}(z_i)$, $(i = 1, 2, ..., n)$ be two sets of elements of an MVmNS. If $X(z_i) \geq \acute{X}(z_i)$ for all $i = 1, 2, ..., n$ then*

$$MVmNWBM^{p,q}(X(z_1), X(z_2), ..., X(z_n)) \geq MVmNWBM^{p,q}\left(\acute{X}(z_1), \acute{X}(z_2), ..., \acute{X}(z_n)\right).$$

Proof. Let

$X(z_i) = \left(\left(\cup\{\gamma_1^{(z_i)}\}, ..., \cup\{\gamma_m^{(z_i)}\}\right), \left(\cup\{\mu_1^{(z_i)}\}, ..., \cup\{\mu_m^{(z_i)}\}\right), \left(\cup\{\varphi_1^{(z_i)}\}, ..., \cup\{\varphi_m^{(z_i)}\}\right)\right)$

and $\acute{X}(z_i) = \left(\left(\cup\{\acute{\gamma}_1^{(z_i)}\}, ..., \cup\{\acute{\gamma}_m^{(z_i)}\}\right), \left(\cup\{\acute{\mu}_1^{(z_i)}\}, ..., \cup\{\acute{\mu}_m^{(z_i)}\}\right), \left(\cup\{\acute{\varphi}_1^{(z_i)}\}, ..., \cup\{\acute{\varphi}_m^{(z_i)}\}\right)\right)$.

(1) $X(z_i) \geq \acute{X}(z_i) \Longrightarrow \gamma_1^{(z_i)} \geq \acute{\gamma}_1^{(z_i)} \Longrightarrow 1 - \left(1 - \gamma_1^{(z_i)}\right)^{w_i} \geq 1 - \left(1 - \acute{\gamma}_1^{(z_i)}\right)^{w_i}$

$$\Longrightarrow \left(1 - \left(1 - \gamma_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \gamma_1^{(z_j)}\right)^{wj}\right)^q \geq \left(1 - \left(1 - \acute{\gamma}_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \acute{\gamma}_1^{(z_j)}\right)^{wj}\right)^q$$

for all $j = 1, 2, \ldots, m$,

$$\Longrightarrow 1 - \left(1 - \left(1 - \acute{\gamma}_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \acute{\gamma}_1^{(z_j)}\right)^{wj}\right)^q \leq 1 - \left(1 - \left(1 - \gamma_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \gamma_1^{(z_j)}\right)^{wj}\right)^q$$

$$\Longrightarrow \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(1 - \left(1 - \acute{\gamma}_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \acute{\gamma}_1^{(z_j)}\right)^{wj}\right)^q\right) \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(1 - \left(1 - \gamma_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \gamma_1^{(z_j)}\right)^{wj}\right)^q\right)$$

$$\Longrightarrow 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\gamma_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(1-\gamma_1^{(z_j)}\right)^{wj}\right)^{q}\right)\geq$$

$$1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\acute{\gamma}_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(1-\acute{\gamma}_1^{(z_j)}\right)^{wj}\right)^{q}\right)$$

$$\Longrightarrow\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\gamma_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(1-\gamma_1^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\geq$$

$$\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\acute{\gamma}_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(1-\acute{\gamma}_1^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

Similarly

$$\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\gamma_k^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(1-\gamma_k^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\geq$$

$$\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\acute{\gamma}_k^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(1-\acute{\gamma}_k^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

for $k=1,2,...,m$.

(2) $X(z_i)\geq \acute{X}(z_i)\Longrightarrow \mu_1^{(z_i)}\leq \acute{\mu}_1^{(z_i)}\Longrightarrow\left(\mu_1^{(z_i)}\right)^{w_i}\leq\left(\acute{\mu}_1^{(z_i)}\right)^{w_i}$

$\Longrightarrow\left(1-\left(\mu_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(\mu_1^{(z_j)}\right)^{w_j}\right)^{q}\geq\left(1-\left(\acute{\mu}_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(\acute{\mu}_1^{(z_j)}\right)^{w_j}\right)^{q}$

$$\Longrightarrow 1-\left(1-\left(\mu_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{(z_j)}\right)^{w_j}\right)^q \leq 1-\left(1-\left(\mu_1^{\prime(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{\prime(z_j)}\right)^{w_j}\right)^q$$

$$\Longrightarrow \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{(z_j)}\right)^{w_j}\right)^q\right) \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_1^{\prime(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{\prime(z_j)}\right)^{w_j}\right)^q\right)$$

$$\Longrightarrow 1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq$$

$$1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_1^{\prime(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{\prime(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

Similarly

$$1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_k^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_k^{(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq$$

$$1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_k^{\prime(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_k^{\prime(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

for $k = 1, 2, ..., m$.

$$(3) 1 - \left(1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \left(\varphi_k^{(z_i)} \right)^{w_i} \right)^p \left(1 - \left(\varphi_k^{(z_j)} \right)^{w_j} \right)^q \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \leq$$

$$1 - \left(1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \left(\varphi_k^{(z_i)} \right)^{w_i} \right)^p \left(1 - \left(\varphi_k^{(z_j)} \right)^{w_j} \right)^q \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}$$

for $k = 1, 2, ..., m$.

From (1)-(3), the required result is obtained. □

Theorem 3.12. ***(Boundedness)*** *Let $X(z_i)$, $(i = 1, 2, ..., n)$ be a set of n elements of an MVmNS, then*

$$\min \left(X \left(z_1 \right), X \left(z_2 \right), ..., X \left(z_n \right) \right) \leq MVmNWBM \left(X \left(z_1 \right), X \left(z_2 \right), ..., X \left(z_n \right) \right) \leq \max \left(X \left(z_1 \right), X \left(z_2 \right), ..., X \left(z_n \right) \right)$$

Proof. Let $m = \min \left(X \left(z_1 \right), X \left(z_2 \right), ..., X \left(z_n \right) \right)$ and $M = \max \left(X \left(z_1 \right), X \left(z_2 \right), ..., X \left(z_n \right) \right)$

Since $m \leq X(z_i) \leq M$ so by using previous theorem

$$m \leq MVmNWBM \left(X \left(z_1 \right), X \left(z_2 \right), ..., X \left(z_n \right) \right), \quad MVmNWBM \left(X \left(z_1 \right), X \left(z_2 \right), ..., X \left(z_n \right) \right) \leq M.$$

Hence the result.

3.3. *Multi-Valued Multi-Polar Neutrosophic Soft Set*

Definition 3.13. Let Z be a universal set and E be a set of parameters with $X \subseteq E$. Define $\omega : X \to P(Z)$, where $P(Z)$ is the collection of all MVmN subsets of the set Z. Then (ω, X) is said to be an multi-valued m-polar neutrosophic soft set (MVmNSS) over Z which is represented as $\Omega_X = (\omega, X) = \{e, \omega_X(e) : e \in E, \omega_X(e) \in P(Z)\}$ and $\omega_X(e)$ is an MVmNS over Z.

Table 1. Representation of a MV3NSS.

	C_1	C_2	C_3
A_1	$\begin{pmatrix} (\{0.6\},\{0.4,0.5\},\{0.4,0.5\}), \\ (\{0.8,\},\{0.1,0.4\},\{0.5\}), \\ (\{0.2\},\{0.8\},\{0.3,0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6\},\{0.7\},\{0.4,0.5\}), \\ (\{0.3,0.7\},\{0.6\},\{0.8,0.9\}), \\ (\{0.4\},\{0.8,0.9\},\{0.5\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3,0.5\},\{0.7\},\{0.9\}), \\ (\{0.6,0.9\},\{0.8\},\{0.6\}), \\ (\{0.4,0.5\},\{0.4\},\{0.7\}) \end{pmatrix}$
A_2	$\begin{pmatrix} (\{0.4,0.6\},\{0.5\},\{0.2\}), \\ (\{0.6,0.7\},\{0.7\},\{0.2,0.4\}), \\ (\{0.4\},\{0.6\},\{0.6,0.8\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.4,0.6\},\{0.6\},\{0.6,0.9\}), \\ (\{0.6\},\{0.6,0.8\},\{0.7\}), \\ (\{0.4\},\{0.6\},\{0.5,0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6,0.8\},\{0.6\},\{0.5,0.7\}), \\ (\{0.2,0.6\},\{0.3,0.4\},\{0.4\}), \\ (\{0.4\},\{0.5\},\{0.5,0.7\}) \end{pmatrix}$
A_3	$\begin{pmatrix} (\{0.4\},\{0.3\},\{0.8,0.9\}), \\ (\{0.4\},\{0.3\},\{0.9\}), \\ (\{0.7,0.9\},\{0.8\},\{0.5\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6\},\{0.4,0.6\},\{0.8\}), \\ (\{0.7,0.8\},\{0.9\},\{0.5\}), \\ (\{0.3,0.5\},\{0.8\},\{0.7,0.8\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\},\{0.7\},\{0.8,0.9\}), \\ (\{0.4\},\{0.7,0.9\},\{0.8\}), \\ (\{0.4\},\{0.8\},\{0.7,0.9\}) \end{pmatrix}$
A_4	$\begin{pmatrix} (\{0.6\},\{0.6\},\{0.5,0.7\}), \\ (\{0.5,0.7\},\{0.7\},\{0.9\}), \\ (\{0.4\},\{0.8,0.9\},\{0.5,0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6\},\{0.4\},\{0.1,0.3\}), \\ (\{0.5\},\{0.8\},\{0.6,0.7\}), \\ (\{0.3\},\{0.6,0.7\},\{0.6,0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.7\},\{0.3,0.6\},\{0.7\}), \\ (\{0.7\},\{0.9\},\{0.4,0.5\}), \\ (\{0.4\},\{0.3\},\{0.6\}) \end{pmatrix}$

Following example shows an MV3NS where three poles represent three different opinion leaders and decision makers are considered as opinion followers. Opinion leaders have an influence power for updating process of opinion followers' opinions [40].

Example 3.14. Let $\{A_1, A_2, A_3, A_4\}$ be a set of four companies where an investor wants to invest a suitable amount and $\{C_1, C_2, C_3\}$ be a set of criteria, then an MV3NSS is represented in Table 1.

3.3.1. *Operations on Multi-Valued m-Neutrosophic Soft Set*

Some operations in MVmNSS are defined in this section.

Definition 3.15. Let Z be a universal set and E be a set of parameters with $U, V \subseteq E$. For two MVmNSSs Ω_U and Ψ_V, $\Omega_U \breve{\subseteq} \Psi_V$ if

(1) $U \subseteq V$,
(2) $\Omega_U(e) \subseteq \Psi_V(e)$ for all $e \in U$ *i.e.* $s(\Omega_U(e)(z)) \leq s(\Psi_V(e)(z))$ for all $e \in U$, $z \in Z$.

Example 3.16. Let $Z = \{z_1, z_2\}$ and $E = \{e_1, e_2, e_3\}$. $U = \{e_1, e_2\}$ and $V = \{e_1, e_2\}$ be subsets of E. Let Ω_U and Ψ_V be two MV3NSSs defined as:

$$\begin{aligned}\Omega_U = \ & \{(e_1,(z_1,(\{0.3,0.4\},\{0.4,0.6,0.7\},\{0.2,0.5\}),(\{0.5,0.7\},\{0.6,0.8,0.9\},\{0.7,0.8\}),\\ & (\{0.4,0.6\},\{0.5,0.7\},\{0.5,0.7,0.8\})),(z_2,(\{0.3,0.4\},\{0.6,0.9\},\{0.8,0.9\}),\\ & (\{0.4,0.5\},\{0.6,0.8\},\{0.4,0.6,0.7\}),(\{0.1,0.3,0.5\},\{0.6,0.7\},\{0.7,0.8\}))),\\ & (e_2,(z_1,(\{0.4,0.5\},\{0.4,0.6\},\{0.6,0.9\}),(\{0.2,0.4,0.5\},\{0.6,0.7\},\{0.7,0.8\}),\\ & (\{0.3,0.5\},\{0.6,0.7,0.8\},\{0.7,0.9\})),(z_2,(\{0.4,0.6\},\{0.4,0.6,0.7\},\{0.6,0.7\}),\\ & (\{0.1,0.2,0.4\},\{0.5,0.6,0.7\},\{0.6,0.8\}),(\{0.4,0.5\},\{0.5,0.6\},\{0.5,0.7\}))))\}\\ \Psi_V = \ & \{(e_1,(z_1,(\{0.8,0.9\},\{0.3,0.5\},\{0.5,0.6\}),(\{0.5,0.6\},\{0.2,0.3\},\{0.4,0.5\}),\\ & (\{0.8,0.9\},\{0.3,0.4,0.5\},\{0.3,0.4,0.5\})),(z_2,(\{0.7,0.8,0.9\},\{0.4,0.5\},\{0.5,0.6\}),\\ & (\{0.6,0.7\},\{0.6,0.7\},\{0.3,0.5\}),(\{0.7,0.8\},\{0.4,0.6\},\{0.2,0.5\}))),\\ & (e_2,(z_1,(\{0.6,0.8\},\{0.4,0.5\},\{0.4,0.5,0.6\}),(\{0.6,0.8,0.9\},\{0.4,0.6\},\{0.6,0.7\}),\\ & (\{0.7,0.8\},\{0.6,0.7\},\{0.5,0.7\})),(z_2,(\{0.6,0.8\},\{0.4,0.5\},\{0.4,0.5\}),\\ & (\{0.5,0.8\},\{0.4,0.5\},\{0.1,0.4\}),(\{0.9\},\{0.4,0.5\},\{0.5,0.7\}))))\}.\end{aligned}$$

Since $s\left(\Omega_U(e)(z)\right) \leq s\left(\Psi_V(e)(z)\right)$ for all $e \in U$, $z \in Z \Rightarrow \Omega_U \breve{\subseteq} \Psi_V$ (one of the different choices of e and z is explained as: $s(\Omega_U(e_1)(z_1)) = -0.3288 \leq -0.083 = s(\Psi_U(e_1)(z_1))$).

Definition 3.17. Let Z be a universal set and Ω_U, Ψ_V be two MVmNS sets, where U and V are subsets of E. Ω_U and Ψ_V are said to be equal if $\Omega_U \breve{\subseteq} \Psi_V$ and $\Psi_V \breve{\subseteq} \Omega_U$.

4. **Distance Measures**

Let $Z = \{z_1, z_2, ..., z_n\}$ be a universal set, $E = \{e_1, e_2, ..., e_p\}$ be a set of attributes and $U, V \subseteq E$. Let Ω_U and Ψ_V be two MVmNS sets over Z with their respective MVmN mappings:

$$\omega_U\left(e_j\right) = \left\{\left(z_k, s_i \circ \widetilde{T}_U\left(e_j\right)\left(z_k\right), s_i \circ \widetilde{I}_U\left(e_j\right)\left(z_k\right), s_i \circ \widetilde{F}_U\left(e_j\right)\left(z_k\right)\right)\right\},$$
$$\psi_V\left(e_j\right) = \left\{\left(z_k, s_i \circ \widetilde{T}_V\left(e_j\right)\left(z_k\right), s_i \circ \widetilde{I}_V\left(e_j\right)\left(z_k\right), s_i \circ \widetilde{F}_V\left(e_j\right)\left(z_k\right)\right)\right\},$$

for all $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, p$ and $k = 1, 2, \ldots, n$, then the distance measures between Ω_U and Ψ_V are defined as:

4.1. *Hamming Distance*

$$d_H\left(\Omega_U, \Psi_V\right) = \frac{1}{3mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}\left(\begin{array}{c}\left|s_i \circ \widetilde{T}_U\left(e_j\right)\left(z_k\right) - s_i \circ \widetilde{T}_V\left(e_j\right)\left(z_k\right)\right| + \\ \left|s_i \circ \widetilde{I}_U\left(e_j\right)\left(z_k\right) - s_i \circ \widetilde{I}_V\left(e_j\right)\left(z_k\right)\right| + \\ \left|s_i \circ \widetilde{F}_U\left(e_j\right)\left(z_k\right) - s_i \circ \widetilde{F}_V\left(e_j\right)\left(z_k\right)\right|\end{array}\right)\right\}.$$

4.2. *Normalized Hamming Distance*

$$d_{NH}\left(\Omega_U,\Psi_V\right)=\frac{1}{3mpn}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}\left(\begin{array}{c}\left|s_i\circ\widetilde{T}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{T}_V\left(e_j\right)\left(z_k\right)\right|+\\ \left|s_i\circ\widetilde{I}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{I}_V\left(e_j\right)\left(z_k\right)\right|+\\ \left|s_i\circ\widetilde{F}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{F}_V\left(e_j\right)\left(z_k\right)\right|\end{array}\right)\right\}.$$

4.3. *Euclidean Distance*

$$d_{E}\left(\Omega_U,\Psi_V\right)=\frac{1}{3mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}\left(\begin{array}{c}\left(s_i\circ\widetilde{T}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{T}_V\left(e_j\right)\left(z_k\right)\right)^2+\\ \left(s_i\circ\widetilde{I}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{I}_V\left(e_j\right)\left(z_k\right)\right)^2+\\ \left(s_i\circ\widetilde{F}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{F}_V\left(e_j\right)\left(z_k\right)\right)^2\end{array}\right)\right\}^{\frac{1}{2}}.$$

4.4. *Normalized Euclidean Distance*

$$d_{NE}\left(\Omega_U,\Psi_V\right)=\frac{1}{3mpn}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}\left(\begin{array}{c}\left(s_i\circ\widetilde{T}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{T}_V\left(e_j\right)\left(z_k\right)\right)^2+\\ \left(s_i\circ\widetilde{I}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{I}_V\left(e_j\right)\left(z_k\right)\right)^2+\\ \left(s_i\circ\widetilde{F}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{F}_V\left(e_j\right)\left(z_k\right)\right)^2\end{array}\right)\right\}^{\frac{1}{2}}.$$

Some distance measures are defined with weight vector $W=(w_1,w_2,...,w_p)^T$ satisfying $w_j\geq 0$ and $\sum\limits_{j=1}^{p}w_j=1$.

4.5. *Weighted Hamming Distance*

$$d_{WH}\left(\Omega_U,\Psi_V\right)=\frac{1}{3mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}w_j\left(\begin{array}{c}\left|s_i\circ\widetilde{T}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{T}_V\left(e_j\right)\left(z_k\right)\right|+\\ \left|s_i\circ\widetilde{I}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{I}_V\left(e_j\right)\left(z_k\right)\right|+\\ \left|s_i\circ\widetilde{F}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{F}_V\left(e_j\right)\left(z_k\right)\right|\end{array}\right)\right\}.$$

4.6. *Weighted Euclidean Distance*

$$d_{E}\left(\Omega_U,\Psi_V\right)=\frac{1}{3mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}w_j\left(\begin{array}{c}\left(s_i\circ\widetilde{T}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{T}_V\left(e_j\right)\left(z_k\right)\right)^2+\\ \left(s_i\circ\widetilde{I}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{I}_V\left(e_j\right)\left(z_k\right)\right)^2+\\ \left(s_i\circ\widetilde{F}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{F}_V\left(e_j\right)\left(z_k\right)\right)^2\end{array}\right)\right\}^{\frac{1}{2}}.$$

5. MCDM Based on MVmNSS by using TOPSIS

Ordering of the elements of MVmNS and formulation of distance measures between them leads us to develop a stepwise algorithm of TOPSIS.

Step 1: Construct a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$ and a set of attributes $E = \{e_1, e_2, \ldots, e_p\}$.

Step 2: A decision matrix is constructed by a decision maker which is the representation of an MVmNSS. In case of group decision, decision matrices are obtained from the experts and then an aggregated matrix D is obtained by using MVmNBM (Definition 3.6), and is represented as:

$$D(x_k) = \left\{ \left(e_j, s_i \circ \widetilde{T}_D(e_j), s_i \circ \widetilde{I}_D(e_j), s_i \circ \widetilde{F}_D(e_j) \right) \right\},$$

for an alternative x_k, $k = 1, 2, \ldots, n$.

Step 3: Choose the positive and negative ideal solutions by calculating the score values of the entries of decision matrices,

$$\begin{aligned} PIS &= \left\{ \left(e_j, s_i \circ \widetilde{T}_P(e_j), s_i \circ \widetilde{I}_P(e_j), s_i \circ \widetilde{F}_P(e_j) \right) \right\}, \\ NIS &= \left\{ \left(e_j, s_i \circ \widetilde{T}_N(e_j), s_i \circ \widetilde{I}_N(e_j), s_i \circ \widetilde{F}_N(e_j) \right) \right\}. \end{aligned}$$

Step 4: Find the distances of the elements of the aggregated matrix from PIS and NIS for each alternative x_k, $k = 1, 2, \ldots, n$ by using one of the following group of distance measures:

- $$d_{WH}(D_{x_k}, PIS) = \frac{1}{3mp} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{p} w_j \begin{pmatrix} \left| s_i \circ \widetilde{T}_D(e_j) - s_i \circ \widetilde{T}_P(e_j) \right| + \\ \left| s_i \circ \widetilde{I}_D(e_j) - s_i \circ \widetilde{I}_P(e_j) \right| + \\ \left| s_i \circ \widetilde{F}_D(e_j) - s_i \circ \widetilde{F}_P(e_j) \right| \end{pmatrix} \right\},$$

  $$d_{WH}(D_{x_k}, NIS) = \frac{1}{3mp} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{p} w_j \begin{pmatrix} \left| s_i \circ \widetilde{T}_D(e_j) - s_i \circ \widetilde{T}_N(e_j) \right| + \\ \left| s_i \circ \widetilde{I}_D(e_j) - s_i \circ \widetilde{I}_N(e_j) \right| + \\ \left| s_i \circ \widetilde{F}_D(e_j) - s_i \circ \widetilde{F}_N(e_j) \right| \end{pmatrix} \right\},$$

- $$d_{WE}(D_{x_k}, PIS) = \frac{1}{3mp} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{p} w_j \begin{pmatrix} \left(s_i \circ \widetilde{T}_D(e_j) - s_i \circ \widetilde{T}_P(e_j) \right)^2 + \\ \left(s_i \circ \widetilde{I}_D(e_j) - s_i \circ \widetilde{I}_P(e_j) \right)^2 + \\ \left(s_i \circ \widetilde{F}_D(e_j) - s_i \circ \widetilde{F}_P(e_j) \right)^2 \end{pmatrix} \right\}^{\frac{1}{2}},$$

$$d_{WE}\left(D_{x_k}, NIS\right) = \frac{1}{3mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p} w_j \left(\begin{array}{c} \left(s_i \circ \widetilde{T}_D\left(e_j\right) - s_i \circ \widetilde{T}_N\left(e_j\right)\right)^2 + \\ \left(s_i \circ \widetilde{I}_D\left(e_j\right) - s_i \circ \widetilde{I}_N\left(e_j\right)\right)^2 + \\ \left(s_i \circ \widetilde{F}_D\left(e_j\right) - s_i \circ \widetilde{F}_N\left(e_j\right)\right)^2 \end{array}\right)\right\}^{\frac{1}{2}}.$$

Step 5: Calculate the co-efficients of relative closeness (RC) for the alternatives by using one of the following formulae:

$$RC(x_k) = \frac{d_{WH}\left(D_{x_k}, NIS\right)}{d_{WH}\left(D_{x_k}, NIS\right) + d_{WH}\left(D_{x_k}, PIS\right)},$$

or

$$RC(x_k) = \frac{d_{WE}\left(D_{x_k}, NIS\right)}{d_{WE}\left(D_{x_k}, NIS\right) + d_{WE}\left(D_{x_k}, PIS\right)},$$

$k = 1, 2, ..., n$, according to the distance measure used in step 4.

Step 6: Rank the alternatives.

5.1. *An Application Example*

Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of alternatives, $E = \{e_1, e_2, e_3\}$ be a set of attributes and $D = \{d_1, d_2, d_3\}$ be the set of decision makers. Ranking of alternatives by the experts and observation of their attitudes is done here by two techniques:

(1) MVmNBM

(2) TOPSIS

By using first technique, stepwise procedure is as under:

Step 1: Obtain the MV2NSSs from the decision makers d_1, d_2 and d_3 which can be represented in Table 2, Table 3 and Table 4 respectively.

Step 2: Obtain an MVmNSS d^{agg} by calculating MV2NBM (Definition 3.6) for the respective values of Table 2, Table 3 and Table 4.

Step 3: Let $W_1 = \begin{pmatrix} 0.3 & 0.5 & 0.2 \end{pmatrix}$, $W_2 = \begin{pmatrix} 0.2 & 0.4 & 0.4 \end{pmatrix}$ and $W_3 = \begin{pmatrix} 0.7 & 0.1 & 0.2 \end{pmatrix}$ be three weight vectors for the attributes provided by three decision makers d_1, d_2 and d_3 respectively. Their weighted aggregated values are obtained from Definition 3.7 and are shown in Table 6, Table 7 and Table 8.

Step 4: Now by using the score function (Definition 3.4), find the single values for each alternative.

Score values for d_1:

$S(x_1) = -0.3128$

$S(x_2) = -0.3341$

$S(x_3) = -0.3009$

$S(x_4) = -0.3147$

TABLE 2. Decision matrix from the decision maker d_1.

1	e_1	e_2	e_3
x_1	$\begin{pmatrix} (\{0.3,0.4\},\{0.4\}), \\ (\{0.5\},\{0.4\}), \\ (\{0.6\},\{0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6,0.7\},\{0.8\}), \\ (\{0.5,0.7\},\{0.4,0.6\}), \\ (\{0.9\},\{0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\},\{0.6,0.8\}), \\ (\{0.4,0.5\},\{0.7\}), \\ (\{0.3\},\{0.6\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.1\},\{0.5\}), \\ (\{0.5\},\{0.9\}), \\ (\{0.6,0.7\},\{0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.7,0.8\},\{0.5\}), \\ (\{0.6,0.7\},\{0.6\}), \\ (\{0.6,0.8\},\{0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3\},\{0.7\}), \\ (\{0.4\},\{0.9\}), \\ (\{0.2\},\{0.3,0.5\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.6\},\{0.8\}), \\ (\{0.7,0.8\},\{0.1\}), \\ (\{0.6\},\{0.1\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\},\{0.8\}), \\ (\{0.7\},\{0.6\}), \\ (\{0.5,0.6\},\{0.5\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.5\},\{0.6\}), \\ (\{0.4\},\{0.9\}), \\ (\{0.1,0.4\},\{0.5\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.5\},\{0.9\}), \\ (\{0.2,0.6\},\{0.4\}), \\ (\{0.7\},\{0.4\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.1,0.4\},\{0.5\}), \\ (\{0.7\},\{0.2\}), \\ (\{0.5\},\{0.3,0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3\},\{0.8\}), \\ (\{0.4\},\{0.5,0.6\}), \\ (\{0.4,0.6\},\{0.7\}) \end{pmatrix}$

TABLE 3. Decision matrix from the decision maker d_2.

2	e_1	e_2	e_3
x_1	$\begin{pmatrix} (\{0.1\},\{0.3\}), \\ (\{0.2\},\{0.1,0.5\}), \\ (\{0.4\},\{0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.7\},\{0.5\}), \\ (\{0.4\},\{0.5\}), \\ (\{0.3,0.5\},\{0.4\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3,0.5\},\{0.5,0.6\}), \\ (\{0.4\},\{0.8\}), \\ (\{0.3\},\{0.7\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.4,0.6\},\{0.9\}), \\ (\{0.8\},\{0.5\}), \\ (\{0.6\},\{0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.1\},\{0.5,0.7\}), \\ (\{0.6\},\{0.7\}), \\ (\{0.1\},\{0.3\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.4\},\{0.9\}), \\ (\{0.5\},\{0.6\}), \\ (\{0.7\},\{0.8\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.2\},\{0.4\}), \\ (\{0.5\},\{0.6,0.7\}), \\ (\{0.2\},\{0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.4\},\{0.7,0.8\}), \\ (\{0.2\},\{0.1\}), \\ (\{0.2\},\{0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.2\},\{0.8,0.9\}), \\ (\{0.5\},\{0.8\}), \\ (\{0.1\},\{0.5,0.9\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.5,0.7\},\{0.4\}), \\ (\{0.4,0.8\},\{0.9\}), \\ (\{0.1\},\{0.5\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\},\{0.8\}), \\ (\{0.6\},\{0.7,0.9\}), \\ (\{0.3\},\{0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3\},\{0.4,0.5\}), \\ (\{0.3,0.4\},\{0.6\}), \\ (\{0.7\},\{0.2,0.3\}) \end{pmatrix}$

Score values for d_2:
$S(x_1) = -0.3084$
$S(x_2) = -0.3326$
$S(x_3) = -0.2952$
$S(x_4) = -0.3167$
Score values for d_3:

TABLE 4. Decision matrix from the decision maker d_3.

3	e_1	e_2	e_3
x_1	$\begin{pmatrix} (\{0.6\},\{0.7\}), \\ (\{0.8,0.9\},\{0.4\}), \\ (\{0.3\},\{0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.4,0.6\},\{0.7\}), \\ (\{0.4\},\{0.6\}), \\ (\{0.5\},\{0.7,0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.1,0.3\},\{0.4\}), \\ (\{0.6\},\{0.5,0.8\}), \\ (\{0.3,0.5\},\{0.4\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.4\},\{0.5\}), \\ (\{0.6\},\{0.7\}), \\ (\{0.8\},\{0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3\},\{0.4\}), \\ (\{0.5,0.6\},\{0.6\}), \\ (\{0.4\},\{0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.5,0.8\},\{0.5\}), \\ (\{0.2\},\{0.3\}), \\ (\{0.4\},\{0.9\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.2\},\{0.3,0.6\}), \\ (\{0.4\},\{0.5\}), \\ (\{0.7\},\{0.8\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6\},\{0.7,0.8\}), \\ (\{0.7\},\{0.9\}), \\ (\{0.4,0.5\},\{0.8\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\},\{0.7\}), \\ (\{0.4\},\{0.6,0.7\}), \\ (\{0.6\},\{0.8\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.2\},\{0.6\}), \\ (\{0.4,0.6\},\{0.7\}), \\ (\{0.6\},\{0.8,0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6\},\{0.4\}), \\ (\{0.4\},\{0.9\}), \\ (\{0.3\},\{0.6,0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.2\},\{0.3,0.5\}), \\ (\{0.7,0.8\},\{0.9\}), \\ (\{0.4\},\{0.1\}) \end{pmatrix}$

TABLE 5. Aggregated matrix d^{agg}.

Agg1	e_1	e_2	e_3
x_1	$\begin{pmatrix} (\{0.3039,0.3437\},\{0.4539\}), \\ (\{0.5187,0.56\},\{0.3025,0.4338\}), \\ (\{0.4362\},\{0.6346\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.5638,0.6666\},\{0.6666\}), \\ (\{0.4338,0.5050\},\{0.5022,0.5677\}), \\ (\{0.5891,0.6507\},\{0.6961,0.7911\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3691,0.5488\},\{0.4978,0.5955\}), \\ (\{0.4687,0.5022\},\{0.6774,0.7690\}), \\ (\{0.3,0.3672\},\{0.5740\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.2860,0.3437\},\{0.6246\}), \\ (\{0.6418\},\{0.7204\}), \\ (\{0.6722,0.7050\},\{0.7140\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3268,0.3484\},\{0.4659,0.5285\}), \\ (\{0.5677,0.6346\},\{0.6346\}), \\ (\{0.3732,0.4454\},\{0.6732\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3966,0.4806\},\{0.7003\}), \\ (\{0.3693\},\{0.6299\}), \\ (\{0.4404\},\{0.7207,0.7607\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.3068\},\{0.4806,0.5955\}), \\ (\{0.5387,0.5775\},\{0.4130,0.4512\}), \\ (\{0.5194\},\{0.5824\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6268\},\{0.7334,0.8\}), \\ (\{0.5607\},\{0.5803\}), \\ (\{0.3693,0.4419\},\{0.6774\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.5095\},\{0.7,0.7334\}), \\ (\{0.4338\},\{0.7832,0.8081\}), \\ (\{0.2430,0.3732\},\{0.6088,0.7607\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.3912,0.4524\},\{0.6268\}), \\ (\{0.3346,0.6722\},\{0.6961\}), \\ (\{0.4936\},\{0.6193\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.5169,0.6268\},\{0.5581\}), \\ (\{0.5740\},\{0.6516,0.7608\}), \\ (\{0.3672\},\{0.5080,0.6682\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.2648\},\{0.4806,0.5947\}), \\ (\{0.4724,0.5432\},\{0.6842,0.7140\}), \\ (\{0.5050,0.5740\},\{0.32,0.3656\}) \end{pmatrix}$

$S(x_1) = -0.3444$
$S(x_2) = -0.3663$
$S(x_3) = -0.3365$
$S(x_4) = -0.3503$

$S(x_2) < S(x_4) < S(x_1) < S(x_3)$ is the ranking of alternatives which is similar for all three decision makers. Alternative x_3 is the best one to select.

Step 5: To analyze the future attitude of the decision makers, system of differential equations (2) is developed by selecting $a^j_{p_i}$, $i, j = 1, 2, 3$ from the score values.$a^1_{p_1} = 0.6991, a^1_{p_2} = 0.7048, a^2_{p_1} = 0.6991, a^2_{p_2} = 0.7048$

$$\begin{aligned} \frac{dP_1}{dt} &= 0.6991P_1 + 0.7048P_2 \\ \frac{dP_2}{dt} &= 0.6991P_1 + 0.7048P_2 \end{aligned} \tag{3}$$

TABLE 6. Weighted aggregated values for d_1.

x_1	$\begin{pmatrix} (\{0.6455, 0.7069\}, \{0.7202, 0.7360\}), \\ (\{0.7883, 0.8137\}, \{0.7889, 0.8375\}), \\ (\{0.7778, 0.8019\}, \{0.8684, 0.8858\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.5922, 0.6256\}, \{0.7444, 0.7571\}), \\ (\{0.8186, 0.8313\}, \{0.8771\}), \\ (\{0.8, 0.8185\}, \{0.8924, 0.8968\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.6843\}, \{0.7723, 0.8112\}), \\ (\{0.8109, 0.8172\}, \{0.8381, 0.8473\}), \\ (\{0.7346, 0.7749\}, \{0.8617, 0.8789\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.6379, 0.6756\}, \{0.7298, 0.7465\}), \\ (\{0.7832, 0.8494\}, \{0.8824, 0.9030\}), \\ (\{0.7760, 0.7859\}, \{0.7963, 0.8345\}) \end{pmatrix}$

TABLE 7. Weighted aggregated values for d_2.

x_1	$\begin{pmatrix} (\{0.6527, 0.7225\}, \{0.7263, 0.7470\}), \\ (\{0.7825, 0.8081\}, \{0.8027, 0.8508\}), \\ (\{0.7683, 0.7942\}, \{0.8634, 0.8808\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.6019, 0.6362\}, \{0.7520, 0.7644), \\ (\{0.8064, 0.8201\}, \{0.8716\}), \\ (\{0.7848, 0.8033\}, \{0.8912, 0.8976\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.7001\}, \{0.7867, 0.8221\}), \\ (\{0.8033, 0.8084\}, \{0.8527, 0.8623\}), \\ (\{0.7179, 0.7647\}, \{0.8595, 0.8840\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.6304, 0.6637\}, \{0.7270, 0.7496\}), \\ (\{0.7826, 0.8418\}, \{0.8802, 0.9015\}), \\ (\{0.7730, 0.7872\}, \{0.7815, 0.8234\}) \end{pmatrix}$

Line graph for the system (3) (Figure 2) shows the same future behaviour of the decision makers d_1 and d_2, since lines are overlapping and phase portrait (Figure 1) shows that the system is unstable. It means that the experts may change their attitudes in future. A similar conclusion can be observed between d_3 and d_2 or d_1 and d_3. Future attitudes of d_1 and d_2 can also be analyzed (Figure 3) with the following fuzzy initial conditions (FICs):

$P_1(0) = (-1, 0, 1)$,

$P_2(0) = (-1, 0, 1)$,

or (α-cut representation)

$P_1(0) = (-1 + \alpha, 1 - \alpha) \qquad \alpha \in [0, 1]$,

$P_2(0) = (-1 + \alpha, 1 - \alpha) \qquad \alpha \in [0, 1]$.

Table 8. Weighted aggregated values for d_3.

x_1	$\begin{pmatrix} (\{0.5941, 0.6529\}, \{0.6708, 0.6887\}), \\ (\{0.8239, 0.8432\}, \{0.8310, 0.8709\}), \\ (\{0.7917, 0.8105\}, \{0.8813, 0.8881\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.5629, 0.6\}, \{0.7245, 0.7304\}), \\ (\{0.8437, 0.8488\}, \{0.8980\}), \\ (\{0.8431, 0.8567\}, \{0.9106, 0.9155\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.6299\}, \{0.7240, 0.7646\}), \\ (\{0.8336, 0.8407\}, \{0.8720, 0.8806\}), \\ (\{0.7762, 0.8090\}, \{0.8767, 0.8985\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.5891, 0.6212\}, \{0.6886, 0.7115\}), \\ (\{0.8066, 0.8744\}, \{0.9019, 0.9135\}), \\ (\{0.8179, 0.8283\}, \{0.8261, 0.8461\}) \end{pmatrix}$

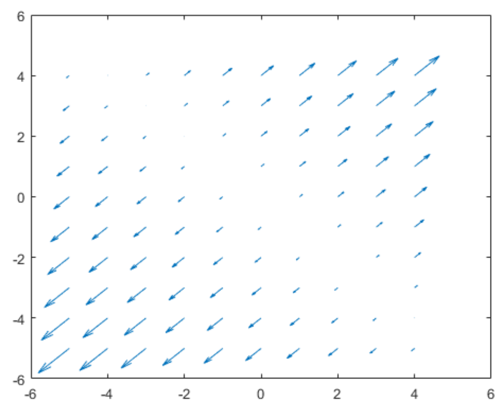

Figure 1. Phase portrait for the system (3).

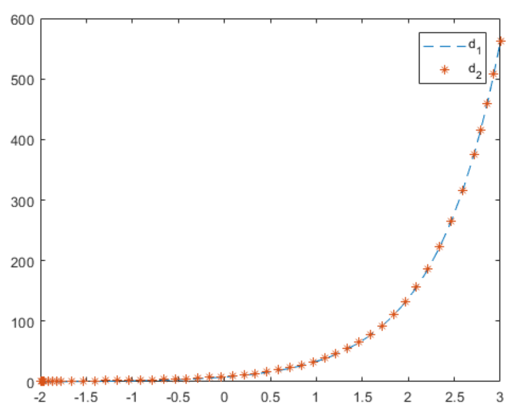

Figure 2. Line graph for the system (3).

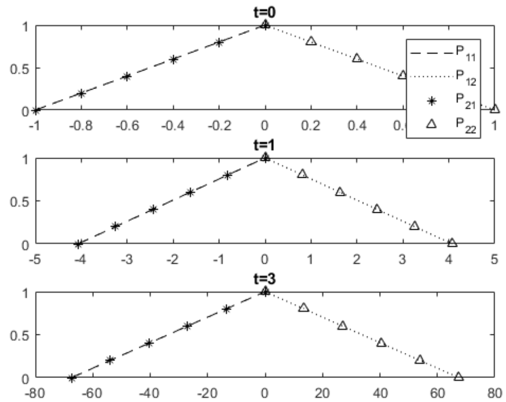

FIGURE 3. Line graph for the system (3) with FICs.

TABLE 9. Score values from the decision maker d_1.

d_1	e_1	e_2	e_3
x_1	-0.2416	-0.2416	-0.075
x_2	-0.3916	-0.2666	-0.15
x_3	-0.025	-0.1083	-0.1583
x_4	-0.0833	-0.1916	-0.175

TABLE 10. Score values from the decision maker d_2.

d_2	e_1	e_2	e_3
x_1	-0.1833	-0.0833	-0.2083
x_2	-0.1833	-0.1666	-0.2166
x_3	-0.2416	-0.0083	-0.175
x_4	-0.1833	-0.1	-0.1916

Now the stepwise procedure of the second technique is as under:

Step 1 Same as in first technique

Step 2 Same as in first technique.

Step 3 Find the score values of the entries of Table 2, Table 3 and Table 4 by Definition 3.4. Respective score values are represented in Table 9, Table 10 and Table 11.

Step 4 By comparing the score values of the alternatives in Table 9, Table 10 and Table 11, select the PIS and NIS from Table 2, Table 3 and Table 4.

Step 5 Find the weighted distances between the entries of Table 5 and Table 12 as described in section 4.5 with $W_1 = \begin{pmatrix} 0.3 & 0.5 & 0.2 \end{pmatrix}$ and $W_2 = \begin{pmatrix} 0.2 & 0.4 & 0.4 \end{pmatrix}$. Here weighted Hamming distance is utilized.

Table 11. Score values from the decision maker d_3.

d_3	e_1	e_2	e_3
x_1	-0.1416	-0.1833	-0.2416
x_2	-0.3	-0.2583	-0.1
x_3	-0.2916	-0.25	-0.1416
x_4	-0.3083	-0.2083	-0.2583

Table 12. Positive and negative ideal solution.

	e_1	e_2	e_3
PIS	$\begin{pmatrix} (\{0.6\},\{0.8\}), \\ (\{0.7,0.8\},\{0.1\}), \\ (\{0.6\},\{0.1\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.4\},\{0.7,0.8\}), \\ (\{0.2\},\{0.1\}), \\ (\{0.2\},\{0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\},\{0.6,0.8\}), \\ (\{0.4,0.5\},\{0.7\}), \\ (\{0.3\},\{0.6\}) \end{pmatrix}$
NIS	$\begin{pmatrix} (\{0.1\},\{0.5\}), \\ (\{0.5\},\{0.9\}), \\ (\{0.6,0.7\},\{0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.7,0.8\},\{0.5\}), \\ (\{0.6,0.7\},\{0.6\}), \\ (\{0.6,0.8\},\{0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.2\},\{0.3,0.5\}), \\ (\{0.7,0.8\},\{0.9\}), \\ (\{0.4\},\{0.1\}) \end{pmatrix}$

$$
\begin{array}{r|l}
d_{W_1H}\left(D_{x_1}, PIS\right) = 0.4059 & d_{W_2H}\left(D_{x_1}, PIS\right) = 0.3875 \\
d_{W_1H}\left(D_{x_2}, PIS\right) = 0.4284 & d_{W_2H}\left(D_{x_2}, PIS\right) = 0.4131 \\
d_{W_1H}\left(D_{x_3}, PIS\right) = 0.4048 & d_{W_2H}\left(D_{x_3}, PIS\right) = 0.3886 \\
d_{W_1H}\left(D_{x_4}, PIS\right) = 0.4523 & d_{W_2H}\left(D_{x_4}, PIS\right) = 0.4485 \\
d_{W_1H}\left(D_{x_1}, NIS\right) = 0.3838 & d_{W_2H}\left(D_{x_1}, NIS\right) = 0.4064 \\
d_{W_1H}\left(D_{x_2}, NIS\right) = 0.3928 & d_{W_2H}\left(D_{x_2}, NIS\right) = 0.3824 \\
d_{W_1H}\left(D_{x_3}, NIS\right) = 0.4060 & d_{W_2H}\left(D_{x_3}, NIS\right) = 0.4109 \\
d_{W_1H}\left(D_{x_4}, NIS\right) = 0.4122 & d_{W_2H}\left(D_{x_4}, NIS\right) = 0.4597
\end{array}
$$

Step 6 Find the Coefficients of relative closeness for each alternative and rank the alternatives.

$$
\begin{array}{c|c}
RC_{W_1}(x_1) = 0.4860 & RC_{W_2}(x_1) = 0.5119 \\
RC_{W_1}(x_2) = 0.4783 & RC_{W_2}(x_2) = 0.4807 \\
RC_{W_1}(x_3) = 0.5007 & RC_{W_2}(x_3) = 0.5139 \\
RC_{W_1}(x_4) = 0.4768 & RC_{W_2}(x_4) = 0.5061 \\
S(x_4) < S(x_2) < S(x_1) < S(x_3) & S(x_2) < S(x_4) < S(x_1) < S(x_3)
\end{array}
$$

Both experts select the same alternative and their future attitude is same as discussed in previous technique.

6. Conclusion

MVmNSS can model the problems of MCDM with undetermined information better than MVNSS and mNSS. It engages not only the multi-polar information but also multi-valued data. The multi-valued neutrosophic set has the membership, non-membership and indeterminacy values which can be treated as in hesitant fuzzy set or dual hesitant fuzzy set when operational laws (Definition 2.3) are defined. An analysis of experts' attitudes after their decisions can also be done by utilizing the MVmNBM. This study has also been carried out by Beg et al. [41] with a fuzzy soft matrix as the initial data which does not captivate the degrees of falsity-membership and indeterminacy-membership. MVmNSS handles these complicated uncertainties and can be aggregated by MVmNBM. In the future, other MCDM methods (TOPSIS, VIKOR, etc.) can be applied in group decision problems by defining the distance and similarity measures in MVmNSs. Another aspect of this research is the utilization of differential equations with FICs which does not produce different results.

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