



On Superhyper BCK -Algebras

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Abstract. BCK -algebras are algebraic structures in universal algebra such that are based on logical axioms and have some applications. This paper introduces the concept of super hyper BCK -algebras as a generalization of BCK -algebras and investigates some properties of this novel concept.

Keywords: BCK -algebra, hyper BCK -algebra, super hyper BCK -algebra, generalized operation.

1. Introduction

Smarandache introduced a new concept in neutrosophy branches as neutro-algebra as a generalization of partial algebra. A neutro algebra is an algebra which has at least one neutro-operation (an operation that is partially well-defined, partially indeterminate, and partially outer-defined) or one neutro-axiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements). A partial algebra is an algebra that has at least one partial operation, and all its axioms are classical (i.e. axioms true for all elements). Through a theorem he proved that Neutro-algebra is a generalization of partial algebra, and he gave examples of neutro-algebras that are not partial algebras. He also introduced the neutro-function (and neutro-operation). Recently, Smarandache, introduced a new concept as a generalization of hypergraphs to n -super hypergraph, plithogenic n -super hypergraph {with super-vertices (that are groups of vertices) and hyper-edges {defined on power-set of power-set...} that is the most general form of graph as today}, and n -ary hyperalgebra, n -ary neutro hyperalgebra, n -ary anti hyperalgebra respectively, which have several properties and are connected with the real world [2,8]. Recently in the scope of neutro logical (hyper) algebra, Hamidi, et al. introduced the concept of neutro BCK -subalgebras [4], neutro d -subalgebras [3] and single-valued neutro hyper BCK -subalgebras [5] as a generalization of BCK -algebras and hyper BCK -subalgebras, respectively and presented the main results in this regard. Also

Smarandache a novel concept as super hyperalgebra with its super hyperoperations and super hyperaxioms, then is introduced some concepts such as super hypertopology and especially the super hyperfunction and neutrosophic super hyperfunction [10, 11].

Regarding these points, we try to develop the notation of *BCK*-algebras to the concept of super hyper *BCK*-algebras and so we want to seek the connection between *BCK*-algebras and super hyper *BCK*-algebras.

2. Preliminaries

In this section, we recall some concepts that need to our work.

Definition 2.1. [6] Let $X \neq \emptyset$. Then a universal algebra $(X, \vartheta, 0)$ of type $(2, 0)$ is called a *BCK-algebra*, if $\forall x, y, z \in X$:

$$(BCI-1) ((x\vartheta y)\vartheta (x\vartheta z))\vartheta (z\vartheta y) = 0,$$

$$(BCI-2) (x\vartheta (x\vartheta y))\vartheta y = 0,$$

$$(BCI-3) x\vartheta x = 0,$$

$$(BCI-4) x\vartheta y = 0 \text{ and } y\vartheta x = 0 \text{ imply } x = y,$$

$$(BCK-5) 0\vartheta x = 0,$$

where $\vartheta(x, y)$ is denoted by $x\vartheta y$.

Definition 2.2. [1, 7] Let $X \neq \emptyset$ and $P^*(X) = \{Y \mid \emptyset \neq Y \subseteq X\}$. Then for a map $\varrho : X^2 \rightarrow P^*(X)$ a hyperalgebraic system $(X, \varrho, 0)$ is called a *hyper BCK-algebra*, if $\forall x, y, z \in X$:

$$(H1) (x \varrho z) \varrho (y \varrho z) \ll x \varrho y,$$

$$(H2) (x \varrho y) \varrho z = (x \varrho z) \varrho y,$$

$$(H3) x \varrho X \ll x,$$

$$(H4) x \ll y \text{ and } y \ll x \text{ imply } x = y,$$

where $x \ll y$ is defined by $0 \in x \varrho y$, $\forall W, Z \subseteq X$, $W \ll Z \Leftrightarrow \forall a \in W \exists b \in Z \text{ s.t } a \ll b$,

$$(W \varrho Z) = \bigcup_{a \in W, b \in Z} (a \varrho b) \text{ and } \varrho(x, y) \text{ is denoted by } x \varrho y.$$

We will call X is a *weak commutative hyper BCK-algebra* if, $\forall x, y \in X$, $(x \varrho (x \varrho y)) \cap (y \varrho (y \varrho x)) \neq \emptyset$.

Theorem 2.3. [7] Let $(X, \varrho, 0)$ be a hyper *BCK-algebra*. Then $\forall x, y, z \in X$ and $W, Z \subseteq X$,

$$(i) (0 \varrho 0) = 0, 0 \ll x, (0 \varrho x) = 0, x \in (x \varrho 0) \text{ and } (W \ll 0 \Rightarrow W = 0),$$

$$(ii) x \ll x, x \varrho y \ll x \text{ and } (y \ll z \Rightarrow x \varrho z \ll x \varrho y),$$

$$(iii) W \varrho Z \ll W, W \ll W \text{ and } (W \subseteq Z \Rightarrow W \ll Z).$$

Definition 2.4. [10, 11] Let X be a nonempty set and $0 \in X$. Then $(X, \circ_{(m,n)}^*)$ is called an (m, n) -super hyperalgebra, where $\circ_{(m,n)}^* : X^m \rightarrow P_*^n(X)$ is called an (m, n) -super hyperoperation, $P_*^n(X)$ is the n^{th} powerset of the set $X, \emptyset \notin P_*^n(X)$, for any $A \in P_*^n(X)$, we identify $\{A\}$ with $A, m, \geq 2, n \geq 0, X^m = \underbrace{X \times X \times \dots \times X}_{m\text{-times}}$ and $P_*^0(X) = X$.

3. Superhyper BCK-subalgebra

In this section, we make the concept of superhyper BCK-subalgebras as an extension of BCK-subalgebras and seek some of their properties.

Proposition 3.1. *Let $(X, \vartheta, 0)$ be a BCK-algebra. Then for all $x, y, z \in X$,*

- (i) $\vartheta(\vartheta(x, y), \vartheta(x, z)) = \vartheta(\vartheta(\vartheta(x, y), \vartheta(x, z)), 0)$.
- (ii) $\vartheta(\vartheta(x), \vartheta(x, y)) = \vartheta(\vartheta(\vartheta(x), \vartheta(x, y)), 0)$.

Proof. Since for all $x \in X, \vartheta(x, 0) = x$, results are clear. \square

By Proposition 3.1, we define the concept of (m, n) -super hyper BCK-subalgebras.

Definition 3.2. Let X be a nonempty set and $0 \in X$ and $\alpha = \underbrace{0, 0, \dots, 0}_{(m-1)\text{-times}}$. Then $(X, \circ_{(m,n)}^*)$

is called an (m, n) -super hyper BCK-subalgebra, if

- (i) $0 \in \circ_{(m,n)}^* \left(\circ_{(m,n)}^* (x_1^1, x_2^1, \dots, x_m^1), \dots, \circ_{(m,n)}^* (x_1^m, x_2^m, \dots, x_m^m), \alpha, \circ_{(m,n)}^* (x_m^m, x_{m-1}^{m-1}, \dots, x_1^1) \right)$,
- (ii) $0 \in \circ_{(m,n)}^* \left(\circ_{(m,n)}^* (x_1^1, \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, \circ_{(m,n)}^* (x_1^1, x_2^1, \dots, x_m^1)), \underbrace{0, 0, \dots, 0}_{(m-1)\text{-times}}, x_m^1 \right)$,
- (iii) $0 \in \circ_{(m,n)}^* (x, x, \dots, x)$,
- (iv) if $0 \in \circ_{(m,n)}^* (x_1, x_2, \dots, x_m)$ and $0 \in \circ_{(m,n)}^* (x_m, x_{m-1}, \dots, x_1)$, then $x_i = x_j$, where $i + j = m + 1$,
- (v) $0 \in \circ_{(m,n)}^* (0, 0, \dots, x)$,

Example 3.3. (i) Let $(X, \circ_{(m,n)}^*)$ be a (m, n) -super hyper BCK-subalgebra. Then $(X, \circ_{(2,0)}^*)$ is a BCK-subalgebra.

(ii) Let $(X, \circ_{(m,n)}^*)$ be a (m, n) -super hyper BCK-subalgebra. Then $(X, \circ_{(2,1)}^*)$ is a hyper BCK-subalgebra.

Example 3.4. Let $X = \{0, a\}$.

(i) Then (X, \circ^*) is a $(3, 3)$ -super hyper BCK-subalgebra as follows:

$$\circ_{(3,3)}^*(x, y, z) = \begin{cases} P_*^3(\{0, x, z\}) & \text{if } x = z \\ P_*^3(\{0, z\}) & \text{if } x = y = 0, \\ P_*^3(\{a\}) & o.w \end{cases}$$

where

$$\begin{aligned}
 P_*({a}) &= P_*^2({a}) = P_*^3({a}) = {a}, P_*({0, a}) = {0, a, {0, a}}, \\
 P_*^2({0, a}) &= {0, a, {0, a}, {0, {0, a}}, {a, {0, a}}}, \\
 P_*^3({0, a}) &= {0, a, {0, a}, {0, {0, a}}, {a, {0, a}}, {0, {0, {0, a}}}, {0, {a, {0, a}}}, {a, {0, {0, a}}}, \\
 &{a, {a, {0, a}}}, {{0, a}, {0, {0, a}}}, {{0, a}, {a, {0, a}}}, {{0, {0, a}}, {a, {0, a}}}.
 \end{aligned}$$

(i) By definition,

$\circ_{(3,3)}^* (\circ_{(3,3)}^* (\circ_{(3,3)}^* (x, y, z), \circ_{(3,3)}^* (x', y', z'), \circ_{(3,3)}^* (x'', y'', z'')), 0, \circ_{(3,3)}^* (z'', z', z)) \subseteq \{0, a\}$. (ii) It is similar to item (i).

(iii) By definition, $\circ_{(3,3)}^* (a, a, a) = \{0, a\}$.

(iv) By definition, if $0 \in \circ_{(3,3)}^* (x, y, z)$ and $0 \in \circ_{(3,3)}^* (z, y, x)$, then $x = z$ and so $(x, y, z) = (z, y, x)$.

(v) By definition, $\circ_{(3,3)}^* (0, 0, a) = \{0, a\}$.

(ii) Then (X, \circ^*) is a $(3, 0)$ -super hyper BCK-subalgebra as follows:

$$\circ_{(3,1)}^* (x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ x & \text{o.w} \end{cases},$$

Theorem 3.5. Let $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra. Then for any $k \geq n$, $(X, \circ_{(m,n)}^*)$ is an (m, k) -super hyper BCK-subalgebra.

Proof. Let $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra and $k \geq n$. Since $P_*^n(X) \subseteq P_*^k(X)$, for any $x_1, x_2, \dots, x_m \in X$, $\circ_{(m,n)}^* (x_1, x_2, \dots, x_m) \subseteq \circ_{(m,k)}^* (x_1, x_2, \dots, x_m)$. Thus $0 \in \circ_{(m,n)}^* (x_1, x_2, \dots, x_m)$ implies that $0 \in \circ_{(m,k)}^* (x_1, x_2, \dots, x_m)$ and all axioms are valid. \square

Example 3.6. Let $X = \{0, a\}$. Then for any $n \geq 3$, by Theorem 3.5, (X, \circ^*) is a $(3, n)$ -super hyper BCK-subalgebra as follows:

$$\circ_{(3,3)}^* (x, y, z) = \begin{cases} P_*^n(\{0, x, z\}) & \text{if } x = z \\ P_*^n(\{0, z\}) & \text{if } x = y = 0. \\ P_*^n(\{a\}) & \text{o.w} \end{cases}$$

Let m be an even and $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra. For any given $x_1, x_2, \dots, x_m \in X$, define $(x_1, x_2, \dots, x_{\frac{m}{2}}) \leq (x_{\frac{m}{2}+1}, x_{\frac{m}{2}+2}, \dots, x_m)$ if and only if $0 \in \circ_{(m,n)}^* (x_1, x_2, \dots, x_m)$.

Theorem 3.7. Let m be an even and $x_1, x_2, \dots, x_m \in X$. Then $(X, \circ_{(m,n)}^*)$ is an (m, n) -super hyper BCK-subalgebra if and only if

$$(i) \circ_{(m,n)}^* (\circ_{(m,n)}^* (x_1^1, x_2^1, \dots, x_m^1), \dots, \circ_{(m,n)}^* (x_1^m, x_2^m, \dots, x_m^m)) \leq \circ_{(m,n)}^* (x_m^m, x_m^{m-1}, \dots, x_m^1),$$

- (ii) $\circ_{(m,n)}^*(x_1^1, \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, \circ_{(m,n)}^*(x_1^1, x_2^1, \dots, x_m^1)) \leq \circ_{(m,n)}^*(\underbrace{0, 0, \dots, 0}_{(m-1)\text{-times}}, x_m^1)$,
- (iii) $\underbrace{(x, x, \dots, x)}_{(\frac{m}{2})\text{-times}} \leq \underbrace{(x, x, \dots, x)}_{(\frac{m}{2})\text{-times}}$,
- (iv) if $(x_1, x_2, \dots, x_{\frac{m}{2}}) \leq (x_{\frac{m}{2}+1}, x_{\frac{m}{2}+2}, \dots, x_m)$ and $(x_{\frac{m}{2}+1}, x_{\frac{m}{2}+2}, \dots, x_m) \leq (x_1, x_2, \dots, x_{\frac{m}{2}})$, then $x_i = x_j$, where $|i - j| = 2$,
- (v) $\underbrace{(0, 0, \dots, 0)}_{(\frac{m}{2})\text{-times}} \leq (x_{\frac{m}{2}+1}, x_{\frac{m}{2}+2}, \dots, x_m)$,
- (vi) $(x_1, x_2, \dots, x_{\frac{m}{2}}) \leq (x_{\frac{m}{2}+1}, x_{\frac{m}{2}+2}, \dots, x_m)$ if and only if $0 \in \circ_{(m,n)}^*(x_1, x_2, \dots, x_m)$.

Proof. Immediate by definition. \square

Theorem 3.8. Let m be an even and $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra and $x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}} \in X$. If $0 \in \circ_{(m,n)}^*(x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}})$, then $0 \in \circ_{(m,n)}^*(\circ_{(m,n)}^*(z_1, z_2, \dots, z_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}), \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, \circ_{(m,n)}^*(z_1, z_2, \dots, z_{\frac{m}{2}}, x_1, x_2, \dots, x_{\frac{m}{2}}))$.

Proof. Let $x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}} \in X$. Clearly,

$$\begin{aligned} & \circ_{(m,n)}^*(\circ_{(m,n)}^*(z_1, z_2, \dots, z_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}), \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, \circ_{(m,n)}^*(z_1, z_2, \dots, z_{\frac{m}{2}}, x_1, x_2, \dots, x_{\frac{m}{2}})) \\ & \leq \circ_{(m,n)}^*(x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}). \end{aligned}$$

Since $0 \in \circ_{(m,n)}^*(x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}})$, we get that

$$0 \in \circ_{(m,n)}^*(\circ_{(m,n)}^*(z_1, z_2, \dots, z_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}), \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, \circ_{(m,n)}^*(z_1, z_2, \dots, z_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}})).$$

\square

Theorem 3.9. Let m be an even and $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra and $x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}} \in X$. If

$$0 \in \circ_{(m,n)}^*(x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}) \cap \circ_{(m,n)}^*(y_1, y_2, \dots, y_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}}),$$

then $0 \in \circ_{(m,n)}^*(x_1, x_2, \dots, x_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}})$.

Proof. Let $x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}} \in X$. Since

$$0 \in \circ_{(m,n)}^*(x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}) \cap \circ_{(m,n)}^*(y_1, y_2, \dots, y_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}}),$$

by Theorem 3.8, we get that

$$0 \in \circ_{(m,n)}^*(\circ_{(m,n)}^*(z_1, z_2, \dots, z_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}), \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, \circ_{(m,n)}^*(z_1, z_2, \dots, z_{\frac{m}{2}}, x_1, x_2, \dots, x_{\frac{m}{2}}))$$

and

$$0 \in \circ_{(m,n)}^* \left(\circ_{(m,n)}^* (x_1, x_2, \dots, x_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}}), \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, \circ_{(m,n)}^* (x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}) \right).$$

It follows that $0 \in \circ_{(m,n)}^* (x_1, x_2, \dots, x_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}})$. \square

Let $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra and $A, B \subseteq X$. If $\circ_{(m,n)}^*(A) \cap \circ_{(m,n)}^*(B) \neq \emptyset$, will denote it by $A \approx B$.

Theorem 3.10. *Let m be an even and $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra*

and $x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}} \in X$. If $\alpha = \underbrace{0, \dots, 0}_{(\frac{m}{2}-1)\text{-times}}$, then

$$\circ_{(m,n)}^* (\circ_{(m,n)}^* (x_1, \dots, x_{\frac{m}{2}}, y_1, \dots, y_{\frac{m}{2}}), \alpha, z_1, \dots, z_{\frac{m}{2}}) \approx \circ_{(m,n)}^* (\circ_{(m,n)}^* (x_1, \dots, x_{\frac{m}{2}}, z_1, \dots, z_{\frac{m}{2}}), \alpha, y_1, \dots, y_{\frac{m}{2}}).$$

Proof. Let $x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}} \in X$. Since

$0 \in \circ_{(m,n)}^* \left(\circ_{(m,n)}^* (x_1, x_2, \dots, x_{\frac{m}{2}}, (\circ_{(m,n)}^* (x_1, x_2, \dots, x_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}})), z_1, z_2, \dots, z_{\frac{m}{2}}) \right)$, we get that

$$\begin{aligned} & \circ_{(m,n)}^* \left(\circ_{(m,n)}^* (x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}), \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, z_1, z_2, \dots, z_{\frac{m}{2}} \right) \\ & \leq \circ_{(m,n)}^* \left(\circ_{(m,n)}^* (x_1, x_2, \dots, x_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}}), \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, y_1, y_2, \dots, y_{\frac{m}{2}} \right) \end{aligned}$$

and in similar to

$$\begin{aligned} & \circ_{(m,n)}^* \left(\circ_{(m,n)}^* (x_1, x_2, \dots, x_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}}), \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, y_1, y_2, \dots, y_{\frac{m}{2}} \right) \\ & \leq \circ_{(m,n)}^* \left(\circ_{(m,n)}^* (x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}), \underbrace{0, 0, \dots, 0}_{(m-2)\text{-times}}, z_1, z_2, \dots, z_{\frac{m}{2}} \right). \end{aligned}$$

It follows that

$$\circ_{(m,n)}^* (\circ_{(m,n)}^* (x_1, \dots, x_{\frac{m}{2}}, y_1, \dots, y_{\frac{m}{2}}), \alpha, z_1, \dots, z_{\frac{m}{2}}) \approx \circ_{(m,n)}^* (\circ_{(m,n)}^* (x_1, \dots, x_{\frac{m}{2}}, z_1, \dots, z_{\frac{m}{2}}), \alpha, y_1, \dots, y_{\frac{m}{2}}).$$

\square

Corollary 3.11. *Let m be an even and $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra*

and $x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}} \in X$. If

$$0 \approx \circ_{(m,n)}^* (\circ_{(m,n)}^* (x_1, \dots, x_{\frac{m}{2}}, y_1, \dots, y_{\frac{m}{2}}), \underbrace{0, \dots, 0}_{(\frac{m}{2}-1)\text{-times}}, z_1, \dots, z_{\frac{m}{2}})$$

then

$$0 \approx \circ_{(m,n)}^* (\circ_{(m,n)}^* (x_1, \dots, x_{\frac{m}{2}}, z_1, \dots, z_{\frac{m}{2}}), \underbrace{0, \dots, 0}_{(\frac{m}{2}-1)\text{-times}}, y_1, \dots, y_{\frac{m}{2}}).$$

Example 3.12. Consider the (3, 3)-super hyper BCK-subalgebra in Example 3.4. Clearly

$$\begin{aligned} \circ_{(3,3)}^*(\circ_{(3,3)}^*(0, a, 0), 0, a) &= \circ_{(3,3)}^*(P_*^3(\{0\}), 0, a) = \circ_{(3,3)}^*(0, 0, a) = P_*^3(\{0, a\}) \\ &= \circ_{(3,3)}^*(a, 0, a) = \circ_{(3,3)}^*(P_*^3(\{a\}), 0, a) = \circ_{(3,3)}^*(\circ_{(3,3)}^*(0, a, a), 0, a). \end{aligned}$$

Thus $\circ_{(3,3)}^*(\circ_{(3,3)}^*(0, a, 0), 0, a) = \circ_{(3,3)}^*(\circ_{(3,3)}^*(0, a, a), 0, a)$, while m is an odd. It follows that the converse of Theorem 3.10, is not necessarily true.

Theorem 3.13. Let m be an even and $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra and $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m \in X$. If $\alpha = \underbrace{0, \dots, 0}_{(\frac{m}{2}-1)\text{-times}}$, then

(i)

$$\circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{\frac{m}{2}}, x_1, \dots, x_{\frac{m}{2}}), \alpha, y_1, \dots, y_{\frac{m}{2}}) \approx \circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{\frac{m}{2}}, y_1, \dots, y_{\frac{m}{2}}), \alpha, x_1, \dots, x_{\frac{m}{2}}).$$

(ii)

$$\circ_{(m,n)}^*(\underbrace{0, \dots, 0}_{(\frac{m}{2})\text{-times}}, y_1, \dots, y_{\frac{m}{2}}) \approx \circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{\frac{m}{2}}, y_1, \dots, y_{\frac{m}{2}}), \underbrace{0, \dots, 0}_{(\frac{m}{2}-1)\text{-times}}, x_1, \dots, x_{\frac{m}{2}}).$$

(iii)

$$\circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{\frac{m}{2}}, x_1, \dots, x_{\frac{m}{2}}), \underbrace{0, \dots, 0}_{(\frac{m}{2}-1)\text{-times}}, y_1, \dots, y_{\frac{m}{2}}) \approx \circ_{(m,n)}^*(\underbrace{0, \dots, 0}_{(\frac{m}{2})\text{-times}}, y_1, \dots, y_{\frac{m}{2}}).$$

Proof. (i), (ii), (iii) Let $x_1, x_2, \dots, x_{\frac{m}{2}}, y_1, y_2, \dots, y_{\frac{m}{2}}, z_1, z_2, \dots, z_{\frac{m}{2}} \in X$. Using Corollary 3.11, we get that

$$0 \approx \circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{\frac{m}{2}}, y_1, \dots, y_{\frac{m}{2}}), \underbrace{0, \dots, 0}_{(\frac{m}{2}-1)\text{-times}}, x_1, \dots, x_{\frac{m}{2}})$$

and

$$0 \approx \circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{\frac{m}{2}}, x_1, \dots, x_{\frac{m}{2}}), \underbrace{0, \dots, 0}_{(\frac{m}{2}-1)\text{-times}}, y_1, \dots, y_{\frac{m}{2}}).$$

In addition, by definition we get that $0 \approx \circ_{(m,n)}^*(\underbrace{0, \dots, 0}_{(\frac{m}{2})\text{-times}}, y_1, \dots, y_{\frac{m}{2}})$, hence the proof is completed. \square

Corollary 3.14. Let m be an even and $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra and $x_1, x_2, \dots, x_{m-1}, y_1, y_2, \dots, y_{m-1} \in X$. Then

(i)

$$\circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{\frac{m}{2}}, x_1, \dots, x_{\frac{m}{2}}), y_1, \dots, y_{m-1}) \approx \circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{\frac{m}{2}}, y_1, \dots, y_{\frac{m}{2}}), x_1, \dots, x_{m-1}).$$

(ii)

$$\circ_{(m,n)}^*(0, y_1, \dots, y_{m-1}) \approx \circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{\frac{m}{2}}, y_1, \dots, y_{\frac{m}{2}}), x_1, \dots, x_{m-1}).$$

(iii)

$$\circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{\frac{m}{2}}, x_1, \dots, x_{\frac{m}{2}}), y_1, \dots, y_{m-1}) \approx \circ_{(m,n)}^*(0, y_1, \dots, y_{m-1}).$$

Theorem 3.15. Let m be an even and $(X, \circ_{(m,n)}^*)$ be an (m, n) -super hyper BCK-subalgebra and $x_1, x_2, \dots, x_{m-1} \in X$. Then $(x_1, \dots, x_{m-1}) \approx \circ_{(m,n)}^*(x_1, \dots, x_{m-1}, 0)$.

Proof. Let $x_1, x_2, \dots, x_m \in X$. Then $0 \approx \circ_{(m,n)}^*(x_1, x_2, \dots, x_{m-1}, \circ_{(m,n)}^*(x_1, x_2, \dots, x_{m-1}, 0))$. Moreover by Theorem 3.13, we have $0 \approx \circ_{(m,n)}^*(\circ_{(m,n)}^*(x_1, \dots, x_{m-1}, 0), x_1, \dots, x_{m-1})$. Thus we conclude that $(x_1, \dots, x_{m-1}) \approx \circ_{(m,n)}^*(x_1, \dots, x_{m-1}, 0)$. \square

4. Conclusion

The concept of super hyper BCK-algebras as a generalization of BCK-algebras is introduced in this paper such that for special cases, we can obtain the concepts of BCK-algebras and hyper BCK-algebras. We wish this research is important for the next studies in logical super hyperalgebras. In our future studies, we hope to obtain more results regarding single-valued neutrosophic super(hyper)BCK-subalgebras and their applications in handing information regarding various aspects of uncertainty, non-classical mathematics (fuzzy mathematics or great extension and development of classical mathematics) that are considered to be a more powerful technique than classical mathematics.

Conflicts of Interest: "The authors declare no conflict of interest."

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