Neutrosophic Bitopological Spaces

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Abstract: In this study, bitopological structure which is a more general structure than topological spaces is built on neutrosophic sets. The necessary arguments which are pairwise neutrosophic open set, pairwise neutrosophic closed set, pairwise neutrosophic closure, pairwise neutrosophic interior are defined and their basic properties are presented. The relations of these concepts with their counterparts in neutrosophic topological spaces are given and many examples are presented.

Keywords: Neutrosophic set; neutrosophic bitopological space; pairwise neutrosophic open (closed) set; pairwise neutrosophic interior; pairwise neutrosophic closure; pairwise neutrosophic neighbourhood.

1. Introduction

In recent years, the major factor in the progress of natural sciences and its sub-branches is the construction of new set structures in mathematics. It is the fuzzy set theory defined by Zadeh [19] that leads to these set structures. This set structure is followed by intuitionistic set theory [7], intuitionistic fuzzy set theory [1] and soft set theory [15]. Later, as a generalization of fuzzy set and intuitionistic fuzzy set, Samarandache [17] introduced neutrosophic set. Neutrosophic set N consist of three independent object called truth-membership $T_N(x)$, interminancy-membership $I_N(x)$ and falsity-memebership $F_N(x)$ whose values are real standard or non-standard subset of unit interval $]^{-}0,1^{+}[$. Scientists continue to intensively study in different fields with this set structure [3, 4, 8, 14, 15, 17, 18, 19, 20, 21, 22]. These set structures have been studied by some authors in topology [2, 5, 6, 16, 18].

The concept of bitopological spaces was introduced by Kelly [13] as an extension of topological spaces in 1963. This concept has been studied with interest in other set structures [10, 12]. Therefore, we find it necessary and important to construct a bitopological spaces on the neutrosophic set structure.

In this study, we presented bitopological spaces on neutrosophic set structure and some basic notions of this spaces, open (closed) set, closure, interior, neighbourhood systems are defined. In addition, the theorems required for this structure are proved and their relations with neutrosophic topological spaces are investigated.

2. Preliminary

In this section, we will give some preliminary information for the present study.

Definition 2.1 [23] Let X be a non empty set, then $N=\{\langle x,T_N(x),I_N(x),F_N(x)\rangle : x\in X\}$ is called a neutrosophic set on X, where $^{-}0\le T_N(x)+I_N(x)+F_N(x)\le 3^{+}$ for all $x\in X$, $T_N(x)$, $I_N(x)$ and $F_N(x)\in]^{-}0,1^{+}[$ are the degree of membership (namely $T_N(x)$), the degree of indeterminacy (namely

$I_N(x)$) and the degree of non membership (namely $F_N(x)$) of each $x \in X$ to the set N respectively. For X, $\aleph(X)$ denotes the collection of all neutrosophic sets of X.

Definition 2.2 [23] The following statements are true for neutrosophic sets N and M on X:
i) $T_N(x) \leq T_M(x)$, $I_N(x) \leq I_M(x)$ and $F_N(x) \geq F_M(x)$ for all $x \in X$ iff $N \subseteq M$.
ii) $N \subseteq M$ and $M \subseteq N$ iff $N = M$.
iii) $N \cap M = \{\langle x, \min\{T_N(x), T_M(x)\}, \min\{I_N(x), I_M(x)\}, \max\{F_N(x), F_M(x)\}\rangle : x \in X\}$.
iv) $N \cup M = \{\langle x, \max\{T_N(x), T_M(x)\}, \max\{I_N(x), I_M(x)\}, \min\{F_N(x), F_M(x)\}\rangle : x \in X\}$.
More generally, the intersection and the union of a collection of neutrosophic sets $\{N_i\}_{i \in I}$, are defined by:

$$\bigcap_{i \in I} N_i = \left\{\left\langle x, \inf\{T_{N_i}(x)\}, \inf\{I_{N_i}(x)\}, \sup\{F_{N_i}(x)\}\right\rangle : x \in X\right\},$$
$$\bigcup_{i \in I} N_i = \left\{\left\langle x, \sup\{T_{N_i}(x)\}, \sup\{I_{N_i}(x)\}, \inf\{F_{N_i}(x)\}\right\rangle : x \in X\right\}.$$

v) N is called neutrosophic universal set, denoted by 1_X, if $T_N(x) = 1$, $I_N(x) = 1$ and $F_N(x) = 0$ for all $x \in X$.

vi) N is called neutrosophic empty set, denoted by 0_X, if $T_N(x) = 0$, $I_N(x) = 0$ and $F_N(x) = 1$ for all $x \in X$.

vii) $N \backslash M = \{\langle x, |T_N(x) - T_M(x)|, |I_N(x) - I_M(x)|, 1 - |F_N(x) - F_M(x)|\rangle : x \in X\}$. Clearly, the neutrosophic complements of 1_X and 0_X are defined:

$$(1_X)^c = 1_X \backslash 1_X = \langle x, 0,0,1\rangle = 0_X,$$
$$(0_X)^c = 1_X \backslash 0_X = \langle x, 1,1,0\rangle = 1_X.$$

Proposition 2.1 [23] Let N_1, N_2, N_3 and $N_4 \in \aleph(X)$. Then followings hold:
i) $N_1 \cap N_3 \subseteq N_2 \cap N_4$ and $N_1 \cup N_3 \subseteq N_2 \cup N_4$, if $N_1 \subseteq N_2$ and $N_3 \subseteq N_4$,
ii) $(N_1^c)^c = N_1$ and $N_1 \subseteq N_2$, if $N_2^c \subseteq N_1^c$,
iii) $(N_1 \cap N_2)^c = N_1^c \cup N_2^c$ and $(N_1 \cup N_2)^c = N_1^c \cap N_2^c$.

Definition 2.3 [22] Let X be a non empty set. A neutrosophic topology on X is a subfamily τ^N of $\aleph(X)$ such that 1_X and 0_X belong to τ^n, τ^n is closed under arbitrary union and τ^n is closed finite intersection. Then (X, τ^n) is called neutrosophic topological space, members of τ^n are known as neutrosophic open sets and their complements are neutrosophic closed sets. For a neutrosophic set N over X, the neutrosophic interior and the neutrosophic closure of N are defined as: $\mathrm{int}^n(N) = \cup \{G : G \subseteq N, G \in \tau^n\}$ and $\mathrm{cl}^n(N) = \cap \{F : N \subseteq F, F^c \in \tau^n\}$.

Definition 2.4 [9] Let X be a non empty set. If α, β, γ be real standard or non standard subsets of $]^-0, 1^+[$, then the neutrosophic set $x_{\alpha,\beta,\gamma}$ is called a neutrosophic point in given by

$$x_{\alpha,\beta,\gamma}(y) = \begin{cases} (\alpha, \beta, \gamma), & \text{if } x = y \\ (0,0,1), & \text{if } x \neq y \end{cases}$$

for $y \in X$ is called the support of $x_{\alpha,\beta,\gamma}$.
It is clear that every neutrosophic set is the union of its neutrosophic points.

Definition 2.5 [9] Let $N \in \aleph(X)$. We say that $x_{\alpha,\beta,\gamma} \in N$ read as belonging to the neutrosophic set N whenever $\alpha \leq T_N(x)$, $\beta \leq I_N(x)$ and $\gamma \geq F_N(x)$.

Definition 2.6 [11] A subcollection τ_n^* of neutrosophic sets on a non empty set X is said to be a neutrosophic supra topology on X if the sets 1_X, $0_X \in \tau_n^*$ and $\bigcup_{i=1}^{\infty} N_i \in \tau_n^*$ for $\{N_i\}_{i=1}^{\infty} \in \tau_n^*$. Then (X, τ_n^*) is called neutrosophic supra topological space on X.

3. Neutrosophic Bitopological Spaces

Definition 3.1 Let (X,τ_1^n) and (X,τ_2^n) be the two different neutrosophic topologies on X. Then (X,τ_1^n,τ_2^n) is called a neutrosophic bitopological space.

Definition 3.2 Let (X,τ_1^n,τ_2^n) be a neutrosophic bitopological space. A neutrosophic set $N=\{\langle x,T_N(x),I_N(x),F_N(x)\rangle : x\in X\}$ over X is said to be a pairwise neutrosophic open set in (X,τ_1^n,τ_2^n) if there exist a neutrosophic open set $N_1=\{\langle x,T_{N_1}(x),I_{N_1}(x),F_{N_1}(x)\rangle : x\in X\}$ in τ_1^n and a neutrosophic open set $N_2=\{\langle x,T_{N_2}(x),I_{N_2}(x),F_{N_2}(x)\rangle : x\in X\}$ in τ_2^n such that $N=N_1\cup N_2=\{\langle x,\max\{T_{N_1}(x),T_{N_2}(x)\},\max\{I_{N_1}(x),I_{N_2}(x)\},\min\{F_{N_1}(x),F_{N_2}(x)\}\rangle : x\in X\}$.

Definition 3.3 Let (X,τ_1^n,τ_2^n) be a neutrosophic bitopological space. A neutrosophic set N over X is said to be a pairwise neutrosophic closed set in (X,τ_1^n,τ_2^n) if its neutrosophic complement is a pairwise neutrosophic open set in (X,τ_1^n,τ_2^n). Obviously, a neutrosophic set $C=\{\langle x,T_C(x),I_C(x),F_C(x)\rangle : x\in X\}$ over X is a pairwise neutrosophic closed set in (X,τ_1^n,τ_2^n) if there exist a neutrosophic closed set $C_1=\{\langle x,T_{C_1}(x),I_{C_1}(x),F_{C_1}(x)\rangle : x\in X\}$ in $(\tau_1^n)^c$ and a neutrosophic closed set $C_2=\{\langle x,T_{C_2}(x),I_{C_2}(x),F_{C_2}(x)\rangle : x\in X\}$ in $(\tau_2^n)^c$ such that $C=C_1\cap C_2=\{\langle x,\min\{T_{C_1}(x),T_{C_2}(x)\},\min\{I_{C_1}(x),I_{C_2}(x)\},\max\{F_{C_1}(x),F_{C_2}(x)\}\rangle : x\in X\}$, where

$$(\tau_i^n)^c=\{N^c\in\aleph(X): N\in\tau_i^n\}, i=1,2.$$

The family of all pairwise neutrosophic open (closed) sets in (X,τ_1^n,τ_2^n) is denoted by $PNO(X,\tau_1^n,\tau_2^n)$ $[PNC(X,\tau_1^n,\tau_2^n)]$, respectively.

Example 3.1 Let $X=\{a,b,c\}$. We think that following neutrosophic set over X.

$$N_1=\{\langle a,0.3,0.2,0.5\rangle,\langle b,0.6,0.5,0.3\rangle,\langle c,0.7,0.1,0.9\rangle\},$$
$$N_2=\{\langle a,0.4,0.1,0.3\rangle,\langle b,0.2,0.6,0.7\rangle,\langle c,0.1,0.3,0.4\rangle\},$$
$$N_3=\{\langle a,0.3,0.1,0.5\rangle,\langle b,0.2,0.5,0.7\rangle,\langle c,0.1,0.1,0.9\rangle\},$$
$$N_4=\{\langle a,0.4,0.2,0.3\rangle,\langle b,0.6,0.6,0.3\rangle,\langle c,0.7,0.3,0.4\rangle\}$$

and

$$M_1=\{\langle a,0.1,0.2,0.3\rangle,\langle b,0.2,0.1,0.4\rangle,\langle c,0.5,0.2,0.4\rangle\},$$
$$M_2=\{\langle a,0.7,0.3,0.1\rangle,\langle b,0.7,0.8,0.2\rangle,\langle c,0.9,0.8,0.3\rangle\}.$$

Then (X,τ_1^n,τ_2^n) is a neutrosophic bitopological space, where

$$\tau_1^n=\{0_X,1_X,N_1,N_2,N_3,N_4\},$$
$$\tau_2^n=\{0_X,1_X,M_1,M_2\}.$$

Obviously,

$$\tau_{12}^n=\tau_1^n\cup\tau_2^n\cup\{N_1\cup M_1,N_2\cup M_1,N_3\cup M_1\}$$

because the neutrosophic sets $N_1\cup M_1$, $N_2\cup M_1$ and $N_3\cup M_1$ not belong to either τ_1^n nor τ_2^n.

Theorem 3.1 Let (X,τ_1^n,τ_2^n) be a neutrosophic bitopological space. Then,

1. 0_X and 1_X are pairwise neutrosophic open sets and pairwise neutrosophic closed sets.

2. An arbitrary neutrosophic union of pairwise neutrosophic open sets is a pairwise neutrosophic open set.

3. An arbitrary neutrosophic intersection of pairwise neutrosophic closed sets is a pairwise neutrosophic closed set.

Proof. 1. Since $0_X\in\tau_1^n$, τ_2^n and $0_X\cup 0_X=0_X$, then 0_X is a pairwise neutrosophic open set. Similarly, 1_X is a pairwise neutrosophic open set.

2. Let $\{(N_i): i\in I\}\subseteq PNO(X,\tau_1^n,\tau_2^n)$. Then N_i is a pairwise neutrosophic open set for all $i\in I$, therefore there exist $N_i^1\in\tau_1^n$ and $N_i^2\in\tau_2^n$ such that $N_i=N_i^1\cup N_i^2$ for all $i\in I$ which implies that

$$\bigcup_{i\in I} N_i=\bigcup_{i\in I}[N_i^1\cup N_i^2]=\left[\bigcup_{i\in I} N_i^1\right]\cup\left[\bigcup_{i\in I} N_i^2\right].$$

Now, since τ_1^n and τ_2^n are neutrosophic topologies, then $\left[\bigcup_{i\in I} N_i^1\right]\in\tau_1^n$ and $\left[\bigcup_{i\in I} N_i^2\right]\in\tau_2^n$. Therefore, $\bigcup_{i\in I} N_i$ is a pairwise neutrosophic open set.

3. It is immediate from the Definition 9, Proposition 1.

Corollary 3.1 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space. Then, the family of all pairwise neutrosophic open sets is a supra neutrosophic topology on X. This supra neutrosophic topology we denoted by τ_{12}^n.

Remark 3.1 The Example 1 show that:
1. τ_{12}^n is not neutrosophic topology in general.
2. The finite neutrosophic intersection of pairwise neutrosophic open sets need not be a pairwise neutrosophic open set.
3. The arbitrary neutrosophic union of pairwise neutrosophic closed sets need not be a pairwise neutrosophic closed set.

Theorem 3.2 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space. Then,

1. Every τ_i^n −open neutrosophic set is a pairwise neutrosophic open set $i = 1,2$, i.e., $\tau_1^n \cup \tau_2^n \subseteq \tau_{12}^n$.
2. Every τ_i^n −closed neutrosophic set is a pairwise neutrosophic closed set $i = 1,2$, i.e., $(\tau_1^n)^c \cup (\tau_2^n)^c \subseteq (\tau_{12}^n)^c$.
3. If $\tau_1^n \subseteq \tau_2^n$, then $\tau_{12}^n = \tau_2^n$ and $(\tau_{12}^n)^c = (\tau_2^n)^c$.

Proof. Straightforward.

Definition 3.4 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space and $N \in \aleph(X)$. The pairwise neutrosophic closure of N, denoted by $cl_p^n(N)$, is the neutrosophic intersection of all pairwise neutrosophic closed super sets of N, i.e.,

$$cl_p^n(N) = \cap \{C \in (\tau_{12}^n)^c : N \subseteq C\}.$$

It is clear that $cl_p^n(N)$ is the smallest pairwise neutrosophic closed set containing N.

Example 3.2 Let (X, τ_1^n, τ_2^n) be the same as in Example 1 and
$G = \{\langle a, 0.7,0.8,0.7\rangle, \langle b, 0.5,0.4,0.6\rangle, \langle c, 0.8,0.7,0.5\rangle\}$ be a neutrosophic set over X.
Now, we need to determine pairwise neutrosophic closed sets in (X, τ_1^n, τ_2^n) to find $cl_p^n(G)$. Then,

$$N_1^c = \{\langle a, 0.7,0.8,0.5\rangle, \langle b, 0.4,0.5,0.7\rangle, \langle c, 0.3,0.9,0.1\rangle\},$$
$$N_2^c = \{\langle a, 0.6,0.9,0.7\rangle, \langle b, 0.8,0.4,0.3\rangle, \langle c, 0.9,0.7,0.6\rangle\},$$
$$N_3^c = \{\langle a, 0.7,0.9,0.5\rangle, \langle b, 0.8,0.5,0.3\rangle, \langle c, 0.9,0.9,0.1\rangle\},$$
$$N_4^c = \{\langle a, 0.6,0.8,0.7\rangle, \langle b, 0.4,0.4,0.7\rangle, \langle c, 0.3,0.7,0.6\rangle\},$$

$$M_1^c = \{\langle a, 0.9,0.8,0.7\rangle, \langle b, 0.8,0.9,0.6\rangle, \langle c, 0.5,0.8,0.6\rangle\},$$
$$M_2^c = \{\langle a, 0.3,0.7,0.9\rangle, \langle b, 0.3,0.2,0.8\rangle, \langle c, 0.1,0.2,0.7\rangle\}.$$

and

$$(N_1 \cup M_1)^c = \{\langle a, 0.7,0.8,0.7\rangle, \langle b, 0.4,0.5,0.7\rangle, \langle c, 0.3,0.8,0.6\rangle\}$$
$$(N_2 \cup M_1)^c = \{\langle a, 0.6,0.8,0.7\rangle, \langle b, 0.8,0.4,0.6\rangle, \langle c, 0.5,0.7,0.6\rangle\}$$
$$(N_3 \cup M_1)^c = \{\langle a, 0.7,0.8,0.7\rangle, \langle b, 0.8,0.5,0.6\rangle, \langle c, 0.5,0.8,0.6\rangle\}$$

In here, the pairwise neutrosophic closed sets which contains G are N_3^c and 1_X it follows that $cl_p^n(G) = N_3^c \cap 1_X$. Therefore, $cl_p^n(G) = N_3^c$.

Theorem 3.3 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space and N, $M \in \aleph(X)$. Then,
1. $cl_p^n(0_X) = 0_X$ and $cl_p^n(1_X) = 1_X$.
2. $N \subseteq cl_p^n(N)$.
3. N is a pairwise neutrosophic closed set iff $cl_p^n(N) = N$.
4. $N \subseteq M \Rightarrow cl_p^n(N) \subseteq cl_p^n(M)$.
5. $cl_p^n(N) \cup cl_p^n(M) \subseteq cl_p^n(N \cup M)$.
6. $cl_p^n[cl_p^n(N)] = cl_p^n(N)$, i.e., $cl_p^n(N)$ is a pairwise neutrosophic closed set.

Proof. Straightforward.

Theorem 3.4 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space and $N \in \aleph(X)$. Then,

$$x_{\alpha,\beta,\gamma} \in cl_p^n(N) \Leftrightarrow U_{x_{\alpha,\beta,\gamma}} \cap N \neq 0_X, \forall U_{x_{\alpha,\beta,\gamma}} \in \tau_{12}^n(x_{\alpha,\beta,\gamma}),$$

where $U_{x_{\alpha,\beta,\gamma}}$ is any pairwise neutrosophic open set contains $x_{\alpha,\beta,\gamma}$ and $\tau_{12}^n(x_{\alpha,\beta,\gamma})$ is the family of all pairwise neutrosophic open sets contains $x_{\alpha,\beta,\gamma}$.

Proof. Let $x_{\alpha,\beta,\gamma} \in cl_p^n(N)$ and suppose that there exists $U_{x_{\alpha,\beta,\gamma}} \in \tau_{12}^n(x_{\alpha,\beta,\gamma})$ such that $U_{x_{\alpha,\beta,\gamma}} \cap N = 0_X$. Then $N \subseteq \left(U_{x_{\alpha,\beta,\gamma}}\right)^c$, thus $cl_p^n(N) \subseteq cl_p^n\left(U_{x_{\alpha,\beta,\gamma}}\right)^c = \left(U_{x_{\alpha,\beta,\gamma}}\right)^c$ which implies $cl_p^n(N) \cap U_{x_{\alpha,\beta,\gamma}} = 0_X$, a contradiction.
Conversely, assume that $x_{\alpha,\beta,\gamma} \notin cl_p^n(N)$, then $x_{\alpha,\beta,\gamma} \in [cl_p^n(N)]^c$. Thus, $[cl_p^n(N)]^c \in \tau_{12}^n(x_{\alpha,\beta,\gamma})$, so, by hypothesis, $[cl_p^n(N)]^c \cap N \neq 0_X$, a contradiction.

Theorem 3.5 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space. A neutrosophic set N over X is a pairwise neutrosophic closed set iff $N = cl_{\tau_1}^n(N) \cap cl_{\tau_2}^n(N)$.

Proof. Suppose that N is a pairwise neutrosophic closed set and $x_{\alpha,\beta,\gamma} \notin N$. Then, $x_{\alpha,\beta,\gamma} \notin cl_p^n(N)$. Thus, [by Theorem 4], there exists $U_{x_{\alpha,\beta,\gamma}} \in \tau_{12}^n(x_{\alpha,\beta,\gamma})$ such that $U_{x_{\alpha,\beta,\gamma}} \cap N = 0_X$. Since $U_{x_{\alpha,\beta,\gamma}} \in \tau_{12}^n(x_{\alpha,\beta,\gamma})$, then there exists $M_1 \in \tau_1^n$ and $M_2 \in \tau_2^n$ such that $U_{x_{\alpha,\beta,\gamma}} = M_1 \cup M_2$. Hence, $(M_1 \cup M_2) \cap N = 0_X$ it follows that $M_1 \cap N = 0_X$ and $M_2 \cap N = 0_X$. Since $x_{\alpha,\beta,\gamma} \in U_{x_{\alpha,\beta,\gamma}}$, then $x_{\alpha,\beta,\gamma} \in M_1$ or $x_{\alpha,\beta,\gamma} \in M_2$ implies, $x_{\alpha,\beta,\gamma} \notin cl_{\tau_1}^n(N)$ or $x_{\alpha,\beta,\gamma} \notin cl_{\tau_2}^n(N)$. Therefore, $x_{\alpha,\beta,\gamma} \notin cl_{\tau_1}^n(N) \cap cl_{\tau_2}^n(N)$. Thus, $cl_{\tau_1}^n(N) \cap cl_{\tau_2}^n(N) \subseteq N$. On the other hand, we have $N \subseteq cl_{\tau_1}^n(N) \cap cl_{\tau_2}^n(N)$. Hence, $N = cl_{\tau_1}^n(N) \cap cl_{\tau_2}^n(N)$.
Conversely, suppose that $N = cl_{\tau_1}^n(N) \cap cl_{\tau_2}^n(N)$. Since, $cl_{\tau_1}^n(N)$ is a neutrosophic closed set in (X, τ_1^n) and $cl_{\tau_2}^n(N)$ is a neutrosophic closed set in (X, τ_2^n), then, [by Definition 9], $cl_{\tau_1}^n(N) \cap cl_{\tau_2}^n(N)$ is a pairwise neutrosophic closed set in (X, τ_1^n, τ_2^n), so N is a pairwise neutrosophic closed set.

Corollary 3.2 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space. Then,

$$cl_p^n(N) = cl_{\tau_1}^n(N) \cap cl_{\tau_2}^n(N), \forall N \in \aleph(X).$$

Definition 3.5 An operator $\Psi: \aleph(X) \to \aleph(X)$ is called a neutrosophic supra closure operator if it satisfies the following conditions for all $N, M \in \aleph(X)$.

1. $\Psi(0_X) = 0_X$,
2. $N \subseteq \Psi(N)$,
3. $\Psi(N) \cup \Psi(M) \subseteq \Psi(N \cup M)$
4. $\Psi(\Psi(N)) = \Psi(N)$.

Theorem 3.6 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space. Then, the operator $cl_p^n: \aleph(X) \to \aleph(X)$ which defined by

$$cl_p^n(N) = cl_{\tau_1}^n(N) \cap cl_{\tau_2}^n(N)$$

is neutrosophic supra closure operator and it is induced, a unique neutrosophic supra topology given by $\{N \in \aleph(X): cl_p^n(N^c) = N^c\}$ which is precisely τ_{12}^n.

Proof. Straightforward.

Definition 3.6 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space and $N \in \aleph(X)$. The pairwise neutrosophic interior of N, denoted by $int_p^n(N)$, is the neutrosophic union of all pairwise neutrosophic open subsets of N, i.e.,

$$int_p^n(N) = \cup \{M \in \tau_{12}^n : M \subseteq N\}.$$

Obviously, $int_p^n(N)$ is the biggest pairwise neutrosophic open set contained in N.

Example 3.3 Let (X, τ_1^n, τ_2^n) be the same as in Example 1 and $M = \{\langle a, 0.3, 0.4, 0.2\rangle, \langle b, 0.5, 0.7, 0.1\rangle, \langle c, 0.8, 0.7, 0.3\rangle\}$ be a neutrosophic set over X. Then the pairwise neutrosophic open sets which containing in M are N_3, M_1, $N_3 \cup M_1$ and 0_X. Therefore,

$$\begin{aligned} int_p^n(M) &= N_3 \cup M_1 \cup (N_3 \cup M_1) \cup 0_X \\ &= N_3 \cup M_1 \\ &= \{\langle a, 0.3, 0.2, 0.3\rangle, \langle b, 0.2, 0.5, 0.4\rangle, \langle c, 0.5, 0.2, 0.4\rangle\}. \end{aligned}$$

Theorem 3.7 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space and $N, M \in \aleph(X)$. Then,

1. $int_p^n(0_X) = 0_X$ and $int_p^n(1_X) = 1_X$,
2. $int_p^n(N) \subseteq N$,
3. N is a pairwise neutrosophic open set iff $int_p^n(N) = N$,
4. $N \subseteq M \Rightarrow int_p^n(N) \subseteq int_p^n(M)$,
5. $int_p^n(N \cap M) \subseteq int_p^n(N) \cap int_p^n(M)$,
6. $int_p^n[int_p^n(N)] = int_p^n(N)$.

Proof. Starightforward.

Theorem 3.8 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space and $N \in \aleph(X)$. Then, $x_{\alpha,\beta,\gamma} \in int_p^n(N) \Leftrightarrow \exists U_{x_{\alpha,\beta,\gamma}} \in \tau_{12}^n(x_{\alpha,\beta,\gamma})$ such that $U_{x_{\alpha,\beta,\gamma}} \subseteq N$.

Proof. Starightforward.

Theorem 3.9 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space. A neutrosophic set N over X is a pairwise neutrosophic open set iff $N = int_{\tau_1}^n(N) \cup int_{\tau_2}^n(N)$.

Proof. Let N be a pairwise neutrosophic open set. Since, $int_{\tau_i}^n(N) \subseteq N$, $i = 1,2$, then $int_{\tau_1}^n(N) \cup int_{\tau_2}^n(N) \subseteq N$. Now, let $x_{\alpha,\beta,\gamma} \in N$. Then, there exists $U^1_{x_{\alpha,\beta,\gamma}} \in \tau_1^n$ such that $U^1_{x_{\alpha,\beta,\gamma}} \subseteq N$ or there exists $U^2_{x_{\alpha,\beta,\gamma}} \in \tau_2^n$ such that $U^2_{x_{\alpha,\beta,\gamma}} \subseteq N$, thus $x_{\alpha,\beta,\gamma} \in int_{\tau_1}^n(N)$ or $x_{\alpha,\beta,\gamma} \in int_{\tau_2}^n(N)$. Hence, $x_{\alpha,\beta,\gamma} \in int_{\tau_1}^n(N) \cup int_{\tau_2}^n(N)$. Therefore, $N = int_{\tau_1}^n(N) \cup int_{\tau_2}^n(N)$.
Coversely, since $int_{\tau_1}^n(N)$ is a neutrosophic open set in (X, τ_1^n) and $int_{\tau_2}^n(N)$ is a neutrosophic open set in (X, τ_2^n), then, [by Definition 8], $int_{\tau_1}^n(N) \cup int_{\tau_2}^n(N)$ is a pairwise neutrosophic open set in (X, τ_1^n, τ_2^n). Thus, N is a pairwise neutrosophic open set.

Corollary 3.3 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space. Then,

$$int_p^n(N) = int_{\tau_1}^n(N) \cup int_{\tau_2}^n(N).$$

Definition 3.7 An operator $I: \aleph(X) \to \aleph(X)$ is called a neutrosophic supra interior operator if it satisfies the following conditions for all N, $M \in \aleph(X)$.

1. $I(0_X) = 0_X$,
2. $I(N) \subseteq N$,
3. $I(N \cap M) \subseteq I(N) \cap I(M)$
4. $I(I(N)) = I(N)$.

Theorem 3.10 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space. Then, the operator $int_p^n: \aleph(X) \to \aleph(X)$ which defined by

$$int_p^n(N) = int_{\tau_1}^n(N) \cup int_{\tau_2}^n(N)$$

is neutrosophic supra interior operator and it is induced, a unique neutrosophic supra topology given by $\{N \in \aleph(X): int_p^n(N) = N\}$ which is precisely τ_{12}^n.

Proof. Straightforward.

Theorem 3.11 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space and $N \in \aleph(X)$. Then,

1. $int_p^n(N) = \left(cl_p^n(N^c)\right)^c$.
2. $cl_p^n(N) = \left(int_p^n(N^c)\right)^c$.

Proof. Starightforward.

Definition 3.8 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space, $N \in \aleph(X)$ and $x_{\alpha,\beta,\gamma} \in \aleph(X)$. Then N is said to be a pairwise neutrosophic neighborhood of $x_{\alpha,\beta,\gamma}$, if there exists a pairwise neutrosophic open set U such that $x_{\alpha,\beta,\gamma} \in U \subseteq N$. The family of pairwise neutrosophic neighborhood of neutrosophic point $x_{\alpha,\beta,\gamma}$ denoted by $N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})$.

Theorem 3.12 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space and $N \in \aleph(X)$. Then N is pairwise neutrosophic open set iff N is a pairwise neutrosophic neighborhood of its neutrosophic points.

Proof. Let N be a pairwise neutrosophic open set and $x_{\alpha,\beta,\gamma} \in N$. Then $x_{\alpha,\beta,\gamma} \in N \subseteq N$. Therefore N is a pairwise neutrosophic neighborhood of $x_{\alpha,\beta,\gamma}$ for each $x_{\alpha,\beta,\gamma} \in N$.
Conversely, suppose that N is a pairwise neutrosophic neighborhood of its neutrosophic points and $x_{\alpha,\beta,\gamma} \in N$. Then there exists a pairwise neutrosophic open set U such that $x_{\alpha,\beta,\gamma} \in U \subseteq N$. Since

$$N = \bigcup_{x_{\alpha,\beta,\gamma} \in N} \{x_{\alpha,\beta,\gamma}\} \subseteq \bigcup_{x_{\alpha,\beta,\gamma} \in N} U \bigcup_{x_{\alpha,\beta,\gamma} \in N} N = N$$

it follows that N is an union of pairwise neutrosophic open sets. Hence, N is a pairwise neutrosophic open set.

Proposition 3.2 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space and $\{N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma}): x_{\alpha,\beta,\gamma} \in \aleph(X)\}$ be a system of pairwise neutrosophic neighborhoods. Then,

1. For every $N \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})$, $x_{\alpha,\beta,\gamma} \in N$;
2. $N \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})$ and $N \subseteq M \Rightarrow M \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})$;
3. $N \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma}) \Rightarrow \exists M \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})$ such that $M \subseteq N$ and $M \in N_{\tau_{12}^n}\left(y_{\alpha',\beta',\gamma'}\right)$, for every $y_{\alpha',\beta',\gamma'} \in M$.

Proof. Proofs of 1 and 2 are straightforward.
3. Let N be a pairwise neutrosophic neighborhood of $x_{\alpha,\beta,\gamma}$, then there exists a pairwise neutrosophic open set $M \in \tau_{12}^n$ such that $x_{\alpha,\beta,\gamma} \in M \subseteq N$. Since $x_{\alpha,\beta,\gamma} \in M \subseteq M$ can be written, then $M \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})$. From the Theorem 12, if M is pairwise neutrosophic open set then N is a pairwise neutrosophic neighborhood of its neutrosophic points, i.e., $M \in N_{\tau_{12}^n}\left(y_{\alpha',\beta',\gamma'}\right)$, for every $y_{\alpha',\beta',\gamma'} \in M$.

Remark 3.2 Generally, $N, M \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma}) \Rightarrow N \cap M \notin N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})$. Actually, if $N, M \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})$, there exist $U_1, U_2 \in \tau_{12}^n$ such that $x_{\alpha,\beta,\gamma} \in U_1 \subseteq N$ and $x_{\alpha,\beta,\gamma} \in U_2 \subseteq M$. But $U_1 \cap U_2$ need not be a pairwise neutrosophic open set . Therefore, $N \cap M$ need not be a pairwise neutrosophic neighborhood of $x_{\alpha,\beta,\gamma}$.

Theorem 3.13 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space. Then

$$N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma}) = N_{\tau_1^n}(x_{\alpha,\beta,\gamma}) \cup N_{\tau_2^n}(x_{\alpha,\beta,\gamma})$$

for each $x_{\alpha,\beta,\gamma} \in \aleph(X)$.

Proof. Let $x_{\alpha,\beta,\gamma} \in \aleph(X)$ be any neutrosophic point and $N \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})$. Then there exists a pairwise neutrosophic open set $M \in \tau_{12}^n$ such that $x_{\alpha,\beta,\gamma} \in M \subseteq N$. If $M \in \tau_{12}^n$, there exist $M_1 \in \tau_1^n$ and $M_2 \in \tau_2^n$ such that $M = M_1 \cup M_2$. Since $x_{\alpha,\beta,\gamma} \in M = M_1 \cup M_2$, then $x_{\alpha,\beta,\gamma} \in M_1$ or $x_{\alpha,\beta,\gamma} \in M_2$. So, $x_{\alpha,\beta,\gamma} \in M_1 \subseteq M \subseteq N$ or $x_{\alpha,\beta,\gamma} \in M_2 \subseteq M \subseteq N$. In this case, $N \in N_{\tau_1^n}(x_{\alpha,\beta,\gamma})$ or $N \in N_{\tau_2^n}(x_{\alpha,\beta,\gamma})$, i.e., $N \in N_{\tau_1^n}(x_{\alpha,\beta,\gamma}) \cup N_{\tau_2^n}(x_{\alpha,\beta,\gamma})$.
Conversely, suppose that $N \in N_{\tau_1^n}(x_{\alpha,\beta,\gamma}) \cup N_{\tau_2^n}(x_{\alpha,\beta,\gamma})$. Then $N \in N_{\tau_1^n}(x_{\alpha,\beta,\gamma})$ or $N \in N_{\tau_2^n}(x_{\alpha,\beta,\gamma})$. Hence, there exists $x_{\alpha,\beta,\gamma} \in M_1 \in \tau_1^n$ or $x_{\alpha,\beta,\gamma} \in M_2 \in \tau_2^n$ such that $x_{\alpha,\beta,\gamma} \in M_1 \subseteq N$ and $x_{\alpha,\beta,\gamma} \in M_2 \subseteq N$. As a result, $x_{\alpha,\beta,\gamma} \in M_1 \cup M_2 = M \subseteq N$ such that $M \in \tau_{12}^n$ i.e., $N \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})$.

Definition 3.9 An operator $\nu: \aleph(X) \to \aleph(X)$ is called a neutrosophic supra neighborhood operator if it satisfies the following conditions for all $N,\ M \in \aleph(X)$.

1. $\forall N \in \nu(x_{\alpha,\beta,\gamma}),\ x_{\alpha,\beta,\gamma} \in N$;
2. $N \in \nu(x_{\alpha,\beta,\gamma})$ and $N \subseteq M \Rightarrow M \in \nu(x_{\alpha,\beta,\gamma})$;
3. $N \in \nu(x_{\alpha,\beta,\gamma}) \Rightarrow \exists M \in \nu(x_{\alpha,\beta,\gamma})$ such that $N \subseteq M$ and $M \in \nu\left(y_{\alpha',\beta',\gamma'}\right),\ y_{\alpha',\beta',\gamma'} \in M$.

Theorem 3.14 Let (X, τ_1^n, τ_2^n) be a neutrosophic bitopological space. Then, the operator $N_{\tau_{12}^n}: \aleph(X) \to \aleph(X)$ which defined by

$$N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma}) = N_{\tau_1^n}(x_{\alpha,\beta,\gamma}) \cup N_{\tau_2^n}(x_{\alpha,\beta,\gamma})$$

is neutrosophic supra neighboorhod operator and it is induced, a unique neutrosophic supra topology given by $\{N \in \aleph(X): \forall x_{\alpha,\beta,\gamma} \in N for N \in N_{\tau_{12}^n}(x_{\alpha,\beta,\gamma})\}$ which is precisely τ_{12}^n.

4. Conclusions

In this paper, neutrosophic bitopological spaces are presented. By defining open (closed) sets, interior, closure and neighbourhood systems, fundamentals theorems for neutrosophic bitopological spaces are proved and some examples on the subject are given. This paper is just a beginning of a new structure and we have studied a few ideas only, it will be necessary to carry out more theoretical research to establish a general framework for the practical application. In the future, using these notions, various classes of mappings on neutrosophic bitopological space, separation axioms on the neutrosophic bitopological spaces and many researchers can be studied

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Conflicts of Interest

The authors declare no conflict of interest.

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