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On $\mathcal{S}_\theta$-summability in neutrosophic soft normed linear spaces

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Abstract. For any lacunary sequence $\theta = (k_s)$, the aim of the present paper is to define $\mathcal{S}_\theta$-convergence, $\mathcal{S}_\theta$-Cauchy and $\mathcal{S}_\theta$-completeness via neutrosophic soft norm. We study certain properties of these notions and give an important characterization of $\mathcal{S}_\theta$-convergence in neutrosophic soft normed linear spaces (briefly $NSNLS$). We provide examples that shows $\mathcal{S}_\theta$-convergence is a more general method of summability in these spaces.

Keywords: S_θ-convergence, S_θ-Cauchy, soft sets, soft normed linear spaces.

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1. **Introduction**

Statistical convergence was first introduced by Fast [8] and linked with the summability theory by Schoenberg [10]. Later, The idea is developed by Maddox [9], Fridy [12], Connor[13], Mursaleen and Edely [18], Šalát [32], Kumar and mursaleen [35] and many others.

Friday and Orhan [11] used lacunary sequences to define a new kind of statistical convergence as follows. "By a lacunary sequence we mean an increasing integer sequence $\theta = (k_s)$ with $k_0 = 0$ and $h_s = k_s - k_{s-1} \to \infty$ as $s \to \infty$. The intervals determined by θ will be denoted by $I_s = (k_{s-1}, k_s]$ and the ratio $\frac{k_s}{k_{s-1}}$ will be abbreviated as q_s. For $K \subseteq \mathbb{N}$, the number $\delta_\theta(K) = \lim_{s\to\infty} \dfrac{1}{h_s}|\{k \in I_s : k \in K\}|$ is called θ-density of K, provided the limit exists. A sequence $x = (x_k)$ of numbers is said to be lacunary statistically convergent (briefly S_θ-convergent) to x_0 if for every $\epsilon > 0$, $\lim_s \frac{1}{h_s}|\{k \in I_s : |x_k - x_0| \geq \epsilon\}| = 0$ or equivalently, the set $K(\epsilon)$ has θ-density zero, where $K(\epsilon) = \{k \in \mathbb{N} : |x_k - x_0| \geq \epsilon\}$. In this case, we write $\mathcal{S}_\theta - \lim_{k\to\infty} x_k = x_0$."Some further interesting works on lacunary statistical convergence can be found in [4], [19], [25], [34], [36], etc.

Zadeh [16] proposed the theory of fuzzy sets in 1965 as a more convenient tool for handling issues that cannot be modelled via crisp set theory. Atanassov [15] observed that fuzzy sets need more modification to handle problems in a time domain and therefore he introduced the intuitionistic fuzzy sets. After the introduction of intuitionistic fuzzy sets, a progressive development is made in this field. For instance, intuitionistic fuzzy metric spaces were introduced by Park [14], intuitionistic fuzzy topological spaces by Saadati and Park [26], etc.

The neutrosophic sets were initially introduced by Smarandache[7] as a generalization of fuzzy sets and intuitionistic fuzzy sets to avoid the complexity arising from uncertainty in settling many practical challenges in real-world activities. Kirişçi and Şimşek[17] defined neutrosophic norm and studied statistical convergence in neutrosophic normed spaces(NNS). For a broad view in this direction, we recommend to the reader [1], [2], [3], [20], [21], [22], [33].

Many approaches discussed above to minimize the uncertainty have their own drawbacks due to the inadequacy of the parametrization. In view of this, Molodtsov[6] proposed a new theory, called soft set theory to reduce the uncertainty during mathematical modelling. These sets turn out very useful tools in many areas of engineering and medical sciences. For instance: Maji et al [23] applied the theory of soft sets to decision-making problems. Kong et al.[39] presented a heuristic algorithm of normal parameter reduction of soft sets. Zou and Xiao[38] presented a data analysis approach of soft sets under incomplete information. Yuksel et al.[30] applied soft set theory to diagnose the prostate cancer risk in human beings whereas Çelik and Yamak[37] applied fuzzy soft set theory for medical diagnosis using fuzzy arithmetic operations.

Maji [24] presented a combined concept of Neutrosophic soft sets in 2013. Recently, Bera and Mahapatra [31] defined a generalized norm and called it a neutrosophic soft norm. They also studied some properties of $NSNLS$ and developed fundamental concepts of sequences in these spaces. In this article, we develop and study the concept of $\mathcal{S}_\theta$-convergence in $NSNLS$. We also introduce the concepts of $\mathcal{S}_\theta$-Cauchy sequence, $\mathcal{S}_\theta$-completeness and develop some of their properties.

2. **Preliminaries**

This section starts with a brief information on soft sets, soft vector spaces and neutrosophic soft normed spaces. We begin with the following notations and definitions.

Throughout this work, $\mathbb{N}$ will denote the set of positive integers, $\mathbb{R}$ the set of reals and $\mathbb{R}^+$ the set of positive real numbers.

Definition 2.1 [5] A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $\circ$ satisfies the following conditions:

(i) $d \circ e = e \circ d$ and $d \circ (e \circ f) = (d \circ e) \circ f$.

(ii) $\circ$ is continuous.

(iii) $d \circ 1 = 1 \circ d = d$ for all $d \in [0,1]$.

(iv) $d \circ e \leq f \circ g$ if $d \leq f, e \leq g$ with $d, e, f, g \in [0,1]$.

Definition 2.2 [5] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm(s-norm) if $\diamond$ satisfies the following conditions:

(i) $d \diamond e = e \diamond d$ and $d \diamond (e \diamond f) = (d \diamond e) \diamond f$.

(ii) $\diamond$ is continuous.

(iii) $d \diamond 0 = 0 \diamond d =$ for all $d \in [0,1]$.

(iv) $d \diamond e \leq f \diamond g$ if $d \leq f, e \leq g$ with $d, e, f, g \in [0,1]$.

For any universe set U and the set E of the parameters, the soft set is defined as follows:

Definition 2.3 [6] A pair (H, E) is called a soft set over U if and only if H is a mapping of E into the set of all subsets of the set U. i.e., the soft set is a parametrized family of subsets of the set U.

Moreover, every set $H(\epsilon), \epsilon \in E$, from this family may be considered as the set of ϵ-elements of the soft set (H, E), or as the set of ϵ-approximate elements of the set.

Definition 2.4 [6] A soft set (H, E) over U is said to be absolute soft set if for all $\epsilon \in E$, $H(\epsilon) = U$. We will denote it by $\widetilde{U}$.

Definition 2.5 [27] Let $\mathbb{R}$ be the set of real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E taken as a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, then it is called a soft real number and denoted by $\widetilde{r}, \widetilde{s}, \widetilde{t}$, etc. $\widetilde{0}, \widetilde{1}$ are the soft real numbers where $\widetilde{0}\ (e) = 0, \widetilde{1}\ (e) = 1$ for all $e \in E$ respectively.

Let $\mathbb{R}(E)$ and $\mathbb{R}^+(E)$ respectively denote the sets of all soft real numbers and all positive soft real numbers.

Definition 2.6 [28] Let (H, E) be a soft set over U. The set (H, E) is said to be a soft point, denoted by H_e^u if there is exactly one $e \in E$ s.t $H(e) = \{u\}$ for some $u \in U$ and $H(e^{'}) = \phi$ for all $e^{'} \in E - \{e\}$.

Two soft points $H_e^u, H_{e'}^w$ are said to be equal if $e = e^{'}$ and $u = w$. Let $\Delta_{\widetilde{U}}$ denotes the set of all soft points on $\widetilde{U}$.

In case U is a vector space over $\mathbb{R}$ and the parameter set $E = \mathbb{R}$, the soft point is called a soft vector.

Soft vector spaces are used to define soft norm as follows:

Definition 2.7 [29] Let $\widetilde{U}$ be a absolute soft vector space. Then a mapping $\| \cdot \| : \widetilde{U} \to \mathbb{R}^+(E)$ is said to be a soft norm on $\widetilde{U}$, if $\| \cdot \|$ satisfies the following conditions:

(i) $\|u_e\| \geq \widetilde{0}$ for all $u_e \in \widetilde{U}$ and $\|u_e\| = \widetilde{0} \Leftrightarrow u_e = \widetilde{\theta}_0$ where $\widetilde{\theta}_0$ denotes the zero element of $\widetilde{U}$.

(ii) $\| \widetilde{\alpha}\ u_e\| = |\widetilde{\alpha}|\|u_e\|$ for all $u_e \in \widetilde{U}$ and for every soft scalar $\widetilde{\alpha}$.

(iii) $\|u_e + v_{e'}\| \leq \|u_e\| + \|v_{e'}\|$ for all $u_e, v_{e'} \in \widetilde{U}$.

(iv) $\|u_e \cdot v_{e'}\| = \|u_e\| \; \|v_{e'}\|, \forall \; u_e, v_{e'} \in \widetilde{U}$.
The soft vector space $\widetilde{U}$ with a soft norm $\|\cdot\|$ on $\widetilde{U}$ is said to be a soft normed linear space and is denoted by $(\widetilde{U}, \|\cdot\|)$.

We now recall the definition of neutrosophic soft normed linear spaces and the convergence structure in these spaces.

Definition 2.8 [31] Let $\widetilde{U}$ be a soft linear space over the field F and $\mathbb{R}(E), \Delta_{\widetilde{U}}$ denote respectively, the set of all soft real numbers and the set of all soft points on $\widetilde{U}$. Then a neutrosophic subset N over $\Delta_{\widetilde{U}} \times \mathbb{R}(E)$ is called a neutrosophic soft norm on $\widetilde{U}$ if for $u_e, v_{e'} \in \widetilde{U}$ and $\widetilde{\alpha} \in F$ ($\widetilde{\alpha}$ being soft scalar), the following conditions hold.
(i) $0 \leq G_N(u_e, \widetilde{\eta_1}), B_N(u_e, \widetilde{\eta_1}), Y_N(u_e, \widetilde{\eta_1}) \leq 1, \forall \; \widetilde{\eta_1} \in \mathbb{R}(E)$.
(ii) $0 \leq G_N(u_e, \widetilde{\eta_1}) + B_N(u_e, \widetilde{\eta_1}) + Y_N(u_e, \widetilde{\eta_1}) \leq 3, \forall \; \widetilde{\eta_1} \in \mathbb{R}(E)$.
(iii) $G_N(u_e, \widetilde{\eta_1}) = 0$ with $\widetilde{\eta_1} \leq \widetilde{0}$.
(iv) $G_N(u_e, \widetilde{\eta_1}) = 1$, with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.
(v) $G_N(\widetilde{\alpha} \; u_e, \widetilde{\eta_1}) = G_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall \; \widetilde{\alpha} \; (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$.
(vi) $G_N(u_e, \widetilde{\eta_1}) \circ G_N(v_{e'}, \widetilde{\eta_2}) \leq G_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2}), \forall \; \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$
(vii) $G_N(u_e, \cdot)$ is continuous non-decreasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} G_N(u_e, \widetilde{\eta_1}) = 1$.
(viii) $B_N(u_e, \widetilde{\eta_1}) = 1$ with $\widetilde{\eta_1} \leq \widetilde{0}$.
(ix) $B_N(u_e, \widetilde{\eta_1}) = 0$, with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.
(x) $B_N(\widetilde{\alpha} \; u_e, \widetilde{\eta_1}) = B_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall \; \widetilde{\alpha} \; (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$.
(xi) $B_N(u_e, \widetilde{\eta_1}) \diamond B_N(v_{e'}, \widetilde{\eta_2}) \geq B_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2}) \; \forall \; \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$.
(xii) $B_N(u_e, \cdot)$ is continuous non-increasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} B_N(u_e, \widetilde{\eta_1}) = 0$.
(xiii) $Y_N(u_e, \widetilde{\eta_1}) = 0$ with $\widetilde{\eta_1} \leq \widetilde{0}$.
(xiv)$Y_N(u_e, \widetilde{\eta_1}) = 0$, with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.
(xv) $Y_N(\widetilde{\alpha} \; u_e, \widetilde{\eta_1}) = Y_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall \; \widetilde{\alpha} \; (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$.
(xvi) $Y_N(u_e, \widetilde{\eta_1}) \diamond Y_N(v_{e'}, \widetilde{\eta_2}) \geq Y_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2}) \; \forall \; \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$.
(xvii) $Y_N(u_e, \cdot)$ is continuous non-increasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} B_N(u_e, \widetilde{\eta_1}) = 0$.

In this case, $\mathcal{N} = (G_N, B_N, Y_N)$ is called the neutrosophic soft norm and $(\widetilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$ is the neutrosophic soft normed linear space ($NSNLS$ briefly).

Let $(\widetilde{U}, \|\cdot\|)$ be a soft normed space. Take the operations $\circ$ and $\diamond$ as $x \circ y = xy$; $x \diamond y = x + y - xy$. For $\widetilde{\eta} > \widetilde{0}$, define

$$G_N(u_e, \widetilde{\eta}) = \begin{cases} \frac{\widetilde{\eta}}{\widetilde{\eta} + \|u_e\|} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise} \end{cases}$$

$$B_N(u_e,\widetilde{\eta}) = \begin{cases} \frac{\|u_e\|}{\widetilde{\eta}+\|u_e\|} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise} \end{cases}$$

$$Y_N(u_e,\widetilde{\eta}) = \begin{cases} \frac{\|u_e\|}{\widetilde{\eta}} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise,} \end{cases}$$

then $(\widetilde{U}\ (F), G_N, B_N, Y_N, \circ, \diamond)$ is an $NSNLS$. From now onwards, unless otherwise stated by $\widetilde{V}$ we shall denote the $NSNLS$ $(\widetilde{U}\ (F), G_N, B_N, Y_N, \circ, \diamond)$.

Definition 2.9 [31] A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be convergent to a soft point $v_e \in \widetilde{V}$ if for $0 < \epsilon < 1$ and $\widetilde{\eta} > \widetilde{0}$ $\exists$ $n_0 \in \mathbb{N}$ s.t $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon,\ B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon,\ Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon$. In this case, we write $\lim_{k\to\infty} v^k_{e_k} = v_e$.

Definition 2.10 [31] A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be cauchy sequence if for $0 < \epsilon < 1$ and $\widetilde{\eta} > \widetilde{0}$ $\exists$ $n_0 \in \mathbb{N}$ s.t for all $k, p \geq n_0$ $G_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) > 1-\epsilon,\ B_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) < \epsilon,\ Y_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) < \epsilon$.

3. **Lacunary statistical convergence in NSNLS**

In this section, we define $\mathcal{S}_\theta$-convergence in neutrosophic soft normed linear spaces and develop some of its properties.

Definition 3.1 A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be lacunary statistical convergent or $\mathcal{S}_\theta$-convergent to a soft point v_e in $\widetilde{V}$ w.r.t neutrosophic soft norm-(G_N, B_N, Y_N) if for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$,

$$\lim_{s\to\infty} \frac{1}{h_s} \Bigg| \Bigg\{ k \in I_s : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon \Bigg\} \Bigg| = 0,$$

i.e., $\delta_\theta(A) = 0$ where

$$A = \{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\}.$$

In this case, we write $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$.

Let, $S_\theta(G_N, B_N, Y_N)$ denotes the set of all sequences of soft points in $\widetilde{V}$ which are S_θ-convergent with respect to the neutrosophic soft norm (G_N, B_N, Y_N).

Definition 3.1 together with the property of θ-density, we have the following lemma.

Lemma 3.1 For any sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$, the following statements are equivalent:

(i) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$;

(ii) $\delta_\theta\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\epsilon\} = \delta_\theta\{k \in \mathbb{N} : B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\} = \delta_\theta\{k \in \mathbb{N} : Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\} = 0$;

(iii) $\delta_\theta\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = 1$;

(iv)$\delta_\theta\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon\} = \delta_\theta\{k \in \mathbb{N} : B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = \delta_\theta\{k \in \mathbb{N} : Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = 1$;

(v) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 1$ and $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 0$, $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 0$.

Theorem 3.1 Let $\theta = (k_s)$ be a lacunary sequence and $v = (v^k_{e_k})$ be any sequence in $\widetilde{V}$. If $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k}$ exists, then it is unique.

Proof. Suppose that $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n\to\infty} v^k_{e_k} = v_{e_1}$ and $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n\to\infty} v^k_{e_k} = v^{'}_{e_2}$, where $v_{e_1} \neq v^{'}_{e_2}$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\varrho > 0$ s.t.

$$(1-\varrho) \circ (1-\varrho) > 1-\epsilon \text{ and } \varrho \diamondsuit \varrho < \epsilon \tag{1}$$

Define the following sets:

$$H_{G_N,1}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : G_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \leq 1-\varrho\right\}.$$

$$H_{G_N,2}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : G_N\left(v^k_{e_k} \ominus v^{'}_{e_2}, \frac{\widetilde{\eta}}{2}\right) \leq 1-\varrho\right\}.$$

$$H_{B_N,1}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : B_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

$$H_{B_N,2}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : B_N\left(v^k_{e_k} \ominus v^{'}_{e_2}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

$$H_{Y_N,1}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : Y_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

$$H_{Y_N,2}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : Y_N\left(v^k_{e_k} \ominus v^{'}_{e_2}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

Since $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_{e_1}$, then using lemma 3.1, we have

$\delta_\theta\{H_{G_N,1}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H_{B_N,1}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H_{Y_N,1}(\varrho, \widetilde{\eta})\} = 0$ and therefore $\delta_\theta\{H^C_{G_N,1}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H^C_{B_N,1}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H^C_{Y_N,1}(\varrho, \widetilde{\eta})\} = 1$.

Further, $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v^{'}_{e_2}$, so

$\delta_\theta\{H_{G_N,2}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H_{B_N,2}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H_{Y_N,2}(\varrho, \widetilde{\eta})\} = 0$ and therefore $\delta_\theta\{H^C_{G_N,2}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H^C_{B_N,2}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H^C_{Y_N,2}(\varrho, \widetilde{\eta})\} = 1$ for all $\widetilde{\eta} > \widetilde{0}$. Now define

$$\begin{aligned} K_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta}) = &\{H_{G_N,1}(\varrho, \widetilde{\eta}) \cup H_{G_N,2}(\varrho, \widetilde{\eta})\} \\ &\cap\{H_{B_N,1}(\varrho, \widetilde{\eta}) \cup H_{B_N,2}(\varrho, \widetilde{\eta})\} \cap \{H_{Y_N,1}(\varrho, \widetilde{\eta}) \cup H_{Y_N,2}(\varrho, \widetilde{\eta})\}, \end{aligned}$$

then $\delta_\theta\{K_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\} = 0$ and therefore, $\delta_\theta\{K^C_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\} = 1$. Let $m \in K^C_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})$, then we have following possibilities.

1. $m \in \left\{H_{G_N,1}(\varrho,\widetilde{\eta}) \cup H_{G_N,2}(\varrho,\widetilde{\eta})\right\}^C$; or
2. $m \in \left\{H_{B_N,1}(\varrho,\widetilde{\eta}) \cup H_{B_N,2}(\varrho,\widetilde{\eta})\right\}^C$; or
3. $m \in \left\{H_{Y_N,1}(\varrho,\widetilde{\eta}) \cup H_{Y_N,2}(\varrho,\widetilde{\eta})\right\}^C$.

Case 1: Let $m \in \left\{H_{G_N,1}(\varrho,\widetilde{\eta}) \cup H_{G_N,2}(\varrho,\widetilde{\eta})\right\}^C$, then $m \in H^C_{G_N,1}(\varrho,\widetilde{\eta})$ and $m \in H^C_{G_N,2}(\varrho,\widetilde{\eta})$ and therefore,

$$G_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) > 1-\varrho \text{ and } G_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) > 1-\varrho. \quad (2)$$

Now

$$\begin{aligned} G_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= G_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\ &> (1-\varrho) \circ (1-\varrho) \qquad \text{by (2)} \\ &> 1-\epsilon. \qquad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $G_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 1$ for all $\widetilde{\eta} > \widetilde{0}$, which gives $v_{e_1} = v'_{e_2}$.

Case 2: Let $m \in \left\{H_{B_N,1}(\varrho,\widetilde{\eta}) \cup H_{B_N,2}(\varrho,\widetilde{\eta})\right\}^C$, then $m \in H^C_{B_N,1}(\varrho,\widetilde{\eta}))$ and $m \in H^C_{B_N,2}(\varrho,\widetilde{\eta})$ and therefore,

$$B_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) < \varrho \text{ and } B_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) < \varrho. \quad (3)$$

Now

$$\begin{aligned} B_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= B_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq B_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \qquad \text{by (3)} \\ &< \epsilon. \qquad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $B_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 0$ for all $\widetilde{\eta} > \widetilde{0}$, which gives $v_{e_1} = v'_{e_2}$.

Case 3: Let $m \in \left\{H_{Y_N,1}(\varrho,\widetilde{\eta}) \cup H_{Y_N,2}(\varrho,\widetilde{\eta})\right\}^C$, then $m \in H^C_{Y_N,1}(\varrho,\widetilde{\eta})$ and $m \in H^C_{Y_N,2}(\varrho,\widetilde{\eta})$ and therefore,

$$Y_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) < \varrho \text{ and } Y_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) < \varrho. \quad (4)$$

Now

$$\begin{aligned} Y_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= Y_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq Y_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \qquad \text{by (4)} \\ &< \epsilon. \qquad\qquad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $Y_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 0$ for all $\widetilde{\eta} > \widetilde{0}$, which gives $v_{e_1} = v'_{e_2}$. Hence, in all cases we have $v_{e_1} = v'_{e_2}$, i.e., $\mathcal{S}_\theta(G_N, B_N, Y_N)$-limit of $(v^k_{e_k})$ is unique.□

Theorem 3.2 Let $\theta = (k_s)$ be a lacunary sequence and $v = (v^k_{e_k})$ be any sequence in $\widetilde{V}$. If $(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$, then $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$.

Proof. Let $(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$. Then for each $\epsilon > 0$ and $\eta > 0, \exists$ positive integers $k_0 \in \mathbb{N}$ s.t $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1 - \epsilon$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\ \forall\ k > k_0$. Hence, the set

$$A = \big\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\big\}$$

has finite number of terms. Since every finite subset of $\mathbb{N}$ has θ-density zero and hence

$$\delta_\theta\Big(\big\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\big\}\Big) = 0.$$

Therefore, $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$. □

The following example shows that the converse of the above theorem need not be true.

Example 3.1 Let $(\widetilde{\mathbb{R}}, \|\cdot\|)$ be a soft normed linear space. For v_e in $\widetilde{\mathbb{R}}$ and $\widetilde{\eta} > \widetilde{0}$, if we define

$$G_N(v_e, \widetilde{\eta}) = \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus \|v_e\|},\ B_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta} \oplus \|v_e\|},\ Y_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta}},$$

$x \circ y = xy$ and $x \diamond y = \min\{x+y, 1\}$, then it is easy to see that $\widetilde{V} = (\widetilde{\mathbb{R}}, G_N, B_N, Y_N, \circ, \diamond)\ \forall\, x, y \in [0, 1]$ is a neutrosophic soft normed linear space.

Now define a sequence $v = (v^k_{e_k})$ in $\widetilde{V}$ by

$$v^k_{e_k} = \begin{cases} \widetilde{k} & \text{if } k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s, s \in \mathbb{N} \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

Now, for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, let

$$\begin{aligned} A(\epsilon, \widetilde{\eta}) &= \left\{ k \in I_s : G_N(v^k_{e_k}, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k}, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k}, \widetilde{\eta}) \geq \epsilon \right\} \\ &= \left\{ k \in I_s : \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus \|v^k_{e_k}\|} \leq 1 - \epsilon \text{ or } \frac{\|v^k_{e_k}\|}{\widetilde{\eta} \oplus \|v^k_{e_k}\|} \geq \epsilon, \frac{\|v^k_{e_k}\|}{\widetilde{\eta}} \geq \epsilon \right\} \\ &= \left\{ k \in I_s : \|v^k_{e_k}\| \geq \frac{\widetilde{\eta}\, \epsilon}{1 - \epsilon} \text{ or } \|v^k_{e_k}\| \geq \widetilde{\eta}\, \epsilon \right\} \\ &\subseteq \left\{ k \in I_s : v^k_{e_k} = \widetilde{k} \right\} \\ &= \left\{ k \in I_s : k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s, s \in \mathbb{N} \right\} \end{aligned}$$

and so we get

$$\frac{1}{h_s} |A(\epsilon, \widetilde{\eta})| \leq \frac{1}{h_s} |\{ k \in I_s : k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s \}| \leq \frac{\sqrt{h_s}}{h_s}.$$

Taking $s \to \infty$,
$\lim\limits_{s\to\infty} \frac{1}{h_s} |A(\epsilon, \widetilde{\eta})| \leq \lim\limits_{s\to\infty} \frac{\sqrt{h_s}}{h_s} = 0$, i.e., $\delta_\theta(A(\epsilon, \widetilde{\eta})) = 0$.
This shows that, $v = (v^k_{e_k})$ is $\mathcal{S}_\theta(G_N, B_N, Y_N)$- convergent to $\widetilde{0}$. But by the structure of the sequence, $v = (v^k_{e_k})$ is not convergent to $\widetilde{0}$ w.r.t (G_N, B_N, Y_N).

Theorem 3.3 Let $\theta = (k_s)$ be a lacunary sequence and let $u = (u^k_{e_k})$ and $v = (v^k_{e_k})$ be any two sequences in $\widetilde{V}$ s.t $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim\limits_{k\to\infty} (u^k_{e_k}) = u_{e_1}$ and $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim\limits_{k\to\infty} (v^k_{e_k}) = v_{e_2}$. Then

(i) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim\limits_{k\to\infty} (u^k_{e_k} \oplus v^k_{e_k}) = u_{e_1} \oplus v_{e_2}$

(ii) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim\limits_{k\to\infty} (\widetilde{\alpha}\, u^k_{e_k}) = \widetilde{\alpha}\, u_{e_1}$, where $\widetilde{0} \neq \widetilde{\alpha} \in F$.

Proof. The proof of the theorem can be obtained as the proof of theorem 3.1, so omitted.□

Theorem 3.4 Let $\theta = (k_s)$ be a lacunary sequence. A sequence $v = (v^k_{e_k})$ in $\widetilde{V}$ is $\mathcal{S}_\theta(G_N, B_N, Y_N)$-convergent to v_e, if and only if $\exists$ a subset $K = \{k_1, k_2, ...\}$ of $\mathbb{N}$ s.t $\delta_\theta(K) = 1$ and $(G_N, B_N, Y_N) - \lim\limits_{\substack{k\in K \\ k\to\infty}} v^k_{e_k} = v_e$.

Proof. First suppose that $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim\limits_{k\to\infty} v^k_{e_k} = v_e$. For $\widetilde{\eta} > \widetilde{0}$ and $\beta \in \mathbb{N}$, define the set

$$\begin{aligned} K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta}) = \Big\{ k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1 - \frac{1}{\beta} \text{ and } \\ B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \frac{1}{\beta}, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \frac{1}{\beta} \Big\} \text{ and} \\ K^C_{G_N,B_N,Y_N}(\beta, \widetilde{\eta}) = \Big\{ k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \frac{1}{\beta} \text{ or } \\ B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \frac{1}{\beta}, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \frac{1}{\beta} \Big\}. \end{aligned}$$

Since $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$, it follows that $\delta_\theta(K^C_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})) = 0$. Furthermore, for $\widetilde{\eta} > \widetilde{0}$ and $\beta \in \mathbb{N}$, we observe $K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta}) \supset K_{G_N,B_N,Y_N}(\beta+1, \widetilde{\eta})$ and

$$\delta_\theta(K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})) = 1. \tag{5}$$

Now, we have to show that, for $k \in K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})$, $(G_N, B_N, Y_N) - \lim_{\substack{k\in K\\ k\to\infty}} v^k_{e_k} = v_e$. Suppose for $k \in K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta}), (v^k_{e_k})$ is not convergent to v_e w.r.t (G_N, B_N, Y_N). Then $\exists$ some $\xi > 0$ and a +ve integer k_0 s.t $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\xi$ or $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \xi, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \xi \ \forall\ k > k_0$. Let $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\xi$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \xi, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \xi \ \forall\ k < k_0$. Then

$$\delta_\theta\big(\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\xi \text{ and } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \xi, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \xi\}\big) = 0.$$

Since $\xi > \frac{1}{\beta}$ where $\beta \in \mathbb{N}$, we have $\delta_\theta(K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})) = 0$. In this way we obtained a contradiction to (5) as $\delta_\theta(K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})) = 1$. Hence, $(G_N, B_N, Y_N) - \lim_{\substack{k\in K\\ k\to\infty}} v^k_{e_k} = v_e$.

Conversely, Suppose that $\exists$ a subset $K = \{k_1, k_2, ..., k_j, ...\}$ of $\mathbb{N}$ with $\delta_\theta(K) = 1$ and $(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$ over K i.e., $(G_N, B_N, Y_N) - \lim_{\substack{k\in K\\ k\to\infty}} v^k_{e_k} = v_e$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, $\exists$ $k_{j_0} \in \mathbb{N}$ s.t for all $k_j \geq k_{j_0}$, $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon$. So if we consider the set

$$T_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta}) = \Big\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\Big\},$$

then it is easy to see that $T_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta}) \subset \mathbb{N} - \{k_{j_0+1}, k_{j_0+2}, ...\}$. This immediately implies that $\delta_\theta\big(T_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta})\big) \leq \delta_\theta(\mathbb{N}) - \delta_\theta(\{k_{j_0+1}, k_{j_0+2}, ...\}) = 1 - 1 = 0$ and therefore $\delta_\theta\big(T_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta})\big) = 0$ as $\delta_\theta\big(T_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta})\big)$ can not be negative. This shows that $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n\to\infty} v^k_{e_k} = v_e$. □

4. **Lacunary statistical completeness in NSNLS**

Definition 4.1 A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be lacunary statistically cauchy (or $\mathcal{S}_\theta$-Cauchy) w.r.t neutrosophic soft norm (G_N, B_N, Y_N) if for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, $\exists$ $p \in \mathbb{N}$ s.t

$$\lim_{s\to\infty} \frac{1}{h_s} \left|\Big\{k \in I_s : G_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \leq 1-\epsilon \text{ or } B_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon\Big\}\right| = 0,$$

or equivalently, the θ-density of the set K is zero, i.e., $\delta_\theta(K) = 0$ where

$$K = \{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon\}.$$

Theorem 4.1 Let $\theta = (k_s)$ be any lacunary sequence. If a sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is $\mathcal{S}_\theta(G_N, B_N, Y_N)$-convergent, then it is $\mathcal{S}_\theta(G_N, B_N, Y_N)$ cauchy.

Proof. Let $v = (v^k_{e_k})$ be any lacunary statistically convergent sequence with $\mathcal{S}_\theta(G_N, B_N, Y_N)-\lim_{k\to\infty} v^k_{e_k} = v_e$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\varrho > 0$ s.t (1) is satisfied. Define a set,

$$M(\varrho, \widetilde{\eta}) = \Big\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \frac{\widetilde{\eta}}{2}) \leq 1 - \varrho \text{ or } B_N(v^k_{e_k} \ominus v_e, \frac{\widetilde{\eta}}{2}) \geq \varrho, Y_N(v^k_{e_k} \ominus v_e, \frac{\widetilde{\eta}}{2}) \geq \varrho\Big\},$$

then

$$M^C(\varrho, \widetilde{\eta}) = \Big\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \frac{\widetilde{\eta}}{2}) > 1 - \varrho \text{ and } B_N(v^k_{e_k} \ominus v_e, \frac{\widetilde{\eta}}{2}) < \varrho, Y_N(v^k_{e_k} \ominus v_e, \frac{\widetilde{\eta}}{2}) < \varrho\Big\}.$$

Since $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n\to\infty} v^k_{e_k} = v_e$, so $\delta_\theta(M(\varrho, \widetilde{\eta})) = 0$ and $\delta_\theta(M^C(\varrho, \widetilde{\eta})) = 1$. Let $p \in M^C(\varrho, \widetilde{\eta})$, then

$$G_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) > 1 - \varrho \text{ and } B_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) < \varrho, Y_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) < \varrho. \tag{6}$$

Now, let $T(\epsilon, \widetilde{\eta}) = \{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \leq 1 - \epsilon$ or $B_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon\}$, then we have to show that $T(\epsilon, \widetilde{\eta}) \subseteq M(\varrho, \widetilde{\eta})$. Let $m \in T(\epsilon, \widetilde{\eta})$, then

$$G_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon, Y_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon. \tag{7}$$

Case 1: If $G_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) \leq 1 - \epsilon$, then $G_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \leq 1 - \varrho$ and therefore $m \in M(\varrho, \widetilde{\eta})$. As otherwise i.e., if $G_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) > 1 - \varrho$, then by (1), (6) and (7) we get

$$\begin{aligned} 1 - \epsilon \geq G_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) &= G_N\left(v^m_{e_m} \ominus v_e \oplus\ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &> (1 - \varrho) \circ (1 - \varrho) \\ &> 1 - \epsilon, \end{aligned}$$

which is impossible. Thus, $T(\epsilon,\widetilde{\eta}) \subseteq M(\varrho,\widetilde{\eta})$.

Case 2: If $B_N(v^m_{e_m} \ominus v^p_{e_p},\widetilde{\eta}) \geq \epsilon$, then $B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \varrho$ and therefore $m \in M(\varrho,\widetilde{\eta})$. As otherwise i.e., if $B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) < \varrho$, then by (1), (6) and (7) we get

$$\begin{aligned}\epsilon \leq B_N(v^m_{e_m} \ominus v^p_{e_p},\widetilde{\eta}) &= B_N\left(v^m_{e_m} \ominus v_e \oplus\ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right)\\ &\leq B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right)\\ &< \varrho \diamond \varrho\\ &< \epsilon,\end{aligned}$$

which is impossible.

Also, If $Y_N(v^m_{e_m} \ominus v^p_{e_p},\widetilde{\eta}) \geq \epsilon$, then $Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \varrho$ and therefore $m \in M(\varrho,\widetilde{\eta})$. As otherwise i.e., if $Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) < \varrho$, then by (1), (6) and (7) we get

$$\begin{aligned}\epsilon \leq Y_N(v^m_{e_m} \ominus v^p_{e_p},\widetilde{\eta}) &= Y_N\left(v^m_{e_m} \ominus v_e \oplus\ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right)\\ &\leq Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right)\\ &< \varrho \diamond \varrho\\ &< \epsilon,\end{aligned}$$

which is impossible. Thus, $T(\epsilon,\widetilde{\eta}) \subseteq M(\varrho,\widetilde{\eta})$.

Hence in all cases, $T(\epsilon,\widetilde{\eta}) \subseteq M(\varrho,\widetilde{\eta})$. Since $\delta_\theta(M(\varrho,\widetilde{\eta})) = 0$, so $\delta_\theta(T(\epsilon,\widetilde{\eta})) = 0$, and therefore $v = (v^k_{e_k})$ is $\mathcal{S}_\theta(G_N, B_N, Y_N)$ Cauchy.□

Definition 4.2 A $NSNLS$ $\widetilde{V}$ is said to be $\mathcal{S}_\theta$-complete if every $\mathcal{S}_\theta$-Cauchy sequence in $\widetilde{V}$ w.r.t neutrosophic soft norm-(G_N, B_N, Y_N) is $\mathcal{S}_\theta$- convergent w.r.t neutrosophic soft norm-(G_N, B_N, Y_N).

Theorem 4.2 Let $\theta = (k_s)$ be any lacunary sequence. Then every $NSNLS$ $\widetilde{V}$ is $\mathcal{S}_\theta$-complete but not complete in general.

Proof. Let $v = (v^k_{e_k})$ be $\mathcal{S}_\theta$-Cauchy but not $\mathcal{S}_\theta$-convergent w.r.t neutrosophic soft norm-(G_N, B_N, Y_N). For a given $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\varrho > 0$ s.t (1) is satisfied. Now

$$\begin{aligned}G_N(v^k_{e_k} \ominus v^p_{e_p},\widetilde{\eta}) \geq G_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \circ G_N(v^p_{e_p} \ominus v_e,\widetilde{\eta}) > (1-\varrho) \circ (1-\varrho) > 1-\epsilon\\ B_N(v^k_{e_k} \ominus v^p_{e_p},\widetilde{\eta}) \leq B_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \diamond B_N(v^p_{e_p} \ominus v_e,\widetilde{\eta}) < \varrho \diamond \varrho < \epsilon\\ Y_N(v^k_{e_k} \ominus v^p_{e_p},\widetilde{\eta}) \leq Y_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \diamond Y_N(v^p_{e_p} \ominus v_e,\widetilde{\eta}) < \varrho \diamond \varrho < \epsilon.\end{aligned}$$

Since $v = (v^k_{e_k})$ is not $\mathcal{S}_\theta$-convergent w.r.t neutrosophic soft norm-(G_N, B_N, Y_N). Therefore $\delta_\theta(H^C(\varrho, \widetilde{\eta})) = 0$, where

$$H(\varrho, \widetilde{\eta}) = \{k \in \mathbb{N} : \mathcal{B}_{v^k_{e_k} \ominus v^p_{e_p}}(\varrho) \leq 1 - \epsilon\}$$

and so $\delta_\theta(H(\varrho, \widetilde{\eta})) = 1$ which is a contradiction, since $v = (v^k_{e_k})$ was $\mathcal{S}_\theta$-cauchy w.r.t neutrosophic soft norm-(G_N, B_N, Y_N). So $v = (v^k_{e_k})$ must be $\mathcal{S}_\theta$-convergent w.r.t neutrosophic soft norm-(G_N, B_N, Y_N). Hence every $NSNLS$ $\widetilde{V}$ is $\mathcal{S}_\theta$-complete.

The following example demonstrates that $NSNLS$ is not complete in general:

Example 4.1[26] Let $\widetilde{U}= (0, 1]$ and $G_N(v, \widetilde{\eta}) = \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus |v|}$, $B_N(v, \widetilde{\eta}) = \frac{|v|}{\widetilde{\eta} \oplus |v|}$, $Y_N(v, \widetilde{\eta}) = \frac{|v|}{\widetilde{\eta}}$ for all $v \in \widetilde{U}$. Then $\widetilde{V}= (\widetilde{U}, G_N, B_N, Y_N, \min, \max)$ is $NSNLS$ but not complete, since the sequence of soft points $\left(\widetilde{\frac{1}{k}} \right)$ is cauchy w.r.t (G_N, B_N, Y_N) but not convergent w.r.t (G_N, B_N, Y_N).

Theorem 4.3 If every $\mathcal{S}_\theta$-cauchy sequence of soft points in $\widetilde{V}$ has a S_θ-convergent subsequence then $\widetilde{V}$ is S_θ-complete.

Proof. Let $v = (v^k_{e_k})$ be any S_θ-cauchy sequence of soft points in $\widetilde{V}$ which has a $\mathcal{S}_\theta$-convergent subsequence $(v^{k(j)}_{e_{k(j)}})$ i.e., $\mathcal{S}_\theta - \lim_{j\to\infty} v^{k(j)}_{e_{k(j)}} = v_e$ for some v_e in $\widetilde{V}$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\varrho > 0$ s.t (1) is satisfied. Since $v = (v^k_{e_k})$ is $\mathcal{S}_\theta$-cauchy, so $\exists$ $n_0 \in \mathbb{N}$ s.t $\forall$ $k, p \geq n_0$ $\delta_\theta(A) = 0$ where

$$A = \left\{k \in \mathbb{N} : G_N\left(v^k_{e_k} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) \leq 1 - \varrho \text{ or } B_N\left(v^k_{e_k} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho, Y_N\left(v^k_{e_k} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

Again since $\mathcal{S}_\theta - \lim_{j\to\infty} v^{k(j)}_{e_{k(j)}} = v_e$. So we have $\delta_\theta(B) = 0$, where

$$B = \left\{k(j) \in \mathbb{N} : G_N\left(v^{k(j)}_{e_{k(j)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \leq 1 - \varrho \text{ or } B_N\left(v^{k(j)}_{e_{k(j)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \varrho, Y_N\left(v^{k(j)}_{e_{k(j)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

Now define

$$D = \{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\}.$$

We now show that $A^C \cap B^C \subseteq D^C$. Let $m \in A^C \cap B^C$. As $m \in A^C$, so

$$G_N\left(v^m_{e_m} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) > 1 - \varrho \text{ and } B_N\left(v^m_{e_m} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) < \varrho, Y_N\left(v^m_{e_m} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) < \varrho, \quad (8)$$

and since $m \in B^C$, so $m = k(j_0)$ for $j_0 \in \mathbb{N}$ and

$$G_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus \ v_e, \frac{\widetilde{\eta}}{2}\right) > 1 - \varrho \text{ and}$$
$$B_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus \ v_e, \frac{\widetilde{\eta}}{2}\right) < \varrho, Y_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus \ v_e, \frac{\widetilde{\eta}}{2}\right) < \varrho. \tag{9}$$

Now

$$\begin{aligned} G_N(v_{e_m}^m \ominus v_e, \widetilde{\eta}) &= G_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)} \oplus v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)}, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &> (1-\varrho) \circ (1-\varrho) \quad \text{for } p = k(j_0) \\ &> 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} B_N(v_{e_m}^m \ominus v_e, \widetilde{\eta}) &= B_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)} \oplus v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq B_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)}, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \quad \text{for } p = k(j_0) \\ &< \epsilon, \end{aligned}$$

$$\begin{aligned} Y_N(v_{e_m}^m \ominus v_e, \widetilde{\eta}) &= Y_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)} \oplus v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq Y_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)}, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \quad \text{for } p = k(j_0) \\ &< \epsilon, \qquad \text{by (1), (8) and (9)} \end{aligned}$$

which implies that $m \in D^C$, so $A^C \cap B^C \subseteq D^C$ or $D \subseteq A \cup B$. Therefore, $\delta_\theta(D) \leq \delta_\theta(A \cup B) = 0$. This shows that $v = (v_{e_k}^k)$ is $\mathcal{S}_\theta$-convergent and therefore, $\widetilde{V}$ is $\mathcal{S}_\theta$-complete.□

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