On The Classification of The Group of Units of Rational and Real 2-Cyclic Refined Neutrosophic Rings

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Abstract: The objective of this paper is to solve two open problems about the group of units of some 2-cyclic refined neutrosophic rings asked by Sadiq. Where it provides a classification theorem for these rings, and uses this classification property to give a full answer of these open questions.

Also, this work presents a novel algorithm to find all imperfect neutrosophic duplets and triplets in many numerical 2-cyclic refined neutrosophic rings by using the classification isomorphisms.

1. Introduction

Neutrosophic logic as a new generalization of fuzzy logic concerns with indeterminacy in science and real life problems [1]. Neutrosophy was proposed by Smarandache [6] for these logical purposes.

Laterally, neutrosophy was applied to algebra and algebraic structures, were we find many algebraic structures defined by using an indeterminacy element (I) such as neutrosophic rings, neutrosophic spaces, neutrosophic modules, and matrices [2-5].

The concept of n-cyclic refined neutrosophic ring was presented firstly in [7], and studied widely in [8-9].

In [10], Sadiq has studied the group of units problem for 2-CRNR rings, where he proved that it is isomorphic to 3 times direct product of Z_2 . Also, he presented the following open research problems: [10]:

Open problem 1: If the ring R with no zero divisors, then is the group of units of $R_2(I)$ is isomorphic to $U(R) \times U(R) \times U(R)$.

Open problem 2: Find a homomorphism between $R_2(I)$ and the direct product $\times R \times R$.

Open problem 3: Is the group of units of the 2-cyclic refined ring of real numbers isomorphic to $R^* \times R^* \times R^*$.

This motivates us to continuo these efforts to classify the group of units of 2-cyclic refined rings, and to prove the validity of Sadiq's open problems.

On the other hand, we classify all imperfect duplets and triplets in the ring of 2-cyclic refined neutrosophic integers by solving many related Diophantine equations.

We denote the 2-cyclic refined ring by 2-CRNR.

2. Preliminaries

Definition 1.2:

Let (R,+,×) be a ring and $I_k; 1 \le k \le n$ be n sub-indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-cyclic refined neutrosophic ring.

Operations on R_n(I) are defined as:

$$\sum_{i=0}^{n} p_i I_i + \sum_{i=0}^{n} q_i I_i$$

$$= \sum_{i=0}^{n} (p_i + q_i) I_i \,, \sum_{i=0}^{n} p_i I_i$$

$$\times \sum_{i=0}^{n} q_i I_i = \sum_{i,j=0}^{n} \left(p_i \times q_j\right) I_i I_j = \sum_{i,j=0}^{n} \left(p_i \times q_j\right) I_{(i+j \, modn)}$$

Example 2.2:

(a) The 2-CRNR of integers is defined as follows:

$Z_2(I) = \{t_0 + t_1 I_1 + t_2 I_2;\ t_i \in Z\}$.

(b) Addition on $Z_2(I)$:

$(a + bI_1 + cI_2) +\ (m + nI_1 + tI_2) = (a + m) + I_1(b + n) + I_2(c + t)$.

(c) Multiplication on $Z_2(I)$:

$(a + bI_1 + cI_2)(\ m + nI_1 + tI_2)$= $am + anI_1 + atI_2 + bmI_1 + bnI_2 + btI_1 + cmI_2 + cnI_1 + ctI_2$

$= am + I_1(an + bm + bt + cn) + I_2(at + bn + cm + ct)$.

Where $I_1 I_1 = I_{(1+1\ mod 2)} = I_2, I_2 I_2 = I_{(2+2\ mod\ 2)} = I_2, I_1 I_2 = I_{(1+2\ mod\ 2)} = I_1$.

Definition 3.2:

Let R be a ring, a duplet (x, y) is called an imperfect duplet with x acts as an identity if and only if $xy = yx = y$.

A triple (x, y, z) is called an imperfect triplet with x acts as an identity if and only if $xy = yx = y, xz = zx = z, zy = yz = x$.

3. Main discussion

Theorem 1.3 : Let Z be the ring of integers, and $S = \{(b_0, b_1, b_2);\ b_i \in Z\ and\ b_1 - b_2 \in 2Z\}$, then $(S, +, .)$ Is a subring of $Z \times Z \times Z$.

Proof: It is clear that $S \neq \emptyset$

$\forall\ x, y \in S$, $x = (a_0, a_1, a_2), y = (b_0, b_1, b_2)$, where $b_1 - b_2, a_1 - a_2 \in 2Z$

$x + y = (a_0 + b_0, a_1 + b_1, a_2 + b_2)$, $xy = (a_0 b_0, a_1 b_1, a_2 b_2)$

We have: $(a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2) \in 2Z$, thus $x + y \in S$

Also, $a_1 b_1 - a_2 b_2 = a_1 b_1 + a_1 b_2 - a_1 b_2 - a_2 b_2 = a_1(b_1 + b_2) - b_2(a_1 + a_2)$. By the assumption, we have $b_1 - b_2, a_1 - a_2 \in 2Z$, hence

$b_1 + b_2, a_1 + a_2 \in 2Z$, this implies $a_1(b_1 + b_2) - b_2(a_1 + a_2) \in 2Z$ and $x.y \in S$.

Theorem 2.3: Let $Z_2(I)$ be the 2-CRNR of integers, then $Z_2(I) \cong S$.

Proof:

Define $f: Z_2(I) \to S; f(a_0 + a_1 I_1 + a_2 I_2) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2)$.

It's clear that f is well defined. On the other hand we have:

(a). f is injective, $\ker f = \{a_0 + a_1 I_1 + a_2 I_2 \in Z_2(I)\ ;\ f(a_0 + a_1 I_1 + a_2 I_2) = (0,0,0)\}$, hence.

$a_0 = 0,\ a_0 + a_1 + a_2 = 0$, $a_0 - a_1 + a_2 = 0$, thus $a_0 = a_1 = a_2$, this means that $\ker f = \{0_S\}$.

(b). f is surjective, $\forall\ y = (a_0, a_1, a_2) \in S$, we have: $a_1 - a_2 \in 2Z$, hence $x = a_0 + I_1\left(\frac{a_1 - a_2}{2}\right) + I_2\left(\frac{a_1 + a_2 - 2a_0}{2}\right) \in Z_2(I)$.

This is because $a_1 - a_2,\ a_1 + a_2 - 2a_0 \in 2Z$.

Now, we compute $f(x)=\left(a_0,a_0+\frac{a_1-a_2}{2}+\frac{a_1+a_2-2a_0}{2},a_0-\frac{a_1-a_2}{2}+\frac{a_1+a_2-2a_0}{2}\right)=(a_0,a_1,a_2)=y$

(c). $f$is a homomorphism because clearly f preserves addition and multiplication, thus $S\cong Z_2(I)$.

Theorem 3.3: Let R be a of real numbers, $R_2(I)$ be the corresponding 22-CRNR of real numbers, then $R_2(I)\cong R^3$.

Proof. Define $f: R_2(I)\ \to R^3; f(a_0+a_1I_1+a_2I_2)=(a_0,a_0+a_1+a_2,a_0-a_1+a_2)$.

f is well defined and bijective. (the proof is exactly similar to the previous theorem).

f is a homomorphism. $\forall\, x,y\in R_2(I)\,, x=a_0+a_1I_1+a_2I_2, y=b_0+b_1I_1+b_2I_2$.

$x+y=a_0+b_0+(a_1+b_1)I_1+(a_2+b_2)I_2$.

$f(x+y)=(a_0+b_0,a_0+b_0+a_1+b_1+a_2+b_2,a_0+b_0-(a_1+b_1)+a_2+b_2)$

$f(x+y)=(a_0,a_0+a_1+a_2,a_0-a_1+a_2)+(b_0,b_0+b_1+b_2,b_0-b_1+b_2)=f(x)+f(y)$.

$xy=a_0b_0+(a_1b_0+a_0b_1+a_2b_1+a_1b_2)I_1+(a_1b_0+a_0b_2+a_1b_1+a_2b_2)I_2$

$$f(xy)=(a_0b_0,a_0b_0+a_1b_0+a_0b_1+a_1b_1+a_2b_0+a_0b_2+a_1b_1+a_2b_2,a_0b_0 \\ -(a_1b_0+a_0b_1+a_2b_1++a_1b_2)+a_2b_0+a_0b_2+a_1b_1+a_2b_2)$$

$f(xy)=(a_0,a_0+a_1+a_2,a_0-a_1+a_2).(b_0,b_0+b_1+b_2,b_0-b_1+b_2)=f(x).f(y)$, hence $R_2(I)\cong R^3$.

Answers to the open questions

The following theorem answers the open question 3.

Theorem 4.3: Let $\cup\left(R_2(I)\right)$ be the group of units of the 2-CRNR$R_2(I)$, then $\cup\left(R_2(I)\right)\cong R^{*3}$.

Proof.

According to the previous theorem, $R_2(I)\cong R\times R\times R$, hence. $\cup\left(R_2(I)\right)\cong\cup(R)\times\cup(R)\times\cup(R)=R^{*3}$.

The following remark answers the open question 2.

Remark 5.3: If R is a ring, and $R_2(I)$ is the corresponding 2-CRNR, hence the map

$f: R_2(I)\ \to R\times R\times R; f(a_0+a_1I_1+a_2I_2)=(a_0,a_0+a_1+a_2,a_0-a_1+a_2)$, Is a ring homomorphism (the proof is similar to Theorem 3.2). Thus the answer to the open question 2 is yes. Remark that f is not supposed to be an isomorphism, check Theorem 1.3 for example.

The first question is still open, but we can solve the problem in a special case for the ring of integers modulo n, with odd n.

Theorem 6.3: Let R be the ring of integers modulo n, with an odd integer n, then $R_2(I) \cong Z_n \times Z_n \times Z_n$

Proof. . Define $f: R_2(I) \rightarrow Z_n \times Z_n \times Z_n; f(a_0 + a_1 I_1 + a_2 I_2) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2)$.

It's clear that f is a well defined homomorphism, by a similar argument of the previous theorem, we should prove that f is a bijective map.

$\ker f = \{a_0 + a_1 I_1 + a_2 I_2 \in R_2(I)\ ;\ f(a_0 + a_1 I_1 + a_2 I_2) = (0,0,0)\}$, hence.

$$a_0 = 0 \ldots (1)$$

$a_0 + a_1 + a_2 = 0 \ldots (2)$

$a_0 - a_1 + a_2 = 0 \ldots (3)$

From equation (2) and (3), we get $2a_2 = 0$, By the proposition of the theorem, n is odd, this means that gcd(2,n)=1 and 2 cannot be a zero divisor, thus $2a_2 = 0 \Rightarrow a_2 = 0$.

This implies that $a_1 = 0$, and $\ker f = \{(0,0,0)\}$.

f is surjective:

$\forall y = (a_0, a_1, a_2) \in Z_n \times Z_n \times Z_n$, we have $x = a_0 + I_1\big((a_1 + a_2)2^{-1}\big) + I_2\big((a_1 + a_2 - 2a_0)2^{-1}\big) \in R_2(I)$.

That is because 2 is a unit in Z_n and 2^{-1} is existed.

Now, we compute

$$f(x) = (a_0, (a_1 - a_2)2^{-1} + (a_1 + a_2 + 2a_0)2^{-1} + a_0, a_0 + (a_1 - a_2)2^{-1} + (a_1 + a_2 + 2a_0)2^{-1})$$

$$= (a_0, a_1 2^{-1} - a_2 2^{-1} + a_1 2^{-1} + a_2 2^{-1} - 2a_0 2^{-1} + a_0, a_0 - a_1 2^{-1} + a_2 2^{-1} + a_1 2^{-1} + a_2 2^{-1} - 2a_0 2^{-1})$$

$$= (a_0, 2a_1 2^{-1}, 2a_2 2^{-1}) = (a_0, a_1, a_2).$$

So that, $R_2(I) \cong Z_n \times Z_n \times Z_n$.

Theorem 7.3: If $R = Z_n$ the ring of integers modulo n with an odd integer n, we have:

$\cup\left(R_2(I)\right) \cong \cup\,(Z_n) \times \cup\,(Z_n) \times \cup\,(Z_n)$.

The proof holds directly by the previous result.

Theorem 8.3: If $R = Z$ the ring of integers, $Z_2(I)$ be the corresponding 2-CRNR, then $Z_2(I)$ has exactly 8 forms of imperfect duplets.

Proof. We have $Z_2(I) \cong S;\ S = \{(a_0, a_1, a_2);\ a_i \in Z \ and\ a_1 - a_2 \in 2Z\}$.

To find imperfect duplets in $Z_2(I)$, it is sufficient to compute duplets in S:

Let $x = (a_0, a_1, a_2), y = (b_0, b_1, b_2)$ be an imperfect duplet in S, with y acts as an identity, we have.

$$x.y = x \Longrightarrow \begin{cases} a_0 b_0 = a_0 \\ a_1 b_1 = a_1 \\ a_2 b_2 = a_2 \end{cases} \Longrightarrow \begin{cases} a_0 = 0 \ or\ b_0 = 0 \\ a_1 = 0 \ or\ b_1 = 0. \\ a_2 = 0 \ or\ b_2 = 0 \end{cases}$$

The possible imperfect duplets are:

(1).$x = (0, 0,0), y = (b_0, b_1, b_2)$

(With $b_1 - b_2 \in 2Z$)

(2). $x = (0, a_1, a_2), y = (b_0, 1,1)$

(With $a_1 - a_2 \in 2Z$)

(3). $x = (0, 0, a_2), y = (b_0, b_1, 1)$

(With $a_2\ is\ even\ and\ b_1\ is\ odd$)

(4). $x = (a_0, 0, a_2), y = (1, b_1, 1)$

(With $a_2\ is\ even\ and\ b_1\ is\ odd$)

(5). $x = (a_0, a_1, 0), y = (1, 1, b_2)$

(With $a_1\ is\ even\ and\ b_2\ is\ odd$)

(6). $x = (a_0, a_1, a_2), y = (1, 1,1)$

(With $a_1 - a_2 \in 2Z$)

(7). $x = (a_0, 0,0), y = (1, b_1, b_2)$

(With $b_1 - b_2 \in 2Z$)

(8). $x = (0, a_1, 0), y = (b_0, 1, b_2)$

(With a_1 *is even and* b_2 *is odd*)

Thus, the imperfect duplets in $Z_2(I)$ are the converse image of the duplets in S, according to the isomorphism

$f(a_0 + a_1I_1 + a_2I_2) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2)$.

$f^{-1}(a_0 + a_1I_1 + a_2I_2) = \left(a_0 + \frac{a_1-a_2}{2}I_1 + \frac{a_1+a_2-2a_0}{2}I_2\right)$, so that the duplets of $Z_2(I)$are:

(1). $x = 0, y = b_0 + b_1I_1 + b_2I_2$

(With $b_1 - b_2 \in 2Z$)

(2). $x = \frac{a_1-a_2}{2}I_1 + \frac{a_1+a_2}{2}I_2, y = b_0 + \frac{2-2b_0}{2}I_2$

(With $a_1 - a_2 \in 2Z$)

(3). $x = \frac{-a_2}{2}I_1 + \frac{a_2}{2}I_2, y = b_0 + \frac{b_1-1}{2}I_1 + \frac{b_1+1-2b_0}{2}I_2$

(With a_2 *is even and* b_1 *is odd*)

(4). $x = a_0 + \frac{-a_2}{2}I_1 + \frac{a_2-2a_0}{2}I_2, y = 1 + \frac{b_1-1}{2}I_1 + \frac{b_1+1-2(1)}{2}I_2$

(With a_2 *is even and* b_1 *is odd*)

(5). $x = a_0 + \frac{a_1}{2}I_1 + \frac{a_1-2a_0}{2}I_2, y = 1 + \frac{1-b_2}{2}I_1 + \frac{1+b_2-2(1)}{2}I_2$

(With a_1 *is even and* b_2 *is odd*)

(6). $x = a_0 + a_1I_1 + a_2I_2, y = 1$

(With $a_1 - a_2 \in 2Z$)

(7). $x = a_0 - a_0I_2, y = 1 + \frac{b_1-b_2}{2}I_1 + \frac{b_1+b_2-2}{2}I_2$

(With $b_1 - b_2 \in 2Z$)

(8). $x = \frac{a_1}{2}I_1 + \frac{a_1}{2}I_2, y = b_0 + \frac{1-b_2}{2}I_1 + \frac{1++b_2-2b_0}{2}I_2$

(With a_1 *is even and* b_2 *is odd*)

Theorem 9.3: Let R be the ring of real numbers, $R_2(I)$ be its 2-CRNR, then $R_2(I)$ has exactly 8 forms of imperfect duplets.

Proof. We have $R_2(I) \cong R \times R \times R$ with the isomorphism:

$f: R_2(I) \to R \times R \times R; f(a_0 + a_1I_1 + a_2I_2) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2)$.

For determining the imperfect duplets in $R_2(I)$, it is sufficient to find duplets in $R \times R \times R$ and go back to $R_2(I)$ by the inverse isomorphism.

The imperfect duplets in $R \times R \times R$ are:

(1). $x = (0, 0,0), y = (b_0, b_1, b_2)$

(2). $x = (0, a_1, a_2), y = (b_0, 1,1)$

(3). $x = (0, 0, a_2), y = (b_0, b_1, 1)$

(4). $x = (a_0, 0, a_2), y = (1, b_1, 1)$

(5). $x = (a_0, a_1, 0), y = (1, 1, b_2)$

(6). $x = (a_0, a_1, a_2), y = (1, 1,1)$

(7). $x = (a_0, 0,0), y = (1, b_1, b_2)$

(8). $x = (0, a_1, 0), y = (b_0, 1, b_2)$

Thus$R_2(I)$ has 8 forms of imperfect duplets.

Remark 10.3: To find any imperfect duplets in$R_2(I)$, we should compute the inverse image of the corresponding duplet in $R \times R \times R$as follows:

$$f^{-1}(a_0, a_1, a_2) = a_0 + \frac{a_1 - a_2}{2} I_1 + \frac{a_1 + a_2 - 2a_0}{2} I_2$$

Example 11.3: Let's, take a duplet with form:$x = (2,0,3), y = (1,5,1)$, it is clear $x.y = x$.

The corresponding duplet in $R_2(I)$ is:

$x_1 = f^{-1}(x) = 2 + \frac{-3}{2} I_1 + \frac{-1}{2} I_2, y_1 = f^{-1}(y) = 1 + 2I_1 + 2I_2$.

Remark 12.3: that $x_1.y_1 = 2 + 4I_1 + 4I_2 = \frac{-3}{2} I_1 - 3I_2 - 3I_1 - \frac{1}{2} I_2 - I_1 - I_2 = 2 - \frac{3}{2} I_1 - \frac{1}{2} I_2 = x_1$.

Theorem 13.3: Let $Z_2(I)$ be the 2-CRNR of integers, then it has exactly 14 forms of imperfect triplets.

Proof.

Let x, y, z be a triplet in S, then we have:

$xy = yx = x$, $yz = zy = z$, $xz = zx = y$, so that, (x, y) , (y, z) are imperfect duplets in S.

We discuss the 8 forms of imperfect duplets to find the desired imperfect duplets:

Form 1: $x = (0, 0,0), y = (b_0, b_1, b_2), z = (0, 0,0)$ it is a triplet if and only if $xy = z,$ thus $, y = (0, 0,0)$.

(the first triplet is $x = y = z = (0, 0,0)$).

Form 2: $x = (0, a_1, a_2), y = (b_0, 1,1), z = (0, c_1, c_2)$ it is a triplet if and only if $xz = y,$ thus $, b_0 = 0, a_1c_1 = 1, a_2c_2 = 1$.

the possible triplets are:

$x = (0, 1,1), y = (0, 1,1), z = (0, 1,1)$.

$x = (0, 1, -1), y = (0, 1,1), z = (0, 1, -1)$

$x = (0, 1, -1), y = (0, 1,1), z = (0, 1, -1)$

$x = (0, -1,1), y = (0, 1,1), z = (0, -1, -1)$

Form 3: $x = (0, 0, a_2), y = (b_0, b_1, 1), z = (0, 0, c_2)$ it is a triplet if and only if $xz = y,$ thus $, b_0 = b_1 = 0, , a_2 = c_2 = 1$.

the possible triplets are:

$x = (0, 0,1), y = (0, 0,1), z = (0, 1,1)$.

$x = (0, 0, -1), y = (0, 0,1), z = (0, 1, -1)$

Form 4: $x = (a_0, 0, a_1), y = (1, b_1, 1), z = (c_0, 0, c_1)$ it is a triplet if and only if $xz = y,$ thus $, a_0c_0 = a_1c_1 = 1, b_1 = 0$.

the possible triplets are:

$x = (1, 0,1), y = (1, 0,1), z = (1, 0,1)$

$x = (-1, 0,1), y = (1, 0,1), z = (-1, 0,1)$.

$x = (1, 0, -1), y = (1, 0,1), z = (1, 0, -1)$

$x = (-1, 0, -1), y = (1, 0,1), z = (-1, 0, -1)$

Form 5: $x = (a_0, a_1, 0), y = (1, 1, b_2), z = (c_0, c_1, 0)$ it is a triplet if and only if $xz = y,$ thus $, a_0c_0 = 1, a_1c_1 = 1, b_2 = 0$.

the possible triplets are:

$x = (1, 1,0), y = (1, 1,0), z = (1, 1,0)$

$x = (-1, -1,0), y = (1, 1,0), z = (-1, -1,0)$.

$x = (-1, 1,0), y = (1, 1,0), z = (-1, 1,0)$

$x = (1, 1,0), y = (1, 1, -1), z = (1, -1,0)$

Form 6: $x = (a_0, a_1, a_2), y = (1, 1,1), z = (c_0, c_1, c_2)$ it is a triplet if and only if $xz = y$, thus, $a_0c_0 = a_1c_1 = a_2c_2 = 1$.

the possible triplets are:

$x = (1, 1,1), y = (1, 1,1), z = (1, 1,1)$

$x = (1, 1, -1), y = (1, 1,1), z = (1, 1, -1)$

$x = (1, -1,1), y = (1, 1,1), z = (1, -1,1)$

$x = (-1, 1,1), y = (1, 1,1), z = (-1, 1,1)$

$x = (-1, - 1,1), y = (1, 1,1), z = (-1, - 1,1)$

$x = (1, -1, -1), y = (1, 1,1), z = (1, -1, -1)$

$x = (-1, 1, -1), y = (1, 1,1), z = (-1, 1, -1)$

$x = (-1, -1, -1), y = (1, 1,1), z = (-1, -1, -1)$

Form 7: $x = (a_0, 0,0), y = (1, b_1, b_2), z = (c_0, 0,0)$ it is a triplet if and only if $xz = y$, thus, $a_0c_0 = 1, b_1 = b_2 = 0$.

the possible triplets are:

$x = (1, 0,0), y = (1, 0,0), z = (1, 0,0)$

$x = (-1, 0,0), y = (1, 0,0), z = (-1, 0,0)$

Form 8: $x = (0, a_1, 0), y = (b_0, 1, b_2), z = (0, c_1, 0)$ it is a triplet if and only if $xz = y$, thus, $a_1c_1 = 1, b_0 = b_2 = 0$.

the possible triplets are:

$x = (0, 1,0), y = (0, 1,0), z = (0, 1,0)$

$x = (0, -1,0), y = (0, -1,0), z = (0, 1,0)$.

References

1. Smarandache, F., " n-Valued Refined Neutrosophic Logic and Its Applications in Physics", Progress in Physics, 143-146, Vol. 4, 2013.

2. M. Ibrahim. A. Agboola, B.Badmus and S. Akinleye. On refined Neutrosophic Vector Spaces . International Journal of Neutrosophic Science, Vol. 7, 2020, pp. 97-109.

3. Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021**.**

4. Celik, M., and Olgun, N., " On The Classification Of Neutrosophic Complex Inner Product Spaces", Galoitica Journal Of Mathematical Structures And Applications, 2022.

5. Celik, M., and Olgun, N., " An Introduction To Neutrosophic Real Banach And Hillbert Spaces", Galoitica Journal Of Mathematical Structures And Applications, 2022.

6. Kandasamy, V.W.B., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona, 2006.

7. Abobala, M, "*n*-Cyclic Refined Neutrosophic Algebraic Systems Of Sub-Indeterminacies, An Application To Rings and Modules", International Journal of Neutrosophic Science, Vol. 12, pp. 81-95 . 2020.

8. Von Shtawzen, O., " Conjectures For Invertible Diophantine Equations Of 3-Cyclic and 4-Cyclic Refined Integers", Journal Of Neutrosophic And Fuzzy Systems, Vol.3, 2022.

9. Basheer, A., Ahmad, K., and Ali, R., " Examples On Some Novel Diophantine Equations Derived From The Group Of Units Problem In n-Cyclic Refined Neutrosophic Rings Of Integers", Galoitica Journal Of Mathematical Structures and Applications, 2022.

10. Sadiq, B., " A Contribution To The Group Of Units' Problem In Some 2-Cyclic Refined Neutrosophic Rings", International Journal Of Neutrosophic Science, 2022.

11. Khaldi, A**.,** " A Study On Split-Complex Vector Spaces"**,** Neoma Journal Of Mathematics and Computer Science, 2023.

12. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

13. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces"**,** Neoma Journal Of Mathematics and Computer Science, 2023.

14. Von Shtawzen, O., " On A Novel Group Derived From A Generalization Of Integer Exponents and Open Problems", Galoitica journal Of Mathematical Structures and Applications, Vol 1, 2022.

15. Hatip, A., "An Introduction To Weak Fuzzy Complex Numbers ", Galoitica Journal Of Mathematical Structures and Applications, Vol.3, 2023.

16. Merkepci, H., and Ahmad, K., " On The Conditions Of Imperfect Neutrosophic Duplets and Imperfect Neutrosophic Triplets", Galoitica Journal Of Mathematical Structures And Applications, Vol.2, 2022.

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