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Neutrosophic Pseudo-t-Norm and Its Derived Neutrosophic Residual Implication

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Abstract: First of all, on the basis of complete lattice, the concept of neutrosophic pseudo-t-norm (NPT) is given. Definitions and examples of representable neutrosophic pseudo-t-norms (RNPTs) are given, while unrepresentable neutrosophic pseudo-t-norms (UNPTs) is also given. Secondly, De Morgan neutrosophic triples (DMNTs) consists of three operators: NPTs, neutrosophic negators (NNs) and neutrosophic pseudo-s-norms (NPSs), where NPTs and NPSs are dual about NNs. Again, we study the neutrosophic residual implications (NRIs) of NPTs, as well as their underlying properties. Finally, we give a method to get NPTs from neutrosophic implications (NIs) and construct non-commutative residuated lattices (NCRLs) based on NRIs and NPTs.

Keywords: Neutrosophic set; Neutrosophic pseudo-t-norm; Neutrosophic residual implication; Non-commutative residuated lattices

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1. Introduction

From the perspective of philosophy, Smarandache introduced neutrosophic sets (NSs). NSs is a expansion of fuzzy sets (FSs), and has universality [1]. Although NSs has expanded the expression of uncertain information, there are many inconveniences in practical application. From a scientific standpoint, so as to solve more practical problems, single valued neutrosophic sets (SVNSs) was put forward by Wang [2]. Some multi-attribute decision problems are solved by applying SVNSs. "SVNSs" is simply denoted as "NSs" in this article.

The t-norms, s-norms, negators, pseudo-t-norms, pseudo-s-norms and implications operators are fundamental tools in FS theory. Pseudo-t-norm and pseudo-s-norm was proposed in [3], followed by their residual implication were put forward by Wang in [4]. Pseudo-t-norm has many applications, such as resolution of finite fuzzy relation equations, linear optimization problems of mixed fuzzy relation inequalities and so on [5-10].

NSs has a lot of important neutrosophic logical operators, such as: NPTs, NPSs, NNs, NIs, NRIs and so on. In past few years, Smarandache [11] introduced n-conorm and n-norm in neutrosophic logic. Zhang et al. [12] introduced a new type of relation of inclusion for NSs. A new kind of residuated lattice obtained through neutrosophic t-norms and its derived NRIs was introduced by Hu and Zhang [13]. On the basis of neutrosophic t-norms, fuzzy reasoning triple I method was studied by Luo et al. [14]. Therefore, it is very meaningful to study the NRIs of NPTs.

The basic framework of this paper: Section 2 presents the basics knowledge that will be useful for writing this paper. We defined NPTs, NPSs, NNs and so on in Section 3. Moreover, we also provide some useful typical examples and theorems. In Section 4, the definitions of NRIs generated from NPTs are obtained, and their basic properties are discussed in depth. In addition, this paper provides a new method to generate NPTs from NIs, and at the same time prove that system (D^*; $\wedge_1$, $\vee_1$, $\otimes$, $\rightarrow$, $\rightsquigarrow$, 0_{D^*}, 1_{D^*}) is a NCRL. Section 5 concludes the whole content of this paper.

2. Preliminaries

Definition 2.1 ([3]) A mapping PT: $[0, 1]^2 \rightarrow [0, 1]$ be a pseudo-t-norm iff, $\forall m, n, r \in [0, 1]$:
(PT1) $PT(m, PT(n, r)) = PT(n, PT(m, r))$;
(PT2) if $m \le n$, then $PT(m, r) \le PT(n, r)$, $PT(r, m) \le PT(r, n)$;
(PT3) $PT(1, m) = m$, $PT(m, 1) = m$.

Definition 2.2 ([3]) A mapping PS: $[0, 1]\times[0, 1] \rightarrow [0, 1]$ be a pseudo-s-norm iff, $\forall m, n, r \in [0, 1]$:
(PS1) $PS(m, PS(n, r)) = PS(n, PS(m, r))$;
(PS2) if $m \le n$, then $PS(m, r) \le PS(n, r)$, $PS(r, m) \le PS(r, n)$;
(PS3) $PS(0, m) = m$, $PS(m, 0) = m$.

Definition 2.3 ([15]) An intuitionistic fuzzy set (IFS) W in nonempty set M is depicted through two mappings: $\mu_W(m)$ and $\nu_W(m)$: $M \rightarrow [0, 1]$. W is expressed as, when $\forall m \in M$,

$$W = \{(m, \mu_W(m), \nu_W(m)) \mid m \in M\},$$

satisfy $0 \le \mu_W(m) + \nu_W(m) \le 1$, where $\mu_W(m)$ is affiliation function, $\nu_W(m)$ is non-affiliation function.

Definition 2.4 ([2]) Let the set M be nonempty. A SVNS W in M is depicted through $T_W(m)$, $I_W(m)$, and $F_W(m)$, all of which are functions defined on [0, 1]. Then, W is expressed as, when $\forall m \in M$,

$$W = \{\langle m, T_W(m), I_W(m), F_W(m)\rangle \mid m \in M\},$$

satisfy $0 \le T_W(m)+I_W(m)+F_W(m) \le 3$, where $T_W(m)$ is the function of truth-affiliation, $I_W(m)$ is the function of indeterminacy- affiliation, and $F_W(m)$ is the function of falsity- affiliation.

Proposition 2.5 ([13]) The first type of inclusion relationship is discussed in this article.

Definition 2.6 ([1,17,18]) Let the set M be nonempty. Give two NSs W, N in M, where $W=\{\langle m, T_W(m), I_W(m), F_W(m)\rangle \mid m \in M\}$, $N = \{\langle m, T_N(m), I_N(m), F_N(m)\rangle \mid m \in M\}$. The algebraic operations of the first type of inclusion relation was given as shown below, $\forall m \in M$,
(1) $W \subseteq_1 N \Leftrightarrow T_W(m) \le T_N(m)$, $I_W(m) \ge I_N(m)$, $F_W(m) \ge F_N(m)$;
(2) $W \cup_1 N = \{\langle m, \max(T_W(m), T_N(m)), \min(I_W(m), I_N(m)), \min(F_W(m), F_N(m))\rangle \mid m \in M\}$;
(3) $W \cap_1 N = \{\langle m, \min(T_W(m), T_N(m)), \max(I_W(m), I_N(m)), \max(F_W(m), F_N(m))\rangle \mid m \in M\}$;
(4) $W^c = \{\langle m, I_W(m), 1- F_W(m), T_W(m)\rangle \mid m \in M\}$.

Proposition 2.7 ([13]) We consider that set D^* defined by,

$$D^* = \{m = (m_1, m_2, m_3) \mid m_1, m_2, m_3 \in [0,1]\}.$$

$\forall m, n \in D^*$, the order relation we define $\le_1$ on D^* is shown below:

$$m \le_1 n \Leftrightarrow m_1 \le n_1, m_2 \ge n_2, m_3 \ge n_3.$$

Proposition 2.8 ([13]) (D^*; $\le_1$) is a partially ordered set.

Proposition 2.9 ([13]) $\forall m, n \in D^*$, $m \wedge_1 n$ is called maximum lower bound of m, n, and expressed as inf(m, n); $m \vee_1 n$ is called minimum upper bound of m, n, and expressed as sup(m, n). In other word, (D^*; $\leq_1$) is a lattice.

The content of definition of operators $\wedge_1$ and $\vee_1$ refers to proposition 2 in [13].

Proposition 2.10 ([13]) (D^*; $\leq_1$) is a complete lattice.

The maximun of D^* is indicated as $1_{D^*} = (1,0,0)$, the minimun of D^* is indicated as $0_{D^*} = (0,1,1)$.

Definition 2.11 ([16]) A pseudo-t-norm PT: $L \times L \rightarrow L$ on (L; $\leq_L$) be undecreasing and associative mapping that meets $PT(1_L, m) = m = PT(m, 1_L)$, which $\forall m \in L$. A pseudo-s-norm PS: $L^2 \rightarrow L$ on (L; $\leq_L$) be associative and undecreasing mapping that meets $PS(0_L, m) = m = PS(m, 0_L)$, $\forall m \in L$.

Definition 2.12 ([13]) For every $m \in D^*$, we define a complement of m as follows:

$$m^c = (m_3, 1-m_2, m_1).$$

Proposition 2.13 ([13]) The system (D^*; $\wedge_1$, $\vee_1$, ^{c}, 0_{D^*}, 1_{D^*}) is a De Morgan algebra.

3. NPTs On (D^*; $\leq_1$)

Definition 3.1 A binary function $\boldsymbol{PT}$: $D^* \times D^* \rightarrow D^*$ is called NPT, $\forall m, n, u, v, r \in D^*$, if $\boldsymbol{PT}$ satisfies:
(NPT1) $\boldsymbol{PT}(m, \boldsymbol{PT}(n, r)) = \boldsymbol{PT}(n, \boldsymbol{PT}(m, r))$;
(NPT2) $\boldsymbol{PT}(m, n) \leq_1 \boldsymbol{PT}(u, v)$ and $\boldsymbol{PT}(n, m) \leq_1 \boldsymbol{PT}(v, u)$, where $m \leq_1 u$, $n \leq_1 v$;
(NPT3) $\boldsymbol{PT}(1_{D^*}, m) = m$, $\boldsymbol{PT}(m, 1_{D^*}) = m$.

Definition 3.2 A binary function $\boldsymbol{PS}$: $D^* \times D^* \rightarrow D^*$ is called NPS, $\forall m, n, u, v, r \in D^*$, if $\boldsymbol{PS}$ satisfies:
(NPS1) $\boldsymbol{PS}(m, \boldsymbol{PS}(n, r)) = \boldsymbol{PS}(n, \boldsymbol{PS}(m, r))$;
(NPS2) $\boldsymbol{PS}(m, n) \leq_1 \boldsymbol{PS}(u, v)$ and $\boldsymbol{PS}(n, m) \leq_1 \boldsymbol{PS}(v, u)$, where $m \leq_1 u$, $n \leq_1 v$;
(NPS3) $\boldsymbol{PS}(0_{D^*}, m) = m$, $\boldsymbol{PS}(m, 0_{D^*}) = m$.

Example 3.3 ([3,19]) Table 1 below gives part pseudo-t-norms, and its derived residual implications.

Table 1. Eaxmple of part pseudo-t-norms

Pseudo-t-norms	**Residual implications**
$PT_1(m,n) = \begin{cases} 0 & \text{if } m \in [0,a_1], n \in [0,b_1], \\ \min(m,n) & \text{otherwise,} \end{cases}$ where $0 < a_1 < b_1 < 1$.	$I_{1L}(m,n) = \begin{cases} \max(a_1,n) & \text{if } m \leq b_1, m > n, \\ n & \text{if } m > b_1, m > n, \\ 1 & \text{if } m \leq n. \end{cases}$ $I_{1R}(m,n) = \begin{cases} b_1 & \text{if } m \leq a_1, m > n, \\ n & \text{if } m > a_1, m > n, \\ 1 & \text{if } m \leq n. \end{cases}$
$PT_2(m,n) = \begin{cases} \min(m,n) & \text{if } \sin(\frac{\pi}{2}m) + n > 1, \\ 0 & \text{if } \sin(\frac{\pi}{2}m) + n \leq 1. \end{cases}$	$I_{2L}(m,n) = \begin{cases} 1 & \text{if } m \leq n, \\ \max\{n, \frac{2}{\pi}\arcsin(1-m)\} & \text{if } m > n. \end{cases}$ $I_{2R}(m,n) = \begin{cases} 1 & \text{if } m \leq n, \\ \max\{n, 1-\sin(\frac{\pi}{2}m)\} & \text{if } m > n. \end{cases}$

$PT_3(m,n)=\begin{cases}\min(m,n) & \text{if } m^2+n^3>1,\\ 0 & \text{if } m^2+n^3\le 1.\end{cases}$	$I_{3L}(m,n)=\begin{cases}1 & \text{if } m\le n,\\ \max\{n,\sqrt{1-m^3}\} & \text{if } m>n.\end{cases}$ $I_{3R}(m,n)=\begin{cases}1 & \text{if } m\le n,\\ \max\{n,\sqrt[3]{1-m^2}\} & \text{if } m>n.\end{cases}$
$PT_4(m,n)=\begin{cases}\min(m,n) & \text{if } m+\sqrt{n}>1,\\ 0 & \text{if } m+\sqrt{n}\le 1.\end{cases}$	$I_{4L}(m,n)=\begin{cases}1 & \text{if } m\le n,\\ \max\{n,1-\sqrt{m}\} & \text{if } m>n.\end{cases}$ $I_{4R}(m,n)=\begin{cases}1 & \text{if } m\le n,\\ \max\{n,(1-m)^2\} & \text{if } m>n.\end{cases}$

Example 3.4 Table 2 below gives part pseudo-s-norms, and its derived residual co-implications.

Table 2. Example of part pseudo-s-norms

Pseudo-s-norms	**Residual co-implications**
$PS_1(m,n)=\begin{cases}1 & \text{if } m\ge a_1, n\ge b_1,\\ \max(m,n) & \text{otherwise},\end{cases}$ where $0<a_1<b_1<1$.	$J_{1L}(m,n)=\begin{cases}a_1 & \text{if } m>b_1, m<n,\\ n & \text{if } m\le b_1, m<n,\\ 0 & \text{if } m\ge n.\end{cases}$ $J_{1R}(m,n)=\begin{cases}\min(a_1,n) & \text{if } m>a_1, m<n,\\ n & \text{if } m\le a_1, m<n,\\ 0 & \text{if } m\ge n.\end{cases}$
$PS_2(m,n)=\begin{cases}\max(m,n) & \text{if } \sin(\frac{\pi}{2}m)+n<1,\\ 1 & \text{if } \sin(\frac{\pi}{2}m)+n\ge 1.\end{cases}$	$J_{2L}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,\frac{2}{\pi}\arcsin(1-m)\} & \text{if } m<n.\end{cases}$ $J_{2R}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,1-\sin(\frac{\pi}{2}m)\} & \text{if } m<n.\end{cases}$
$PS_3(m,n)=\begin{cases}\max(m,n) & \text{if } m^2+n^3<1,\\ 1 & \text{if } m^2+n^3\ge 1.\end{cases}$	$J_{3L}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,\sqrt{1-m^3}\} & \text{if } m<n.\end{cases}$ $J_{3R}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,\sqrt[3]{1-m^2}\} & \text{if } m<n.\end{cases}$
$PS_4(m,n)=\begin{cases}\max(m,n) & \text{if } m+\sqrt{n}<1,\\ 1 & \text{if } m+\sqrt{n}\ge 1.\end{cases}$	$J_{4L}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,1-\sqrt{m}\} & \text{if } m<n.\end{cases}$ $J_{4R}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,(1-m)^2\} & \text{if } m<n.\end{cases}$

Example 3.5 Suppose that PT_i (i=1,2,3,4) are pseudo-t-norms as shown in **Example 3.3** and PS_i (i=1,2,3,4) are pseudo-s-norms as shown in **Example 3.4**. Then, the binary function $\boldsymbol{PT}_i$ (i=1,2,3,4,5,6) defined on D^* are NPTs as follows:

(1) $\boldsymbol{PT}_1(m,n)=(PT_1(m_1,n_1),PS_1(m_2,n_2),PS_1(m_3,n_3))$;

(2) $\boldsymbol{PT}_2(m,n) = (PT_2(m_1,n_1), PS_2(m_2,n_2), PS_2(m_3,n_3))$;

(3) $\boldsymbol{PT}_3(m,n) = (PT_3(m_1,n_1), PS_3(m_2,n_2), PS_3(m_3,n_3))$;

(4) $\boldsymbol{PT}_4(m,n) = (PT_4(m_1,n_1), PS_4(m_2,n_2), PS_4(m_3,n_3))$;

(5) $\boldsymbol{PT}_5(m,n) = (PT_1(m_1,n_1), PS_2(m_2,n_2), PS_3(m_3,n_3))$;

(6) $\boldsymbol{PT}_6(m,n) = (PT_1(m_1,n_1), PS_3(m_2,n_2), PS_3(m_3,n_3))$.

Example 3.6 Suppose that PT_i (i=1,2,3,4) are pseudo-t-norms as shown in **Example 3.3** and PS_i (i=1,2,3,4) are pseudo-s-norms as shown in **Example 3.4**. Then, the binary function $\boldsymbol{PS}_i$ (i=1,2,3,4,5,6) defined on D^* are NPSs as follows:

(1) $\boldsymbol{PS}_1(m,n) = (PS_1(m_1,n_1), PT_1(m_2,n_2), PT_1(m_3,n_3))$;

(2) $\boldsymbol{PS}_2(m,n) = (PS_2(m_1,n_1), PT_2(m_2,n_2), PT_2(m_3,n_3))$;

(3) $\boldsymbol{PS}_3(m,n) = (PS_3(m_1,n_1), PT_3(m_2,n_2), PT_3(m_3,n_3))$;

(4) $\boldsymbol{PS}_4(m,n) = (PS_4(m_1,n_1), PT_4(m_2,n_2), PT_4(m_3,n_3))$;

(5) $\boldsymbol{PS}_5(m,n) = (PS_1(m_1,n_1), PT_2(m_2,n_2), PT_3(m_3,n_3))$;

(6) $\boldsymbol{PS}_6(m,n) = (PS_1(m_1,n_1), PT_3(m_2,n_2), PT_3(m_3,n_3))$.

Theorem 3.7 Give a binary function $\boldsymbol{PT}$: $D^*\times D^*\rightarrow D^*$, two pseudo-s-norms PS_i (i=1,2) and a pseudo-t-norm PT. Then, $\forall m, n\in D^*$,

$$\boldsymbol{PT}(m,n) = (PT(m_1,n_1), PS_1(m_2,n_2), PS_2(m_3,n_3))$$

is a NPT.

Proof. $\forall m, u, n, v, r\in D^*$, have following:

(NPT1) According to item (PT1) of **Definition 2.1** and item (PS1) of **Definition 2.2**, it is obvious that $\boldsymbol{PT}(m, \boldsymbol{PT}(n, r)) = \boldsymbol{PT}(m, (PT(n_1, r_1), PS_1(n_2, r_2), PS_2(n_3, r_3))) = (PT(m_1, PT(n_1, r_1)), PS_1(m_2, PS_1(n_2, r_2)), PS_2(m_3, PS_2(n_3, r_3))) = (PT(n_1, PT(m_1, r_1)), PS_1(n_2, PS_1(m_2, r_2)), PS_2(n_3, PS_2(m_3, r_3))) = \boldsymbol{PT}(n, \boldsymbol{PT}(m, r))$;

(NPT2) If $m \leq_1 u$, $n \leq_1 v$, then $PT(m_1, n_1) \leq PT(u_1, v_1)$, $PS_1(m_2, n_2) \geq PS_1(u_2, v_2)$, $PS_2(m_3, n_3) \geq PS_2(u_3, v_3)$. Therefore, $\boldsymbol{PT}(m, n) \leq_1 \boldsymbol{PT}(u, v)$. Likewise, we can also get $\boldsymbol{PT}(n, m) \leq_1 \boldsymbol{PT}(v, u)$.

(NPT3) $\boldsymbol{PT}(m, 1_{D^*}) = (PT(m_1, 1), PS_1(m_2, 0), PS_2(m_3, 0)) = (m_1, m_2, m_3) = m$. Similarly, $\boldsymbol{PT}(1_{D^*}, m) = m$.

Thus, $\boldsymbol{PT}(m, n)$ is a NPT.

Theorem 3.8 Give a binary function $\boldsymbol{PS}$: $D^*\times D^*\rightarrow D^*$, two pseudo-t-norms PT_i (i=1,2) and a pseudo-s-norm PS. Then,

$$\boldsymbol{PS}(m,n) = (PS(m_1,n_1), PT_1(m_2,n_2), PT_2(m_3,n_3))$$

is a NPS, for arbitrary $m, n\in D^*$.

Theorem 3.7 provieds a idea for constructing NPT on D^* with pseudo-s-norm and pseudo-t-norm. However, the reverse is not able to find two pseudo-s-norms PS_i (i=1,2) and a pseudo-t-norm PT to make $\boldsymbol{PT} = (PT, PS_1, PS_2)$.

In order to make a clear distinction between the two types of NPTs, so put forward a concept of RNPT.

Definition 3.9 If $\forall m, n\in D^*$, there exists two pseudo-s-norms PS_i (i=1,2) and a pseudo-t-norm PT such that $\boldsymbol{PT}$ holds with respect to the following equation:

$$\boldsymbol{PT}(m,n) = (PT(m_1,n_1), PS_1(m_2,n_2), PS_2(m_3,n_3)).$$

Then $\boldsymbol{PT}$ is said to be representable.

Definition 3.10 If $\forall m, n \in D^*$, there exists pseudo-s-norm PS and pseudo-t-norm PT such that ***PT*** holds with respect to the following equation:

$$\boldsymbol{PT}(m,n) = (PT(m_1,n_1), PS(m_2,n_2), PS(m_3,n_3)).$$

Then ***PT*** is said to be standard representable.

These NPTs given in **Example 3.5** are representable.

Definition 3.11 If $\forall m, n \in D^*$, there exists two pseudo-t-norms PT_i (i=1,2) and a pseudo-s-norm PS such that ***PS*** holds with respect to the following equation:

$$\boldsymbol{PS}(m,n) = (PS(m_1,n_1), PT_1(m_2,n_2), PT_2(m_3,n_3)).$$

Then ***PS*** is said to be representable.

Definition 3.12 If $\forall m, n \in D^*$, there exists pseudo-s-norm PS and pseudo-t-norm PT such that ***PS*** holds with respect to the following equation:

$$\boldsymbol{PS}(m,n) = (PS(m_1,n_1), PT(m_2,n_2), PT(m_3,n_3)).$$

Then ***PS*** is said to be standard representable.

These NPSs given in **Example 3.6** are representable.

Propositions 3.13 and **3.14** below demonstrate a approach to construct new RNPTs (RNPSs) with intuitionistic fuzzy t-norms (IFTs) and intuitionistic fuzzy s-norms (IFSs).

Proposition 3.13 $\forall x = (x_1, x_3) \in L$, $y = (y_1, y_3) \in L$, $T(x, y) = (t(x_1, y_1), s_2(x_3, y_3))$ is a representable IFT, which t and s_2 are t-norm and s-norm, respectively. If $\forall m, n \in D^*$, there is a pseudo-s-norm ps_1 that makes $0 \le t(m_1, n_1)+ps_1(m_2, n_2)+s_2(m_3, n_3) \le 3$ true, then $\boldsymbol{PT}(m, n) = (t(m_1, n_1), ps_1(m_2, n_2), s_2(m_3, n_3))$ is a RNPT.

Proposition 3.14 $\forall x = (x_1, x_3) \in L$, $y = (y_1, y_3) \in L$, $S(x, y) = (s(x_1, y_1), t_2(x_3, y_3))$ is a representable IFS, where s and t_2 are s-norm and t-norm, respectively. If $\forall m, n \in D^*$, there is a pseudo-t-norm pt_1 that makes $0 \le s(m_1, n_1)+pt_1(m_2, n_2)+t_2(m_3, n_3) \le 3$ true, then $\boldsymbol{PS}(m, n) = (s(m_1, n_1), pt_1(m_2, n_2), t_2(m_3, n_3))$ is a RNPS.

Example 3.15 ([20]) Table 3 below gives part t-norms, and its derived residual implications.

Table 3. Example of the part t-norms

t-norms	Residual implications
$T_M(m,n) = \min(m,n)$	$I_{GD}(m,n) = \begin{cases} 1 & \text{if } m \le n, \\ n & \text{if } m > n. \end{cases}$
$T_P(m,n) = m \cdot n$	$I_{GG}(m,n) = \begin{cases} 1 & \text{if } m \le n, \\ \dfrac{n}{m} & \text{if } m > n. \end{cases}$
$T_{LK}(m,n) = \max(m+n-1,0)$	$I_{LK}(m,n) = \min(1,1-m+n)$

Example 3.16 ([20]) Table 4 below gives part s-norms, and its derived residual co-implications.

Table 4. Example of the part s-norms

s-norms	Residual co-implications

$S_M(m,n)=\max(m,n)$	$J_{GD}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ n & \text{if } m<n.\end{cases}$
$S_P(m,n)=m+n-m\cdot n$	$J_{GG}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \dfrac{n-m}{1-m} & \text{if } m<n.\end{cases}$
$S_{LK}(m,n)=\min(m+n,1)$	$J_{LK}(m,n)=\max(0,n-m)$

Example 3.17 Let PS_i (i=1,2,4) are pseudo-s-norms as shown in **Example 3.4**, T_M, T_P, T_{LK} are t-norms as shown in **Example 3.15**, and S_M, S_P, S_{LK} are s-norms as shown in **Example 3.16**. Then, the binary function $\boldsymbol{PT}_i$ (i=7,8,9) constructed by IFTs defined on D^* are RNPTs as follows:

(1) $\boldsymbol{PT}_7(m,n)=(T_M(m_1,n_1),PS_4(m_2,n_2),S_{LK}(m_3,n_3))$;

(2) $\boldsymbol{PT}_8(m,n)=(T_P(m_1,n_1),PS_2(m_2,n_2),S_M(m_3,n_3))$;

(3) $\boldsymbol{PT}_9(m,n)=(T_{LK}(m_1,n_1),PS_1(m_2,n_2),S_P(m_3,n_3))$.

Example 3.18 Let PT_i (i=1,2,4) are pseudo-t-norms as shown in **Example 3.3**, T_M, T_P, T_{LK} are t-norms as shown in **Example 3.15**, and S_M, S_P, S_{LK} are s-norms as shown in **Example 3.16**. Then, the binary function $\boldsymbol{PS}_i$ (i=7,8,9) constructed by IFSs defined on D^* are RNPSs as follows:

(1) $\boldsymbol{PS}_7(m,n)=(S_M(m_1,n_1),PT_4(m_2,n_2),T_{LK}(m_3,n_3))$;

(2) $\boldsymbol{PS}_8(m,n)=(S_P(m_1,n_1),PT_2(m_2,n_2),T_M(m_3,n_3))$;

(3) $\boldsymbol{PS}_9(m,n)=(S_{LK}(m_1,n_1),PT_1(m_2,n_2),T_P(m_3,n_3))$.

Definition 3.19 ([13]) A mapping $\boldsymbol{N}$: $D^*\to D^*$ be known as NN if satisfies, $\forall m, n\in D^*$:
(NN1) $m \le_1 n$ iff $\boldsymbol{N}(m)\ge_1 \boldsymbol{N}(n)$;
(NN2) $\boldsymbol{N}(1_{D^*}) = 0_{D^*}$;
(NN3) $\boldsymbol{N}(0_{D^*}) = 1_{D^*}$.

If $\boldsymbol{N}(\boldsymbol{N}(m)) = m$ holds with $\forall m\in D^*$, then $\boldsymbol{N}$ is said to be involutive NN.

The function $\boldsymbol{N}_S$: $D^*\to D^*$ defined by, $\forall(m_1, m_2, m_3)\in D^*$,

$$\boldsymbol{N}_S(m_1, m_2, m_3) = (m_3, 1\text{-}m_2, m_1)$$

is a involutive NN, which is also called standard NN. Meanwhile, $\boldsymbol{N}(m) = (m_2, 1\text{-}m_3, m_1)$, $\boldsymbol{N}(m) = (m_2, m_1, m_1)$, $\boldsymbol{N}(m) = (m_2, 1\text{-}m_2, m_1)$ are NNs.

Definition 3.20 Assume that $\boldsymbol{PT}$ is a NPT, $\boldsymbol{N}$ is a NN and $\boldsymbol{PS}$ is a NPS. $\forall m, n\in D^*$, if the triple ($\boldsymbol{PT}$, $\boldsymbol{N}$, $\boldsymbol{PS}$) satisfied the following conditions:

$$\boldsymbol{N}(\boldsymbol{PS}(m, n)) = \boldsymbol{PT}(\boldsymbol{N}(m), \boldsymbol{N}(n)).$$
$$\boldsymbol{N}(\boldsymbol{PT}(m, n)) = \boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(n));$$

Then, we call the triple ($\boldsymbol{PT}$, $\boldsymbol{N}$, $\boldsymbol{PS}$) is a DMNT.

Theorem 3.21 Suppose $\boldsymbol{N}$ is involutive. If exists a NPS $\boldsymbol{PS}$, then such that $\boldsymbol{PT}$ be defined as

$$\boldsymbol{PT}(m, n) = \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(n)))$$

is NPT. Besides, ($\boldsymbol{PT}$, $\boldsymbol{N}$, $\boldsymbol{PS}$) is a DMNT.

Proof. According to known condition, there are as follows, $\forall m, u, n, v, r\in D^*$:
(NPT1)According to item (NPS1) of **Definition 3.2**, naturally there is $\boldsymbol{PT}(m, \boldsymbol{PT}(n, r)) = \boldsymbol{PT}(m, \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{N}(r)))) = \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(\boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{N}(r)))))) = \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{N}(r)))) = \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(r)))) = \boldsymbol{PT}(n, \boldsymbol{PT}(m, r))$.

(NPT2) If $m \leq_1 u$, $n \leq_1 v$, so $\boldsymbol{N}(m) \geq_1 \boldsymbol{N}(u)$, $\boldsymbol{N}(n) \geq_1 \boldsymbol{N}(v)$. From (NPS2) of **Definition 3.2** and (NN1) of **Definition 3.19**, we get $\boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(n)) \geq_1 \boldsymbol{PS}(\boldsymbol{N}(u), \boldsymbol{N}(v))$ and $\boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{N}(m)) \geq_1 \boldsymbol{PS}(\boldsymbol{N}(v), \boldsymbol{N}(u))$. Thus, $\boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(n))) \leq_1 \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(u), \boldsymbol{N}(v)))$ and $\boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{N}(m))) \leq_1 \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(v), \boldsymbol{N}(u)))$, that is, $\boldsymbol{PT}(m, n) \leq_1 \boldsymbol{PT}(u, v)$ and $\boldsymbol{PT}(n, m) \leq_1 \boldsymbol{PT}(v, u)$.

(NPT3) $\boldsymbol{PT}(1_{D^*}, m) = \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(1_{D^*}), \boldsymbol{N}(m))) = \boldsymbol{N}(\boldsymbol{PS}(0_{D^*}, \boldsymbol{N}(m))) = \boldsymbol{N}(\boldsymbol{N}(m)) = m$. Similarly, $\boldsymbol{PT}(m, 1_{D^*}) = m$.

Therefore, the statement that $\boldsymbol{PT}$ is NPT is proved.

Besides, $(\boldsymbol{PT}, \boldsymbol{N}, \boldsymbol{PS})$ is a DMNT.

Theorem 3.22 Assume $\boldsymbol{N}$ is involutive. If exists a NPT $\boldsymbol{PT}$, then such that $\boldsymbol{PS}$ be defined as

$$\boldsymbol{PS}(m, n) = \boldsymbol{N}(\boldsymbol{PT}(\boldsymbol{N}(m), \boldsymbol{N}(n)))$$

being NPS. Moreover, $(\boldsymbol{PT}, \boldsymbol{N}, \boldsymbol{PS})$ is a DMNT.

Example 3.23 A few NPTs and NPSs are dual about $\boldsymbol{N}_S$.

(1) $\boldsymbol{PT}_1(m, n) = (PT_1(m_1, n_1), PS_1(m_2, n_2), PS_1(m_3, n_3))$, $\boldsymbol{PS}_1(m, n) = (PS_1(m_1, n_1), PT_1(m_2, n_2), PT_1(m_3, n_3))$.

Indeed, $\boldsymbol{PT}_1(\boldsymbol{N}_S(m), \boldsymbol{N}_S(n)) = \boldsymbol{PT}_1((m_3, 1\text{-}m_2, m_1), (n_3, 1\text{-}n_2, n_1)) = (PT_1(m_3, n_3), PS_1(1\text{-}m_2, 1\text{-}n_2), PS_1(m_1, n_1))$, then $\boldsymbol{N}_S(\boldsymbol{PT}_1(\boldsymbol{N}_S(m), \boldsymbol{N}_S(n))) = (PS_1(m_1, n_1), 1\text{-}PS_1(1\text{-}m_2, 1\text{-}n_2), PT_1(m_3, n_3)) = (PS_1(m_1, n_1), PT_1(m_2, n_2), PT_1(m_3, n_3)) = \boldsymbol{PS}_1(m, n)$.

(2) $\boldsymbol{PT}_3(m, n) = (PT_3(m_1, n_1), PS_3(m_2, n_2), PS_3(m_3, n_3))$, $\boldsymbol{PS}_3(m, n) = (PS_3(m_1, n_1), PT_3(m_2, n_2), PT_3(m_3, n_3))$.

The theorem about UNPT is given next:

Theorem 3.24 Let $\boldsymbol{PT}$: $D^*\times D^*\rightarrow D^*$ being a function. $\forall m, n\in D^*$,

$$\boldsymbol{PT}(m,n)=\begin{cases} m & n=1_{D^*}, \\ n & m=1_{D^*}, \\ (\min(2m_1,n_1),\max(1-2m_1,1-n_1),\max(m_3,n_3)) & \text{otherwise.} \end{cases}$$

is a UNPT.

Proof. First, we show that $\boldsymbol{PT}$ is a NPT, $\forall m, u, n, v, r\in D^*$.

(NPT1) If $m = 1_{D^*}$ or $n = 1_{D^*}$, then $\boldsymbol{PT}$ satisfies the associative law. If $m \neq 1_{D^*}$, $n \neq 1_{D^*}$, $\boldsymbol{PT}(m, \boldsymbol{PT}(n, r)) = (\min(2m_1, \min(2n_1, r_1)), \max(1\text{-}2m_1, 1\text{-}\min(2n_1, r_1)), \max(m_3, \max(n_3, r_3))) = (\min(2m_1, 2n_1, r_1), \max(1\text{-}2m_1, 1\text{-}2n_1, 1\text{-}r_1,), \max(m_3, n_3, r_3)) = (\min(2n_1, \min(2m_1, r_1)), \max(1\text{-}2n_1, 1\text{-}\min(2m_1, r_1)), \max(n_3, \max(m_3, r_3))) = \boldsymbol{PT}(n, \boldsymbol{PT}(m, r))$.

(NPT2) If $m = 1_{D^*}$ or $n = 1_{D^*}$, we can prove $\boldsymbol{PT}$ is undecreasing in each variable. If $m \neq 1_{D^*}$, $n \neq 1_{D^*}$, at the same time satisfy $m \leq_1 u$, $n \leq_1 v$, and $m_1 \leq u_1$, $n_1 \leq v_1$, $m_3 \geq u_3$, $n_3 \geq v_3$. Thus, $\min(2m_1, n_1) \leq \min(2u_1, v_1)$, $\max(1\text{-}2m_1, 1\text{-}n_1) \geq \max(1\text{-}2u_1, 1\text{-}v_1)$, $\max(m_3, n_3) \geq \max(u_3, v_3)$. That is, $\boldsymbol{PT}(m, n) \leq_1 \boldsymbol{PT}(u, v)$. Likewise, we can also have $\boldsymbol{PT}(n, m) \leq_1 \boldsymbol{PT}(v, u)$.

(NPT3) $\boldsymbol{PT}(m, 1_{D^*}) = m$, $\boldsymbol{PT}(1_{D^*}, m) = m$. Therefore, $\boldsymbol{PT}$ is a NPT.

Second, assume NPT $\boldsymbol{PT}$ is representable, $m = (m_1, m_2, m_3)\in D^*$, $n = (n_1, n_2, n_3)\in D^*$, there are a pseudo-t-norm PT and two pseudo-s-norms PS_i (i=1,2) such that $\boldsymbol{PT}(m, n) = (PT(m_1, n_1), PS_1(m_2, n_2), PS_2(m_3, n_3))$. Let $m = (0.2, 0.5, 0.4)$, $u = (0.4, 0.3, 0.2)$, $n = (0.5, 0.7, 0.6)$. From $\boldsymbol{PT}(m, n) = (0.4, 0.6, 0.6)$ and $\boldsymbol{PT}(u, n) = (0.5, 0.5, 0.6)$, we get $PS_1(m_2, n_2) = 0.6$ and $PS_1(u_2, n_2) = 0.5$, so $PS_1(m_2, n_2) \neq PS_1(u_2, n_2)$. Thus $PS_1(m, n)$ is not independent from m_1, that is to say $\boldsymbol{PT}$ is UNPT.

Moreover, $\forall m, n\in D^*$, the dual of NPT $\boldsymbol{PT}$ about standard NN $\boldsymbol{N}_S$ is NPS $\boldsymbol{PS}$, which be defined as:

$$\boldsymbol{PS}(m,n)=\begin{cases} m & n=0_{D^*}, \\ n & m=0_{D^*}, \\ (\max(m_1,n_1),\min(2m_3,n_3),\min(2m_3,n_3)) & \text{otherwise.} \end{cases}$$

Then, ***PS*** is unrepresentable.

Remark 3.25 On the one hand, suppose ***PT*** and $\boldsymbol{N}_S$ are UNPT and standard NN on *D**, respectively. The dual of ***PT*** about $\boldsymbol{N}_S$ is ***PS***. Then, we have that ***PS*** is UNPS. On the other hand, let ***N*** be involutive NN, the dual NPT about ***N*** of UNPS is unrepresentable.

4. NRI Induced by NPT on *D**

Definition 4.1 ([13]) A NI is a mapping $\boldsymbol{I}$: $D^*\times D^*\rightarrow D^*$, $\forall m, u, n, v\in D^*$, if it satisfies:
(NI1) $m \leq_1 u \Rightarrow \boldsymbol{I}(m, n) \geq_1 \boldsymbol{I}(u, n)$;
(NI2) $n \leq_1 v \Rightarrow \boldsymbol{I}(m, n) \leq_1 \boldsymbol{I}(m, v)$;
(NI3) $\boldsymbol{I}(1_{D^*}, 1_{D^*}) = \boldsymbol{I}(0_{D^*}, 0_{D^*}) = 1_{D^*}$;
(NI4) $\boldsymbol{I}(1_{D^*}, 0_{D^*}) = 0_{D^*}$.

Since NPT without commutativity, we can define left and right NRIs which satisfy the residual property induced by NPT.

Definition 4.2 Let ***PT*** be a NPT. Define two functions $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$: $D^*\times D^*\rightarrow D^*$,

$$\boldsymbol{I}^{(L)}(m, n) = \sup\{k \mid k\in D^*, \boldsymbol{PT}(k, m) \leq_1 n\};$$
$$\boldsymbol{I}^{(R)}(m, n) = \sup\{k \mid k\in D^*, \boldsymbol{PT}(m, k) \leq_1 n\}.$$

Then, $\boldsymbol{I}^{(L)}$ ($\boldsymbol{I}^{(R)}$) is called left NRI (right NRI) induced by ***PT***.

We note that the two NRIs induced by ***PT*** as $\boldsymbol{I}_{PT^{(L)}}$, $\boldsymbol{I}_{PT^{(R)}}$.

Besides, Let ***PT*** be a NPT, then $\forall m, n, k\in D^*$, ***PT*** satisfies the residual criteria iff,

$$\boldsymbol{PT}(k, m) \leq_1 n \text{ iff } k \leq_1 \boldsymbol{I}_{PT^{(L)}}(m, n);$$
$$\boldsymbol{PT}(m, k) \leq_1 n \text{ iff } k \leq_1 \boldsymbol{I}_{PT^{(R)}}(m, n).$$

Likewise, the concept of neutrosophic co-implications (NCIs) and related knowledge are also given as follows:

Definition 4.3 ([13]) A NCI is a binary function $\boldsymbol{J}$: $(D^*)^2\rightarrow D^*$, $\forall m, u, n, v\in D^*$, if it satisfies:
(NJ1) $m \leq_1 u \Rightarrow \boldsymbol{J}(m, n) \geq_1 \boldsymbol{J}(u, n)$;
(NJ2) $n \leq_1 v \Rightarrow \boldsymbol{J}(m, n) \leq_1 \boldsymbol{J}(m, v)$;
(NJ3) $\boldsymbol{J}(0_{D^*}, 0_{D^*}) = \boldsymbol{J}(1_{D^*}, 1_{D^*}) =0_{D^*}$;
(NJ4) $\boldsymbol{J}(0_{D^*}, 1_{D^*}) = 1_{D^*}$.

Analogously, we can also define left and right neutrosophic residual co-implications (NRCIs) which satisfie the residual property induced by NPS.

Definition 4.4 Suppose that ***PS*** is a NPS. Define two functions $\boldsymbol{J}^{(L)}$, $\boldsymbol{J}^{(R)}$: $D^*\times D^*\rightarrow D^*$,

$$\boldsymbol{J}^{(L)}(m, n) = \inf\{k \mid k\in D^*, \boldsymbol{PS}(k, m) \geq_1 n\};$$
$$\boldsymbol{J}^{(R)}(m, n) = \inf\{k \mid k\in D^*, \boldsymbol{PS}(m, k) \geq_1 n\}.$$

Then, $\boldsymbol{J}^{(L)}$ ($\boldsymbol{J}^{(R)}$) is called left NRCI (right NRCI) induced by ***PS***.

We remark that two NRCIs induced by ***PS*** as $\boldsymbol{J}_{PS^{(L)}}$, $\boldsymbol{J}_{PS^{(R)}}$.

Let ***PS*** be a NPS, then $\forall m, n, k\in D^*$, ***PS*** satisfies the residual criteria iff

$$\boldsymbol{PS}(k, m) \geq_1 n \text{ iff } k \geq_1 \boldsymbol{J}_{\boldsymbol{PS}^{(L)}}(m, n);$$
$$\boldsymbol{PS}(m, k) \geq_1 n \text{ iff } k \geq_1 \boldsymbol{J}_{\boldsymbol{PS}^{(R)}}(m, n).$$

Through learning above definitions, we give the NRIs (NRCIs) of NPTs (NPSs) discussed in **Section 3** as follows:

Example 4.5 Suppose that I_{iL} (i=1,2,3,4) and I_{iR} (i=1,2,3,4) are left and right residual implications induced by pseudo-t-norms PT_i (i=1,2,3,4) as shown in **Example 3.3**; J_{iL} (i=1,2,3,4) and J_{iR} (i=1,2,3,4) are left and right residual co-implications induced by pseudo-s-norms PS_i (i=1,2,3,4) as shown in **Example 3.4**. Then, the binary functions $\boldsymbol{I}_{\boldsymbol{PT}(i)^{(L)}}$ (i=1,2,3,4,5,6) and $\boldsymbol{I}_{\boldsymbol{PT}(i)^{(R)}}$ (i=1,2,3,4,5,6) induced by RNPTs $\boldsymbol{PT}_i$ (i=1,2,3,4,5,6) of **Example 3.5** defined on D^* are left and right NRIs as follows:

(1) $\boldsymbol{I}_{\boldsymbol{PT}_1}{}^{(L)}(m,n) = (I_{1L}(m_1,n_1), J_{1L}(m_2,n_2), J_{1L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_1}{}^{(R)}(m,n) = (I_{1R}(m_1,n_1), J_{1R}(m_2,n_2), J_{1R}(m_3,n_3))$;

(2) $\boldsymbol{I}_{\boldsymbol{PT}_2}{}^{(L)}(m,n) = (I_{2L}(m_1,n_1), J_{2L}(m_2,n_2), J_{2L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_2}{}^{(R)}(m,n) = (I_{2R}(m_1,n_1), J_{2R}(m_2,n_2), J_{2R}(m_3,n_3))$;

(3) $\boldsymbol{I}_{\boldsymbol{PT}_3}{}^{(L)}(m,n) = (I_{3L}(m_1,n_1), J_{3L}(m_2,n_2), J_{3L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_3}{}^{(R)}(m,n) = (I_{3R}(m_1,n_1), J_{3R}(m_2,n_2), J_{3R}(m_3,n_3))$;

(4) $\boldsymbol{I}_{\boldsymbol{PT}_4}{}^{(L)}(m,n) = (I_{4L}(m_1,n_1), J_{4L}(m_2,n_2), J_{4L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_4}{}^{(R)}(m,n) = (I_{4R}(m_1,n_1), J_{4R}(m_2,n_2), J_{4R}(m_3,n_3))$;

(5) $\boldsymbol{I}_{\boldsymbol{PT}_5}{}^{(L)}(m,n) = (I_{1L}(m_1,n_1), J_{2L}(m_2,n_2), J_{3L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_5}{}^{(R)}(m,n) = (I_{1R}(m_1,n_1), J_{2R}(m_2,n_2), J_{3R}(m_3,n_3))$;

(6) $\boldsymbol{I}_{\boldsymbol{PT}_6}{}^{(L)}(m,n) = (I_{1L}(m_1,n_1), J_{3L}(m_2,n_2), J_{3L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_6}{}^{(R)}(m,n) = (I_{1R}(m_1,n_1), J_{3R}(m_2,n_2), J_{3R}(m_3,n_3))$.

Example 4.6 Suppose that I_{iL} (i=1,2,3,4) and I_{iR} (i=1,2,3,4) are left and right residual implications induced by pseudo-t-norms PT_i (i=1,2,3,4) as shown in **Example 3.3**; J_{iL} (i=1,2,3,4) and J_{iR} (i=1,2,3,4) are left and right residual co-implications induced by pseudo-s-norms PS_i (i=1,2,3,4) as shown in **Example 3.4**. Then, the binary functions $\boldsymbol{J}_{\boldsymbol{PS}(i)^{(L)}}$ (i=1,2,3,4,5,6) and $\boldsymbol{J}_{\boldsymbol{PS}(i)^{(R)}}$ (i=1,2,3,4,5,6) induced by RNPSs $\boldsymbol{PS}_i$ (i=1,2,3,4,5,6) of **Example 3.6** defined on D^* are left and right NRCIs as follows:

(1) $\boldsymbol{J}_{\boldsymbol{PS}_1}{}^{(L)}(m,n) = (J_{1L}(m_1,n_1), I_{1L}(m_2,n_2), I_{1L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_1}{}^{(R)}(m,n) = (J_{1R}(m_1,n_1), I_{1R}(m_2,n_2), I_{1R}(m_3,n_3))$;

(2) $\boldsymbol{J}_{\boldsymbol{PS}_2}{}^{(L)}(m,n) = (J_{2L}(m_1,n_1), I_{2L}(m_2,n_2), I_{2L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_2}{}^{(R)}(m,n) = (J_{2R}(m_1,n_1), I_{2R}(m_2,n_2), I_{2R}(m_3,n_3))$;

(3) $\boldsymbol{J}_{\boldsymbol{PS}_3}{}^{(L)}(m,n) = (J_{3L}(m_1,n_1), I_{3L}(m_2,n_2), I_{3L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_3}{}^{(R)}(m,n) = (J_{3R}(m_1,n_1), I_{3R}(m_2,n_2), I_{3R}(m_3,n_3))$;

(4) $\boldsymbol{J}_{\boldsymbol{PS}_4}{}^{(L)}(m,n) = (J_{4L}(m_1,n_1), I_{4L}(m_2,n_2), I_{4L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_4}{}^{(R)}(m,n) = (J_{4R}(m_1,n_1), I_{4R}(m_2,n_2), I_{4R}(m_3,n_3))$;

(5) $\boldsymbol{J}_{\boldsymbol{PS}_5}{}^{(L)}(m,n) = (J_{1L}(m_1,n_1), I_{2L}m_2,n_2), I_{3L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_5}{}^{(R)}(m,n) = (J_{1R}(m_1,n_1), I_{2R}(m_2,n_2), I_{3R}(m_3,n_3))$;

(6) $\boldsymbol{J}_{\boldsymbol{PS}_6}{}^{(L)}(m,n) = (J_{1L}(m_1,n_1), I_{3L}(m_2,n_2), I_{3L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_6}{}^{(R)}(m,n) = (J_{1R}(m_1,n_1), I_{3R}(m_2,n_2), I_{3R}(m_3,n_3))$.

Because NPS $\boldsymbol{PS}$ are dual operator of NPT $\boldsymbol{PT}$ about $\boldsymbol{N}_S$, so the NRIs induced by NPT and the NRCIs induced by NPS are dual. For **Examples 3.5** and **3.6** given above, If $\boldsymbol{PS}$ and $\boldsymbol{PT}$ are dual, then the NRCIs $\boldsymbol{J}_{PS}$ derived by $\boldsymbol{PS}$ is the dual operator of the NRIs $\boldsymbol{I}_{PT}$ induced by $\boldsymbol{PT}$.

The following we will show an important theorem which proves sufficient conditions that the residual operator derived by a NPT is always a NI.

Theorem 4.7 Assume $\boldsymbol{PT}$ be a NPT on D^*. Then, $\forall m, n \in D^*$,

$$\boldsymbol{I}_{PT^{(L)}}(m, n) = \sup\{k \mid k \in D^*, \boldsymbol{PT}(k, m) \leq_1 n\};$$
$$\boldsymbol{I}_{PT^{(R)}}(m, n) = \sup\{k \mid k \in D^*, \boldsymbol{PT}(m, k) \leq_1 n\}.$$

are NIs.

Proof. First give the proof that $\boldsymbol{I}_{PT^{(L)}}$ is a NI, $\forall m, u, n, v \in D^*$:

We get $\boldsymbol{I}_{PT^{(L)}}(m, 1_{D^*}) = \sup\{k \mid k \in D^*, \boldsymbol{PT}(k, m) \leq_1 1_{D^*}\} = 1_{D^*}$ by **Definition 4.2**. Thus $\boldsymbol{I}_{PT^{(L)}}(1_{D^*}, 1_{D^*}) = 1_{D^*}$. From (NPT2) in **Definition 3.1**, we get $\boldsymbol{I}_{PT^{(L)}}(1_{D^*}, 0_{D^*}) = \sup\{k \mid k \in D^*, \boldsymbol{PT}(k, 1_{D^*}) \leq_1 0_{D^*}\} = 0_{D^*}$. $\boldsymbol{I}_{PT^{(L)}}(0_{D^*}, 0_{D^*}) = \sup\{k \mid k \in D^*, \boldsymbol{PT}(k, 0_{D^*}) \leq_1 0_{D^*}\} = 1_{D^*}$.

If $m \leq_1 u$. By (NPT2) in **Definition 3.1**, $\{k \mid k \in D^*, \boldsymbol{PT}(k, m) \leq_1 n\} \supseteq_1 \{k \mid k \in D^*, \boldsymbol{PT}(k, u) \leq_1 n\}$, then $\sup\{k \mid k \in D^*, \boldsymbol{PT}(k, m) \leq_1 n\} \geq_1 \sup\{k \mid k \in D^*, \boldsymbol{PT}(k, u) \leq_1 n\}$. Thus, $\boldsymbol{I}_{PT^{(L)}}(m, n) \geq_1 \boldsymbol{I}_{PT^{(L)}}(u, n)$.

If $n \leq_1 v$. Since the undecreasingness of $\boldsymbol{PT}$, we have $\{k \mid k \in D^*, \boldsymbol{PT}(k, m) \leq_1 n\} \subseteq_1 \{k \mid k \in D^*, \boldsymbol{PT}(k, m) \leq_1 v\}$, then $\sup\{k \mid k \in D^*, \boldsymbol{PT}(k, m) \leq_1 n\} \leq_1 \sup\{k \mid k \in D^*, \boldsymbol{PT}(k, m) \leq_1 v\}$. Thus, $\boldsymbol{I}_{PT^{(L)}}(m, n) \leq_1 \boldsymbol{I}_{PT^{(L)}}(m, v)$.

To sum up, $\boldsymbol{I}_{PT^{(L)}}$ is a NI. Likewise, $\boldsymbol{I}_{PT^{(R)}}$ is a NI can also be proved.

Some relevant properties of NRI are given below.

Theorem 4.8 Suppose that $\boldsymbol{PT}$ be a NPT on D^*, $\boldsymbol{I}_{PT^{(L)}}$, $\boldsymbol{I}_{PT^{(R)}}$ are NRIs. Then, $\forall m, n, r \in D^*$,

(1) $\boldsymbol{I}_{PT^{(L)}}(0_{D^*}, n) = 1_{D^*}$;

(2) $\boldsymbol{I}_{PT^{(L)}}(m, 1_{D^*}) = 1_{D^*}$;

(3) $\boldsymbol{I}_{PT^{(L)}}(m, m) = 1_{D^*}$;

(4) $\boldsymbol{I}_{PT^{(L)}}(1_{D^*}, n) = n$;

(5) $\boldsymbol{I}_{PT^{(L)}}(m, n) \geq_1 n$;

(6) $\boldsymbol{I}_{PT^{(L)}}(m, n) = 1_{D^*}$ iff $m \leq_1 n$;

(7) $\boldsymbol{I}_{PT^{(L)}}(\boldsymbol{PT}(m, n), \boldsymbol{PT}(m, r)) \geq_1 \boldsymbol{I}_{PT^{(L)}}(n, r)$;

(8) $m \leq_1 \boldsymbol{I}_{PT^{(L)}}(n, \boldsymbol{PT}(m, n))$.

Similarly, $\boldsymbol{I}_{PT^{(R)}}$ also satisfies the properties (1)-(7) in **Theorem 4.8**. However, it should be noted that NI induced by NPT, because pseudo-t-norm removes commutativity, leads to the difference in property (8) in the corresponding **Theorem 4.8** of $\boldsymbol{I}_{PT^{(R)}}$, as shown below:

(8) $m \leq_1 \boldsymbol{I}_{PT^{(R)}}(n, \boldsymbol{PT}(n, m))$.

Proof. The proofs of (1)-(4) are straightforward to obtain by **Definition 4.2**, so the proof is ignored.

(5) From (NI1) in **Definition 4.1,** we get that $\boldsymbol{I}_{PT^{(L)}}(m, n) \geq_1 \boldsymbol{I}_{PT^{(L)}}(1_{D^*}, n) = n$.

(6) ($\Rightarrow$) if $\boldsymbol{I}_{PT^{(L)}}(m, n) = 1_{D^*}$, then $\boldsymbol{PT}(1_{D^*}, m) \leq_1 n$. Thus, $m \leq_1 n$. ($\Leftarrow$) since $m \leq_1 n$, $\boldsymbol{PT}(1_{D^*}, m) \leq_1 n$. Thus, $\boldsymbol{I}_{PT^{(L)}}(m, n) \geq_1 1_{D^*}$, that is $\boldsymbol{I}_{PT^{(L)}}(m, n) = 1_{D^*}$.

(7) $\boldsymbol{I}_{PT^{(L)}}(\boldsymbol{PT}(m, n), \boldsymbol{PT}(m, r)) = \sup\{k \mid k \in D^*, \boldsymbol{PT}(k, \boldsymbol{PT}(m, n)) \leq_1 \boldsymbol{PT}(m, r)\} = \sup\{k \mid k \in D^*, \boldsymbol{PT}(m, \boldsymbol{PT}(k, n)) \leq_1 \boldsymbol{PT}(m, r)\} \geq_1 \sup\{k \mid k \in D^*, \boldsymbol{PT}(k, n) \leq_1 r\} = \boldsymbol{I}_{PT^{(L)}}(n, r)$.

(8) Since $\boldsymbol{PT}(m, n) \leq_1 \boldsymbol{PT}(m, n)$, so we get $m \leq_1 \boldsymbol{I}_{PT^{(L)}}(n, \boldsymbol{PT}(m, n))$.

The proof which $\boldsymbol{I}_{PT^{(R)}}$ satisfies the properties (1)-(8) is similar to the proof of $\boldsymbol{I}_{PT^{(L)}}$.

In the same way, we give two theorems about NPS on D^*.

Theorem 4.9 Let $\boldsymbol{PS}$ be a NPS on D^*. Then, $\forall m, n \in D^*$,

$$\boldsymbol{J}_{PS^{(L)}}(m, n) = \inf\{k \mid k \in D^*, \boldsymbol{PS}(k, m) \geq_1 n\};$$
$$\boldsymbol{J}_{PS^{(R)}}(m, n) = \inf\{k \mid k \in D^*, \boldsymbol{PS}(m, k) \geq_1 n\}.$$

are NCIs.

Proof. According to the **Definition 4.4**, we can use the proof of **Theorem 4.7** method to prove it.

Theorem 4.10 Let $\boldsymbol{PS}$ is a NPS on D^*, $\boldsymbol{J}_{PS^{(L)}}$, $\boldsymbol{J}_{PS^{(R)}}$ are NRCIs. Then, $\forall m, n, r \in D^*$,
(1) $\boldsymbol{J}_{PS^{(L)}}(1_{D^*}, n) = 0_{D^*}$;
(2) $\boldsymbol{J}_{PS^{(L)}}(m, 0_{D^*}) = 0_{D^*}$;
(3) $\boldsymbol{J}_{PS^{(L)}}(m, m) = 0_{D^*}$;
(4) $\boldsymbol{J}_{PS^{(L)}}(0_{D^*}, n) = n$;
(5) $\boldsymbol{J}_{PS^{(L)}}(m, n) \leq_1 n$;
(6) $\boldsymbol{J}_{PS^{(L)}}(m, n) = 0_{D^*}$ iff $m \geq_1 n$;
(7) $\boldsymbol{J}_{PS^{(L)}}(\boldsymbol{PS}(m, n), \boldsymbol{PS}(m, r)) \leq_1 \boldsymbol{J}_{PS^{(L)}}(n, r)$;
(8) $m \geq_1 \boldsymbol{J}_{PS^{(L)}}(n, \boldsymbol{PS}(m, n))$.

Similarly, $\boldsymbol{J}_{PS^{(R)}}$ also satisfies the properties (1)-(7) in **Theorem 4.10**. However, it should be noted that NCI induced by NPS, because pseudo-s-norm removes commutativity, leads to the difference in property (8) in the corresponding Theorem 4.10 of $\boldsymbol{J}_{PS^{(R)}}$, as shown below:

(8) $m \geq_1 \boldsymbol{J}_{PS^{(R)}} (n, \boldsymbol{PS} (n, m))$.

Definition 4.11 Let $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$: $D^* \times D^* \rightarrow D^*$ are NIs. $\forall m, n \in D^*$, the induced operators $\boldsymbol{PT}_{I^{(L)}}$, $\boldsymbol{PT}_{I^{(R)}}$ by $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$ are defined as follows:

$$\boldsymbol{PT}_{I^{(L)}}(m, n) = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\};$$
$$\boldsymbol{PT}_{I^{(R)}}(m, n) = \inf\{k \mid k \in D^*, n \leq_1 \boldsymbol{I}^{(R)}(m, k)\}.$$

Theorem 4.12 Let $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$ are NIs on D^*. $\forall m, n, r \in D^*$, if $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$ satisfies below conditions:

(a) $r \leq_1 \boldsymbol{I}^{(L)}(n, m)$ iff $n \leq_1 \boldsymbol{I}^{(R)}(r, m)$;
(b) $\boldsymbol{I}^{(L)}(m, \boldsymbol{I}^{(L)}(n, r)) = \boldsymbol{I}^{(L)}(n, \boldsymbol{I}^{(L)}(m, r))$; $\boldsymbol{I}^{(R)}(m, \boldsymbol{I}^{(R)}(n, r)) = \boldsymbol{I}^{(R)}(n, \boldsymbol{I}^{(R)}(m, r))$;
(c) $\boldsymbol{I}^{(L)}(m, n) = 1_{D^*}$ iff $m \leq_1 n$; $\boldsymbol{I}^{(R)} (m, n) = 1_{D^*}$ iff $m \leq_1 n$;
(d) $\boldsymbol{I}^{(L)}(1_{D^*}, m) = m$; $\boldsymbol{I}^{(R)}(1_{D^*}, m) = m$.

Then, the induced operators $\boldsymbol{PT}_{I^{(L)}}$, $\boldsymbol{PT}_{I^{(R)}}$ by $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$ in Definition 4.11 are NPTs.

Proof. $\forall m, u, n, v \in D^*$, there are below:

(NPT1) From (a) and (b), $\boldsymbol{PT}_{I^{(L)}}(m, \boldsymbol{PT}_{I^{(L)}}(n, r)) = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(\boldsymbol{PT}_{I^{(L)}}(n, r), k)\} = \inf\{k \mid k \in D^*, \boldsymbol{PT}_{I^{(L)}}(n, r) \leq_1 \boldsymbol{I}^{(R)}(m, k)\} = \inf\{k \mid k \in D^*, r \leq_1 \boldsymbol{I}^{(R)}(n, \boldsymbol{I}^{(R)}(m, k))\} = \inf\{k \mid k \in D^*, r \leq_1 \boldsymbol{I}^{(R)}(m, \boldsymbol{I}^{(R)}(n, k))\} = \inf\{k \mid k \in D^*, \boldsymbol{PT}_{I^{(L)}}(m, r) \leq_1 \boldsymbol{I}^{(R)}(n, k)\} = \inf\{k \mid k \in D^*, n \leq_1 \boldsymbol{I}^{(L)}(\boldsymbol{PT}_{I^{(L)}}(m, r), k)\} = \boldsymbol{PT}_{I^{(L)}}(n, \boldsymbol{PT}_{I^{(L)}}(m, r))$.

(NPT2) If $m \leq_1 u$, $n \leq_1 v$. So $\boldsymbol{I}^{(L)}(v, k) \leq_1 \boldsymbol{I}^{(L)}(n, k)$ for $\forall k \in D^*$. $\forall k_0 \in \{k \mid k \in D^*, u \leq_1 \boldsymbol{I}^{(L)}(v, k)\}$, it can be concluded that $u \leq_1 \boldsymbol{I}^{(L)}(v, k_0)$. Since $m \leq_1 u$, and $\boldsymbol{I}^{(L)}(v, k_0) \leq_1 \boldsymbol{I}^{(L)}(n, k_0)$, $m \leq_1 \boldsymbol{I}^{(L)}(n, k_0)$, namely $k_0 \in \{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\}$. Thus, $\{k \mid k \in D^*, u \leq_1 \boldsymbol{I}^{(L)}(v, k)\} \subseteq_1 \{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\}$. Hence, $\inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\} \leq_1 \inf\{k \mid k \in D^*, u \leq_1 \boldsymbol{I}^{(L)}(v, k)\}$, that is, $\boldsymbol{PT}_{I^{(L)}}(m, n) \leq_1 \boldsymbol{PT}_{I^{(L)}}(u, v)$. Likewise, we can prove that $\boldsymbol{PT}_{I^{(L)}}(n, m) \leq_1 \boldsymbol{PT}_{I^{(L)}}(v, u)$.

(NPT3) $\boldsymbol{PT}_{I^{(L)}}(1_{D^*}, m) = \inf\{k \mid k \in D^*, 1_{D^*} \leq_1 \boldsymbol{I}^{(L)}(m, k)\} = \inf\{k \mid k \in D^*, \boldsymbol{I}^{(L)}(m, k) = 1_{D^*}\} = \inf\{k \mid k \in D^*, m \leq_1 k\} = m$; $\boldsymbol{PT}_{I^{(L)}} (m, 1_{D^*}) = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(1_{D^*}, k)\} = \inf\{k \mid k \in D^*, m \leq_1 k\} = m$.

Therefore $\boldsymbol{PT}_{I^{(L)}}$ is a NPT, and in the same way, we can also show that $\boldsymbol{PT}_{I^{(R)}}$ is a NPT.

Theorem 4.13 If $\boldsymbol{PT}$ is a NPT on D^*, so there is $\boldsymbol{PT} = \boldsymbol{PT}_{I^{(L)}} = \boldsymbol{PT}_{I^{(R)}}$.

Proof. $\forall m, n \in D^*$, from **Definitions 4.2** and **4.11**, we get $\boldsymbol{PT}_{I^{(L)}}(m, n) = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\} = \inf\{k \mid k \in D^*, \boldsymbol{PT}(m, n) \leq_1 k\} = \boldsymbol{PT}(m, n)$ and $\boldsymbol{PT}_{I^{(R)}}(m, n) = \inf\{k \mid k \in D^*, n \leq_1 \boldsymbol{I}^{(R)}(m, k)\} = \inf\{k \mid k \in D^*, \boldsymbol{PT}(m, n) \leq_1 k\} = \boldsymbol{PT}(m, n)$. Thus, $\boldsymbol{PT} = \boldsymbol{PT}_{I^{(L)}} = \boldsymbol{PT}_{I^{(R)}}$.

Definition 4.14 ([21]) An algebraic system $S=(S; \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ is said to be a NCRL, $\forall m, n, r \in S$, if S satisfies:

(1) $(S; \wedge, \vee, 0, 1)$ be a bounded lattice on S, its corresponding order is $\leq$, 0 is minimal element and 1 is maximal element of S;

(2) $(S; \otimes, 1)$ be non-commutative monoid and its neutral element is 1;

(3) $m \otimes n \leq r \Leftrightarrow m \leq n \rightarrow r \Leftrightarrow n \leq m \rightsquigarrow r$.

Sections 3 and **4** focus on NPTs and their NRIs. Next, a NCRL is established, which is constructed from three neutrosophic logic operators.

Theorem 4.15 Suppose $(D^*; \wedge_1, \vee_1, {}^c, 0_{D^*}, 1_{D^*})$ is a system and $\boldsymbol{PT}$ is a NPT on D^*. $\forall m, n \in D^*$, define the following three equations:

$$m \otimes n = \boldsymbol{PT}(m, n);\ m \rightarrow n = \boldsymbol{I}_{PT^{(L)}}(m, n);\ m \rightsquigarrow n = \boldsymbol{I}_{PT^{(R)}}(m, n).$$

Then, $(D^*; \wedge_1, \vee_1, \otimes, \rightarrow, \rightsquigarrow, 0_{D^*}, 1_{D^*})$ is NCRL.

Proof. First, by **Proposition 2.9**, we get that $(D^*; \wedge_1, \vee_1, 0_{D^*}, 1_{D^*})$ be a bounded lattice on D^*.

Second, the fact that $(D^*; \otimes, 1_{D^*})$ is non-commutative monoid is proved. (1) $m \otimes 1_{D^*} = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(1_{D^*}, k)\} = \inf\{k \mid k \in D^*, m \leq_1 k\} = m$ and $1_{D^*} \otimes m = \inf\{k \mid k \in D^*, 1_{D^*} \leq_1 \boldsymbol{I}^{(L)}(m, k)\} = \inf\{k \mid k \in D^*, \boldsymbol{I}^{(L)}(m, k) = 1_{D^*}\} = \inf\{k \mid k \in D^*, m \leq_1 k\} = m$, i.e. $\forall m \in D^*$, the equation $1_{D^*} \otimes m = m \otimes 1_{D^*} = m$ is true. (2) Theorem 4.13 proves that $\boldsymbol{PT}_{I^{(L)}} = \boldsymbol{PT}$ is a NPT. Thus, $\boldsymbol{PT}$ does not satisfy the commutative law. (3) From (NPT1) of **Definition 3.1**, $\otimes$ satisfies the associative law.

Finally, $\forall m, n, k \in D^*$, we prove the below equivalence relation

$$m \otimes n \leq_1 k \Leftrightarrow m \leq n \rightarrow k \Leftrightarrow n \leq m \rightsquigarrow k$$

holds. On the one hand, by what we know about $\otimes$, there are $m \otimes n = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\}$, $m \otimes n \leq_1 k$. Thus, there are $n \leq m \rightsquigarrow k$ and $m \leq n \rightarrow k$. On the other hand, by what we know about $\rightarrow$, we get $n \rightarrow k = \sup\{t \mid t \in D^*, \boldsymbol{PT}(t, n) \leq_1 k\}$. Since $m \leq n \rightarrow k$, therefore $m \otimes n \leq_1 k$. Likewise, there are $n \leq m \rightsquigarrow k \Rightarrow m \otimes n \leq_1 k$.

Thus, $(D^*; \wedge_1, \vee_1, \otimes, \rightarrow, \rightsquigarrow, 0_{D^*}, 1_{D^*})$ is NCRL.

5. Conclusions

Neutrosophic logic is an important part of NS theory. Common neutrosophic logic operators are: NPTs, NPSs NIs, NNs and so on. On the basis of complete lattice $(D^*; \leq_1)$, We define NPTs and NPSs. In addition, DMNTs are defined, describing that NPT and NPS are dual with regard to the standard NN. Then, on the basis of complete lattice $(D^*; \leq_1)$, the concepts of NRI and NRCI is given, and we present a theorem which states that residual operators derived by NPTs must be NIs, and further study their fundamental properties. Finally, we provide a method to get NPT from NI and construct NCRLs. In the future, we will investigate neutrosophic inference methods and neutrosophic pseudo overlap functions based on some new results [22-36], and further study their fundamental properties.

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