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Pythagorean m-polar Fuzzy Neutrosophic Metric Spaces

Atiqa Siraj[1], Khalid Naeem[2,*] and Broumi Said[3]

[1]Department of Mathematics and Statistics, The University of Lahore, Pakistan; atiqasiraj60@gmail.com

[2,*]Department of Mathematics, Federal Government Degree College, Lahore Cantt., Pakistan; khalidnaeem333@gmail.com

[3]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, Casablanca, Morocco; broumisaid78@gmail.com

[*]Correspondence: khalidnaeem333@gmail.com

Abstract. Neutrosophy deals with the study of neutrosophic logic, set and probability. A Pythagorean m-polar neutrosophic set is indeed an expansion of crisp, fuzzy, intuitionistic fuzzy, neutrosophic and Pythagorean m-polar fuzzy sets. In this paper, we develop the perception of Pythagorean m-polar fuzzy neutrosophic metric space defined over Pythagorean m-polar fuzzy neutrosophic set relying on the classical definition of metric spaces defined on a crisp set. We present some related results and illustrations to perceive the conceptions. We present many examples of metrics which hold true for classical sets but fail to make sense in Pythagorean m-polar fuzzy environment. We also render a practical utility of the proposed metrics in pattern recognition.

Keywords: Pythagorean m-polar fuzzy neutrosophic set; Pythagorean m-polar fuzzy neutrosophic subset; Pythagorean m-polar fuzzy neutrosophic metric spaces; Pattern recognition

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1. Introduction

In the wake of advancement of classical sets to fuzzy sets by Zadeh [33], the scientists around the globe started working on diverse aspects of fuzzy sets and its expansions. Contrary to classical sets, an element is allowed to partially belong to the set, as specified in fuzzy set. In [2,3] Atanassov unveiled the notion of intuitionistic fuzzy sets (IFSs) by including the so called non-membership grade to already included membership grade in a fuzzy set. Yager [32] comforted the decision makers by enhancing the space for the the choice of association and dissociation grades prevailing in the IFSs and called the resulting model as Pythagorean fuzzy set. Naeem *et al.* [21] expanded the conception given by Yager to Pythagorean m-polar fuzzy sets and rendered a fascinating practical implementation to advertisement mode selection problem. Later, Riaz *et al.* [24] further generalized the thought to Pythagorean fuzzy soft sets. Naeem *et al.* further explored the chief characteristics of Pythagorean m-polar fuzzy sets in [22].

Maurice René Fréchet, a French mathematician, floated the idea of metric spaces in 1906. Deng [12], Diamond and Kloden [13], Atefi and Jehadi [4], Chaudhuri and Rosenfeld [5], George and Veeramani [14], and Gregori and Romaguera [15] are among the mathematicians who studied and explored different aspects of fuzzy metric spaces. The scientists who explored metric spaces in the framework of IFSs mainly include Gregori and Romaguera [16], Li *et al.* [19], and Park *et al.* [23].

Smarandache [28] presented yet another expansion of fuzzy sets called Neutrosophic sets. He made further explorations in [29] and [30]. The series of fascinating explorations by Smarandache is continued. Wang *et al.* [31] presented single valued neutrosophic sets. Arockiarani *et al.* [1] studied fuzzy neutrosophic soft topological spaces. Şimşek and Kirşci [27] explored fixed points in the context of neutrosophic metric spaces. Ishtiaq *et al.* [17] presented fixed points results in orthogonal neutrosophic metric spaces. Jansi and Mohana [18] studied, in recent times, pairwise Pythagorean neutrosophic P-spaces (with dependent neutrosophic components between T and F). In recent times, Siraj *et al.* [25] unveiled the apprehension of Pythagorean m-polar fuzzy neutrosophic topology with applications towards handling economic crises caused due to COVID-19 and the root cause behind scarcity of water in Thar desert of Pakistan.

Das *et al.* [6] presented the notion of neutrosophic fuzzy matrices with their algebraic operations. Das and Tripathy [7] studied neutrosophic multiset topological space. Mukherjee and Das [20] explored neutrosophic bipolar vague soft set and its application. Das *et al.* [8] unveiled the notions of neutro algebra and neutro group. Das and Das [9] presented neutrosophic separation axioms. Recently, Das *et al.* [10] rendered the idea of pentapartitioned neutrosophic probability distributions. Das *et al.* [11] studied topology on ultra neutrosophic set.

There arise many situations in real life where we have to think time and again before reaching at some decision–a decision that may be thought as flawless. It is in fact the process of multipolarity. The ever-expanding applications of neutrosophic sets are not concealed from the world. Pythagorean neutrosophic environment provides the enhanced facility of choosing values for the three membership functions (truth, indeterminacy and falsity) from a broader space.

In this article, we explore some notions of Pythagorean m-polar fuzzy neutrosophic metric spaces. Section 2 presents some basic notions necessary to conceive the main topic of this study. The third section presents main study of this article. In this section, the notion of

Pythagorean m-polar fuzzy neutrosophic metric spaces has been put forward. A large number of examples and illustrations are presented to conceive the perception. Section 4 presents a practical implementation of the proposed metrics in pattern recognition. Section 5 concludes the paper.

2. Preliminaries

Definition 2.1. [28, 29] A *neutrosophic set* $\mathbb{N}$ on the underlying set X is specified as

$$\mathbb{N} = \{< \Upsilon, T_{\mathbb{N}}(\Upsilon), I_{\mathbb{N}}(\Upsilon), F_{\mathbb{N}}(\Upsilon) >: \Upsilon \in X\}$$

where $T, I, F : X \mapsto]^{-}0, 1^{+}[$ accompanied by the constraint $^{-}0 \leq T_{\mathbb{N}}(\Upsilon)+I_{\mathbb{N}}(\Upsilon)+F_{\mathbb{N}}(\Upsilon) \leq 3^{+}$. Here $T_{\mathbb{N}}(\Upsilon)$, $I_{\mathbb{N}}(\Upsilon)$ and $F_{\mathbb{N}}(\Upsilon)$ are the degrees of membership, indeterminacy and falsity (non-membership) of members of the given set, respectively. T, I and F are acknowledged as the neutrosophic components.

Definition 2.2. [1] A *fuzzy neutrosophic set* ($\mathfrak{fn}$-set) over X is delineated as

$$A = \{< \Upsilon, T_A(\Upsilon), I_A(\Upsilon), F_A(\Upsilon) >: \Upsilon \in X\}$$

where $T, I, F : X \mapsto [0, 1]$ in such a way that $0 \leq T_A(\Upsilon) + I_A(\Upsilon) + F_A(\Upsilon) \leq 3$.

Definition 2.3. [25] A *Pythagorean m-polar fuzzy neutrosophic set* (PmFNS) $\Im$ over a basic set X is marked by three mappings $T^{(i)}_{\Im} : X \to [0, 1]^m$, $I^{(i)}_{\Im} : X \to [0, 1]^m$ and $F^{(i)}_{\Im} : X \to [0, 1]^m$, where m is a natural number, $\forall i = 1, 2, \cdots, m$, with the limitation that

$$0 \leq \left(T^{(i)}_{\Im}(\Upsilon)\right)^2 + \left(I^{(i)}_{\Im}(\Upsilon)\right)^2 + \left(F^{(i)}_{\Im}(\Upsilon)\right)^2 \leq 2$$

for all $\Upsilon \in X$.

A PmFNS may be expressed as

$$\begin{aligned}
\Im &= \left\{(\Upsilon, \left((T^{(1)}_{\Im}(\Upsilon), I^{(1)}_{\Im}(\Upsilon), F^{(1)}_{\Im}(\Upsilon)), \cdots, (T^{(m)}_{\Im}(\Upsilon), I^{(m)}_{\Im}(\Upsilon), F^{(m)}_{\Im}(\Upsilon))\right) : \Upsilon \in X\right\} \\
&= \left\{\frac{\Upsilon}{(T^{(1)}_{\Im}(\Upsilon), I^{(1)}_{\Im}(\Upsilon), F^{(1)}_{\Im}(\Upsilon)), \cdots, (T^{(m)}_{\Im}(\Upsilon), I^{(m)}_{\Im}(\Upsilon), F^{(m)}_{\Im}(\Upsilon))} : \Upsilon \in X\right\} \\
&= \left\{\frac{\Upsilon}{(T^{(i)}_{\Im}(\Upsilon), I^{(i)}_{\Im}(\Upsilon), F^{(i)}_{\Im}(\Upsilon))} : \Upsilon \in X, i = 1, 2, \cdots, m\right\}
\end{aligned}$$

If cardinality of X is l, then tabular structure of $\Im$ is as in Table 1:

TABLE 1. Tabular representation of PmFNS $\Im$

$\Im$				
$\curlyvee_1$	$(T_\Im^{(1)}(\curlyvee_1), I_\Im^{(1)}(\curlyvee_1), F_\Im^{(1)}(\curlyvee_1))$	$(T_\Im^{(2)}(\curlyvee_1), I_\Im^{(2)}(\curlyvee_1), F_\Im^{(2)}(\curlyvee_1))$	$\cdots$	$(T_\Im^{(m)}(\curlyvee_1), I_\Im^{(m)}(\curlyvee_1), F_\Im^{(m)}(\curlyvee_1))$
$\curlyvee_2$	$(T_\Im^{(1)}(\curlyvee_2), I_\Im^{(1)}(\curlyvee_2), F_\Im^{(1)}(\curlyvee_2))$	$(T_\Im^{(2)}(\curlyvee_2), I_\Im^{(2)}(\curlyvee_2), F_\Im^{(2)}(\curlyvee_2))$	$\cdots$	$(T_\Im^{(m)}(\curlyvee_2), I_\Im^{(m)}(\curlyvee_2), F_\Im^{(m)}(\curlyvee_2))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\curlyvee_l$	$(T_\Im^{(1)}(\curlyvee_l), I_\Im^{(1)}(\curlyvee_l), F_\Im^{(1)}(\curlyvee_l))$	$(T_\Im^{(2)}(\curlyvee_l), I_\Im^{(2)}(\curlyvee_l), F_\Im^{(2)}(\curlyvee_l))$	$\cdots$	$(T_\Im^{(m)}(\curlyvee_l), I_\Im^{(m)}(\curlyvee_l), F_\Im^{(m)}(\curlyvee_l))$

The corresponding matrix format is

$$\Im = \begin{pmatrix} (T_\Im^{(1)}(\curlyvee_1), I_\Im^{(1)}(\curlyvee_1), F_\Im^{(1)}(\curlyvee_1)) & (T_\Im^{(2)}(\curlyvee_1), I_\Im^{(2)}(\curlyvee_1), F_\Im^{(2)}(\curlyvee_1)) & \cdots & (T_\Im^{(m)}(\curlyvee_1), I_\Im^{(m)}(\curlyvee_1), F_\Im^{(m)}(\curlyvee_1)) \\ (T_\Im^{(1)}(\curlyvee_2), I_\Im^{(1)}(\curlyvee_2), F_\Im^{(1)}(\curlyvee_2)) & (T_\Im^{(2)}(\curlyvee_2), I_\Im^{(2)}(\curlyvee_2), F_\Im^{(2)}(\curlyvee_2)) & \cdots & (T_\Im^{(m)}(\curlyvee_2), I_\Im^{(m)}(\curlyvee_2), F_\Im^{(m)}(\curlyvee_2)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_\Im^{(1)}(\curlyvee_l), I_\Im^{(1)}(\curlyvee_l), F_\Im^{(1)}(\curlyvee_l)) & (T_\Im^{(2)}(\curlyvee_l), I_\Im^{(2)}(\curlyvee_l), F_\Im^{(2)}(\curlyvee_l)) & \cdots & (T_\Im^{(m)}(\curlyvee_l), I_\Im^{(m)}(\curlyvee_l), F_\Im^{(m)}(\curlyvee_l)) \end{pmatrix}$$

This $l \times m$ matrix is known as *PmFN matrix*. The assortment of each PmFNS characterized over universe would be designated by PmFNS(X).

Definition 2.4. [25] Let $\Im_1$ and $\Im_2$ be PmFNSs over X. $\Im_1$ is acknowledged as a *subset* of $\Im_2$, written as $\Im_1 \subseteq \Im_2$, $\forall \Im \in X$ and each values of i ranging from 1 to m, if

1) $T_{\Im_1}^{(i)}(\curlyvee) \leq T_{\Im_2}^{(i)}(\curlyvee)$,
2) $I_{\Im_1}^{(i)}(\curlyvee) \geq I_{\Im_2}^{(i)}(\curlyvee)$,
3) $F_{\Im_1}^{(i)}(\curlyvee) \geq F_{\Im_2}^{(i)}(\curlyvee)$.

$\Im_1$ and $\Im_2$ are said to be *equal* if $\Im_1 \subseteq \Im_2 \subseteq \Im_1$ and is written as $\Im_1 = \Im_2$.

Definition 2.5. [25] A PmFNS $\Im$ over X is known as *null PmFNS* if $T_\Im^{(i)}(\curlyvee) = 0$, $I_\Im^{(i)}(\curlyvee) = 1$ and $F_\Im^{(i)}(\curlyvee) = 1$, $\forall \curlyvee \in X$ and all acceptable values of i. It is designated by Φ.
Thus,

$$\Phi = \begin{pmatrix} (0,1,1) & (0,1,1) & \cdots & (0,1,1) \\ (0,1,1) & (0,1,1) & \cdots & (0,1,1) \\ \vdots & \vdots & \ddots & \vdots \\ (0,1,1) & (0,1,1) & \cdots & (0,1,1) \end{pmatrix}.$$

Definition 2.6. [25] A PmFNS $\Im$ over X is called an *absolute PmFNS* if $T_\Im^{(i)}(\curlyvee) = 1$, $I_\Im^{(i)}(\curlyvee) = 0$, and $F_\Im^{(i)}(\curlyvee) = 0$, $\forall \curlyvee \in X$. It is denoted by $\breve{\chi}$.
Thus,

$$\breve{\chi} = \begin{pmatrix} (1,0,0) & (1,0,0) & \cdots & (1,0,0) \\ (1,0,0) & (1,0,0) & \cdots & (1,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (1,0,0) & (1,0,0) & \cdots & (1,0,0) \end{pmatrix}.$$

Definition 2.7. [25] The *complement* of a PmFNS

$$\Im = \left\{ \frac{\Upsilon}{(T_{\Im}^{(i)}(\Upsilon), I_{\Im}^{(i)}(\Upsilon), F_{\Im}^{(i)}(\Upsilon))} : \Upsilon \in X, i = 1, \cdots, m \right\}$$

over X is defined as

$$\Im^c = \left\{ \frac{\Upsilon}{(F_{\Im}^{(i)}(\Upsilon), 1 - I_{\Im}^{(i)}(\Upsilon), T_{\Im}^{(i)}(\Upsilon))} : \Upsilon \in X, i = 1, \cdots, m \right\}.$$

Definition 2.8. [25] The *union* of any PmFNSs $\Im_1$ and $\Im_2$ expressed over the same universe X is represented as

$$\Im_1 \cup_{\mathfrak{M}} \Im_2 = \left\{ \frac{\Upsilon}{(\max(T_{\Im 1}^{(i)}(\Upsilon), T_{\Im 2}^{(i)}(\Upsilon)), \min(I_{\Im 1}^{(i)}(\Upsilon), I_{\Im 2}^{(i)}(\Upsilon)), \min(F_{\Im 1}^{(i)}(\Upsilon), F_{\Im 2}^{(i)}(\Upsilon))} : \Upsilon \in X, i = 1, \cdots, m \right\}$$

Definition 2.9. [25] The *intersection* of any PmFNSs $\Im_1$ and $\Im_2$ expressed over the same universe X is represented as

$$\Im_1 \cap_{\mathfrak{M}} \Im_2 = \left\{ \frac{\Upsilon}{(\min(T_{\Im 1}^{(i)}(\Upsilon), T_{\Im 2}^{(i)}(\Upsilon)), \max(I_{\Im 1}^{(i)}(\Upsilon), I_{\Im 2}^{(i)}(\Upsilon)), \max(F_{\Im 1}^{(i)}(\Upsilon), F_{\Im 2}^{(i)}(\Upsilon))} : \Upsilon \in X, i = 1, \cdots, m \right\}$$

3. **Pythagorean m-Polar Fuzzy Neutrosophic Metric Spaces**

In this section, we introduce the notion of Pythagorean m-polar fuzzy neutrosophic metric space along with its prime characteristics and illustrations. The superscript i, wherever used, will run from 1 to m, unless stated otherwise.

Definition 3.1. Let $\eth_1$, $\eth_2$ and $\eth_3$ be three PmFNSs on $\underline{\mathrm{X}}$. The mapping $\underline{\mathbb{M}}^s : PmFN(\underline{\mathrm{X}}) \times PmFN(\underline{\mathrm{X}}) \mapsto [0, 2]$ is said to be a *Pythagorean m-polar fuzzy neutrosophic metric* on $PmFN(\underline{\mathrm{X}})$ if it ensures the following postulates:

- $\underline{\mathbb{M}}_1^s$: $0 \leq \underline{\mathbb{M}}^s(\eth_1, \eth_2) \leq 2$
- $\underline{\mathbb{M}}_2^s$: $\underline{\mathbb{M}}^s(\eth_1, \eth_2) = \underline{\mathbb{M}}^s(\eth_2, \eth_1)$
- $\underline{\mathbb{M}}_3^s$: $\underline{\mathbb{M}}^s(\eth_1, \eth_2) = 0 \Leftrightarrow \eth_1 = \eth_2$
- $\underline{\mathbb{M}}_4^s$: $\underline{\mathbb{M}}^s(\eth_1, \eth_3) \leq \underline{\mathbb{M}}^s(\eth_1, \eth_2) + \underline{\mathbb{M}}^s(\eth_2, \eth_3)$
- $\underline{\mathbb{M}}_5^s$: If $\eth_1 \subseteq \eth_2 \subseteq \eth_3$, then $\underline{\mathbb{M}}^s(\eth_1, \eth_2) \leq \underline{\mathbb{M}}^s(\eth_1, \eth_3)$ and $\underline{\mathbb{M}}^s(\eth_2, \eth_3) \leq \underline{\mathbb{M}}^s(\eth_1, \eth_3)$

for all $\eth_1$, $\eth_2$ and $\eth_3 \Subset PmFN(\underline{\mathrm{X}})$.

The pair $(PmFN(\underline{\mathrm{X}}), \underline{\mathbb{M}}^s)$ is said to be the *Pythagorean m-polar fuzzy neutrosophic metric space* (PmFNMS). $PmFN(\underline{\mathrm{X}})$ is known as the *Pythagorean m-polar fuzzy neutrosophic underlying set* (PmFN-underlying set) or the *Pythagorean m-polar fuzzy neutrosophic ground set* (PmFN-ground set). The elements of $PmFN(\underline{\mathrm{X}})$ are called the *Pythagorean m-polar fuzzy neutrosophic points* (PmFN-points) of the PmFNMS $(PmFN(\underline{\mathrm{X}}), \underline{\mathbb{M}}^s)$.

Remark 3.2. If $\eth_1,\ \eth_2,\ \eth_3, \cdots, \eth_{n-1},\ \eth_n$ are n distinct PmFN points of the PmFNMS $(PmFN(\underline{X}),\ \underline{\mathbb{M}}^s)$, then the fourth postulate may be generalized as

$$\underline{\mathbb{M}}^s(\eth_1,\ \eth_n)\ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ +\ \underline{\mathbb{M}}^s(\eth_2,\ \eth_3)\ +\ \underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\ +\ \cdots\ +\underline{\mathbb{M}}^s(\eth_{n-1}, \eth_n)$$

Example 3.3. Let

$$\eth_1\ =\ \{\langle(\underline{\daleth}_1^{(i)},\ \underline{I}_1^{(i)},\ \underline{F}_1^{(i)})\rangle\},$$

and

$$\eth_2\ =\ \{\langle(\underline{\daleth}_2^{(i)},\ \underline{I}_2^{(i)},\ \underline{F}_2^{(i)})\rangle\}$$

be members of $PmFN(\underline{X})$. We establish that

$$\underline{\mathbb{M}}^s(\eth_1,\ \eth_2) = \sqrt[2m]{\sum_{i=1}^{m}\left\{(\underline{\daleth}_1^{(i)} - \underline{\daleth}_2^{(i)})^{2m} + (\underline{I}_1^{(i)} - \underline{I}_2^{(i)})^{2m} + (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^{2m}\right\}}$$

is a PmFNMS on $PmFN(\underline{X})$.

$\underline{\mathbb{M}}_2^s$ and $\underline{\mathbb{M}}_3^s$ of Definition 3.1 are obvious. We establish the remaining requirements.

$\underline{\mathbb{M}}_1^s$: Since $0 \leq (\underline{\daleth}_1^{(i)} - \underline{\daleth}_2^{(i)})^{2m} \leq 1,\ 0 \leq (\underline{I}_1^{(i)} - \underline{I}_2^{(i)})^{2m} \leq 1,\ 0 \leq (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^{2m} \leq 1$

$\Rightarrow 0 \leq \sqrt[2m]{\sum_{i=1}^{m}\left\{(\underline{\daleth}_1^{(i)} - \underline{\daleth}_2^{(i)})^{2m} + (\underline{I}_1^{(i)} - \underline{I}_2^{(i)})^{2m} + (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^{2m}\right\}} \leq 2$

Thus,

$$0 \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ \leq\ 2$$

$\forall\ \eth_1,\ \eth_2 \in PmFN(\underline{X})$.

$\underline{\mathbb{M}}_4^s$: By virtue of Minkowski's inequality, we have

$[(\underline{\daleth}_1^{(i)} - \underline{\daleth}_3^{(i)})^{2m} + (\underline{I}_1^{(i)} - \underline{I}_3^{(i)})^{2m} + (\underline{F}_1^{(i)} - \underline{F}_3^{(i)})^{2m}]^{\frac{1}{2m}}\ \leq\ [(\underline{\daleth}_1^{(i)} - \underline{\daleth}_2^{(i)})^{2m} + (\underline{I}_1^{(i)} - \underline{I}_2^{(i)})^{2m} + (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^{2m}]^{\frac{1}{2m}}\ +\ [(\underline{\daleth}_2^{(i)} - \underline{\daleth}_3^{(i)})^{2m} + (\underline{I}_2^{(i)} - \underline{I}_3^{(i)})^{2m} + (\underline{F}_2^{(i)} - \underline{F}_3^{(i)})^{2m}]^{\frac{1}{2m}}$

$\Rightarrow\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_3)\ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ +\ \underline{\mathbb{M}}^s(\eth_2,\ \eth_3)$

$\forall\ \eth_1,\ \eth_2,\ \eth_3 \in PmFN(\underline{X})$.

$\underline{\mathbb{M}}_5^s$: If $\eth_1\ \subseteq\ \eth_2\ \subseteq\ \eth_3$, then

$$\begin{gathered}\underline{\daleth}_1^{(i)}\ \leq\ \underline{\daleth}_2^{(i)}\ \leq\ \underline{\daleth}_3^{(i)},\\ \underline{I}_1^{(i)}\ \geq\ \underline{I}_2^{(i)}\ \geq\ \underline{I}_3^{(i)},\\ \underline{F}_1^{(i)}\ \geq\ \underline{F}_2^{(i)}\ \geq\ \underline{F}_3^{(i)}\end{gathered}$$

so that

$\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ =\ [(\underline{\daleth}_1^{(i)} - \underline{\daleth}_2^{(i)})^{2m} + (\underline{I}_1^{(i)} - \underline{I}_2^{(i)})^{2m} + (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^{2m}]^{\frac{1}{2m}}$

$\because\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ =\ \underline{\mathbb{M}}^s(\eth_2,\ \eth_1)$, from $\underline{\mathbb{M}}_2^s$

So,

$\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ =\ [(\underline{\daleth}_2^{(i)} - \underline{\daleth}_1^{(i)})^{2m} + (\underline{I}_2^{(i)} - \underline{I}_1^{(i)})^{2m} + (\underline{F}_2^{(i)} - \underline{F}_1^{(i)})^{2m}]^{\frac{1}{2m}}$

and

$\underline{\mathbb{M}}^s(\eth_1,\ \eth_3)\ =\ [(\underline{\daleth}_1^{(i)} - \underline{\daleth}_3^{(i)})^{2m} + (\underline{I}_1^{(i)} - \underline{I}_3^{(i)})^{2m} + (\underline{F}_1^{(i)} - \underline{F}_3^{(i)})^{2m}]^{\frac{1}{2m}}$

again from $\underline{\mathbb{M}}_2^s$, we have

$\underline{\mathbb{M}}^s(\eth_1,\ \eth_3)\ =\ [(\underline{\daleth}_3^{(i)} - \underline{\daleth}_1^{(i)})^{2m} + (\underline{\mathrm{I}}_3^{(i)} - \underline{\mathrm{I}}_1^{(i)})^{2m} + (\underline{F}_3^{(i)} - \underline{F}_1^{(i)})^{2m}]^{\frac{1}{2m}}$

Now, if $\eth_2\ \subseteq\ \eth_3$, then

$$\begin{aligned} \underline{\daleth}_2^{(i)} &\leq \underline{\daleth}_3^{(i)} \\ \Rightarrow \underline{\daleth}_2^{(i)} - \underline{\daleth}_1^{(i)} &\leq \underline{\daleth}_3^{(i)} - \underline{\daleth}_1^{(i)} \\ \Rightarrow (\underline{\daleth}_2^{(i)} - \underline{\daleth}_1^{(i)})^{2m} &\leq (\underline{\daleth}_3^{(i)} - \underline{\daleth}_1^{(i)})^{2m} \end{aligned}$$

Also,

$$\begin{aligned} \underline{\mathrm{I}}_2^{(i)} &\geq \underline{\mathrm{I}}_3^{(i)} \\ \Rightarrow -\underline{\mathrm{I}}_2^{(i)} &\leq -\underline{\mathrm{I}}_3^{(i)} \\ \Rightarrow \underline{\mathrm{I}}_1^{(i)} - \underline{\mathrm{I}}_2^{(i)} &\leq \underline{\mathrm{I}}_1^{(i)} - \underline{\mathrm{I}}_3^{(i)} \\ \Rightarrow (\underline{\mathrm{I}}_1^{(i)} - \underline{\mathrm{I}}_2^{(i)})^{2m} &\leq (\underline{\mathrm{I}}_1^{(i)} - \underline{\mathrm{I}}_3^{(i)})^{2m} \end{aligned}$$

and

$$\begin{aligned} \underline{\mathrm{I}}_2^{(i)} &\geq \underline{\mathrm{I}}_3^{(i)} \\ \Rightarrow -\underline{F}_2^{(i)} &\leq -\underline{F}_3^{(i)} \\ \Rightarrow \underline{F}_1^{(i)} - \underline{F}_2^{(i)} &\leq \underline{F}_1^{(i)} - \underline{F}_3^{(i)} \\ \Rightarrow (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^{2m} &\leq (\underline{F}_1^{(i)} - \underline{F}_3^{(i)})^{2m} \end{aligned}$$

It follows from above inequalities that $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2) \leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_3)$.

The other inclusion may be established on the parallel track.

Thus, $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

Example 3.4. Consider the PmFNSs $\eth_1,\ \eth_2$ and $\eth_3$ given in Example 3.3. Then, none of the following is a PmFNMS on $PmFN(\underline{\mathrm{X}})$:

(1) $\underline{\mathbb{M}}_r^s(\eth_1,\ \eth_2) = \max_i\{\underline{\daleth}_1^{(i)}, \underline{\daleth}_2^{(i)}\} + \max_i\{\underline{\mathrm{I}}_1^{(i)}, \underline{\mathrm{I}}_2^{(i)}\} + \max_i\{\underline{F}_1^{(i)}, \underline{F}_2^{(i)}\}$.

(2) $\underline{\mathbb{M}}_t^s(\eth_1,\ \eth_2) = \min_i\{\underline{\daleth}_1^{(i)}, \underline{\daleth}_2^{(i)}\} + \min_i\{\underline{\mathrm{I}}_1^{(i)}, \underline{\mathrm{I}}_2^{(i)}\} + \min_i\{\underline{F}_1^{(i)}, \underline{F}_2^{(i)}\}$.

(3) $\underline{\mathbb{M}}_b^s(\eth_1, \eth_2) = \sum_{i=1}^m \left(\underline{\daleth}_1^{(i)} + \underline{\daleth}_2^{(i)} + \underline{\mathrm{I}}_1^{(i)} + \underline{\mathrm{I}}_2^{(i)} + \underline{F}_1^{(i)} + \underline{F}_2^{(i)}\right)$.

(4) $\underline{\mathbb{M}}_c^s(\eth_1, \eth_2) = \sum_{i=1}^m \left\{(\underline{\daleth}_1^{(i)} + \underline{\daleth}_2^{(i)})^2 + (\underline{\mathrm{I}}_1^{(i)} + \underline{\mathrm{I}}_2^{(i)})^2 + (\underline{F}_1^{(i)} + \underline{F}_2^{(i)})^2\right\}$.

(5) $\underline{\mathbb{M}}_d^s(\eth_1, \eth_2) = \sqrt{\sum_{i=1}^m \left\{(\underline{\daleth}_1^{(i)} + \underline{\daleth}_2^{(i)})^2 + (\underline{\mathrm{I}}_1^{(i)} + \underline{\mathrm{I}}_2^{(i)})^2 + (\underline{F}_1^{(i)} + \underline{F}_2^{(i)})^2\right\}}$.

(6) $\underline{\mathbb{M}}_e^s(\eth_1, \eth_2) = \sum_{i=1}^m \left\{\left((\underline{\daleth}_1^{(i)} + \underline{\daleth}_2^{(i)})^2 + (\underline{\mathrm{I}}_1^{(i)} + \underline{\mathrm{I}}_2^{(i)})^2 + (\underline{F}_1^{(i)} + \underline{F}_2^{(i)})^2\right)^n\right\}$, where $n \in \mathbb{R}$.

(7) $\underline{\mathbb{M}}_u^s(\eth_1,\ \eth_2) = \max_i\{\underline{\daleth}_1^{(i)} + \underline{\daleth}_2^{(i)}, \underline{\mathrm{I}}_1^{(i)} + \underline{\mathrm{I}}_2^{(i)}, \underline{F}_1^{(i)} + \underline{F}_2^{(i)}\}$.

In case of (1), if $\eth_1 = \eth_2$ i.e. if $\underline{\daleth}_1^{(i)} = \underline{\daleth}_2^{(i)}$, $\underline{\mathrm{I}}_1^{(i)} = \underline{\mathrm{I}}_2^{(i)}$ and $\underline{F}_1^{(i)} = \underline{F}_2^{(i)}$, then it is not necessary that $\underline{\daleth}_1^{(i)} = \underline{\daleth}_2^{(i)} =\ \underline{\mathrm{I}}_1^{(i)} = \underline{\mathrm{I}}_2^{(i)} = \underline{F}_1^{(i)} = \underline{F}_2^{(i)} = 0$.

Therefore, $\eth_1 = \eth_2 \not\Rightarrow \underline{\mathbb{M}}_s^r(\eth_1,\ \eth_2)\ =\ 0$. Hence, $\underline{\mathbb{M}}_r^s(\eth_1,\ \eth_2)$ is not a PmFNMS on $PmFN(\underline{\mathrm{X}})$. Same reasoning holds good for (2).

For (3), it is not guaranteed that the sum on the RHS will not exceed 2. So, $\underline{\mathbb{M}}^s_b$ fails to be a PmFNMS on $PmFN(\underline{\mathrm{X}})$. The same argument is valid for (4), (5) and (6).
The analogous issue arises in case of (7).

Example 3.5. Let $PmFN(\underline{\mathrm{X}}) = \{x_1, x_2\}$ be the universal sets with three P3FNSs as given in Tables 2 - 4.

TABLE 2. P3FNS $\mathbb{M}_1$

$\mathbb{M}_1$			
x_1	(0.417, 0.312, 0.356)	(0.012, 0.374, 0.436)	(0.811, 0.363, 0.272)
x_2	(0.712, 0.117, 0.562)	(0.333, 0.672, 0.891)	(0.068, 0.772, 0.921)

TABLE 3. P3FNS $\mathbb{N}_1$

$\mathbb{N}_1$			
x_1	(0.811, 0.062, 0.211)	(0.312, 0.270, 0.137)	(0.921, 0.266, 0.152)
x_2	(0.932, 0.101, 0.431)	(0.466, 0.352, 0.721)	(0.368, 0.572, 0.900)

and

TABLE 4. P3FNS $\mathbb{O}_1$

$\mathbb{O}_1$			
x_1	(0.932, 0.001, 0.200)	(0.527, 0.170, 0.007)	(1.000, 0.062, 0.008)
x_2	(0.982, 0.001, 0.231)	(0.667, 0.252, 0.421)	(0.766, 0.423, 0.262)

where $\mathbb{M}_1,\ \mathbb{N}_1,\ \mathbb{O}_1 \Subset P3FN(\underline{\mathrm{X}})$ and $\mathbb{M}_1 \subset \mathbb{N}_1 \subset \mathbb{O}_1$. We show that $\underline{\mathbb{M}}^s(\mathbb{M}_1,\ \mathbb{N}_1)$ is a P3FNMS on $PmFN(\underline{\mathrm{X}})$ if

$$\underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{N}_1) \ = \ \sqrt{\sum_i \left\{(\underline{\beth}_1^{(i)} - \underline{\beth}_2^{(i)})^2 + (\underline{\mathrm{I}}_1^{(i)} - \underline{\mathrm{I}}_2^{(i)})^2 + (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^2\right\}}$$

$\underline{\mathbb{M}}^s_1$: $\underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{N}_1) = \sqrt{0.239 + 0.190 + 0.036 + 0.066 + 0.149 + 0.130} = 0.900$
$\Rightarrow\ 0\ \leq \underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{N}_1) \leq 2.$

$\underline{\mathbb{M}}^s_2$: Obvious.

$\underline{\mathbb{M}}^s_3$: Obvious.

$\underline{\mathbb{M}}^s_4$: Since $\underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{O}_1) = 1.680$, $\underline{\mathbb{M}}^s_\alpha(\mathbb{N}_1,\ \mathbb{O}_1) = 0.970$, and $\underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{N}_1) = 0.900$, so $\underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{O}_1)\ \leq\ \underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{N}_1) + \underline{\mathbb{M}}^s_\alpha(\mathbb{N}_1,\ \mathbb{O}_1)$

$\underline{\mathbb{M}}^s_5$: $\mathbb{M}_1 \subset \mathbb{N}_1 \subset \mathbb{O}_1 \Rightarrow \underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{N}_1)\ <\ \underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{O}_1)$ and $\underline{\mathbb{M}}^s_\alpha(\mathbb{N}_1,\ \mathbb{O}_1)\ <\ \underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{O}_1)$ follows from above computations.

Thus, $\underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{N}_1)$ is a P3FNMS on $PmFN(\underline{\mathrm{X}})$, for $\mathbb{M}_1,\ \mathbb{N}_1,\ \mathbb{O}_1 \Subset P3FN(\underline{\mathrm{X}})$.

Proposition 3.6. *Let* $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ *and* $\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)$ *be two PmFNMSs on a PmFNS* $PmFN(\underline{X})$*, then* $\underline{\mathbb{M}}^s_f[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)] = \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) + \underline{\mathbb{M}}^s(\eth_3,\ \eth_4)$ *is not a PmFNMS on* $PmFN(\underline{X}) \times PmFN(\underline{X})$.

Proof. Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ and $\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)$ are two PmFNMSs. Therefore, by definition

$$0 \ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ \leq\ 2 \text{ and } 0\ \leq\ \underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\ \leq\ 2$$
$$\Rightarrow\ 0\ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) + \underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\ \leq\ 4$$
$$\Rightarrow\ 0\ \leq\ \underline{\mathbb{M}}^s_f[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)]\ \leq\ 4$$

So, $\underline{\mathbb{M}}^s_f[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)]$ is not a PmFNMS on $PmFN(\underline{\mathrm{X}}) \times PmFN(\underline{\mathrm{X}})$.
0.1cm□

Remark 3.7. It is interesting to note that the distance defined in the way as in Proposition 3.6 yields metric space in crisp sets but fails to hold in PmFNSs.

Example 3.8. Consider the PmFNSs $PmFN(\underline{\mathrm{X}})$, $\mathbb{M}_1,\ \mathbb{N}_1$ and $\mathbb{O}_1$ given in Example 3.5 and $\mathbb{M}_2,\ \mathbb{N}_2$ and $\mathbb{O}_2$ given in Tables 5, 6 and 7, respectively:

Table 5. P3FNS $\mathbb{M}_2$

$\mathbb{M}_2$			
x_1	(0.444, 0.123, 0.256)	(0.114, 0.274, 0.336)	(0.901, 0.269, 0.117)
x_2	(0.882, 0.107, 0.432)	(0.441, 0.521, 0.742)	(0.172, 0.710, 0.916)

Table 6. P3FNS $\mathbb{N}_2$

$\mathbb{N}_2$			
x_1	(0.844, 0.002, 0.201)	(0.332, 0.260, 0.037)	(0.922, 0.261, 0.109)
x_2	(0.936, 0.006, 0.331)	(0.470, 0.262, 0.621)	(0.468, 0.472, 0.889)

Table 7. P3FNS $\mathbb{O}_2$

$\mathbb{O}_2$			
x_1	(0.992, 0.001, 0.169)	(0.627, 0.070, 0.006)	(1.000, 0.032, 0.006)
x_2	(0.988, 0.001, 0.201)	(0.676, 0.152, 0.411)	(0.862, 0.413, 0.216)

$\forall\ \mathbb{M}_1,\ \mathbb{M}_2,\ \mathbb{N}_1,\ \mathbb{N}_2,\ \mathbb{O}_1,\ \mathbb{O}_2 \Subset P3FN(\underline{X})$. We check whether $\underline{\mathbb{M}}^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] = \sum_{r=1}^{2}\underline{\mathbb{M}}^s(\mathbb{M}_r,\mathbb{N}_r)$ is a PmFNMS on $PmFN(\underline{X}) \times PmFN(\underline{X})$ or not?
Since $\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1) = 0.900$ and $\underline{\mathbb{M}}^s(\mathbb{M}_2,\mathbb{N}_2) = 0.756$, so that $\underline{\mathbb{M}}^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] = 1.656$

$$\Rightarrow\ 0 \le \underline{\mathbb{M}}^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] \le 2$$

But,

$$\begin{aligned}
\underline{\mathbb{M}}^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{O}_1,\mathbb{O}_2)] &= \underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{O}_1) + \underline{\mathbb{M}}^s(\mathbb{M}_2,\mathbb{O}_2)\\
&= \sqrt{0.386 + 0.491 + 0.196 + 0.196 + 0.509 + 1.043}\\
&+ \sqrt{0.323 + 0.414 + 0.095 + 0.076 + 0.301 + 1.054}\\
&= \sqrt{2.821} + \sqrt{2.263}\\
&= 3.182 \nleq 2
\end{aligned}$$

So, $\underline{\mathbb{M}}^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)]$ is not a P3FNMS on $PmFN(\underline{X}) \times PmFN(\underline{X})$.

Proposition 3.9. *Let $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ and $\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)$ be two PmFNMSs on PmFNS $PmFN(\underline{X})$, then*

(i) $\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)] = \max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2),\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\}$
(ii) $\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)] = \min\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2),\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\}$

are PmFNMSs on $PmFN(\underline{X}) \times PmFN(\underline{X})$.

Proof. We prove (i) here. The proof of (ii) may be furnished on the parallel track.

$\underline{\mathbb{M}}^s_1$: Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ and $\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)$ are PmFNMSs on $PmFN(\underline{X})$.
$\Rightarrow\ 0 \le \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) \le 2$ and $0 \le \underline{\mathbb{M}}^s(\eth_3,\ \eth_4) \le 2$
But then, $\max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2),\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\}$, being the maximum of two non-negative and less than or equal to 2 quantities, is also non-negative and less than or equal to 2.
$\underline{\mathbb{M}}^s_2$: Obvious.
$\underline{\mathbb{M}}^s_3$:

$$\begin{aligned}
\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)] = 0 &\Leftrightarrow \max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2),\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\} = 0\\
&\Leftrightarrow \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) = 0,\ \underline{\mathbb{M}}^s(\eth_3,\ \eth_4) = 0\\
&\Leftrightarrow \eth_1 = \eth_2,\ \eth_3 = \eth_4\\
&\Leftrightarrow (\eth_1,\eth_3) = (\eth_2,\eth_4)
\end{aligned}$$

$\underline{\mathbb{M}}^s_4$: $\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_5,\ \eth_6)] = \max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_5),\underline{\mathbb{M}}^s(\eth_3,\ \eth_6)\}$
Let $\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_5,\ \eth_6)] = \underline{\mathbb{M}}^s(\eth_1,\ \eth_5)$. Then,
$\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ \le\ \max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2),\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\}$
and

$\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)\ \leq\ \max\{\underline{\mathbb{M}}^s(\eth_2,\ \eth_5), \underline{\mathbb{M}}^s(\eth_4,\ \eth_6)\}$

Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS. Therefore,

$\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)\ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) + \underline{\mathbb{M}}^s(\eth_2,\ \eth_5)$

$\Rightarrow\ \max\ [\underline{\mathbb{M}}^s(\eth_1,\ \eth_5), \underline{\mathbb{M}}^s(\eth_3,\ \eth_6)]\ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) + \underline{\mathbb{M}}^s(\eth_2,\ \eth_5)\ \leq \max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2), \underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\} + [\underline{\mathbb{M}}^s(\eth_2,\ \eth_5), \underline{\mathbb{M}}^s(\eth_4,\ \eth_6)]$

$\Rightarrow\ \underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3), (\eth_5,\ \eth_6)] \leq \underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3), (\eth_2,\ \eth_4)] + \underline{\mathbb{M}}^s_g[(\eth_2,\ \eth_4), (\eth_5,\ \eth_6)]$

$\underline{\mathbb{M}}^s_5$: Straight forward.

Thus, $\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3), (\eth_2,\ \eth_4)],\ \forall\ \eth_1,\ \eth_2,\ \eth_3,\ \eth_4,\ \eth_5,\ \eth_6 \Subset PmFN(\underline{\mathbb{X}})$ is a PmFNMS on $PmFN(\underline{\mathbb{X}}) \times PmFN(\underline{\mathbb{X}})$.

0.1cm□

Remark 3.10. Proposition 3.9 shows that more than one PmFNMSs can be defined on a single PmFNS.

Example 3.11. Consider the P3FNSs given in Example 3.5 and 3.8. We check whether

(i) $\underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2), (\mathbb{N}_1, \mathbb{N}_2)] = \max\{\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{N}_1), \underline{\mathbb{M}}^s(\mathbb{M}_2, \mathbb{N}_2)\}$

(ii) $\underline{\mathbb{M}}^s_2[(\mathbb{M}_1, \mathbb{M}_2), (\mathbb{N}_1, \mathbb{N}_2)] = \min\{\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{N}_1), \underline{\mathbb{M}}^s(\mathbb{M}_2, \mathbb{N}_2)\}$

where $\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{M}_2)$ and $\underline{\mathbb{M}}^s[(\mathbb{M}_3, \mathbb{M}_4)$ are P3FNMSs on $PmFN(\underline{\mathbb{X}}) \times PmFN(\underline{\mathbb{X}})$ or not?

For (i):

$\underline{\mathbb{M}}^s_1$:

$$\begin{aligned}\underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2), (\mathbb{N}_1, \mathbb{N}_2)] &= \max\{\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{N}_1), \underline{\mathbb{M}}^s(\mathbb{M}_2, \mathbb{N}_2)\} \\ &= \max\{0.900, 0.756\} \\ &= 0.900\end{aligned}$$

$$\Rightarrow\ 0 \leq \underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2), (\mathbb{N}_1, \mathbb{N}_2)] \leq 2$$

$\underline{\mathbb{M}}^s_2$: Obvious.

$\underline{\mathbb{M}}^s_3$: Obvious.

$\underline{\mathbb{M}}^s_4$: $\underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2), (\mathbb{O}_1, \mathbb{O}_2)] = \max\{\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{O}_1), \underline{\mathbb{M}}^s(\mathbb{M}_2, \mathbb{O}_2)\} = \max\{1.680, 1.504\} = 1.680$.

Also,

$\underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2), (\mathbb{N}_1, \mathbb{N}_2)] = \max\{\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{N}_1), \underline{\mathbb{M}}^s(\mathbb{M}_2, \mathbb{N}_2)] = \max\{0.900, 0.756\} = 0.900$,

and

$\underline{\mathbb{M}}^s_1[(\mathbb{N}_1, \mathbb{N}_2), (\mathbb{O}_1, \mathbb{O}_2)] = \max\{\underline{\mathbb{M}}^s(\mathbb{N}_1, \mathbb{O}_1), \underline{\mathbb{M}}^s(\mathbb{N}_2, \mathbb{O}_2)\} = \max\{0.970, 0.973\} = 0.973$.

$\Rightarrow\ \underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2), (\mathbb{N}_1, \mathbb{N}_2)] + \underline{\mathbb{M}}^s_1[(\mathbb{N}_1, \mathbb{N}_2), (\mathbb{O}_1, \mathbb{O}_2)] = 0.900 + 0.973 = 1.873$

Thus,

$$\underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{O}_1,\mathbb{O}_2)] < \underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] + \underline{\mathbb{M}}_1^s[(\mathbb{N}_1,\mathbb{N}_2),(\mathbb{O}_1,\mathbb{O}_2)]$$

$\underline{\mathbb{M}}_5^s$: As given, $\mathbb{M}_1 \subseteq \mathbb{N}_1 \subseteq \mathbb{O}_1$ and $\mathbb{M}_2 \subseteq \mathbb{N}_2 \subseteq \mathbb{O}_2$

$$\underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] = 0.900,$$
$$\underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{O}_1,\mathbb{O}_2)] = 1.680,$$

and

$$\underline{\mathbb{M}}_1^s[(\mathbb{N}_1,\mathbb{N}_2),(\mathbb{O}_1,\mathbb{O}_2)] = 0.973.$$

It may be observed that

$$\underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] < \underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{O}_1,\mathbb{O}_2)]$$

and

$$\underline{\mathbb{M}}_1^s[(\mathbb{N}_1,\mathbb{N}_2),(\mathbb{O}_1,\mathbb{O}_2)] < \underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{O}_1,\mathbb{O}_2)]$$

Hence, $\underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)]$ is a P3FNMS on $PmFN(\underline{\mathbb{X}}) \times PmFN(\underline{\mathbb{X}})$.

(ii) may be proved likewise.

Example 3.12. Consider the P3FNMS given in Example 3.5, then following are not P3FNMSs on $PmFN(\underline{\mathbb{X}})$.

(i) $\underline{\mathbb{M}}_4^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$

(ii) $\underline{\mathbb{M}}_5^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$

(iii) $\underline{\mathbb{M}}_6^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{3-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$

(iv) $\underline{\mathbb{M}}_7^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{3-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$

(v) $\underline{\mathbb{M}}_8^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$

We prove them one by one as follows:

(i) Since $\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1) = 0.900$, so

$$\underline{\mathbb{M}}_4^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)} = \frac{0.900}{1-0.900} = 9 \nleq 2$$

and hence $\underline{\mathbb{M}}_4^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$ is not a P3FNMS on $PmFN(\underline{\mathbb{X}})$.

(ii) Since

$$\underline{\mathbb{M}}_5^s(\mathbb{M}_1,\mathbb{M}_1) = \frac{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)}{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)} = \frac{1-0}{0} = \frac{1}{0}$$

which is undefined. So, $\underline{\mathbb{M}}_5^s(\mathbb{M}_1,\mathbb{N}_1)$ is not a P3FNMS on $PmFN(\underline{\mathbb{X}})$.

(iii) Since

$$\underline{\mathbb{M}}_6^s(\mathbb{M}_1,\mathbb{M}_1) = \frac{3-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)} = \frac{3-0}{1+0} = 3 \neq 0$$

Hence, $\underline{\mathbb{M}}_6^s(\mathbb{M}_1,\mathbb{N}_1)$ is not a P3FNMS on $PmFN(\underline{\mathbb{X}})$.

(iv) Since

$$\underline{\mathbb{M}}_7^s(\mathbb{M}_1,\mathbb{O}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{O}_1)}{3-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{O}_1)} = \frac{1.680}{3-1.680} = 1.273$$
$$\underline{\mathbb{M}}_7^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{3-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)} = \frac{0.900}{3-0.900} = 0.429$$

and

$$\underline{\mathbb{M}}_7^s(\mathbb{N}_1, \mathbb{O}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{N}_1,\mathbb{O}_1)}{3-\underline{\mathbb{M}}^s(\mathbb{N}_1,\mathbb{O}_1)} = \frac{0.970}{3-0.970} = 0.478$$

so,

$$\underline{\mathbb{M}}_7^s(\mathbb{M}_1, \mathbb{O}_1) \not\leq \underline{\mathbb{M}}_7^s(\mathbb{M}_1, \mathbb{N}_1) + \underline{\mathbb{M}}_7^s(\mathbb{N}_1, \mathbb{O}_1)$$

and hence $\underline{\mathbb{M}}_7^s(\mathbb{M}_1, \mathbb{N}_1)$ is not a P3FNMS on $PmFN(\underline{\mathrm{X}})$.

(v) Since

$$\underline{\mathbb{M}}_8^s(\mathbb{M}_1, \mathbb{M}_1) = \frac{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)} = \frac{1-0}{1+0} = 1 \neq 0$$

Thus, $\underline{\mathbb{M}}_8^s(\mathbb{M}_1, \mathbb{N}_1)$ is not a P3FNMS on $PmFN(\underline{\mathrm{X}})$.

Remark 3.13. Let $\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{N}_1)$ be a PmFNMS on non-empty universal PmFNS $PmFN(\underline{\mathrm{X}})$, then

(1) $\underline{\mathbb{M}}^{s*}(\mathbb{M}_1, \mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{q-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$, where q is any integer, is not a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

(2) $\underline{\mathbb{M}}^{s**}(\mathbb{M}_1, \mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{n+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$, where n is any natural number, is a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

(3) Distance defined in this way yields metric spaces in crisp set but fails to hold in PmFNMSs.

Proposition 3.14. *Let $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ be a PmFNMS on a non-empty universal PmFNS $PmFN(\underline{X})$ then $\underline{\mathbb{M}}_f^s(\eth_1,\ \eth_2) = \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}$, where n is any natural number, is also a PmFNMS on $PmFN(\underline{X})$.*

Proof. $\underline{\mathbb{M}}_1^s$: Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$. So,

$$0 \leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) \leq 2 \Rightarrow 0 \leq \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)} \leq 2 \Rightarrow 0 \leq \underline{\mathbb{M}}_f^s(\eth_1,\ \eth_2) \leq 2$$

$\underline{\mathbb{M}}_2^s$:

$$\begin{aligned} \underline{\mathbb{M}}_f^s(\eth_1,\ \eth_2) &= \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)} \\ &= \frac{\underline{\mathbb{M}}^s(\eth_2,\ \eth_1)}{n+\underline{\mathbb{M}}^s(\eth_2,\ \eth_1)} \ (\because \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) \text{ is a P}m\text{FNMS on } PmFN(\underline{\mathrm{X}})) \\ &= \underline{\mathbb{M}}_f^s(\eth_2,\ \eth_1) \end{aligned}$$

$\underline{\mathbb{M}}_3^s$:

$$\begin{aligned} \underline{\mathbb{M}}_f^s(\eth_1,\ \eth_2) = 0 \ &\Leftrightarrow \ \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)} = 0 \\ &\Leftrightarrow \ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) = 0 \\ &\Leftrightarrow \ \eth_1 = \eth_2 \end{aligned}$$

Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

$\underline{\mathbb{M}}^s_4$: Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$. So,

$$\begin{aligned}
\underline{\mathbb{M}}^s(\eth_1,\ \eth_5) &\leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)\\
n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5) &\leq n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)\\
\frac{1}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)} &\geq \frac{1}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
\frac{-n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)} &\leq \frac{-n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
1-\frac{n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)} &\leq 1-\frac{n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
\frac{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)-n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)} &\leq \frac{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)-n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
\frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)} &\leq \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
&= \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}+\frac{\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
&\leq \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}+\frac{\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}\\
\Rightarrow \underline{\mathbb{M}}^s_f(\eth_1,\ \eth_5) &\leq \underline{\mathbb{M}}^s_f(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s_f(\eth_2,\ \eth_5)
\end{aligned}$$

$\underline{\mathbb{M}}^s_5$: Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$. So, if $\eth_1 \subseteq \eth_2 \subseteq \eth_5$ then $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2) \leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_5)$ and $\underline{\mathbb{M}}^s(\eth_2,\ \eth_5) \leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_5)$ then follows directly from definition.

Hence, $\underline{\mathbb{M}}^s_f(\eth_1,\ \eth_2)$ is also a PmFNMS on $PmFN(\underline{\mathrm{X}})$. 0.1cm□

Example 3.15. Consider the PmFNMS given in Example 3.5. We show that $\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

$\underline{\mathbb{M}}^s_1$: Since $\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_2) = 0.900$ from Example 3.5, so
$\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)} = \frac{0.900}{1+0.900} = \frac{0.900}{1.900} = 0.474 \ \Rightarrow\ 0 \leq \underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1) \leq 2.$

$\underline{\mathbb{M}}^s_2$: Obvious.

$\underline{\mathbb{M}}^s_3$: Obvious.

$\underline{\mathbb{M}}^s_4$: $\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{O}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{O}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{O}_1)} = \frac{1.680}{1+1.680} = 0.627$
$\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1) = 0.474$ and
$\underline{\mathbb{M}}^s_3(\mathbb{N}_1,\mathbb{O}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{N}_1,\mathbb{O}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{N}_1,\mathbb{O}_1)} = \frac{0.970}{1+0.970} = 0.492$
$\therefore\ \underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{O}_1) < \underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1) + \underline{\mathbb{M}}^s_3(\mathbb{N}_1,\mathbb{O}_1)$

$\underline{\mathbb{M}}^s_5$: $\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1) = 0.474 < 0.627 = \underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{O}_1)$
and $\underline{\mathbb{M}}^s_3(\mathbb{N}_1,\mathbb{O}_1) = 0.492 < 0.627 = \underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{O}_1)$

Thus, $\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1)$ is also a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

4. Application of proposed metrics in pattern recognition

In this section, we present an application of suggested metrics in pattern recognition. Pattern recognition is the science endued with diverse utilizations, mainly including speech and fingerprint recognition, medical imaging and diagnosis, aerial photo interpretation, image processing, and optical character recognition in scanned documents such as contracts and photographs.

Example 4.1. Let PmFN($\underline{Z}$) $= \{z_1, z_2, z_3\}$ be the universal set with model P3FNS $\mathbb{M}$ and three P3FNSs $\mathbb{M}_1$, $\mathbb{M}_2$ and $\mathbb{M}_3$ as given in Tables 8, 9, 10 and 11, respectively.

Table 8. Model P3FNS $\mathbb{M}$

$\mathbb{M}$			
z_1	(0.206, 0.101, 0.135)	(0.010, 0.153, 0.215)	(0.600, 0.142, 0.051)
z_2	(0.114, 0.100, 0.215)	(0.080, 0.093, 0.435)	(0.090, 0.002, 0.981)
z_3	(0.087, 0.132, 0.156)	(0.090, 0.123, 0.204)	(0.340, 0.642, 0.131)

Table 9. P3FNS $\mathbb{M}_1$

$\mathbb{M}_1$			
z_1	(0.307, 0.202, 0.246)	(0.002, 0.264, 0.326)	(0.701, 0.253, 0.162)
z_2	(0.542, 0.002, 0.254)	(0.143, 0.876, 0.796)	(0.214, 0.005, 0.214)
z_3	(0.053, 0.007, 0.760)	(0.320, 0.432, 0.324)	(0.530, 0.241, 0.964)

Table 10. P3FNS $\mathbb{M}_2$

$\mathbb{M}_2$			
z_1	(0.701, 0.052, 0.101)	(0.202, 0.160, 0.027)	(0.811, 0.156, 0.042)
z_2	(0.262, 0.001, 0.003)	(0.290, 0.980, 0.017)	(0.041, 0.126, 0.022)
z_3	(0.754, 0.023, 0.100)	(0.192, 0.360, 0.023)	(0.408, 0.134, 0.702)

and

Table 11. P3FNS $\mathbb{M}_3$

$\mathbb{M}_3$			
z_1	(0.822, 0.001, 0.100)	(0.417, 0.060, 0.007)	(1.000, 0.052, 0.008)
z_2	(0.143, 0.084, 0.098)	(0.009, 0.170, 0.037)	(0.000, 0.402, 0.064)
z_3	(0.632, 0.340, 0.132)	(0.128, 0.604, 0.215)	(0.800, 0.322, 0.609)

where $\mathbb{M}, \mathbb{M}_3, \mathbb{M}_4, \mathbb{M}_5 \in P3FN(\underline{Z})$. We use the metrics defined in Example 3.3, which is

$$\underline{\mathbb{M}}^s(\eth_1,\ \eth_2) = \sqrt[2m]{\sum_{i=1}^{m} \{(\underline{\daleth}_1^{(i)} - \underline{\daleth}_2^{(i)})^{2m} + (\underline{\mathrm{I}}_1^{(i)} - \underline{\mathrm{I}}_2^{(i)})^{2m} + (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^{2m}\}},$$

in Example 3.5, which is

$$\underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{N}_1) \ = \ \sqrt{\sum_i \{(\underline{\daleth}_1^{(i)} - \underline{\daleth}_2^{(i)})^2 + (\underline{\mathrm{I}}_1^{(i)} - \underline{\mathrm{I}}_2^{(i)})^2 + (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^2\}}$$

and that in Example 3.14, which is

$$\underline{\mathbb{M}}_f^s(\eth_1,\ \eth_2) = \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n + \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}$$

taking $n = 2$, to determine pattern similarity between $\mathbb{M}$ and $\mathbb{M}_i$'s. The results so computed are tabulated in Table 12.

Table 12. Metrics between $\mathbb{M}$ and $\mathbb{M}_i$'s

Metric	$(\mathbb{M}, \mathbb{M}_1)$	$(\mathbb{M}, \mathbb{M}_2)$	$(\mathbb{M}, \mathbb{M}_3)$
$\underline{\mathbb{M}}^s$	0.969	1.068	**0.948**
$\underline{\mathbb{M}}_\alpha^s$	1.753	1.880	**1.650**
$\underline{\mathbb{M}}_f^s$	0.326	0.348	**0.322**

Above results show that pattern of $\mathbb{M}_3$ is most recognizable with $\mathbb{M}$. These results are depicted in Figure 1.

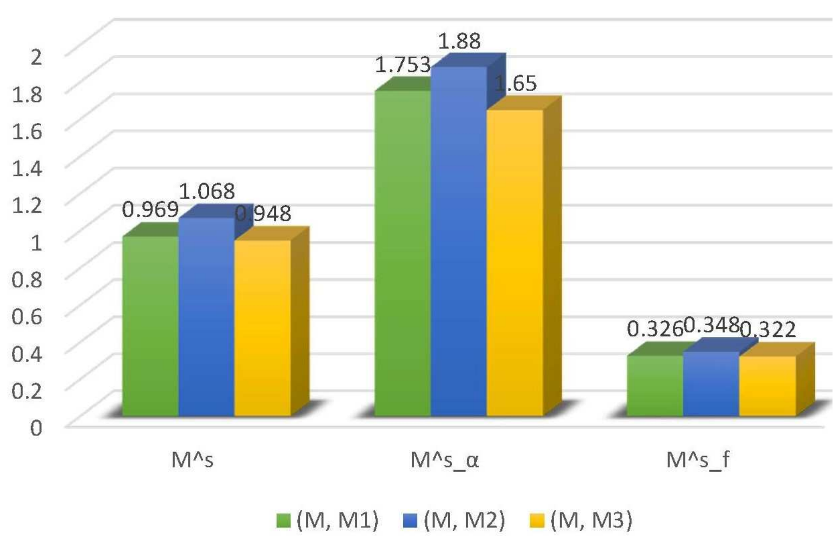

Figure 1. Chart of metrics between $\mathbb{M}$ and $\mathbb{M}_i$'s

5. Conclusion

We have inculcated the axiomatic definition of Pythagorean m-polar fuzzy neutrosophic metric space with the help of Pythagorean m-polar neutrosophic sets and classical metric space in this study. We provided a large number of examples to perceive the notion clearly. The cases which are metrics in classical sets but fail to be so in the environment of Pythagorean m-polar neutrosophic setting have also been made part of the study. The results presented also hold good in the case of Pythagorean fuzzy neutrosophic sets. We presented an application of the proposed metrics in pattern recognition. We computed three metrics there and exhibited that these metrics yield the same optimal choice. The results computed are displayed with the assistance of a statistical chart. We hope that this article will give new ideas to the researchers to promote research in various fields.

Conflicts of Interest: The authors declare that there is no conflict of interests.
Authors' Contributions: The authors contributed to each part of this paper equally.

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