# Plithogenic Cubic Vague Sets

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**Abstract:** To measure the level of accuracy of crisp, fuzzy, intuitionistic fuzzy sets and neutrosophic set various mathematical tools are available. The generalization of these four sets is Plithogenic set. In this paper, for the first time we introduce the concept of plithogenic cubic vague set and its generalization, plithogenic fuzzy cubic vague set, plithogenic intuitionistic fuzzy cubic vague set, plithogenic neutrosophic cubic vague set. It is the combination of cubic vague set and plithogenic set. It aims to address the problems involving multiple attribute decision making. This concept is suitable and the accuracy of the result is precise as the set is described by more value of attributes. An attribute value v has a corresponding (fuzzy, intuitionistic fuzzy, neutrosophic) degree of appurtenance d(x,v) of the element x to the set P, with respect to some given criteria. Its corresponding internal and external sets are also discussed with examples. Further, P-union, P-intersection as well as R-union and R-intersection are introduced for plithogenic cubic vague sets which acts as a tool to study some of their properties. Some examples of the newly developed concepts in our everyday life is offered in this article.

**Keywords:** cubic vague set, plithogenic fuzzy cubic vague set, plithogenic intuitionistic fuzzy cubic vague set, plithogenic neutrosophic cubic vague set.

## 1. Introduction

In real life, there may be an uncertainty about any degree of membership in the variable assumption. Zadeh [24] introduced fuzzy set in which each element is assigned a membership degree in the form of a single crisp value in the interval [0,1]. Fuzzy sets is an extension of crisp set. He also gave the perception of an interval valued fuzzy set as a cause of uncertainty in the membership. Grabisch et.al[9] represent an aggregation operator exhibits a set of mathematical properties, which depends on imposed axiomatic assumptions. A new definition of cardinality of fuzzy sets on the basis of membership value is introduced by Mamonidhar[16]. The generalization of fuzzy set and fuzzy logic to intuitionistic fuzzy sets (IFS) by adding the falsehood (f), the degree of non-membership was introduced by Atanassov [5] to have a better accuracy level. The definition for some operations on intuitionistic fuzzy set and its properties was given by Atanassov[6]. It is based only on membership and non-membership function, but it does not exist in the indeterminacy. The concept of intuitionistic fuzzy topological spaces was put forward by Coker [7]. Norsyahida Zulkifli[18] proposed the interval-valued intuitionistic fuzzy vague sets (IVIFVS) where membership and non-membership of interval-valued intuitionistic fuzzy sets are combined with truth membership and false membership of vague sets. Then the next evaluation of IFS is neutrosophic set introduced by Smarandache[20]. It is a generalization of fuzzy sets and IFS. It deals with membership, indeterminacy and non-membership degree which is highly helpful for dealing with uncertain, inadequate and varying data exist in real life. But it is applicable only on three attribute values. Data needs to be handled with more attribute values so as to raise the accuracy level in the stage of advanced research.

The theory of vague sets was proposed by Gau and Buehrer[8]. It has more powerful ability than fuzzy sets to process fuzzy information to some degree. Jun, Y.B[12] introduced the concept of cubic set and it is characterized by interval valued fuzzy set and fuzzy set, which is a more general tool to capture uncertainty and vagueness. The ideas of internal and external cubic sets and their characteristics were also presented. Cubic interval-valued intuitionistic fuzzy sets was introduced by Jun et.al [13]. Khaleed et.al [14] introduced the novel concept of cubic vague set by incorporating both the ideas of cubic set and vague set. Some new operations on intuitionistic neutrosophic set with examples for the implementation of the operations problems is introduced by Monoranjan et.al.[15]

As an extension of the neutrosophic set Wang et.al [22] proposed the definition of the interval valued neutrosophic set (INS). Interval neutrosophic sets and their application in multi-criteria decision making problems is defined by Hong et.al[11]. Hazwani Hashimcet.et.al [10] introduced the idea Interval Neutrosophic Vague Sets. Shawkat Alkhazaleh [19] introduced the concept of neutrosophic vague set as a combination of neutrosophic set and vague set. Anitha et.al[4] introduced the NGSR closed sets in neutrosophic topological spaces. Wang et.al[23] presented an instance of neutrosophic set called single valued neutrosophic set.

To, increase the preciseness, Smarandache [21] introduced plithogenic. It is a powerful tool which is a generalization of crisp set, fuzzy set, intuitionistic fuzzy set and neutrosophic set is collectively called plithogenic. It is the base for all plithogenic functions such as plithogenic set, plithogenic probability, plithogenic logic and plithogenic statistics. These sets which are characterized by a single appurtenance is plithogenic set. A plithogenic set, in general may have elements characterized by attributes with four or more attributes. It is a set whose components are described by at least one trait and each attribute may have numerous elements. He developed the aggregation operations on plithogenic set and proved that plithogenic set is the most generalized structure that can be efficiently applied to a variety of real life problems.

A procedure to come up with a methodical system to assess the infirmary serving under a framework of plithogenic theory was suggested by Abdel-Basset et al.[1] as an approach to be constructed on the connotation of plithogenic theory. To increase the accuracy of the evaluation Abdel-Basset et al.[2] proposed a method which is a combination of quality function deployment with plithogenic aggregation operations. Nivetha et.al[17] developed a concept of combined plithogenic hypersoft set and its application in multi attribute decision making. Based on the technique in order of preference by similarity to ideal solution and criteria importance through inter-criteria correlation methods to estimation of sustainable supply chain risk management Abdel-Basset et al.[3] proposed a methodology as a combination of plithogenic multi-criteria decision making approach.

In this paper, we introduced the generalization of plithogenic cubic vague sets for fuzzy, intuitionistic fuzzy and neutrosophic sets using the principles of cubic vague set and plithogenic set and its union and intersection. Plithogenic cubic vague set is the combination of cubic vague set and plithogenic set. It aims to address the problems involving multiple attribute decision making. This concept is suitable and the accuracy of the result is precise as the set is described by more value of attributes. The organization of this paper is as follows. Introduction is presented in Section 1. Section 2 provides some preliminaries for the proposed concept is given. Section 3 covers the notion of plithogenic cubic vague set and it is divided into four subsections. In 3.1 definition and examples of plithogenic fuzzy cubic vague set, in 3.2 plithogenic intuitionistic fuzzy cubic vague set, in 3.3 plithogenic neutrosophic cubic vague set and in 3.4 internal and external plithogenic cubic vague sets were presented. In Section 4 we define the basic operations namely, union and intersection of the developed set. Finally, Section 5 conclude this paper and provides the direction for future studies.

## 2. Preliminaries

**Definition: 2.1. [8]** A vague set A (VS) in the universe of discourse U is a characterized by membership functions given by: truth membership function $\mathrm{t_A}: \mathrm{U} \rightarrow [0,1]$ and false membership function $\mathrm{f_A}: \mathrm{U} \rightarrow [0,1]$ where $\mathrm{t_A}(u)$ is a lower bound of the grade of membership of u derived from the evidence of u and $\mathrm{f_A}(u)$ is a lower bound of the negation of u derived from the evidence against u and $\mathrm{t_A}(u) + \mathrm{f_A}(u) \leq 1$. Thus the grade of membership of u in the vague set A is bounded by a sub interval $[\mathrm{t_A}(u), 1 - \mathrm{f_A}(u)]$ of [0,1]. This indicates that if the actual grade of membership is $\mu(u)$ then $\mathrm{t_A}(u) \leq \mu_A(u) \leq 1 - \mathrm{f_A}(u)$. The vague set A is written as A $=\{(u, [\mathrm{t_A}(u), 1 - \mathrm{f_A}(u)]) | u \in U\}$, where the internal $[\mathrm{t_A}(u), 1 - \mathrm{f_A}(u)]$ is called vague value of u in A and denoted by$V_A(u)$.

**Definition: 2.2. [12]** Let X be a non-empty set. A structure $A = \{\langle x, A(x), \lambda(x)\rangle : x \in X\}$ be a cubic set in X in which A is an IVFS and $\lambda$ is a fuzzy set in X.

**Definition: 2.3. [12]** Let X be a universal Set. A cubic vague set $\mathbb{A}^{\mathrm{V}}$ defined over the universal set X is an ordered pair which is defined as follows $\mathbb{A}^{\mathrm{V}} = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X\}$ where $A_V = \langle A_V^t, A_V^{1-f}\rangle = \{\langle x, [t_{A_V}^-(x), t_{A_V}^+(x)], [1 - f_{A_V}^-(x), 1 - f_{A_V}^+(x)]\rangle : x \in X\}$ represents IVVS defined on X while $\lambda_V = \{(x, t_{\lambda V}(x), 1 - f_{\lambda V}(x)) : x \in X\}$ represents VS such that $t_{A_V}^+(x) + f_{A_V}^+(x) \leq 1$ and $t_{\lambda V}(x) + f_{\lambda V}(x) \leq 1$. For clarity we denote the pairs as $\mathbb{A}^{\mathrm{V}} = \langle A_V, \lambda_V\rangle$, where $A_V = \langle [t_{A_V}^-, t_{A_V}^+], [1 - f_{A_V}^-, 1 - f_{A_V}^+]\rangle$ and $\lambda_V = (t_{\lambda V}, 1 - f_{\lambda V})$. $\mathrm{C}_{\mathrm{V}}^{\mathrm{X}}$ denotes the set of all cubic vague sets in X.

**Definition: 2.4. [12]** Let X be a universal set and V be a non-empty vague set. A cubic vague set $\mathbb{A}^{\mathrm{V}} = \langle A_v, \lambda_v\rangle$ is called an internal cubic vague set (brief. ICVS) if $A_V^-(x) \leq \lambda_V(x) \leq A_V^+(x)$ for all$x \in X$.

**Definition: 2.5. [12]** Let X be a universal set and V be a non-empty vague set. A cubic vague set $\mathbb{A}^{\mathrm{V}} = \langle A_v, \lambda_v\rangle$ is called an external cubic vague set (brief. ECVS) if$\lambda_V(x) \notin (A_V^-(x), A_V^+(x))$ for all $x \in X$.

**Definition: 2.6. [8]** An interval valued vague sets $\widetilde{A}^V$ over a universe of discourse X is defined as an object of the form $\widetilde{A}^V = \{(x_i, [\mathrm{T}_{\widetilde{A}^V}(x_i), F_{\widetilde{A}^V}(x_i)]) | x_i \in X\}$, where $\mathrm{T}_{\widetilde{A}^V}: X \rightarrow \mathrm{D}[0,1]$ and $\mathrm{F}_{\widetilde{A}^V}: X \rightarrow \mathrm{D}[0,1]$ are called truth membership function and false membership function respectively and where D[0,1] is the set of all intervals within [0,1].

**Definition: 2.7. [19]** A neutrosophic vague set $A_{NV}$ (NVS in short) on the universe of discourse X written as $A_{NV} = \{\langle x, \hat{T}_{A_{NV}}, \hat{I}_{A_{NV}}, \hat{F}_{A_{NV}}\rangle x \in X\}$ whose truth membership, indeterminacy membership and falsity membership functions are defined as $\hat{T}_{A_{NV}}(x) = [T^-, T^+]$, $\hat{I}_{A_{NV}}(x) = [I^-, I^+]$, $\hat{F}_{A_{NV}}(x) = [F^-, F^+]$, where $T^+ = 1 - F^-$, $F^+ = 1 - T^-$ and $0^- \leq T^- + I^- + F^- \leq 2^+$.

**Definition: 2.8. [10]** An interval valued neutrosophic vague set $A_{INV}$ also known as INVS in the universe of discourse E. An IVNVS is characterized by truth membership, indeterminacy membership and falsity membership functions is defined as: $A_{INV} = \{< e, [\hat{V}_A^L(e), \hat{V}_A^U(e)], [\widehat{W}_A^L(e), \widehat{W}_A^U(e)], [\hat{X}_A^L(e), \hat{X}_A^U(e)] > | e \in E\}$,
$\hat{V}_A^L(e) = [V^{L^-}, V^{L^+}]$, $\hat{V}_A^U(e) = [V^{U^-}, V^{U^+}]$, $\widehat{W}_A^L(e) = [W^{L^-}, W^{L^+}]$, $\widehat{W}_A^L(e) = [W^{U^-}, W^{U^+}]$, $\hat{X}_A^L(e) = [X^{L^-}, X^{L^+}]$, $\hat{X}_A^U(e) = [X^{U^-}, X^{U^+}]$ where $V^{L^+} = 1 - X^{L^-}$, $X^{L^+} = 1 - V^{L^-}$, $V^{U^+} = 1 - X^{U^-}$, $X^{U^+} = 1 - V^{U^-}$ and
$0^- \leq V^{L^-} + V^{U^-} + W^{L^-} + W^{U^-} + X^{L^-} + X^{U^-} \leq 4^+$, $0^- \leq V^{L^+} + V^{U^+} + W^{L^+} + W^{U^+} + X^{L^+} + X^{U^+} \leq 4^+$.

**Definition: 2.9 [12]** $\mathcal{A} = \langle A, \lambda\rangle$ and $\mathcal{B} = \langle B, \mu\rangle$ be cubic sets in X. Then we define

(a) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow A = B$ and $\lambda = \mu$

(b) (P-order) $\mathcal{A} = \mathcal{B} \Leftrightarrow A \subseteq B$ and $\lambda \leq \mu$

(c) (R-order) $\mathcal{A} = \mathcal{B} \Leftrightarrow A \subseteq B$ and $\lambda \geq \mu$

## 3. PLITHOGENIC CUBIC VAGUE SETS

**Definition: 3.1**
Interval valued plithogenic fuzzy vague set (IPFVS) is defined as $\forall x \in P, d_v: P \times Q_v \rightarrow P([0,1])$ and $\forall q \in Q, d(x,q)$ is an (open, semi-open, closed) interval included in [0,1].

**Definition: 3.2**
Interval valued plithogenic intuitionistic fuzzy vague set (IPIFVS) is defined as $\forall x \in P, d_v: P \times Q_v \rightarrow P([0,1]^2)$ and $\forall q \in Q, d(x,q)$ is an (open, semi-open, closed) interval included in [0,1].

**Definition: 3.3**
Interval valued plithogenic Neutrosophic vague set (IPIFVS) is defined as $\forall x \in P, d_v: P \times Q_v \rightarrow P([0,1]^3)$ and $\forall q \in Q, d(x,q)$ is an (open, semi-open, closed) interval included in [0,1].

**Definition: 3.4**
A fuzzy vague set $A_{FV}$ (FVS in short) on the universe of discourse X written as $A_{FV} = \{\langle x, \hat{T}_{A_{FV}} \rangle\, x \in X\}$ whose truth membership is defined as $\hat{T}_{A_{FV}}(x) = [T^-, T^+]$ where $0 \leq T^- \leq T^+ \leq 1$.

**Definition: 3.5**
A Intuitionistic fuzzy vague set $A_{IFV}$ (IFVS in short) on the universe of discourse X written as $A_{IFV} = \{\langle x, \hat{T}_{A_{IFV}}, \hat{F}_{A_{IFV}} \rangle\, x \in X\}$ whose truth membership and falsity membership functions are defined as $\hat{T}_{A_{NV}}(x) = [T^-, T^+]$, $\hat{F}_{A_{NV}}(x) = [F^-, F^+]$.

### 3.1. Plithogenic Fuzzy Cubic Vague Sets

**Definition: 3.1.1**
Let U be a universal set. The set $\mathbb{A}_p{}^v = \{\langle x, A_V(x), \lambda_V(x) \rangle : x \in X\}$ is called plithogenic fuzzy cubic vague set in which $A_V$ is an interval valued plithogenic fuzzy vague set in X and $\lambda_V$ is the fuzzy vague set in X.

**Example: 3.1.2**
Let the attribute be "size" and the attribute values are {small, medium, big, very big}. Let's consider the dominant value of attribute size be small.
The attribute value contradictory degrees are:
c(small, small) = 0, c(small, medium) = 0.50, c(small, big) = 0.75, c(small, very big) = 1.
The degree of appurtenance $A_V$: {small, medium, big, very big} → [0,1].
$A_V$(small) = ([0.3,0.4],[0.4,0.6]), $A_V$(medium) = ([0.2,0.3],[0.4,0.5]),
$A_V$(big) = ([0.1,0.3],[0.3,0.6]), $A_V$(very big) = ([0.3,0.4],[0.4,0.5]).

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | small | Medium | Big | very big |
| Degrees of appurtenance $A_V(x)$ | ([0.3,0.4], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | ([0.3,0.4], [0.4,0.5]) |
| $\lambda_V(x)$ | [0.3,0.5] | [0.2,0.4] | [0.3,0.5] | [0.4,0.5] |

### 3.2 Plithogenic Intuitionistic Fuzzy Cubic Vague Sets

**Definition: 3.2.1**

Let U be a universal set. The set $\mathbb{A}_p{}^v = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X\}$ is called plithogenic intuitionistic fuzzy cubic vague set in which $A_V$ is an interval valued plithogenic intuitionistic fuzzy vague set in X and $\lambda_V$ is the intuitionistic fuzzy vague set in X.

**Example: 3.2.2**
Let the attribute be "Car Brands" and the attribute values are {Ford, Audi, Benz, BMW}. Let's consider the dominant value of attribute car brands be Ford.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | Ford | Audi | Benz | BMW |
| Degrees of appurtenance $A_V(x)$ | {([0.4,0.6],[0.5,0.5]),([ 0.4,0.6],[0.5.0.5])} | {([0.3,0.8],[0.5,0.6]),([ 0.2,0.7],[0.4.0.5])} | {([0.2,0.6],[0.3,0.6]),([ 0.4,0.8],[0.4.0.7])} | {([0.1,0.7],[0.2,0.4]),([ 0.3,0.9],[0.6.0.8])} |
| $\lambda_V(x)$ | [0.3,0.6],[0.4,0.7] | [0.2,0.5],[0.5,0.8] | [0.4,0.7],[0.3,0.6] | [0.5,0.6],[0.4,0.5] |

**3.3 Plithogenic Neutrosophic Cubic Vague Sets**

**Definition: 3.3.1**
Let U be a universal set. The set $\mathbb{A}_p{}^v = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X\}$ is called plithogenic neutrosophic cubic vague set in which $A_V$ is an interval valued plithogenic neutrosophic vague set in X and $\lambda_V$ is the neutrosophic vague set in X.

**Example: 3.3.2**

Let the attribute be "Colleges" and the attribute values are {Arts, Engineering, Medical, Agriculture}. Let's consider the dominant value of attribute colleges be Arts.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | Arts | Engineering | Medical | Agriculture |
| Degrees of appurtenance $A_V(x)$ | {([0.4,0.6],[0.5,0.5]),([0. 1,0.8],[0.2,0.6]),([0.4,0. 6],[0.5.0.5])} | {([0.3,0.8],[0.5,0.6]),([0. 3,0.5],[0.4,0.6]),([0.2,0. 7],[0.4.0.5])} | {([0.2,0.6],[0.3,0.6]),([0. 4,0.8],[0.2,0.5]),([0.4,0. 8],[0.4.0.7])} | {([0.1,0.7],[0.2,0.4 ]),([0.4,0.5],[0.3,0. 6]),([0.3,0.9],[0.6. 0.8])} |
| $\lambda_V(x)$ | [0.2,0.5],[0.3,0.4],[0.5,0. 8] | [0.4,0.7],[0.2,0.3],[0.3,0. 6] | [0.3,0.6],[0.4,0.5],[0.4,0. 7] | [0.5,0.6],[0.5,0.6],[ 0.4,0.5] |

**3.4. Internal Plithogenic Cubic Vague Set and External Plithogenic Cubic Vague Set**

**Definition: 3.4.1**
Let X be a universal set and V be a non-empty vague set. The plithogenic fuzzy cubic vague set $\mathbb{A}_p{}^v = \langle A_v, \lambda_v\rangle$ in X is called an internal plithogenic fuzzy cubic vague set (IPFCVS) if $A_{v_{d_i}}{}^-(x) \le \lambda_{v_i}(x) \le A_{v_{d_i}}{}^+(x)$ for all $x \in X$ and $d_i$ denotes the contradictory degree and its attribute values.

**Example: 3.4.2**
Let the attribute be "Continents" and the attribute values are {Asia, Africa, Europe, Australia}. Let's consider the dominant value of attribute continents as Asia.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|

| Attribute values | Asia | Africa | Europe | Australia |
|---|---|---|---|---|
| Degrees of appurtenance $A_V(x)$ | ([0.3,0.4], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | ([0.3,0.4], [0.4,0.5]) |
| $\lambda_V(x)$ | [0.3,0.5] | [0.2,0.4] | [0.1,0.5] | [0.3,0.4] |

**Definition: 3.4.3**

Let X be a universal set and V be a non-empty vague set. The plithogenic intuitionistic fuzzy cubic vague set $\mathbb{A}_p{}^v$ = $\langle A_v, \lambda_v \rangle$ in X is called an internal plithogenic intuitionistic fuzzy cubic vague set (IPIFCVS) if $A_{v_{d_i}}{}^{-}(x) \leq \lambda_{v_i}(x) \leq A_{v_{d_i}}{}^{+}(x)$ for all $x \in X$ and $d_i$ denotes the contradictory degree and its attribute values.

**Example: 3.4.4**

Let the attribute be "Vehicles" and the attribute values are {Car, Bus, Bicycle, Scooter}. Let's consider the dominant value of the attribute Vehicles as Car.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | Car | Bus | Bicycle | Scooter |
| Degrees of appurtenance $A_V(x)$ | {([0.3,0.8],[0.5,0.6]),([ 0.2,0.7],[0.4.0.5])} | {([0.4,0.7],[0.3,0.5]),([ 0.3,0.6],[0.5.0.7])} | {([0.2,0.6],[0.3,0.6]),([ 0.4,0.8],[0.4.0.7])} | {([0.4,0.6],[0.5,0.5]),([ 0.4,0.6],[0.5.0.5])} |
| $\lambda_V(x)$ | [0.4,0.6],[0.3,0.4] | [0.5,0.5],[0.3,0.6] | [0.3,0.4],[0.5,0.5] | [0.2,0.5],[0.4,0.5] |

**Definition: 3.4.5**

Let X be a universal set and V be a non-empty vague set. The plithogenic neutrosophic cubic vague set $\mathbb{A}_p{}^v$ = $\langle A_v, \lambda_v \rangle$ in X is said to be internal plithogenic neutrosophic cubic vague set (IPNCVS) in X if it satisfies the following equations.

(i) truth-internal (briefly, T-internal) if $A_{v_{d_i}}{}^{-T}(x) \leq \lambda_{v_i}{}^{T}(x) \leq A_{v_{d_i}}{}^{+T}(x)$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 1)

(ii) indeterminacy-internal (briefly, I-internal) if $A_{v_{d_i}}{}^{-I}(x) \leq \lambda_{v_i}{}^{I}(x) \leq A_{v_{d_i}}{}^{+I}(x)$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 2)

(iii) falsity-internal (briefly, F-internal) if $A_{v_{d_i}}{}^{-F}(x) \leq \lambda_{v_i}{}^{F}(x) \leq A_{v_{d_i}}{}^{+F}(x)$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 3)

**Example: 3.4.6**

Let the attribute be "Weather" and the attribute values are {Sunny, Cloudy, Rain, Snow}. Let's consider the dominant value of the attribute Weather as Sunny.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | Sunny | Cloudy | Rain | Snow |
| Degrees of appurtenance $A_V(x)$ | {([0.1,0.7],[0.2,0.4]),([0. 4,0.5],[0.3,0.6]),([0.3,0. 9],[0.6.0.8])} | {([0.2,0.6],[0.3,0.6]),([0. 4,0.8],[0.2,0.5]),([0.4,0. 8],[0.4.0.7])} | {([0.3,0.8],[0.5,0.6]),([0. 3,0.5],[0.4,0.6]),([0.2,0. 7],[0.4.0.5])} | {([0.4,0.6],[0.5,0. 5]),([0.1,0.8],[0.2 ,0.6]),([0.4,0.6],[ 0.5.0.5])} |
| $\lambda_V(x)$ | [0.2,0.3],[0.4,0.4],[0.5,0. 8] | [0.3,0.5],[0.5,0.5],[0.5,0. 6] | [0.4,0.6],[0.3,0.5],[0.2,0. 4] | [0.5,0.5],[0.2,0.6] ,[0.4,0.5] |

**Definition: 3.4.7**

Let X be a universal set and V be a non-empty vague set. The plithogenic fuzzy cubic vague set $\mathbb{A}_p^{\ v} = \langle A_v, \lambda_v \rangle$ in X is called an external plithogenic fuzzy cubic vague set (EPFCVS) if $\lambda_{v_i}(x) \notin (A_{v_{d_i}}{}^{-}(x), A_{v_{d_i}}{}^{+}(x))$ for all $x \in X$ and $d_i$ denotes the contradictory degree and its attribute values.

**Example: 3.4.8**

Let the attribute be "Languages" and the attribute values are {Tamil, French, Malayalam, English}. Let's consider the dominant value of the attribute Languages as Tamil.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | Tamil | French | Malayalam | English |
| Degrees of appurtenance $A_V(x)$ | ([0.3,0.4], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.5]) | ([0.3,0.5], [0.4,0.5]) |
| $\lambda_V(x)$ | [0.2,0.7] | [0.1,0.6] | [0.4,0.6] | [0.2,0.7] |

**Definition: 3.4.9**

Let X be a universal set and V be a non-empty vague set. The plithogenic intuitionistic fuzzy cubic vague set $\mathbb{A}_p^{\ v}$ = $\langle A_v, \lambda_v \rangle$ in X is called an external plithogenic intuitionistic fuzzy cubic vague set (IPIFCVS) if $\lambda_{v_i}(x) \notin (A_{v_{d_i}}{}^{-}(x), A_{v_{d_i}}{}^{+}(x))$ for all $x \in X$ and $d_i$ denotes the contradictory degree and its attribute values.

**Example: 3.4.10**

Let the attribute be "Country" and the attribute values are {India, Japan, Malaysia, Korea}. Let's consider the dominant value of the attribute country as India.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | India | Japan | Malaysia | Korea |
| Degrees of appurtenance $A_V(x)$ | {([0.3,0.8],[0.5,0.6]),([ 0.2,0.7],[0.4.0.5])} | {([0.4,0.7],[0.3,0.5]),([ 0.3,0.6],[0.5.0.7])} | {([0.2,0.6],[0.3,0.6]),([ 0.4,0.8],[0.4.0.7])} | {([0.4,0.6],[0.5,0.5]),([ 0.4,0.6],[0.5.0.5])} |
| $\lambda_V(x)$ | [0.2,0.8],[0.1,0.7] | [0.3,0.6],[0.2,0.8] | [0.1,0.7],[0.2,0.3] | [0.1,0.6],[0.2,0.6] |

**Definition: 3.4.11**

Let X be a universal set and V be a non-empty vague set. The plithogenic neutrosophic cubic vague set $\mathbb{A}_p^{\ v}$ = $\langle A_v, \lambda_v \rangle$ in X is said to be external plithogenic neutrosophic cubic vague set (EPNCVS) in X if it satisfies the following equations.

(i) truth-external (briefly, T- external) if $\lambda_{v_i}{}^{T}(x) \notin (A_{v_{d_i}}{}^{-T}(x), A_{v_{d_i}}{}^{+T}(x))$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 4)

(ii) indeterminacy- external (briefly, I- external) if $\lambda_{v_i}{}^{I}(x) \notin (A_{v_{d_i}}{}^{-I}(x), A_{v_{d_i}}{}^{+I}(x))$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 5)

(iii) falsity- external (briefly, F- external) if $\lambda_{v_i}{}^{F}(x) \notin (A_{v_{d_i}}{}^{-F}(x), A_{v_{d_i}}{}^{+F}(x))$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 6)

**Example: 3.4.12**

Let the attribute be "Month" and the attribute values are {July, May, April, March}. Let's consider the dominant value of the attribute Month as July.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | July | May | April | March |

| | | | | |
|---|---|---|---|---|
| Degrees of appurtenance $A_V(x)$ | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.2,0.5]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.6],[0.5,0.5]),([0.3,0.8],[0.2,0.6]),([0.4,0.6],[0.5.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.4,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.1,0.2]),([0.4,0.5],[0.3,0.6]),([0.4,0.6],[0.8.0.9])} |
| $\lambda_V(x)$ | [0.1,0.7],[0.3,0.8],[0.2,0.8] | [0.3,0.7],[0.2,0.9],[0.3,0.6] | [0.1,0.9],[0.2,0.7],[0.1,0.6] | [0.1,0.5],[0.2,0.8],[0.2,0.5] |

**4. Union and Intersection**

In this section, we introduce the definitions of union and intersection of plithogenic cubic vague sets with examples.

**Definition:4.1** Let $\mathbb{A}_p{}^{v} = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X, v \in V\}$ and $\mathbb{B}_p{}^{v} = \{\langle x, B_V(x), \mu_V(x)\rangle : x \in X, v \in V\}$ be two plithogenic cubic vague sets in X then we have

1. $\mathbb{A}_p{}^{v} = \mathbb{B}_p{}^{v}$ if and only if $A_V(x) = B_V(x)$ and $\lambda_V(x) = \mu_V(x)$.
2. $\mathbb{A}_p{}^{v}$ and $\mathbb{B}_p{}^{v}$ are two plithogenic cubic vague sets in X then we define and denote P-order as $\mathbb{A}_p{}^{v} \leq_P \mathbb{B}_p{}^{v}$ if and only if $A_V(x) \subseteq B_V(x)$ and $\lambda_V(x) \leq \mu_V(x)$ for all $x \in X$.
3. $\mathbb{A}_p{}^{v}$ and $\mathbb{B}_p{}^{v}$ are two plithogenic cubic vague sets in X then we define and denote R-order as $\mathbb{A}_p{}^{v} \leq_R \mathbb{B}_p{}^{v}$ if and only if $A_V(x) \subseteq B_V(x)$ and $\lambda_V(x) \geq \mu_V(x)$ for all $x \in X$.

**Definition:4.2** Let $\mathbb{A}_p{}^{v} = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X, v \in V\}$ and $\mathbb{B}_p{}^{v} = \{\langle x, B_V(x), \mu_V(x)\rangle : x \in X, v \in V\}$ be two plithogenic cubic vague sets in X. Then we have

1. $\mathbb{A}_p{}^{v} \cup_P \mathbb{B}_p{}^{v} = \{\langle (x), \sup(A_V(x), B_V(x)), \sup(\lambda_V(x), \mu_V(x))\rangle \mid x \in X, v \in V\}$ (P-union)
2. $\mathbb{A}_p{}^{v} \cap_P \mathbb{B}_p{}^{v} = \{\langle (x), \inf(A_V(x), B_V(x)), \inf(\lambda_V(x), \mu_V(x))\rangle \mid x \in X, v \in V\}$ (P-intersection)
3. $\mathbb{A}_p{}^{v} \cup_R \mathbb{B}_p{}^{v} = \{\langle (x), \sup(A_V(x), B_V(x)), \inf(\lambda_V(x), \mu_V(x))\rangle \mid x \in X, v \in V\}$ (R-union)
4. $\mathbb{A}_p{}^{v} \cap_R \mathbb{B}_p{}^{v} = \{\langle (x), \inf(A_V(x), B_V(x)), \sup(\lambda_V(x), \mu_V(x))\rangle \mid x \in X, v \in V\}$ (R-intersection)

**4.1 Plithogenic Fuzzy Cubic Vague Union and Intersection**

**Example: 4.1.1 (P-Order)**

The expert values between "color" and "height" and their values are Color = {red, blue, green} and Height = {tall, medium} then the object elements are characterized by the Cartesian product

Color × Height $= \{(\text{red}, \text{tall}), (\text{red}, \text{medium}), (\text{blue}, \text{tall}), (\text{blue}, \text{medium}), (\text{green}, \text{tall}), (\text{green}, \text{medium})\}$.

Let's consider the dominant value of attribute "Color" be "red" and of attribute "Height" be "tall". The attribute value contradiction degrees are:

c(red, red) = 0, c(red, blue) = $\frac{1}{3}$, c(red, green) = $\frac{2}{3}$, c(tall, tall) = 0, c(tall, medium) = $\frac{1}{3}$.

We have two plithogenic cubic vague sets A and B and we consider the fuzzy, intuitionistic fuzzy and neutrosophic appurtenance degree of attribute values to the sets.

$\mathbb{A}_p{}^{v}$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $A_V(x)$ | ([0.2,0.3], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.4,0.5]) |
| $\lambda_V(x)$ | [0.2,0.5] | [0.2,0.4] | [0.1,0.8] | | [0.3,0.5] | [0.1,0.4] |

$\mathbb{B}_p{}^{v}$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |

| | | | | | | |
|---|---|---|---|---|---|---|
| Degrees of appurtenance $B_V(x)$ | ([0.3,0.4], [0.4,0.6]) | ([0.3,0.4], [0.4,0.5]) | ([0.2,0.3], [0.4,0.6]) | | ([0.3,0.4], [0.4,0.5]) | ([0.3,0.4], [0.4,0.5]) |
| $\mu_V(x)$ | [0.5,0.5] | [0.3,0.4] | [0.2,0.8] | | [0.4,0.5] | [0.4,0.4] |

Then P-Union denoted by $\mathbb{A}_p{}^v \cup_{\text{P}} \mathbb{B}_p{}^v$ and P-Intersection denoted by $\mathbb{A}_p{}^v \cap_{\text{P}} \mathbb{B}_p{}^v$

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| $A_V \cup B_V$ | ([0.3,0.4], [0.4,0.6]) | ([0.3,0.4], [0.4,0.5]) | ([0.2,0.3], [0.4,0.6]) | | ([0.3,0.4], [0.4,0.5]) | ([0.3,0.4], [0.4,0.5]) |
| $\lambda_V \cup \mu_V$ | [0.5,0.5] | [0.3,0.4] | [0.2,0.8] | | [0.4,0.5] | [0.4,0.4] |
| $A_V \cap B_V$ | ([0.2,0.3], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.4,0.5]) |
| $\lambda_V \cap \mu_V$ | [0.2,0.5] | [0.2,0.4] | [0.1,0.8] | | [0.3,0.5] | [0.1,0.4] |

**Example: 4.1.2 (R-Order)**

$\mathbb{A}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $A_V(x)$ | ([0.2,0.3], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.4,0.5]) |
| $\lambda_V(x)$ | [0.4,0.5] | [0.3,0.4] | [0.2,0.8] | | [0.4,0.5] | [0.4,0.4] |

$\mathbb{B}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $B_V(x)$ | ([0.3,0.4], [0.4,0.6]) | ([0.3,0.4], [0.4,0.5]) | ([0.2,0.3], [0.4,0.6]) | | ([0.3,0.4], [0.4,0.5]) | ([0.3,0.4], [0.4,0.5]) |
| $\mu_V(x)$ | [0.3,0.5] | [0.2,0.4] | [0.1,0.8] | | [0.3,0.5] | [0.1,0.4] |

Then R-Union denoted by $\mathbb{A}_p{}^v \cup_{\text{R}} \mathbb{B}_p{}^v$ and R-Intersection denoted by $\mathbb{A}_p{}^v \cap_{\text{R}} \mathbb{B}_p{}^v$

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| $A_V \cup B_V$ | ([0.3,0.4], [0.4,0.6]) | ([0.3,0.4], [0.4,0.5]) | ([0.2,0.3], [0.4,0.6]) | | ([0.3,0.4], [0.4,0.5]) | ([0.3,0.4], [0.4,0.5]) |
| $\lambda_V \cup \mu_V$ | [0.3,0.5] | [0.2,0.4] | [0.1,0.8] | | [0.3,0.5] | [0.1,0.4] |
| $A_V \cap B_V$ | ([0.2,0.3], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.4,0.5]) |
| $\lambda_V \cap \mu_V$ | [0.4,0.5] | [0.3,0.4] | [0.2,0.8] | | [0.4,0.5] | [0.4,0.4] |

**4.2 Plithogenic Intuitionistic Fuzzy Cubic Vague Union and Intersection**

**Example: 4.2.1 (P-Order)**

$\mathbb{A}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $A_V(x)$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V(x)$ | [0.2,0.5],[0.5,0.8] | [0.4,0.5],[0.5,0.6] | [0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.5,0.9] | [0.2,0.4],[0.6,0.8] |

$\mathbb{B}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $B_V(x)$ | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5.0.5]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.1,0.4],[0.5.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\mu_V(x)$ | [0.2,0.5],[0.5,0.8] | [0.4,0.5],[0.5,0.6] | [0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.5,0.9] | [0.2,0.4],[0.6,0.8] |

Then P-Union denoted by $\mathbb{A}_p{}^v \cup_{\mathrm{P}} \mathbb{B}_p{}^v$ and P-Intersection denoted by $\mathbb{A}_p{}^v \cap_{\mathrm{P}} \mathbb{B}_p{}^v$

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| $A_V \cup B_V$ | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0.5]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.1,0.4],[0.5.0.8]) | {([0.2,0.6],[0.2,0.3]),([0.4,0.8],[0.7.0.8])} |
| $\lambda_V \cup \mu_V$ | [0.2,0.5], [0.5,0.8] | [0.4,0.5],[0.5,0.6] | [0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.5,0.9] | [0.2,0.4],[0.6,0.8] |
| $A_V \cap B_V$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V \cap \mu_V$ | [0.2,0.5],[0.5,0.8] | [0.4,0.5],[0.5,0.6] | [0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.5,0.9] | [0.2,0.4],[0.6,0.8] |

**Example: 4.2.2 (R-Order)**

$\mathbb{A}_p{}^v$:

| | | | | | | |
|---|---|---|---|---|---|---|
| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $A_V(x)$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V(x)$ | [0.4,0.7],[0.3,0.6] | [0.1,0.3],[0.7,0.9] | [0.7,0.8],[0.2,0.3] | | [0.5,0.6],[0.4,0.5] | [0.3,0.5],[0.5,0.7] |

$\mathbb{B}_p{}^v$:

| | | | | | | |
|---|---|---|---|---|---|---|
| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $B_V(x)$ | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5.0.5]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.1,0.4],[0.5.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\mu_V(x)$ | [0.4,0.7],[0.3,0.6] | [0.1,0.3],[0.7,0.9] | [0.7,0.8],[0.2,0.3] | | [0.5,0.6],[0.4,0.5] | [0.3,0.5],[0.5,0.7] |

Then R-Union denoted by $\mathbb{A}_p{}^v \cup_R \mathbb{B}_p{}^v$ and R-Intersection denoted by $\mathbb{A}_p{}^v \cap_R \mathbb{B}_p{}^v$

| | | | | | | |
|---|---|---|---|---|---|---|
| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
| Attribute values | red | blue | green | | tall | medium |
| $A_V \cup B_V$ | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5.0.5]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.1,0.4],[0.5.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V \cup \mu_V$ | [0.4,0.7],[0.3,0.6] | [0.1,0.3],[0.7,0.9] | [0.7,0.8],[0.2,0.3] | | [0.5,0.6],[0.4,0.5] | [0.3,0.5],[0.5,0.7] |
| $A_V \cap B_V$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V \cap \mu_V$ | [0.4,0.7],[0.3,0.6] | [0.1,0.3],[0.7,0.9] | [0.7,0.8],[0.2,0.3] | | [0.5,0.6],[0.4,0.5] | [0.3,0.5],[0.5,0.7] |

**4.3 Plithogenic Neutrosophic Cubic Vague Union and Intersection**

**Example: 4.3.1 (P-Order)**

$\mathbb{A}_p{}^v$:

| | | | | | | |
|---|---|---|---|---|---|---|
| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
| Attribute values | red | blue | green | | Tall | Medium |
| Degrees of appurtenance $A_V(x)$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.6],[0.2,0.6]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.8],[0.2,0.6]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.4,0.7],[0.3,0.6]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6]),([0.5,0.8],[0.7.0.8])} |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| $\lambda_V(x)$ | [0.2,0.5],[0.2,0.3],[0.5,0.8] | [0.4,0.5],[0.5,0.5],[0.5,0.6] | [0.2,0.4],[0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.3,0.4],[0.5,0.9] | [0.2,0.4],[0.3,0.5],[0.6,0.8] |

$\mathbb{B}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | Medium |
| Degrees of appurtenance $B_V(x)$ | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.1,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.2,0.5]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0.5]),([0.1,0.5],[0.1,0.3]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.3,0.5],[0.2,0.5]),([0.1,0.4],[0.5.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6]),([0.5,0.8],[0.7.0.8])} |
| $\mu_V(x)$ | [0.2,0.5],[0.3,0.4],[0.5,0.8] | [0.4,0.5],[0.6,0.6],[0.5,0.6] | [0.2,0.4],[0.3,0.5],[0.6,0.8] | | [0.1,0.5],[0.5,0.6],[0.5,0.9] | [0.2,0.4],[0.7,0.8],[0.6,0.8] |

Then P-Union denoted by $\mathbb{A}_p{}^v \cup_P \mathbb{B}_p{}^v$ and P-Intersection denoted by $\mathbb{A}_p{}^v \cap_P \mathbb{B}_p{}^v$ are

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | Green | | tall | medium |
| $A_V \cup B_V$ | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.1,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.2,0.5]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0.5]),([0.1,0.5],[0.1,0.3]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.3,0.5],[0.2,0.5]),([0.1,0.4],[0.5.0.8]) | {([0.2,0.6],[0.2,0.3]),([0.4,0.7],[0.3,0.5]),([0.4,0.8],[0.7.0.8])} |
| $\lambda_V \cup \mu_V$ | [0.2,0.5],[0.2,0.3],[0.5,0.8] | [0.4,0.5],[0.5,0.5],[0.5,0.6] | [0.2,0.4],[0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.3,0.4],[0.5,0.9] | [0.2,0.4],[0.3,0.5],[0.6,0.8] |
| $A_V \cap B_V$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.6],[0.2,0.6]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.8],[0.2,0.6]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.4,0.7],[0.3,0.6]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V \cap \mu_V$ | [0.2,0.5],[0.3,0.4],[0.5,0.8] | [0.4,0.5],[0.6,0.6],[0.5,0.6] | [0.2,0.4],[0.3,0.5],[0.6,0.8] | | [0.1,0.5],[0.5,0.6],[0.5,0.9] | [0.2,0.4],[0.7,0.8],[0.6,0.8] |

**Example: 4.3.2 (R-Order)**

$\mathbb{A}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | Medium |
| Degrees of appurtenance $A_V(x)$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.6],[0. | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.8],[0.2, | | {([0.1,0.7],[0.2,0.4]),([0.4,0.7],[0.3, | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 2,0.6]),([0.4,0.6 8],[0.5.0.5])} | | 0.6]),([0.5,0.6],[0. 6.0.9])} | | 0.6]),([0.3,0.9],[0. 6.0.8])} | ]),([0.5,0.8],[0.7.0. 8])} |
| $\lambda_V(x)$ | [0.4,0.7],[0.3,0.4 ],[0.3,0.6] | [0.1,0.3],[0.5,0.6],[0 .7,0.9] | [0.7,0.8],[0.3,0.5] ,[0.2,0.3] | | [0.5,0.6],[0.5,0.6] ,[0.4,0.5] | [0.3,0.5],[0.6,0.6],[ 0.5,0.7] |

$\mathbb{B}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | Medium |
| Degrees of appurtenance $B_V(x)$ | {([0.3,0.8],[0.5,0 .6]),([0.3,0.5],[0. 1,0.6]),([0.2,0.7] ,[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]) ,([0.3,0.5],[0.2,0.5]), ([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0. 5]),([0.1,0.5],[0.1, 0.3]),([0.4,0.6],[0. 5.0.5])} | | {([0.6,0.9],[0.2,0. 5]),([0.3,0.5],[0.2, 0.5]),([0.1,0.4],[0. 5.0.8])} | {([0.2,0.5],[0.2,0.3] ),([0.5,0.8],[0.4,0.6 ]),([0.5,0.8],[0.7.0. 8])} |
| $\mu_V(x)$ | [0.4,0.7],[0.2,0.3 ],[0.3,0.6] | [0.1,0.3],[0.3,0.4],[0 .7,0.9] | [0.7,0.8],[0.1,0.3] ,[0.2,0.3] | | [0.5,0.6],[0.2,0.],[ 0.4,0.5] | [0.3,0.5],[0.4,0.5],[ 0.5,0.7] |

Then R-Union denoted by $\mathbb{A}_p{}^v \cup_R \mathbb{B}_p{}^v$ and R-Intersection denoted by $\mathbb{A}_p{}^v \cap_R \mathbb{B}_p{}^v$

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| $A_V \cup B_V$ | {([0.3,0.8],[0.5,0 .6]),([0.3,0.5],[0. 1,0.6]),([0.2,0.7] ,[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]) ,([0.3,0.5],[0.2,0.5]), ([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0. 5]),([0.1,0.5],[0.1, 0.3]),([0.4,0.6],[0. 5.0.5])} | | {([0.6,0.9],[0.2,0. 5]),([0.3,0.5],[0.2, 0.5]),([0.1,0.4],[0. 5.0.8])} | {([0.2,0.5],[0.2,0.3] ),([0.5,0.8],[0.4,0.6 ]),([0.5,0.8],[0.7.0. 8])} |
| $\lambda_V \cup \mu_V$ | [0.4,0.7],[0.2,0.3 ],[0.3,0.6] | [0.1,0.3],[0.3,0.4],[0 .7,0.9] | [0.7,0.8],[0.1,0.3] ,[0.2,0.3] | | [0.5,0.6],[0.2,0.],[ 0.4,0.5] | [0.3,0.5],[0.4,0.5],[ 0.5,0.7] |
| $A_V \cap B_V$ | {([0.2,0.6],[0.5,0 .5]),([0.4,0.6],[0. 2,0.6]),([0.4,0.8] ,[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]) ,([0.4,0.8],[0.4,0.6]), ([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0. 4]),([0.5,0.8],[0.2, 0.6]),([0.5,0.6],[0. 6.0.9])} | | {([0.1,0.7],[0.2,0. 4]),([0.4,0.7],[0.3, 0.6]),([0.3,0.9],[0. 6.0.8])} | {([0.2,0.5],[0.2,0.3] ),([0.5,0.8],[0.4,0.6 ]),([0.5,0.8],[0.7.0. 8])} |
| $\lambda_V \cap \mu_V$ | [0.4,0.7],[0.3,0.4 ],[0.3,0.6] | [0.1,0.3],[0.5,0.6],[0 .7,0.9] | [0.7,0.8],[0.3,0.5] ,[0.2,0.3] | | [0.5,0.6],[0.5,0.6] ,[0.4,0.5] | [0.3,0.5],[0.6,0.6],[ 0.5,0.7] |

**Theorem: 4.3.2**

Let X be a non-empty set and $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ be a PNCVS in X which is not external. Then there exists $x \in X$ such that ${\lambda_{v_i}}^T(x) \in ({A_{v_{d_i}}}^{-T}(x), {A_{v_{d_i}}}^{+T}(x))$ , ${\lambda_{v_i}}^I(x) \in ({A_{v_{d_i}}}^{-I}(x), {A_{v_{d_i}}}^{+I}(x))$ , ${\lambda_{v_i}}^F(x) \in ({A_{v_{d_i}}}^{-F}(x), {A_{v_{d_i}}}^{+F}(x))$ where $d_i$ denotes the contradictory degree and its attribute values.

**Proof:** Straight forward

**Theorem: 4.3.3**

Let X be a non-empty set and $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both T-internal and T-external, then, $(\forall\, x \in X)\left(\lambda_{v_i}{}^T(x) \in \left\{(A_{v_{d_i}}{}^{-T}(x) | x \in X\right\} \cup \left\{A_{v_{d_i}}{}^{+T}(x) | x \in \mathrm{X}\right\}\right)$ where $d_i$ denotes the contradictory degree and its attribute values.

**Proof:** Consider the conditions 1 and 4 which implies that $A_{v_{d_i}}{}^{-T}(x) \leq \lambda_{v_i}{}^T(x) \leq A_{v_{d_i}}{}^{+T}(x)$ and $\lambda_{v_i}{}^T(x) \notin (A_{v_{d_i}}{}^{-T}(x), A_{v_{d_i}}{}^{+T}(x))$ for all $x \in X$. Then it follows that $\lambda_{v_i}{}^T(x) = A_{v_{d_i}}{}^{-T}(x)$ or $\lambda_{v_i}{}^T(x) = A_{v_{d_i}}{}^{+T}(x)$, and hence $\left(\lambda_{v_i}{}^T(x) \in \left\{(A_{v_{d_i}}{}^{-T}(x) | x \in X\right\} \cup \left\{A_{v_{d_i}}{}^{+T}(x) | x \in \mathrm{X}\right\}\right)$ where $d_i$ denotes the contradictory degree and its attribute values. Hence proved.

**Remark:** Similarly the consequent theorems holds for indeterminacy and falsity values.

**Theorem: 4.3.4**

Let X be a non-empty set and $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both I-internal and I-external, then, $(\forall\, x \in X)\left(\lambda_{v_i}{}^I(x) \in \left\{(A_{v_{d_i}}{}^{-I}(x) | x \in X\right\} \cup \left\{A_{v_{d_i}}{}^{+I}(x) | x \in \mathrm{X}\right\}\right)$ where $d_i$ denotes the contradictory degree and its attribute values.

**Theorem: 4.3.5**

Let X be a non-empty set and $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both F-internal and F-external, then, $(\forall\, x \in X)\left(\lambda_{v_i}{}^F(x) \in \left\{(A_{v_{d_i}}{}^{-F}(x) | x \in X\right\} \cup \left\{A_{v_{d_i}}{}^{+F}(x) | x \in \mathrm{X}\right\}\right)$ where $d_i$ denotes the contradictory degree and its attribute values.

**Definition: 4.3.6**

Let X be a non-empty set. The complement of $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ is said to be PNCVS, $(\mathbb{A}_p{}^v)^c = \langle A^C{}_v, \lambda^C{}_v \rangle$ where $A_v{}^c = \{(A^C{}_{v_{d_i}}{}^T(x),\ A^C{}_{v_{d_i}}{}^I(x), A^C{}_{v_{d_i}}{}^F(x))\}$ is an interval valued PNCVS in X and $\lambda^C{}_v = \{(\lambda^C{}_v{}^T(x), \lambda^C{}_v{}^I(x), \lambda^C{}_v{}^F(x))\}$ is a neutrosophic set in Y.

**Theorem: 4.3.7**

Let X be a non-empty set and $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both T-internal and T-external, then the complement $(\mathbb{A}_p{}^v)^c = \langle A^C{}_v, \lambda^C{}_v \rangle$ of $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ is a T-internal and T-external PNCVS in X.

**Proof:**

Let X be a non-empty set and if $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ is an T-internal and T-external PNCVS in X, then $A_{v_{d_i}}{}^{-T}(x) \leq \lambda_{v_i}{}^T(x) \leq A_{v_{d_i}}{}^{+T}(x)$ and $\lambda_{v_i}{}^T(x) \notin (A_{v_{d_i}}{}^{-T}(x), A_{v_{d_i}}{}^{+T}(x))$ for all $x \in X$. It follows that $1 - A_{v_{d_i}}{}^{-T}(x) \leq 1 - \lambda_{v_i}{}^T(x) \leq 1 - A_{v_{d_i}}{}^{+T}(x)$ and $1 - \lambda_{v_i}{}^T(x) \notin (1 - A_{v_{d_i}}{}^{-T}(x), 1 - A_{v_{d_i}}{}^{+T}(x))$. Therefore $(\mathbb{A}_p{}^v)^c = \langle A^C{}_v, \lambda^C{}_v \rangle$ of $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ is a T-internal and T-external PNCVS in X.

**Remark:** Similarly the consequent theorems holds for indeterminacy and falsity values.

**Theorem: 4.3.8**

Let X be a non-empty set and $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both I-internal and I-external, then the complement $(\mathbb{A}_p{}^v)^c = \langle A^C{}_v, \lambda^C{}_v \rangle$ of $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ is a I-internal and I-external PNCVS in X.

**Theorem: 4.3.9**

Let X be a non-empty set and $\mathbb{A}_p{}^v = \langle A_v , \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both F-internal and F-external, then the complement $(\mathbb{A}_p{}^v)^c = \langle A^C{}_v, \lambda^C{}_v \rangle$ of $\mathbb{A}_p{}^v = \langle A_v , \lambda_v \rangle$ is a F-internal and F-external PNCVS in X.

**5 Conclusion and Future Work**

The objective of this paper is to define the concept of plithogenic cubic vague set and its generalization; plithogenic fuzzy cubic vague set, plithogenic intuitionistic fuzzy cubic vague set, plithogenic neutrosophic cubic vague set with examples. Its corresponding internal and external cubic vague sets are also defined. We studied the union and intersection of P and R order also some basic properties. The work presented in this paper delivers the theoretical framework for further study on plithogenic cubic vague set. In the future work, we will study AND and OR operations, similarity measures of plithogenic cubic vague set.

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