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Solution and Analysis of System of Differential Equation with Initial Condition as $\mathbf{TrapN_{number}}$

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Abstract: In this present study, we have analyzed the solution of system of first order simultaneous differential equations with initial condition as a neutrosophic environment. Here, we consider the initial values as Trapezoidal Neutrosophic Numbers ($TrapN_{number}$). The solution procedure of the system of first order ODE is developed using $TrapN_{number}$ and $(\alpha,\beta,\gamma)_{cut}$ of a TN_{number}. Furthermore, a numerical example is illustrated to validate the efficiency and feasibility of the proposed neutrosophic method, and the solutions are compared with the crisp values. The numerical solutions $x_1(t,\alpha), x_2(t,\alpha), x_1'(t,\beta), x_2'(t,\beta), x_1''(t,\gamma), x_2''(t,\gamma), y_1(t,\alpha), y_2(t,\alpha), y_1'(t,\beta), y_2'(y,\beta), y_1''(t,\gamma)$ and $y_2''(t,\gamma)$ for the different values of α,β and γ at $t = 0.5$ are examined via tables and graphs. The numerical solutions delight the conditions of strong solution.

Keywords: *Difference equation;* $TrapN_{number}$; $(\alpha,\beta,\gamma)_{cut}$ *of a* $TrapN_{number}$

1. Introduction

The concept of Neutrosophic set (N_{set}) was introduced by Smarandache [1]. N_{set} is a wide-ranging context of the C_{set}, F_{set} [2], IF_{set} [3-6], and the IVF_{set} [7-10], respectively. N_{set} is used to characterize the uncertainty and indeterminacy in any multicriteria decision making problems. N_{set} is a proposition of three different components namely, T_{value}, $I_{value,}$ and F_{value}, and the grade of these membership values are defined within $^{-}]0,1[^{+}$. Researchers in different field of Engineers have applied N_{set} on various applications. The multifaceted factors of N_{logic}, SVN_{number}, $TriN_{number}$, and $TrapN_{number}$ have been applied in the differential equations are analyzed in [11-15]. The system of first order simultaneous differential equations (SFOSDE) with an initial condition plays an important role in various field of science and engineering like fluid mechanics, thermodynamics, heat and electromagnetism, rate of chemical reaction, bacteria/ plants and organisms growth rates, and population/economic growth rates, etc. Almost all the modern scientific analysis associates differential equations. There are many ways to solve SFOSD. Mondal et al [16] solving the system of differential equation and its application with intuitionistic fuzzy environment. Sadeghi et al. [17] discussed the necessary and sufficient condition for the existence of solution of fuzzy differential equations. Keshavarz et al. [18] investigated the application of differential equations in Newton's law of cooling, distribution of a drug in the human body and harmonic oscillator problem using fuzzy logic. Karpagappriya et al. [19] examined the solution of fuzzy initial value problems using cubic spline function. Several other researchers [21–23] have also discussed the significance of Neutrosophic in Agricultural Water Management, Healthcare Waste to Achieve Cost Effectiveness and Transportation Problem.

In the above literature studies, researchers explored several numerical/analytical solutions of differential equations. In almost all cases authors find the solutions of fuzzy differential equations using fuzzy environments. However, no attempt has been made to find the solutions of System of Differential Equation with Initial Condition as trapezoidal Neutrosophic number. The purpose of the present study is to investigate the Neutrosophic solutions of first order system of differential equations. The efficiency and feasibility of the present approach is illustrated by numerical examples and the results are for better perceptive of our investigation.

2. Preliminary

Definition 2.1. $\boldsymbol{N_{set}}$: [24] Let X be a universe set. A N_{set} $\widetilde{A_N}$ on X is defined as $\widetilde{A_N} = \{\langle x, T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)\rangle : x \in X\}$, where $T_{\widetilde{A_N}}(x): X \to ^{-}]0,1[^{+}$ is said to be the $T_{membership}$, which represents the degree of confidence, $I_{\widetilde{A_N}}(x): X \to ^{-}]0,1[^{+}$is said to be the $I_{membership}$, which represents the degree of uncertainty, and $F_{\widetilde{A_N}}(x): X \to ^{-}]0,1[^{+}$ is said to be the $F_{membership}$, which represents the degree of skepticism, respectively of the element $x \in X$ in $\widetilde{A_N}$, such that $^{-}0 \leq T_{\widetilde{A_N}}(x) + I_{\widetilde{A_N}}(x) + F_{\widetilde{A_N}}(x) \leq 3^{+}$.

Definition 2.2. $\boldsymbol{SVN_{set}}$ [24] A N_{set} $\widetilde{A_N}$ on X (Definition 2.1) is said to be SVN_{set} $(\widetilde{SA_N})$ if x is a single-valued independent variable. $\widetilde{SA_N} = \{\langle x, T_{\widetilde{SA_N}}(x), I_{\widetilde{SA_N}}(x), F_{\widetilde{SA_N}}(x)\rangle : x \in X\}$, where $T_{\widetilde{SA_N}}(x), I_{\widetilde{SA_N}}(x), F_{\widetilde{SA_N}}(x): X \to ^{-}]0,1[^{+}$ represents the concept of $T_{membership}, I_{membership}, F_{membership}$, functions, respectively of the element $x \in X$ in $\widetilde{SA_N}$, such that $^{-}0 \leq T_{\widetilde{SA_N}}(x) + I_{\widetilde{SA_N}}(x) + F_{\widetilde{SA_N}}(x) \leq 3^{+}$.

Definition 2.3. $\boldsymbol{(\alpha, \beta, \gamma)_{cut}}$: [24] The $(\alpha, \beta, \gamma)_{cut}$ N_{set} is $\widetilde{A_N}$ defined as $\widetilde{A_N}_{(\alpha,\beta,\gamma)} = \{\langle T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)\rangle : x \in X, T_{\widetilde{A_N}}(x) \geq \alpha, I_{\widetilde{A_N}}(x) \leq \beta, F_{\widetilde{A_N}}(x) \leq \gamma\}$, where $\alpha, \beta, \gamma \in [0,1]$, such that $\alpha + \beta + \gamma \leq 3$.

Definition 2.4. $\boldsymbol{N_{number}}$: [24] A N_{set} $\widetilde{A_N}$ over the real numbers set $\mathbb{R}$ is a neutrosophic number if it satisfies the following conditions:

i. $\widetilde{A_N}$ is normal: $\exists x_0 \in \mathbb{R}$ such that $T_{\widetilde{A_N}}(x_0) = 1$. $\left(I_{\widetilde{A_N}}(x_0) = F_{\widetilde{A_N}}(x_0) = 0\right)$.

ii. $\widetilde{A_N}$ is convex for the truth function $T_{\widetilde{A_N}}(x)$.
(ie) $T_{\widetilde{A_N}}(\mu x_1 + (1-\mu)x_2) \geq \min\left(T_{\widetilde{A_N}}(x_1), T_{\widetilde{A_N}}(x_2)\right)$, for all $x_1, x_2 \in \mathbb{R}$ and $\mu \in [0,1]$.

iii. $\widetilde{A_N}$ is concave set for the falsity, indeterministic functions namely, $F_{\widetilde{A_N}}(x)$ and $I_{\widetilde{A_N}}(x)$.
(ie) $I_{\widetilde{A_N}}(\mu x_1 + (1-\mu)x_2) \geq \max\left(I_{\widetilde{A_N}}(x_1), I_{\widetilde{A_N}}(x_2)\right)$ and $F_{\widetilde{A_N}}(\mu x_1 + (1-\mu)x_2) \geq \max\left(F_{\widetilde{A_N}}(x_1), F_{\widetilde{A_N}}(x_2)\right)$, for all $x_1, x_2 \in \mathbb{R}$ and $\mu \in [0,1]$.

Definition 2.5. $\boldsymbol{TrapN_{number}}$: [25] A subset of N_{number} , $TrapN_{number}$, $\widetilde{A_N}$ in $\mathbb{R}$ with the following $T_{function}, I_{function}, and\ F_{function}$ is defined as

$$T_{\widetilde{A_N}}(x) = \begin{cases} \left(\dfrac{x-a_1}{a_2-a_1}\right) u_{\widetilde{A_N}}\ for\ a_1 \leq x \leq a_2 \\ u_{\widetilde{A_N}} \quad for\ a_2 \leq x \leq a_3 \\ \left(\dfrac{a_4-x}{a_4-a_3}\right) u_{\widetilde{A_N}}\ for a_3 \leq x \leq a_4 \\ 0 \quad otherwise \end{cases};$$

$$I_{\widetilde{A_N}}(x) = \begin{cases} \left(\frac{a_2-x}{a_2-a_1}\right) v_{\widetilde{A_N}}\ for\ a_1 \leq x \leq a_2 \\ v_{\widetilde{A_N}} \quad for\ a_2 \leq x \leq a_3 \\ \left(\frac{a_4-x}{a_4-a_3}\right) v_{\widetilde{A_N}}\ for a_3 \leq x \leq a_4 \\ 1 \quad otherwise \end{cases} \qquad (1)$$

$$F_{\widetilde{A_N}}(x) = \begin{cases} \left(\dfrac{a_2-x}{a_2-a_1}\right) w_{\widetilde{A_N}}\ for\ a_1 \leq x \leq a_2 \\ w_{\widetilde{A_N}} \quad for\ a_2 \leq x \leq a_3 \\ \left(\dfrac{a_4-x}{a_4-a_3}\right) w_{\widetilde{A_N}}\ for a_3 \leq x \leq a_4 \\ 1 \quad otherwise \end{cases}$$

where $^{-}0 \leq T_{\widetilde{A_N}}(x) + I_{\widetilde{A_N}}(x) + F_{\widetilde{A_N}}(x) \leq 3^{+}, x \in \widetilde{A_N}$.

Definition 2.6. $\boldsymbol{(\alpha, \beta, \gamma)_{cut}\ of\ a\ TrapN_{number}}$: The $(\alpha, \beta, \gamma)_{cut}$ $of\ a\ TrapN_{number}$ $\widetilde{A_N} = \langle (a_1, a_2, a_3, a_4); u_{\widetilde{A_N}}, v_{\widetilde{A_N}}, w_{\widetilde{A_N}} \rangle$ is defined as follows:

$(\widetilde{A_N})_{\alpha,\beta,\gamma} = \left[T_{\widetilde{A_{N1}}}(\alpha), T_{\widetilde{A_{N2}}}(\alpha); I_{\widetilde{A_{N1}}}(\beta), I_{\widetilde{A_{N2}}}(\beta); F_{\widetilde{A_{N1}}}(\gamma), F_{\widetilde{A_{N2}}}(\gamma)\right]$, where

$T_{\widetilde{A_{N1}}}(\alpha) = [a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}}, T_{\widetilde{A_{N2}}}(\alpha) = [a_4 - \alpha(a_4 - a_3)]u_{\widetilde{A_N}}$

$I_{\widetilde{AN1}}(\beta) = [a_2 - \beta(a_2 - a_1)]v_{\widetilde{AN}}, I_{\widetilde{AN2}}(\beta) = [a_3 + \beta(a_4 - a_3)]v_{\widetilde{AN}}$ (2)
$F_{\widetilde{AN1}}(\gamma) = [a_2 - \gamma(a_2 - a_1)]w_{\widetilde{AN}}, F_{\widetilde{AN2}}(\gamma) = [a_3 + \gamma(a_4 - a_3)]w_{\widetilde{AN}}$

here $0 < \alpha \leq 1, 0 < \beta \leq 1, 0 < \gamma \leq 1$ *and* $^{-}0 < \alpha + \beta + \gamma \leq 3^{+}$.

Definition 2.7. $\boldsymbol{Strong_{solution}}$: Let the solution of the neutrosophic differential equations be $x(t)$ and $y(t)$, and its $(\alpha,\beta,\gamma)_{cut}$ be $[x(t,\alpha,\beta,\gamma)] = [x_1(t,\alpha), x_2(t,\alpha), x_1'(t,\beta), x_2'(t,\beta), x_1''(t,\gamma), x_2''(t,\gamma)]$ and $[y(t,\alpha,\beta,\gamma)] = [y_1(t,\alpha), y_2(t,\alpha), y_1'(t,\beta), y_2'(t,\beta), y_1''(t,\gamma), y_2''(t,\gamma)]$.
The solution is a strong solution if,

i. $\frac{dx_1(t,\alpha)}{d\alpha} > 0, \frac{dx_2(t,\alpha)}{d\alpha} < 0$ *and* $\frac{dy_1(t,\alpha)}{d\alpha} > 0, \frac{dy_2(t,\alpha)}{d\alpha} < 0 \forall \alpha \in [0,1], x_1(t,1) \leq x_2(t,1)$ *and* $y_1(t,1) \leq y_2(t,1)$.

ii. $\frac{dx_1'(t,\beta)}{d\alpha} < 0, \frac{dx_2'(t,\beta)}{d\alpha} > 0$ *and* $\frac{dy_1'(t,\beta)}{d\alpha} < 0, \frac{dy_2'(t,\beta)}{d\alpha} > 0 \forall \beta \in [0,1], x_1'(t,0) \leq x_2'(t,0)$ *and* $y_1'(t,0) \leq y_2'(t,0)$. (3)

iii. $\frac{dx_1''(t,\gamma)}{d\alpha} < 0, \frac{dx_2''(t,\gamma)}{d\alpha} > 0$ *and* $\frac{dy_1''(t,\gamma)}{d\alpha} < 0, \frac{dy_2''(t,\gamma)}{d\alpha} > 0 \forall \gamma \in [0,1], x_1''(t,0) \leq x_2''(t,0)$ *and* $y_1''(t,0) \leq y_2''(t,0)$.

3. Solution of Neutrosophic boundary value problem

In this section, we discuss the solution of first-order ODE with a neutrosophic initial value conditions.

3.1 Solution of System of First-Order ODE using $TrapN_{number}$

Let us consider the system of first order differential equation

$$\frac{dx}{dt} = k_1 y \quad (4)$$

and

$$\frac{dy}{dt} = k_2 x \quad (5)$$

with boundary condition $x(t_0) = \tilde{a}$ and $y(t_0) = \tilde{b}$ where $\tilde{a} = \langle(a_1, a_2, a_3, a_4); u_{\widetilde{AN}}, v_{\widetilde{AN}}, w_{\widetilde{AN}}\rangle$ and $\tilde{b} = \langle(b_1, b_2, b_3, b_4); u_{\widetilde{AN}}, v_{\widetilde{AN}}, w_{\widetilde{AN}}\rangle$ are $TrapN_{number}$.

Case $(\boldsymbol{i})$ when k_1 *and* k_2 are positive constant $(ie) k_1, k_2 > 0$
The (α,β,γ)-cut of Eq. (4)& (5) are

$$\frac{d}{dt}[x_1(t,\alpha), x_2(t,\alpha); x_1'(t,\beta), x_2'(t,\beta); x_1''(t,\gamma), x_2''(t,\gamma)]$$
$$= k_1[y_1(t,\alpha), y_2(t,\alpha); y_1'(t,\beta), y_2'(t,\beta); y_1''(t,\gamma), y_2''(t,\gamma)] \quad (6)$$

$$\frac{d}{dt}[y_1(t,\alpha),y_2(t,\alpha);y_1'(t,\beta),y_2'(t,\beta);y_1''(t,\gamma),y_2''(t,\gamma)]$$
$$= k_2[x_1(t,\alpha),x_2(t,\alpha);x_1'(t,\beta),x_2'(t,\beta);x_1''(t,\gamma),x_2''(t,\gamma)] \quad (7)$$

with the initial condition

$$x(t_0;\alpha,\beta,\gamma) = \langle [a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}}, [a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}};\ [a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}}, [a_3 + \beta(a_4-a_3)]v_{\widetilde{A_N}}; [a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}}, [a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}}\rangle \quad (8)$$

and

$$y(t_0;\alpha,\beta,\gamma) = \langle [b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}, [b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}};\ [b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}, [b_3 + \beta(b_4-b_3)]v_{\widetilde{A_N}}; [b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}}, [b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\rangle \quad (9)$$

From (6) we get

$$\left.\begin{aligned} \frac{dx_1(t,\alpha)}{dt} &= k_1 y_1(t,\alpha); \frac{dx_2(t,\alpha)}{dt} = k_1 y_2(t,\alpha) \\ \frac{dx_1'(t,\beta)}{dt} &= k_1 y_1'(t,\beta); \frac{dx_2'(t,\beta)}{dt} = k_1 y_2'(t,\beta) \\ \frac{dx_1''(t,\gamma)}{dt} &= k_1 y_1''(t,\gamma); \frac{dx_2''(t,\gamma)}{dt} = k_1 y_2''(t,\gamma) \end{aligned}\right\} \quad (10)$$

From (7) we get

$$\left.\begin{aligned} \frac{dy_1(t,\alpha)}{dt} &= k_2 x_1(t,\alpha); \frac{dy_2(t,\alpha)}{dt} = k_2 x_2(t,\alpha) \\ \frac{dy_1'(t,\beta)}{dt} &= k_2 x_1'(t,\beta); \frac{dy_2'(t,\beta)}{dt} = k_2 x_2'(t,\beta) \\ \frac{dy_1''(t,\gamma)}{dt} &= k_2 y_1''(t,\gamma); \frac{dy_2''(t,\gamma)}{dt} = k_2 x_2''(t,\gamma) \end{aligned}\right\} \quad (11)$$

with initial conditions

$$\left.\begin{aligned} x_1(t_0,\alpha) &= [a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}}; x_2(t_0,\alpha) = [a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}} \\ x_1'(t_0,\beta) &= [a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}}; x_2'(t_0,\beta) = [a_3+\beta(a_4-a_3)]v_{\widetilde{A_N}} \\ x_1''(t_0,\gamma) &= [a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}};\ x_2''(t_0,\gamma) = [a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} \end{aligned}\right\} \quad (12)$$

and

$$\left.\begin{aligned} y_1(t_0,\alpha) &= [b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}; y_2(t_0,\alpha) = [b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}} \\ y_1'(t_0,\beta) &= [b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}; y_2'(t_0,\beta) = [b_3+\beta(b_4-b_3)]v_{\widetilde{A_N}} \\ y_1''(t_0,\gamma) &= [b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}};\ b_2''(t_0,\gamma) = [b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}} \end{aligned}\right\} \quad (13)$$

From Eqs. (10) and (11) we have

$$\frac{d^2x_1(t,\alpha)}{dt^2} = k_1k_2x_1(t,\alpha) \quad (14)$$

The solution of Eq.(14) is

$$x_1(t,\alpha) = Ae^{\sqrt{k_1k_2}t} + Be^{-\sqrt{k_1k_2}t} \quad (15)$$

substituting Eq.(15) in Eq.(11) , we get

$$Ae^{\sqrt{k_1k_2}t} - Be^{-\sqrt{k_1k_2}t} = \sqrt{\frac{k_1}{k_2}}y_1(t,\alpha) \quad (16)$$

Using initial condition, we get

$$Ae^{\sqrt{k_1k_2}t} + Be^{-\sqrt{k_1k_2}t} = [a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}}$$

and $Ae^{\sqrt{k_1k_2}t} - Be^{-\sqrt{k_1k_2}t} = \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}$

Therefore,

$$\left.\begin{aligned} A &= \frac{1}{2}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}t_0} \\ B &= \frac{1}{2}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}t_0} \end{aligned}\right\} \quad (17)$$

substituting Eq.(17) in Eqs.(12) and (13), we get

$$x_1(t,\alpha) = \frac{1}{2}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (18)$$

$$y_1(t,\alpha) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (19)$$

similarly,

$$x_2(t,\alpha) = \frac{1}{2}\left\{[a_4 - \alpha(a_4 - a_3)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_4 - \alpha(b_4 - b_3)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_4 - \alpha(a_4 - a_3)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_4 - \alpha(b_4 - b_3)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (20)$$

$$y_2(t,\alpha) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_4 - \alpha(a_4 - a_3)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_4 - \alpha(b_4 - b_3)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_4 - \alpha(a_4 - a_3)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_4 - \alpha(b_4 - b_3)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (21)$$

$$x_1'(t,\beta) = \frac{1}{2}\left\{[a_2 - \beta(a_2 - a_1)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2 - \beta(b_2 - b_1)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_2 - \beta(a_2 - a_1)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2 - \beta(b_2 - b_1)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (22)$$

$$y_1'(t,\beta) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2 - \beta(a_2 - a_1)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2 - \beta(b_2 - b_1)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2 - \beta(a_2 - a_1)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2 - \beta(b_2 - b_1)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (23)$$

$$x_2'(t,\beta) = \frac{1}{2}\left\{[a_3 + \beta(a_4 - a_3)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3 + \beta(b_4 - b_3)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_3 + \beta(a_4 - a_3)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3 + \beta(b_4 - b_3)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (24)$$

$$y_2'(t,\beta) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3 + \beta(a_4 - a_3)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3 + \beta(b_4 - b_3)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3 + \beta(a_4 - a_3)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3 + \beta(b_4 - b_3)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (25)$$

$$x_1''(t,\gamma) = \frac{1}{2}\left\{[a_2 - \gamma(a_2 - a_1)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2 - \gamma(b_2 - b_1)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_2 - \gamma(a_2 - a_1)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2 - \gamma(b_2 - b_1)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (26)$$

$$y_1''(t,\gamma) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2 - \gamma(a_2 - a_1)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2 - \gamma(b_2 - b_1)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2 - \gamma(a_2 - a_1)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2 - \gamma(b_2 - b_1)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (27)$$

$$x_2''(t,\gamma) = \frac{1}{2}\left\{[a_3 + \gamma(a_4 - a_3)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3 + \gamma(b_4 - b_3)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_3 + \gamma(a_4 - a_3)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3 + \gamma(b_4 - b_3)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (28)$$

$$y_2''(t,\gamma)=\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}}+\sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}}-\sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (29)$$

Case (ii) when $k_1 and\ k_2$ are negative constants $(ie)k_1,k_2<0$
The general solution of the system of solutions are as follows:

$$x_1(t,\alpha)=\frac{1}{2}\left\{[a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}}-\sqrt{\frac{k_1}{k_2}}[b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}}+\sqrt{\frac{k_1}{k_2}}[b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (30)$$

$$y_2(t,\alpha)=-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}}-\sqrt{\frac{k_1}{k_2}}[b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}}+\sqrt{\frac{k_1}{k_2}}[b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (31)$$

$$x_2(t,\alpha)=\frac{1}{2}\left\{[a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}}-\sqrt{\frac{k_1}{k_2}}[b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}}+\sqrt{\frac{k_1}{k_2}}[b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (32)$$

$$y_1(t,\alpha)=-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}}-\sqrt{\frac{k_1}{k_2}}[b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}}+\sqrt{\frac{k_1}{k_2}}[b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (33)$$

$$x_1'(t,\beta)=\frac{1}{2}\left\{[a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}}-\sqrt{\frac{k_1}{k_2}}[b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}}+\sqrt{\frac{k_1}{k_2}}[b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (34)$$

$$y_2'(t,\beta) = -\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (35)$$

$$x_2'(t,\beta) = \frac{1}{2}\left\{[a_3+\beta(a_4-a_3)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3+\beta(b_4-b_3)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_3+\beta(a_4-a_3)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3+\beta(b_4-b_3)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (36)$$

$$y_1'(t,\beta) = -\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\beta(a_4-a_3)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3+\beta(b_4-b_3)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\beta(a_4-a_3)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3+\beta(b_4-b_3)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (37)$$

$$x_1''(t,\gamma) = \frac{1}{2}\left\{[a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (38)$$

$$y_2''(t,\gamma) = -\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (39)$$

$$x_2''(t,\gamma) = \frac{1}{2}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (40)$$

$$y_1''(t,\gamma) = -\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (41)$$

3.2 Numerical Example:

Consider a system of differential equation $\frac{dx}{dt} = 4y$ and $\frac{dy}{dt} = 5x$ with initial conditions $x(t_0) = \tilde{a} = \langle(3,4,5,6); 0.7, 0.6, 0.4\rangle$ and $y(t_0) = \tilde{b} = \langle(5,6,7,8); 0.5, 0.3, 0.2\rangle$.

Solution:

The solution is given by the equations

$$x_1(t,\alpha) = \frac{1}{2}\left\{[3+\alpha]0.7 + \sqrt{\frac{4}{5}}[5+\alpha]0.5\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[3+\alpha]0.7 - \sqrt{\frac{4}{5}}[5+\alpha]0.5\right\}e^{-\sqrt{20}t}$$

$$y_1(t,\alpha) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[3+\alpha]0.7 + \sqrt{\frac{4}{5}}[5+\alpha]0.5\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[3+\alpha]0.7 - \sqrt{\frac{4}{5}}[5+\alpha]0.5\right\}e^{-\sqrt{20}t}$$

$$x_2(t,\alpha) = \frac{1}{2}\left\{[6-\alpha]0.7 + \sqrt{\frac{4}{5}}[8-\alpha]0.5\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[6-\alpha]0.7 - \sqrt{\frac{4}{5}}[8-\alpha]0.5\right\}e^{-\sqrt{20}t}$$

$$y_2(t,\alpha) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[6-\alpha]0.7 + \sqrt{\frac{4}{5}}[8-\alpha]0.5\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[6-\alpha]0.7 - \sqrt{\frac{4}{5}}[8-\alpha]0.5\right\}e^{-\sqrt{20}t}$$

$$x_1'(t,\beta) = \frac{1}{2}\left\{[4-\beta]0.6 + \sqrt{\frac{4}{5}}[6-\beta]0.3\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[4-\beta]0.6 - \sqrt{\frac{4}{5}}[6-\beta]0.3\right\}e^{-\sqrt{20}t}$$

$$y_1'(t,\beta) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[4-\beta]0.6 + \sqrt{\frac{4}{5}}[6-\beta]0.3\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[4-\beta]0.6 - \sqrt{\frac{4}{5}}[6-\beta]0.3\right\}e^{-\sqrt{20}t}$$

$$x_2'(t,\beta) = \frac{1}{2}\left\{[5+\beta]0.6 + \sqrt{\frac{4}{5}}[7+\beta]0.3\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[5+\beta]0.6 - \sqrt{\frac{4}{5}}[7+\beta]0.3\right\}e^{-\sqrt{20}t}$$

$$y_2'(t,\beta) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[5+\beta]0.6 + \sqrt{\frac{4}{5}}[7+\beta]0.3\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[5+\beta]0.6 - \sqrt{\frac{4}{5}}[7+\beta]0.3\right\}e^{-\sqrt{20}t}$$

$$x_1''(t,\gamma) = \frac{1}{2}\left\{[4-\gamma]0.4 + \sqrt{\frac{4}{5}}[6-\gamma]0.2\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[4-\gamma]0.4 - \sqrt{\frac{4}{5}}[6-\gamma]0.3\right\}e^{-\sqrt{20}t}$$

$$y_1''(t,\gamma) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[4-\gamma]0.4 + \sqrt{\frac{4}{5}}[6-\gamma]0.3\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[4-\gamma]0.4 - \sqrt{\frac{4}{5}}[6-\gamma]0.3\right\}e^{-\sqrt{20}t}$$

and

$$x_2''(t,\gamma) = \frac{1}{2}\left\{[5+\gamma]0.4 + \sqrt{\frac{4}{5}}[7+\gamma]0.2\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[5+\gamma]0.4 - \sqrt{\frac{4}{5}}[7+\gamma]0.3\right\}e^{-\sqrt{20}t}$$

$$y_2''(t,\gamma) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[5+\gamma]0.4 + \sqrt{\frac{4}{5}}[7-\gamma]0.3\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[5+\gamma]0.4 - \sqrt{\frac{4}{5}}[7+\gamma]0.3\right\}e^{-\sqrt{20}t}$$

Table 1. $\boldsymbol{TrapN_{number}}$ **solution of** $\boldsymbol{x(t,\ \alpha,\beta,\gamma)}$ **at** $\boldsymbol{t = 0.5}$

α	$x_1(t,\alpha)$	$x_2(t,\alpha)$	β	$x_1'(t,\beta)$	$x_2'(t,\beta)$	γ	$x_1''(t,\gamma)$	$x_2''(t,\gamma)$
0.1	52.45255	90.30831	0.1	46.23416	58.51406	0.1	30.82277	39.00937
0.2	53.80454	88.95632	0.2	45.21083	59.53738	0.2	30.14056	39.69159
0.3	55.15653	87.60432	0.3	44.18751	60.56071	0.3	29.45834	40.3738
0.4	56.50853	86.25233	0.4	43.16419	61.58403	0.4	28.77612	41.05602
0.5	57.86052	84.90034	0.5	42.14086	62.60736	0.5	28.09391	41.73824
0.6	59.21251	83.54835	0.6	41.11754	63.63068	0.6	27.41169	42.42045
0.7	60.5645	82.19369	0.7	40.09421	64.654	0.7	26.72947	43.10267
0.8	61.91649	80.84437	0.8	39.07089	65.67733	0.8	26.04726	43.78489
0.9	63.26848	79.49238	0.9	38.04756	66.70065	0.9	25.36504	44.4671
1	64.62047	78.14039	1	37.02424	67.72398	1	24.68282	45.14932

Table 2. $\boldsymbol{TrapN_{number}}$ **solution of** $\boldsymbol{y(t,\ \alpha,\beta,\gamma)}$ **at** $\boldsymbol{t = 0.5}$

α	$y_1(t,\alpha)$	$y_2(t,\alpha)$	β	$y_1'(t,\beta)$	$y_2'(t,\beta)$	γ	$y_1''(t,\gamma)$	$y_2''(t,\gamma)$
0.1	46.91499	80.7742	0.1	41.35309	52.33656	0.1	33.13125	41.01924
0.2	48.12425	79.56495	0.2	40.4378	53.25185	0.2	32.42677	41.15803
0.3	49.3335	78.35569	0.3	39.52251	54.16714	0.3	31.7223	41.29683
0.4	50.54276	77.14643	0.4	38.60722	55.08243	0.4	31.01783	41.43562
0.5	51.75202	75.93717	0.5	37.69193	55.99772	0.5	30.31335	41.57441
0.6	52.96128	74.72792	0.6	36.77664	56.91301	0.6	29.60888	41.71321
0.7	54.17054	73.51866	0.7	35.86135	57.8283	0.7	28.90441	41.852
0.8	55.37979	72.3094	0.8	34.94606	58.74359	0.8	28.19994	41.99079
0.9	56.58905	71.10014	0.9	34.03077	59.65888	0.9	27.49546	42.12959
1	57.79831	69.89089	1	33.11548	60.57417	1	26.79099	42.26838

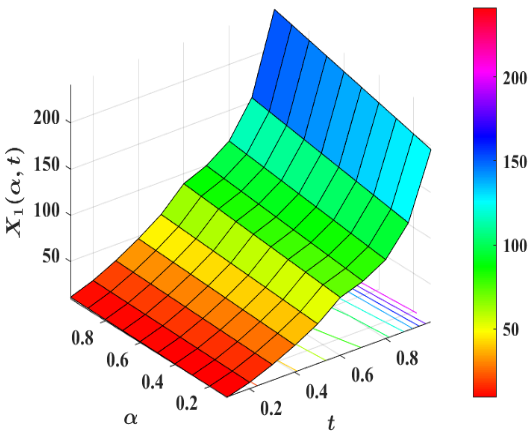

Fig. 1. $x_1(\alpha, t)$ **for increasing values of** ***t***

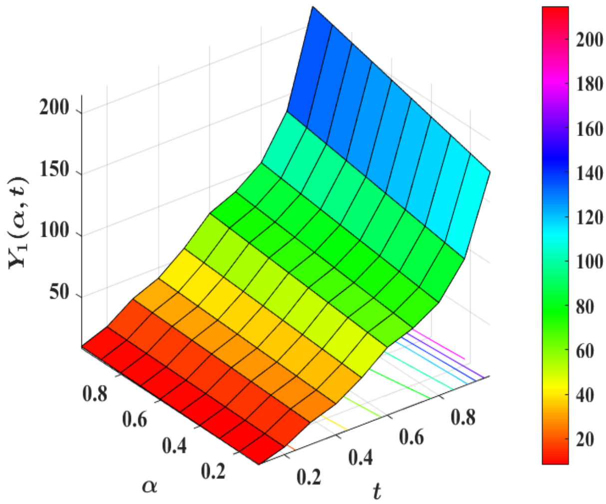

Fig. 2. $y_1(\alpha, t)$ **for increasing values of** ***t***

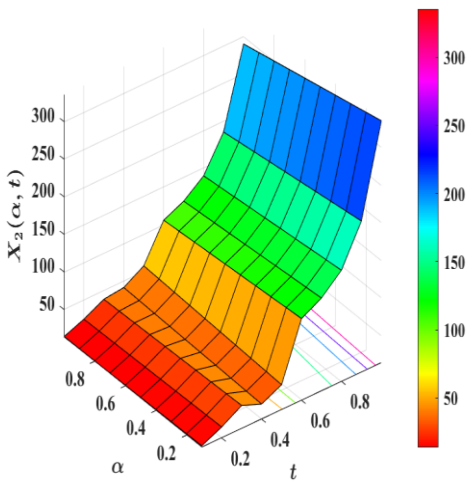

Fig. 3. $x_2(\alpha, t)$ **for increasing values of** ***t***

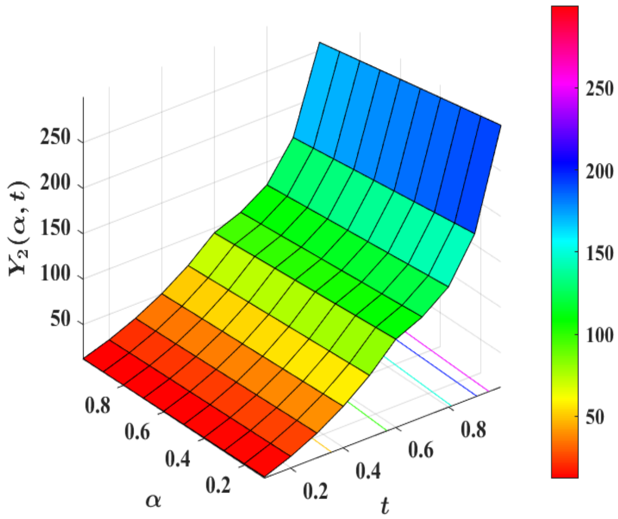

Fig. 4. $y_2(\alpha, t)$ **for increasing values of** ***t***

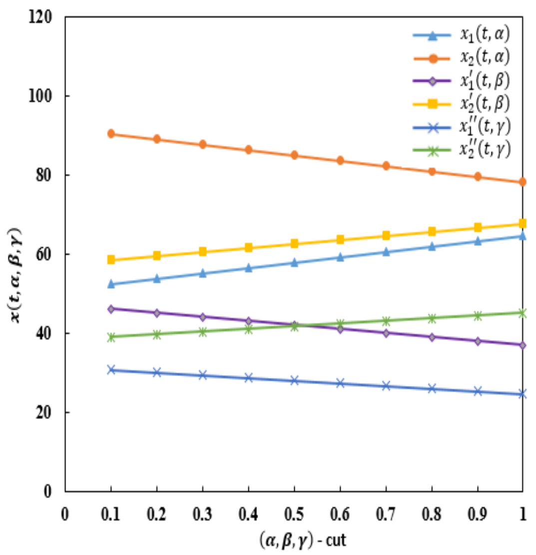

Fig. 5. Solution of $x(t,\ \alpha,\beta,\gamma)$ at $t = 0.5$

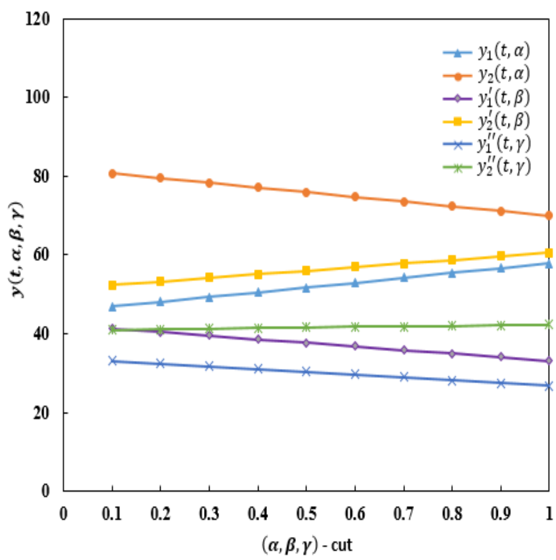

Fig. 6. Solution of $y(t,\ \alpha,\beta,\gamma)$ at $t = 0.5$

3.3 Results and Discussion

The system of first-order simultaneous differential equations (SFOSDE) with initial conditions is a powerful tool used in various scientific and engineering disciplines to model and analyze dynamic systems. These equations capture relationships between different variables and how they change over time, providing insights into the behavior of complex systems. SFOSDEs are used to describe the flow of fluids in various scenarios, like incompressible or compressible flows, turbulence, and boundary layer analysis. These equations can model the heat transfer, energy exchange, and temperature changes in systems governed by the laws of thermodynamics. SFOSDEs can be employed to model reaction rates, concentrations, and reaction pathways. Differential equations help in understanding growth rates, interactions between different species, population dynamics, and the spread of diseases in biological and ecological systems. Uncertainty is a common aspect in various scientific and engineering applications, and it's important to consider it when modeling real-world systems. In mathematical modeling, uncertainty can be dealt with in various ways, depending on the specific context and the type of uncertainty being considered. Solving SFOSDEs with neutrosophic inputs would involve extending traditional solution methods to handle neutrosophic values. This might involve developing new numerical techniques or analytical approaches that can accommodate the three aspects of membership in neutrosophic sets.

The solutions of a system of differential equation $\frac{dx}{dt} = 4y$ and $\frac{dy}{dt} = 5x$ with initial conditions $x(t_0) = \tilde{a} = \langle(3,4,5,6); 0.7, 0.6, 0.4\rangle$ and $y(t_0) = \tilde{b} =$

$\langle(5,6,7,8); 0.5, 0.3, 0.2\rangle$ for various t and $0 < \alpha, \beta, \gamma \leq 1$ are depicted in Tables 1 and 2 Figures 1-4. It is observed that, the increasing values of α, β, γ increase $x_1(\alpha, t)$, $y_1(\alpha, t)$ and decrease $x_2(\alpha, t)$, $y_2(\alpha, t)$ where as $x_1'(\alpha, t)$, $y_1'(\alpha, t)$and $x_1''(\alpha, t)$, $y_1''(\alpha, t)$ are diminishing functions and $x_2'(\alpha, t)$, $y_2'(\alpha, t)$and $x_2''(\alpha, t)$, $y_2''(\alpha, t)$ are escalation functions. Therefore, the numerical example of system of differential equation satisfies the conditions of the $Strong_{solution}$ of a neutrosophic difference equation. Hence the obtained solution is a strong solution. In addition, the graphs for different values for $T_{membership}$ $(T_{\widetilde{A_N}}(x))$, $I_{membership}$ $(I_{\widetilde{A_N}}(x))$, and $F_{membership}$ $(F_{\widetilde{A_N}}(x))$ functions with $(\alpha, \beta, \gamma)_{cut}$ at time $t = 0$ are displayed in Figures 5 and 6. As the α_{cut} value enhances and $\beta_{cut}, \gamma_{cut}$ values diminishes the solutions of $x(t, \alpha, \beta, \gamma)$ and $y(t, \alpha, \beta, \gamma)$ which approaches to the exact solution.

Conclusion

A technique for approximating the solution of system of first order simultaneous differential equations with initial condition as a neutrosophic environment is presented in this proposed work. We used the initial conditions as trapezoidal neutrosophic numbers. Furthermore, numerical examples are illustrated for better understanding of solving SFOSD utilizing $TrapN_{number}$ via MATLAB software. The simulation results are shown in Tables and Figures. From these it is noticed that the new technique using neutrosophic are effective and more flexible to estimate the solutions of SFOSD. This technique is used to develop the solution of highly nonlinear coupled ordinary differential equations in the field of Computational Fluid Mechanics.

Future Work:

In future, one can extend this technique to solve higher-order linear and nonlinear neutrosophic initial value problems. Also, we will focus on higher-order nonlinear coupled partial differential equations and their applications in neutrosophic environments.

Table 3. Notations

C_{set}	Classical set
F_{set}	Fuzzy Set
IF_{set}	Intuitionistic Fuzzy Set
IVF_{set}	Interval Valued Fuzzy Set
N_{set}	Neutrosophic set
N_{logic}	Neutrosophic logic

SVN_{set}	Single Values Neutrosophic set
SVN_{number}	Single Values Neutrosophic number
$TriN_{number}$	Triangular Neutrosophic number
$TrapN_{number}$	Trapezoidal Neutrosophic number
$T_{value}, I_{value,} and\ F_{value}$	membership, indeterminacy, and non-membership values
$T_{membership}, I_{membership}, F_{membership}$	truth, indeterminacy, and falsity membership
$T_{function}, I_{function}, and\ F_{functuion}$	truth, indeterminacy, and falsity function
$(\alpha,\beta,\gamma)_{cut}$	$(\alpha,\beta,\gamma) - cut$
$(\alpha,\beta,\gamma)_{cut}\ of\ a\ TrapN_{number}$	$(\alpha,\beta,\gamma) - cut$ of a Trapezoidal Neutrosophic number

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