NSS

Neutrosophic Sets and Systems, Vol. 58, 2023

University of New Mexico

Neutrosophic $\varkappa$-structures in an AG-groupoid

G. Muhiuddin[1], **K. Porselvi**[2*] **and B. Elavarasan**[3]

[1]Department of Mathematics, University of Tabuk, P.O. Box-741, Tabuk-71491, Saudi Arabia.

[1]E-mail:chishtygm@gmail.com

[2,3]Department of Mathematics, Karunya Institute of Technology and Sciences, Coimbatore - 641 114, India.

[2] E-mail:porselvi94@yahoo.co.in; porselvi@karunya.edu

[3]E-mail:belavarasan@gmail.com; elavarasan@karunya.edu.

[*]Correspondence: porselvi94@yahoo.co.in

Abstract. An AG-groupoid is the midway between commutative semigroup and groupoid. The core structure of Flock theory is an AG-groupoid, which focuses on motion replication and distance optimization and has numerous applications in physics and biology. Unfortunately, in many cases, modelling real-world problems in domains like computer science, operations research, artificial intelligence, control engineering, and robotics can be risky. Different theories, such as fuzzy sets, intuitionistic fuzzy sets, probability, soft sets, neutrosophic sets, and others, have been created to deal with similar situations. In this paper, We define the notions of neutrosophic $\varkappa$-ideal structures in an AG-groupoid and investigate their properties. We also obtain equivalent assertion of neutrosophic $\varkappa$-ideals and product of neutrosophic $\varkappa$-structures in AG-groupoid.

Keywords: AG-groupoid;neutrosophic $\varkappa$-structures; ideals; neutrosophic $\varkappa$-ideals; neutrosophic $\varkappa$-interior ideals.

---

1. Introduction

In [1], Zadeh pioneered the fuzzy set theory to model imprecise ideas in the globe. Atanassov expanded fuzzy set theory principles and termed it Intuitionistic fuzzy set in [2]. In his opinion, there are two types of degrees of freedom in an universe: non-membership in a specific subset and membership in a vague subset. In [3], Rosenfeld proposed the notion of fuzziness in groups and produced a number of results. Recently, several authors studied their research in this field, and similar notions are used in variety of algebraic structures, including semigroups, semiring, ordered semigroups, rings (refer, [4] - [14], [18]- [21]).

To deal with the uncertainty that exists everywhere, Smarandache suggested the notions of neutrosophic sets in [15]. It's a combination of fuzzy sets and intuitionistic fuzzy sets that's been generalized. Neutrosophic sets are defined using these three properties, which include membership functions for truth (T), indeterminacy (I), and falsity (F). These sets can be used in a variety of fields to deal with the difficulties that result from ambiguous data. The relative and absolute membership functions can be distinguished by a neutrosophic set. Smarandache employed neutrosophic sets in non-standard analysis, such as control theory, decision making theory, sports decision (winning/losing/tie), and so on.

In BCK-algebra, Muhiuddin et al. discovered an association between (ϵ, ϵ)-neutrosophic subalgebra and (ϵ, ϵ)-neutrosophic ideal in [16], and Muhiuddin et al. created and investigated neutrosophic implicative $\varkappa$-ideal in [17]. Additionally, the connection between several neutrosophic implicative $\varkappa$-ideals were examined.

In semigroup, neutrosophic $\varkappa$-subsemigroup and the ϵ-neutrosophic $\varkappa$-subsemigroup were defined and their different features were covered in [18] by Khan et al. We examined the properties of various neutrosophic $\varkappa$-structure notions, namely neutrosophic $\varkappa$-ideal structures in a semigroup, as inspiration from [18]. A neutrosophic $\varkappa$-ideals in a semigroup were suggested by B. Elavarasan et al. in [19] and different features were achieved. The comparable claims for the typical neutrosophic $\varkappa$-structure were also given.

Porselvi et al. studied a number of characteristics of the neutrosophic $\varkappa$-bi-ideal in a semigroup in [20], and neutrosophic $\varkappa$-interior ideal in [21]. We have established equivalent claims for regular semigroup. In [22], Elavarasan et al. presented and studied neutrosophic $\varkappa$-filters in semigroups. In [23], Muhiuddin and others proposed the concepts of neutrosophic $\varkappa$-structures in ordered semigroup, and examined their properties. Smarandache proposed neutrosophic topologies in [26], Runu Dhar studied compactness and neutrosophic topological space in [27], Sudeep Dey et al. presented neutrosophic composite relation in [28].

We present the ideas of neutrosophic $\varkappa$-ideal structures in an AG-groupoid in this paper. We prove that the product of two neutrosophic $\varkappa$-right-ideal is a neutrosophic $\varkappa$-bi-ideal, and neutrosophic $\varkappa$-right-ideal is equivalent to neutrosophic $\varkappa$-interior-ideal, under certain condition.

2. Preliminaries

Unless otherwise specified, $\mathscr{M}$ denotes an AG-groupoid throughout this paper. Here is a glossary of the definitions we have already used for your perusal.

For $M_1, M_2 \subseteq \mathscr{M}$, we denote $(M_1] = \{k \in \mathscr{M} \ : \ k \leq m$ for some $m \in M_1\}$ and $M_1M_2 = \{k_1k_2 :$ for all $k_1 \in M_1$ and $k_2 \in M_2\}$. Following [24] and [25], an AG-groupoid, $\mathscr{M}$, is a groupoid whose elements hold the left invertive law: $(m_1m_2)k_3 = (k_3m_2)m_1$ for all $m_1, m_2, k_3 \in \mathscr{M}$. An AG-groupoid structure lies between a commutative semigroup and a groupoid. In $\mathscr{M}$, the medial law $(m_1m_2)(k_3k_4) = (m_1k_3)(m_2k_4)$ for all $m_1, m_2, k_3, k_4 \in \mathscr{M}$ holds. If there is an element $e \in \mathscr{M} \ni em = m \ \forall \ m \in \mathscr{M}$, then e is the left identity. If $\mathscr{M}$ has a right identity, then $\mathscr{M}$ is said to be commutative monoid. If $\mathscr{M}$ is having a left identity, then $(m_1m_2)(k_3k_4) = (k_4k_3)(m_2m_1)$ holds for all $m_1, m_2, k_3, k_4 \in \mathscr{M}$. An element $m \in \mathscr{M}$ is said to be idempotent if $m^2 = m$.

Let $\mathscr{M}$ be an AG-groupoid and $\phi \neq M \subseteq \mathscr{M}$. Then M is called a AG-subgroupoid of $\mathscr{M}$ (see [24]) if $M^2 \subseteq M$. A subset $M \neq \phi$ in $\mathscr{M}$ is called a left(respectively, right) ideal if $\mathscr{M}M \subseteq M$ (respectively, $M\mathscr{M} \subseteq M$), and M is said to be an ideal if it is both a right and a left ideal of $\mathscr{M}$. A subset $M \neq \phi$ in $\mathscr{M}$ is said to be an interior ideal if $(\mathscr{M}M)\mathscr{M} \subseteq M$. A subset $M \neq \phi$ in $\mathscr{M}$ is known as bi-ideal if $(M\mathscr{M})M \subseteq M$. A subset $M \neq \phi$ in $\mathscr{M}$ is said to be a idempotent if $MM = M$.

Let $\mathscr{M}$ be an AG-groupoid. Then a function $\nu : \mathscr{M} \to [-1, 0]$ is the $\varkappa$-function on $\mathscr{M}$, and the set of all the $\varkappa$-functions is given by $F(\mathscr{M}, [-1, 0])$. A $\varkappa$-structure is an ordered pair $(\mathscr{M}, h)$ of $\mathscr{M}$ and an $\varkappa$-function ν on $\mathscr{M}$.

Definition 2.1. Let $\mathscr{M}$ be an AG-groupoid. A neutrosophic $\varkappa$-structure in $\mathscr{M}$ is given in the form:

$$\mathscr{M}_\zeta := \frac{\mathscr{M}}{(T_\zeta, I_\zeta, F_\zeta)} = \left\{ \frac{k}{(T_\zeta(k), I_\zeta(k), F_\zeta(k))} \mid k \in \mathscr{M} \right\},$$

where T_ζ, F_ζ and I_ζ are the negative truth, negative falsity and negative indeterminacy membership functions respectively in $\mathscr{M}$ ($\varkappa$-functions). Clearly, $-3 \leq T_\zeta(m) + I_\zeta(m) + F_\zeta(m) \leq 0$ $\forall \ m \in \mathscr{M}$.

Throughout this section, we assume that $\mathscr{M}_\zeta$ and $\mathscr{M}_\xi$ are neutrosophic $\varkappa$-structures in $\mathscr{M}$, unless otherwise stated.

Notation 1. *We denote the set of*

(i) neutrosophic $\varkappa$-left ideal by $\mathscr{M}_l$,

(ii) neutrosophic $\varkappa$-right ideal by $\mathscr{M}_r$,

(iii) neutrosophic $\varkappa$-ideal by $\mathscr{M}_i$,

(iv) neutrosophic $\varkappa$-bi-ideal by $\mathscr{M}_b$,

(v) neutrosophic $\varkappa$-interior ideal by $\mathscr{M}_n$,

(vi) neutrosophic $\varkappa$-AG-subgroupoid by $\mathscr{M}_s$.

(vii) neutrosophic $\varkappa$-idempotent by $\mathscr{M}_d$.

Definition 2.2. Let $\mathscr{M}_\zeta \in \mathscr{M}$. Then $\mathscr{M}_\zeta \in \mathscr{M}_s$ provided the below condition is valid:

$$(\forall m_1, m_2 \in \mathscr{M}) \begin{pmatrix} T_\zeta(m_1m_2) \leq T_\zeta(m_1) \vee T_\zeta(m_2) \\ I_\zeta(m_1m_2) \geq I_\zeta(m_1) \wedge I_\zeta(m_2) \\ F_\zeta(m_1m_2) \leq F_\zeta(m_1) \vee F_\zeta(m_2) \end{pmatrix}.$$

Let $\nu, \gamma, \omega \in [-1, 0]$. Consider the sets:

$$T_\zeta^\nu = \{m_1 \in \mathscr{M} \mid T_\zeta(m_1) \leq \nu\},$$
$$I_\zeta^\gamma = \{m_1 \in \mathscr{M} \mid I_\zeta(m_1) \geq \gamma\},$$
$$F_\zeta^\omega = \{m_1 \in \mathscr{M} \mid F_\zeta(m_1) \leq \omega\}.$$

The set $\mathscr{M}_\zeta(\nu, \gamma, \omega) := \{m_1 \in \mathscr{M} \mid T_\zeta(m_1) \leq \nu,\ I_\zeta(m_1) \geq \gamma,\ F_\zeta(m_1) \leq \omega\}$ is known as (ν, γ, ω)-level set on $\mathscr{M}_\zeta$. Obviously, $\mathscr{M}_\zeta(\nu, \gamma, \omega) = T_\zeta^\nu \cap I_\zeta^\gamma \cap F_\zeta^\omega$.

Definition 2.3. Let $\mathscr{M}_\zeta \in \mathscr{M}$. Then $\mathscr{M}_\zeta \in \mathscr{M}_i$ provided the below conditions are valid:

(i) $(\forall m_1, m_2 \in \mathscr{M}) \begin{pmatrix} T_\zeta(m_1m_2) \leq T_\zeta(m_2) \\ I_\zeta(m_1m_2) \geq I_\zeta(m_2) \\ F_\zeta(m_1m_2) \leq F_\zeta(m_2) \end{pmatrix}$.

(ii) $(\forall m_1, m_2 \in \mathscr{M}) \begin{pmatrix} T_\zeta(m_1m_2) \leq T_\zeta(m_1) \\ I_\zeta(m_1m_2) \geq I_\zeta(m_1) \\ F_\zeta(m_1m_2) \leq F_\zeta(m_1) \end{pmatrix}$.

If condition (i) hold, then $\mathscr{M}_\zeta \in \mathscr{M}_l$. If condition (ii) hold, then $\mathscr{M}_\zeta \in \mathscr{M}_r$.

Definition 2.4. Let $\mathscr{M}_\zeta \in \mathscr{M}_s$.Then $\mathscr{M}_\zeta \in \mathscr{M}_b$ if the below assertion is valid:

$$(\forall a, k_1, k_2 \in \mathscr{M}) \begin{pmatrix} T_\zeta(k_1ak_2) \leq T_\zeta(k_1) \vee T_\zeta(k_2) \\ I_\zeta(k_1ak_2) \geq I_\zeta(k_1) \wedge I_\zeta(k_2) \\ F_\zeta(k_1ak_2) \leq F_\zeta(k_1) \vee F_\zeta(k_2) \end{pmatrix}.$$

It is obvious that for any $\mathscr{M}_\zeta \in \mathscr{M}_i$, we have $\mathscr{M}_\zeta \in \mathscr{M}_b$. The converse need not be true, as shown by an example.

Example 2.5. Suppose $\mathscr{M} := \{x_1, x_2, x_3, x_4, x_5\}$. Then $(\mathscr{M}, .)$ is an AG-groupoid as given below:

.	x_1	x_2	x_3	x_4	x_5
x_1	x_1	x_4	x_1	x_4	x_4
x_2	x_1	x_2	x_1	x_4	x_4
x_3	x_1	x_4	x_3	x_4	x_5
x_4	x_1	x_4	x_1	x_4	x_4
x_5	x_1	x_4	x_3	x_4	x_5

Let

$$\mathscr{M}_\zeta = \left\{ \frac{x_1}{(-0.8,-0.2,-0.6)}, \frac{x_2}{(-0.5,-0.9,-0.1)}, \frac{x_3}{(-0.3,-0.4,-0.5)}, \frac{x_4}{(-0.8,-0.2,-0.6)}, \frac{x_5}{(-0.2,-0.5,-0.1)} \right\}.$$

Then $\mathscr{M}_\zeta \in \mathscr{M}_b$, and $\mathscr{M}_\zeta \notin \mathscr{M}_i$ as $T_{\mathscr{M}}(x_3x_5) = -0.2 > T_{\mathscr{M}}(x_3)$, $I_{\mathscr{M}}(x_3x_5) = -0.5 < I_{\mathscr{M}}(x_3)$ and $F_{\mathscr{M}}(x_3x_5) = -0.1 > F_{\mathscr{M}}(x_3)$.

Definition 2.6. Let $\mathscr{M}_\zeta \in \mathscr{M}_s$.Then $\mathscr{M}_\zeta \in \mathscr{M}_n$ provided the below assertion is valid:

$$(\forall a, m_1, m_2 \in \mathscr{M}) \begin{pmatrix} T_\zeta(m_1 a m_2) \leq T_\zeta(a) \\ I_\zeta(m_1 a m_2) \geq I_\zeta(a) \\ F_\zeta(m_1 a m_2) \leq F_\zeta(a) \end{pmatrix}.$$

It is obvious that for any $\mathscr{M}_\zeta \in \mathscr{M}_i$, we have $\mathscr{M}_\zeta \in \mathscr{M}_n$. The converse is not true, as shown by an example.

Example 2.7. Let $\mathscr{M}$ be the collection of all positive integers with 0 except 1. Then under usual multiplication, $\mathscr{M}$ is an AG-groupoid.

Let

$\mathscr{M}_\zeta = \left\{ \frac{0}{(-0.8,-0.2,-0.8)}, \frac{2}{(-0.3,-0.4,-0.5)}, \frac{5}{(-0.5,-0.6,-0.6)}, \frac{10}{(-0.2,-0.5,-0.3)}, \frac{otherwise}{(-0.8,-0.2,-0.4)} \right\}$.

Then $\mathscr{M}_\zeta \in \mathscr{M}_n$, and $\mathscr{M}_\zeta \notin \mathscr{M}_i$, as $T_{\mathscr{M}}(2.5) = -0.2 > T_{\mathscr{M}}(2)$ and $T_{\mathscr{M}}(2.5) = -0.2 > T_{\mathscr{M}}(5)$.

Definition 2.8. For any $\mathscr{X} \subseteq \mathscr{M}$, the characteristic neutrosophic χ-structure in $\mathscr{M}$ is refered as

$$\chi_{\mathscr{X}}(\mathscr{M}_\zeta) = \frac{\mathscr{M}}{(\chi_{\mathscr{X}}(T)_\zeta, \chi_{\mathscr{X}}(I)_\zeta, \chi_{\mathscr{X}}(F)_\zeta)}$$

where

$$\chi_{\mathscr{X}}(T)_\zeta : \mathscr{M} \to [-1, 0],\ m_1 \mapsto \begin{cases} -1 & if\ m_1 \in \mathscr{X} \\ 0 & otherwise, \end{cases}$$

$$\chi_{\mathscr{X}}(I)_\zeta : \mathscr{M} \to [-1, 0],\ m_1 \mapsto \begin{cases} 0 & if\ m_1 \in \mathscr{X} \\ -1 & otherwise, \end{cases}$$

$$\chi_{\mathscr{X}}(F)_\zeta : \mathscr{M} \to [-1, 0],\ m_1 \mapsto \begin{cases} -1 & if\ m_1 \in \mathscr{X} \\ 0 & otherwise. \end{cases}$$

Definition 2.9. Let $\mathscr{M}_\xi := \frac{\mathscr{M}}{(T_\xi, I_\xi, F_\xi)} \in \mathscr{M}$ and $\mathscr{M}_\zeta := \frac{\mathscr{M}}{(T_\zeta, I_\zeta, F_\zeta)} \in \mathscr{M}$. Then

(i) $\mathscr{M}_\xi$ is said to be a neutrosophic $\varkappa$-substructure in $\mathscr{M}_\zeta$, denote by $\mathscr{M}_\zeta \subseteq \mathscr{M}_\xi$, if $T_\zeta(m_1) \geq T_\xi(m_1), I_\zeta(m_1) \leq I_\xi(m_1), F_\zeta(m_1) \geq F_\xi(m_1)$ for all $m_1 \in \mathscr{M}$.

If $\mathscr{M}_\xi \subseteq \mathscr{M}_\zeta$ and $\mathscr{M}_\zeta \subseteq \mathscr{M}_\xi$, then we write $\mathscr{M}_\xi = \mathscr{M}_\zeta$.

(ii) The union of $\mathscr{M}_\xi$ and $\mathscr{M}_\zeta$ over $\mathscr{M}$ is described as

$$\mathscr{M}_\xi \cup \mathscr{M}_\zeta = \mathscr{M}_{\xi \cup \zeta} = (\mathscr{M}; T_{\xi \cup \zeta}, I_{\xi \cup \zeta}, F_{\xi \cup \zeta}),$$

where $\forall\ m_1 \in \mathscr{M}$,

$$(T_\xi \cup T_\zeta)(m_1) = T_{\xi \cup \zeta}(m_1) = T_\xi(m_1) \wedge T_\zeta(m_1),$$
$$(I_\xi \cup I_\zeta)(m_1) = I_{\xi \cap \zeta}(m_1) = I_\xi(m_1) \vee I_\zeta(m_1),$$
$$(F_\xi \cup F_\zeta)(m_1) = F_{\xi \cup \zeta}(m_1) = F_\xi(m_1) \wedge F_\zeta(m_1).$$

(iii) The intersection of $\mathscr{M}_\xi$ and $\mathscr{M}_\zeta$ over $\mathscr{M}$ is described as

$$\mathscr{M}_\xi \cap \mathscr{M}_\zeta = \mathscr{M}_{\xi\cap\zeta} = (\mathscr{M}; T_{\xi\cap\zeta}, I_{\xi\cap\zeta}, F_{\xi\cap\zeta}),$$

$$\text{where } \forall\ m_1 \in \mathscr{M}, \quad \begin{array}{l} (T_\xi \cap T_\zeta)(m_1) = T_{\xi\cap\zeta}(m_1) = T_\xi(m_1) \vee T_\zeta(m_1), \\ (I_\xi \cap I_\zeta)(m_1) = I_{\xi\cap\zeta}(m_1) = I_\xi(m_1) \wedge I_\zeta(m_1), \\ (F_\xi \cap F_\zeta)(m_1) = F_{\xi\cap\zeta}(m_1) = F_\xi(m_1) \vee F_\zeta(m_1). \end{array} .$$

3. Main Results

We present some characteristics of neutrosophic $\varkappa$-ideal structures in an AG-groupoid $\mathscr{M}$. In $\mathscr{M}$, neutrosophic $\varkappa$-ideals are clearly neutrosophic $\varkappa$-interior ideals, but the converse is true under certain conditions.

Theorem 3.1. *For any $\mathscr{M}$, $(\mathscr{M}_\xi, \odot)$ is an AG-groupoid.*

Proof. It is clear that $(\mathscr{M}_\xi, \odot)$ is closed. Let $\mathscr{M}_\xi, \mathscr{M}_\zeta, \mathscr{M}_\mathscr{R} \in \mathscr{M}$. Then for any $t \in \mathscr{M}$,

$$\begin{aligned}
((T_\xi \circ T_\zeta) \circ T_\mathscr{R})(t) &= \{(T_\xi \circ T_\zeta)(y) \vee T_\mathscr{R}(z)\} \\
&= \bigwedge_{t=yz} \{ \bigwedge_{y=rs} \{T_\xi(r) \vee T_\zeta(s)\} \vee T_\mathscr{R}(z)\} \\
&= \bigwedge_{t=(rs)z} \{T_\xi(r) \vee T_\zeta(s) \vee T_\mathscr{R}(z)\} \\
&= \bigwedge_{t=(zs)r} \{T_\mathscr{R}(z) \vee T_\zeta(s) \vee T_\xi(r)\} \\
&= \bigwedge_{t=ur} \{(T_\mathscr{R} \circ T_\zeta)(u) \vee T_\xi(r)\} \\
&= ((T_\mathscr{R} \circ T_\zeta) \circ T_\xi)(t), \\
((I_\xi \circ I_\zeta) \circ I_\mathscr{R})(t) &= \bigvee_{t=yz} \{(I_\xi \circ I_\zeta)(y) \wedge I_\mathscr{R}(z)\} \\
&= \bigvee_{t=yz} \{ \bigvee_{y=rs} \{I_\xi(r) \wedge I_\zeta(s)\} \wedge I_\mathscr{R}(z)\} \\
&= \bigvee_{t=(rs)z} \{I_\xi(r) \wedge I_\zeta(s) \wedge I_\mathscr{R}(z)\} \\
&= \bigvee_{t=(zs)r} \{I_\mathscr{R}(z) \wedge I_\zeta(s) \wedge I_\xi(r)\} \\
&= \bigwedge_{t=ur} \{(I_\mathscr{R} \circ I_\zeta)(u) \wedge I_\xi(r)\} \\
&= ((I_\mathscr{R} \circ I_\zeta) \circ I_\xi)(t),
\end{aligned}$$

$$\begin{aligned}((F_\xi \circ F_\zeta) \circ F_{\mathscr{R}})(t) &= \bigwedge_{t=yz} \{(F_\xi \circ F_\zeta)(y) \vee F_{\mathscr{R}}(z)\} \\ &= \bigwedge_{t=yz} \{ \bigwedge_{y=rs} \{F_\xi(r) \vee F_\zeta(s)\} \vee F_{\mathscr{R}}(z)\} \\ &= \bigwedge_{t=(rs)z} \{F_\xi(r) \vee F_\zeta(s) \vee F_{\mathscr{R}}(z)\} \\ &= \bigwedge_{t=(zs)r} \{F_{\mathscr{R}}(z) \vee F_\zeta(s) \vee F_\xi(r)\} \\ &= \bigwedge_{t=ur} \{(F_{\mathscr{R}} \circ F_\zeta)(u) \vee F_\xi(r)\} \\ &= ((F_{\mathscr{R}} \circ F_\zeta) \circ F_\xi)(t).\end{aligned}$$

Therefore $(\mathscr{M}_\xi, \odot)$ is an AG-groupoid. □

Corollary 3.2. *For any $\mathscr{M}_\xi, \mathscr{M}_\zeta, \mathscr{M}_{\mathscr{R}}, \mathscr{M}_{\mathscr{Q}} \in \mathscr{M}$, $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_{\mathscr{Q}}) = (\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}}) \odot (\mathscr{M}_\zeta \odot \mathscr{M}_{\mathscr{Q}})$.*

Proof. Let $\mathscr{M}_\xi, \mathscr{M}_\zeta, \mathscr{M}_{\mathscr{R}}, \mathscr{M}_{\mathscr{Q}} \in \mathscr{M}$. Then

$(T_\xi \circ T_\zeta) \circ (T_{\mathscr{R}} \circ T_{\mathscr{Q}}) = ((T_{\mathscr{R}} \circ T_{\mathscr{Q}}) \circ T_\zeta) \circ T_\xi) = ((T_\zeta \circ T_{\mathscr{Q}}) \circ T_{\mathscr{R}}) \circ T_\xi) = (T_\xi \circ T_{\mathscr{R}}) \circ (T_\zeta \circ T_{\mathscr{Q}})$,

$(F_\xi \circ F_\zeta) \circ (F_{\mathscr{R}} \circ F_{\mathscr{Q}}) = ((F_{\mathscr{R}} \circ F_{\mathscr{Q}}) \circ F_\zeta) \circ F_\xi) = ((F_\zeta \circ F_{\mathscr{Q}}) \circ F_{\mathscr{R}}) \circ F_\xi) = (F_\xi \circ F_{\mathscr{R}}) \circ (F_\zeta \circ F_{\mathscr{Q}})$

and

$(I_\xi \circ I_\zeta) \circ (I_{\mathscr{R}} \circ I_{\mathscr{Q}}) = ((I_{\mathscr{R}} \circ I_{\mathscr{Q}}) \circ I_\zeta) \circ I_\xi) = ((I_\zeta \circ I_{\mathscr{Q}}) \circ I_{\mathscr{R}}) \circ I_\xi) = (I_\xi \circ I_{\mathscr{R}}) \circ (I_\zeta \circ I_{\mathscr{Q}})$.

Hence $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_{\mathscr{Q}}) = (\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}}) \odot (\mathscr{M}_\zeta \odot \mathscr{M}_{\mathscr{Q}})$. □

Theorem 3.3. *If $\mathscr{M}$ has left identity, then for any $\mathscr{M}_\xi, \mathscr{M}_\zeta, \mathscr{M}_{\mathscr{R}}, \mathscr{M}_{\mathscr{Q}} \in \mathscr{M}$, we have the following:*

(i) $\mathscr{M}_\xi \odot (\mathscr{M}_\zeta \odot \mathscr{M}_{\mathscr{R}}) = \mathscr{M}_\zeta \odot (\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}})$,

(ii) $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_{\mathscr{Q}}) = (\mathscr{M}_{\mathscr{Q}} \odot \mathscr{M}_{\mathscr{R}}) \odot (\mathscr{M}_\zeta \odot \mathscr{M}_\xi)$.

Proof. (i) Let $m \in \mathscr{M}$. If $m \neq xy$ for any $x, y \in \mathscr{M}$, then

$$(T_\xi \circ (T_\zeta \circ T_{\mathscr{R}}))(m) = 0 = (T_\zeta \circ (T_\xi \circ T_{\mathscr{R}}))(m),$$

$$(I_\xi \circ (I_\zeta \circ I_{\mathscr{R}}))(m) = -1 = (I_\zeta \circ (I_\xi \circ I_{\mathscr{R}}))(m),$$

$$(F_\xi \circ (F_\zeta \circ F_{\mathscr{R}}))(m) = 0 = (F_\zeta \circ (F_\xi \circ F_{\mathscr{R}}))(m).$$

Suppose $m = yz$ for $y, z \in \mathscr{M}$. Then

$$
\begin{aligned}
(T_\xi \circ (T_\zeta \circ T_{\mathscr{R}}))(m) &= \bigwedge_{m=yz} \{T_\xi(y) \vee (T_\zeta \circ T_{\mathscr{R}})(z)\} \\
&= \bigwedge_{m=yz} \{T_\xi(y) \vee \bigwedge_{z=rs} \{T_\zeta(r) \vee T_{\mathscr{R}}(s)\}\} \\
&= \bigwedge_{m=y(rs)} \{T_\xi(y) \vee T_\zeta(r) \vee T_{\mathscr{R}}(s)\} \\
&= \bigwedge_{m=r(ys)} \{T_\zeta(r) \vee T_\xi(y) \vee T_{\mathscr{R}}(s)\} \\
&= \bigwedge_{m=rp} \{T_\zeta(r) \vee \bigwedge_{p=ys} \{T_\xi(y) \vee T_{\mathscr{R}}(s)\}\} \\
&= \bigwedge_{m=rp} \{T_\zeta(r) \vee (T_\xi \circ T_{\mathscr{R}})(p)\} \\
&= (T_\zeta \circ (T_\xi \circ T_{\mathscr{R}}))(m), \\
(I_\xi \circ (I_\zeta \circ I_{\mathscr{R}}))(m) &= \bigvee_{m=yz} \{I_\xi(y) \wedge (I_\zeta \circ I_{\mathscr{R}})(z)\} \\
&= \bigvee_{m=yz} \{I_\xi(y) \wedge \bigvee_{z=rs} \{I_\zeta(r) \wedge I_{\mathscr{R}}(s)\}\} \\
&= \bigvee_{m=y(rs)} \{I_\xi(y) \wedge I_\zeta(r) \wedge I_{\mathscr{R}}(s)\} \\
&= \bigvee_{m=r(ys)} \{I_\zeta(r) \wedge I_\xi(y) \wedge I_{\mathscr{R}}(s)\} \\
&= \bigvee_{m=rp} \{I_\zeta(r) \wedge \bigvee_{p=ys} \{I_\xi(y) \wedge I_{\mathscr{R}}(s)\}\} \\
&= \bigvee_{m=rp} \{I_\zeta(r) \wedge (I_\xi \circ I_{\mathscr{R}})(p)\} \\
&= (I_\zeta \circ (I_\xi \circ I_{\mathscr{R}}))(m), \\
(F_\xi \circ (F_\zeta \circ F_{\mathscr{R}}))(m) &= \bigwedge_{m=yz} \{F_\xi(y) \vee (F_\zeta \circ F_{\mathscr{R}})(z)\} \\
&= \bigwedge_{m=yz} \{F_\xi(y) \vee \bigwedge_{z=rs} \{F_\zeta(r) \vee F_{\mathscr{R}}(s)\}\} \\
&= \bigwedge_{m=y(rs)} \{F_\xi(y) \vee F_\zeta(r) \vee F_{\mathscr{R}}(s)\} \\
&= \bigwedge_{m=r(ys)} \{F_\zeta(r) \vee F_\xi(y) \vee F_{\mathscr{R}}(s)\} \\
&= \bigwedge_{m=rp} \{F_\zeta(r) \vee \bigwedge_{p=ys} \{F_\xi(y) \vee F_{\mathscr{R}}(s)\}\} \\
&= \bigwedge_{m=rp} \{F_\zeta(r) \vee (F_\xi \circ F_{\mathscr{R}})(p)\} \\
&= (F_\zeta \circ (F_\xi \circ F_{\mathscr{R}}))(m).
\end{aligned}
$$

Therefore $\mathscr{M}_\xi \odot (\mathscr{M}_\zeta \odot \mathscr{M}_{\mathscr{R}}) = \mathscr{M}_\zeta \odot (\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}})$.

(ii) Let $m \in \mathscr{M}$. If $m \neq xy$ for any $x, y \in \mathscr{M}$, then

$$((T_\xi \circ T_\zeta) \circ (T_\mathscr{R} \circ T_\mathscr{Q}))(m) = 1 = ((T_\mathscr{Q} \circ T_\mathscr{R}) \circ (T_\zeta \circ T_\xi))(m).$$

Suppose $m = yz$ for any $y, z \in \mathscr{M}$. Then

$$\begin{aligned}
((T_\xi \circ T_\zeta) \circ (T_\mathscr{R} \circ T_\mathscr{Q}))(m) &= \bigwedge_{m=yz} \{(T_\xi \circ T_\zeta)(y) \vee (T_\mathscr{R} \circ T_\mathscr{Q})(z)\} \\
&= \bigwedge_{m=yz} \{ \bigwedge_{y=pq} \{T_\xi(p) \vee T_\zeta(q)\} \vee \bigwedge_{z=rs} \{T_\mathscr{R}(r) \vee T_\mathscr{Q}(s)\}\} \\
&= \bigwedge_{m=(pq)(rs)} \{T_\xi(p) \vee T_\zeta(q) \vee T_\mathscr{R}(r) \vee T_\mathscr{Q}(s)\} \\
&= \bigwedge_{m=(sr)(qp)} \{T_\mathscr{Q}(s) \vee T_\mathscr{R}(r) \vee T_\zeta(q) \vee T_\xi(p)\} \\
&= \bigwedge_{m=vw} \{ \bigwedge_{v=sr} \{T_\mathscr{Q}(s) \vee T_\mathscr{R}(r)\} \vee \bigwedge_{w=qp} \{T_\zeta(q) \vee T_\xi(p)\}\} \\
&= \bigwedge_{m=vw} \{(T_\mathscr{Q} \circ T_\mathscr{R})(v) \vee (T_\zeta \circ T_\xi)(w)\} \\
&= ((T_\mathscr{Q} \circ T_\mathscr{R}) \circ (T_\zeta \circ T_\xi))(m), \\
((I_\xi \circ I_\zeta) \circ (I_\mathscr{R} \circ I_\mathscr{Q}))(m) &= \bigvee_{m=yz} \{(I_\xi \circ I_\zeta)(y) \wedge (I_\mathscr{R} \circ I_\mathscr{Q})(z)\} \\
&= \bigvee_{m=yz} \{ \bigvee_{y=pq} \{I_\xi(p) \wedge I_\zeta(q)\} \wedge \bigvee_{z=rs} \{I_\mathscr{R}(r) \wedge I_\mathscr{Q}(s)\}\} \\
&= \bigvee_{m=(pq)(rs)} \{I_\xi(p) \wedge I_\zeta(q) \wedge I_\mathscr{R}(r) \wedge I_\mathscr{Q}(s)\} \\
&= \bigvee_{m=(sr)(qp)} \{I_\mathscr{Q}(s) \wedge I_\mathscr{R}(r) \wedge I_\zeta(q) \wedge I_\xi(p)\} \\
&= \bigvee_{m=vw} \{ \bigvee_{v=sr} \{I_\mathscr{Q}(s) \wedge I_\mathscr{R}(r)\} \wedge \bigvee_{w=qp} \{I_\zeta(q) \wedge I_\xi(p)\}\} \\
&= \bigvee_{m=vw} \{(I_\mathscr{Q} \circ I_\mathscr{R})(v) \wedge (I_\zeta \circ I_\xi)(w)\} \\
&= ((I_\mathscr{Q} \circ I_\mathscr{R}) \circ (I_\zeta \circ I_\xi))(m), \\
((F_\xi \circ F_\zeta) \circ (F_\mathscr{R} \circ F_\mathscr{Q}))(m) &= \bigwedge_{m=yz} \{(F_\xi \circ F_\zeta)(y) \vee (F_\mathscr{R} \circ F_\mathscr{Q})(z)\} \\
&= \bigwedge_{m=yz} \{ \bigwedge_{y=pq} \{F_\xi(p) \vee F_\zeta(q)\} \vee \bigwedge_{z=rs} \{F_\mathscr{R}(r) \vee F_\mathscr{Q}(s)\}\} \\
&= \bigwedge_{m=(pq)(rs)} \{F_\xi(p) \vee F_\zeta(q) \vee F_\mathscr{R}(r) \vee F_\mathscr{Q}(s)\} \\
&= \bigwedge_{m=(sr)(qp)} \{F_\mathscr{Q}(s) \vee F_\mathscr{R}(r) \vee F_\zeta(q) \vee F_\xi(p)\} \\
&= \bigwedge_{m=vw} \{ \bigwedge_{v=sr} \{F_\mathscr{Q}(s) \vee F_\mathscr{R}(r)\} \vee \bigwedge_{w=qp} \{F_\zeta(q) \vee F_\xi(p)\}\}
\end{aligned}$$

$$= \bigwedge_{m=vw} \{(F_{\mathscr{Q}} \circ F_{\mathscr{R}})(v) \vee (F_\zeta \circ F_\xi)(w)\}$$
$$= ((F_{\mathscr{Q}} \circ F_{\mathscr{R}}) \circ (F_\zeta \circ F_\xi))(m).$$

Therefore $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_{\mathscr{Q}}) = (\mathscr{M}_{\mathscr{Q}} \odot \mathscr{M}_{\mathscr{R}}) \odot (\mathscr{M}_\zeta \odot \mathscr{M}_\xi)$. $\square$

Theorem 3.4. *Let $\mathscr{M}_\xi \in \mathscr{M}$. Then the listed conditions hold:*

(i) $\mathscr{M}_\xi \in \mathscr{M}_s \Leftrightarrow \mathscr{M}_\xi \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$.

(ii) $\mathscr{M}_\xi \in \mathscr{M}_l \Leftrightarrow \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.

(iii) $\mathscr{M}_\xi \in \mathscr{M}_r \Leftrightarrow \mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.

(iv) $\mathscr{M}_\xi \in \mathscr{M}_i \Leftrightarrow \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ and $\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.

Proof. (i) Assume $\mathscr{M}_\xi \in \mathscr{M}_s$. Now, for any $k \in \mathscr{M}$,

$$(T_\xi \circ T_\xi)(k) = \bigwedge_{k=k_1k_2} \{T_\xi(k_1) \vee T_\xi(k_2)\} \geq \bigwedge_{k=k_1k_2} T_\xi(k_1k_2) = T_\xi(k),$$
$$(I_\xi \circ I_\xi)(k) = \bigvee_{k=k_1k_2} \{I_\xi(k_1) \wedge I_\xi(k_2)\} \leq \bigvee_{k=k_1k_2} I_\xi(k_1k_2) = I_\xi(k),$$
$$(F_\xi \circ F_\xi)(k) = \bigwedge_{k=k_1k_2} \{F_\xi(k_1) \vee F_\xi(k_2)\} \geq \bigwedge_{k=k_1k_2} F_\xi(k_1k_2) = F_\xi(k).$$

So $\mathscr{M}_\xi \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$.

Conversely, assume $\mathscr{M}_\xi \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$. Now, for any $k_1, k_2 \in \mathscr{M}$,

$$T_\xi(k_1k_2) \leq (T_\xi \circ T_\xi)(k_1k_2) = \bigwedge_{k_1k_2} \{T_\xi(k_1) \vee T_\xi(k_2)\} \leq T_\xi(k_1) \vee T_\xi(k_2),$$
$$I_\xi(k_1k_2) \geq (I_\xi \circ I_\xi)(k_1k_2) = \bigvee_{k_1k_2} \{I_\xi(k_1) \wedge I_\xi(k_2)\} \geq I_\xi(k_1) \wedge I_\xi(k_2),$$
$$F_\xi(k_1k_2) \leq (F_\xi \circ F_\xi)(k_1k_2) = \bigwedge_{k_1k_2} \{F_\xi(k_1) \vee F_\xi(k_2)\} \leq F_\xi(k_1) \vee F_\xi(k_2).$$

So $\mathscr{M}_\xi \in \mathscr{M}_s$.

(ii) Assuming $\mathscr{M}_\xi \in \mathscr{M}_l$. Now for any $\mathscr{M}_\zeta \in \mathscr{M}$ and $k \in \mathscr{M}$,

$$(\chi_{\mathscr{M}}(T)_\zeta \circ T_\xi)(k) = \bigwedge_{k=k_1k_2} \{\chi_{\mathscr{M}}(T)_\zeta(k_1) \vee T_\xi(k_2)\}$$
$$= \bigwedge_{k=k_1k_2} T_\xi(k_2)$$
$$\geq T_\xi(k_1k_2)$$
$$= T_\xi(k),$$

$$\begin{aligned}(\chi_{\mathscr{M}}(I)_\zeta \circ I_\xi)(k) &= \bigvee_{k=k_1k_2} \{\chi_{\mathscr{M}}(I)_\zeta(k_1) \wedge I_\xi(k_2)\}\\ &= \bigvee_{k=k_1k_2} I_\xi(k_2)\\ &\leq I_\xi(k_1k_2)\\ &= I_\xi(k),\end{aligned}$$

$$\begin{aligned}(\chi_{\mathscr{M}}(F)_\zeta \circ F_\xi)(k) &= \bigwedge_{k=k_1k_2} \{\chi_{\mathscr{M}}(F)_\zeta(k_1) \vee F_\xi(k_2)\}\\ &= \bigwedge_{k=k_1k_2} F_\xi(k_2)\\ &\geq F_\xi(k_1k_2)\\ &= F_\xi(k).\end{aligned}$$

Therefore $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$.

Conversely, suppose $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$. Now for any $k_1, k_2 \in \mathscr{M}$,

$$\begin{aligned}T_\xi(k_1k_2) &\leq (\chi_{\mathscr{M}}(T)_\zeta \circ T_\xi)(k_1k_2)\\ &= \bigwedge_{k=k_1k_2} \{\chi_{\mathscr{M}}(T)_\zeta(k_1) \vee T_\xi(k_2)\}\\ &\leq \chi_{\mathscr{M}}(T)_\zeta(k_1) \vee T_\xi(k_2)\\ &= T_\xi(k_2),\\ I_\xi(k_1k_2) &\geq (\chi_{\mathscr{M}}(I)_\zeta \circ I_\xi)(k_1k_2)\\ &= \bigvee_{k=k_1k_2} \{\chi_{\mathscr{M}}(T)_\zeta(k_1) \wedge I_\xi(k_2)\}\\ &\geq \chi_{\mathscr{M}}(I)_\zeta(k_1) \wedge I_\xi(k_2)\\ &= I_\xi(k_2),\\ F_\xi(k_1k_2) &\leq (\chi_{\mathscr{M}}(F)_\zeta \circ F_\xi)(k_1k_2)\\ &= \bigwedge_{k=k_1k_2} \{\chi_{\mathscr{M}}(F)_\zeta(k_1) \vee F_\xi(k_2)\}\\ &\leq \chi_{\mathscr{M}}(F)_\zeta(k_1) \vee F_\xi(k_2)\\ &= F_\xi(k_2).\end{aligned}$$

Hence $\mathscr{M}_\xi \in \mathscr{M}_l$.

The proof of (iii) and (iv) is left to the reader. □

Lemma 3.5. *(i) If $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_s$, then $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_s$.*

(ii) If $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_l$, then $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_l$.

(iii) If $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_r$, then $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_r$.

(iv) If $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_i$, then $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_i$.

Proof. (i) Let $\mathscr{M}_\xi$ and $\mathscr{M}_\zeta$ be two neutrosophic $\varkappa$-AG-subgroupoids in $\mathscr{M}$. Now for $k_1, k_2 \in \mathscr{M}$,

$$\begin{aligned}
(T_\xi \cap T_\zeta)(k_1k_2) &= T_\xi(k_1k_2) \vee T_\zeta(k_1k_2) \\
&\leq (T_\xi(k_1) \vee T_\xi(k_2)) \vee (T_\zeta(k_1) \vee T_\zeta(k_2)) \\
&= (T_\xi(k_1) \vee T_\zeta(k_1)) \vee (T_\xi(k_2) \vee T_\zeta(k_2)) \\
&= (T_\xi \cap T_\zeta)(k_1) \vee (T_\xi \cap T_\zeta)(k_2), \\
(I_\xi \cap I_\zeta)(k_1k_2) &= I_\xi(k_1k_2) \wedge I_\zeta(k_1k_2) \\
&\geq (I_\xi(k_1) \wedge I_\xi(k_2)) \wedge (I_\zeta(k_1) \wedge I_\zeta(k_2)) \\
&= (I_\xi(k_1) \wedge I_\zeta(k_1)) \wedge (I_\xi(k_2) \wedge I_\zeta(k_2)) \\
&= (I_\xi \cap I_\zeta)(k_1) \wedge (I_\xi \cap I_\zeta)(k_2), \\
(F_\xi \cap F_\zeta)(k_1k_2) &= F_\xi(k_1k_2) \vee F_\zeta(k_1k_2) \\
&\leq (F_\xi(k_1) \vee F_\xi(k_2)) \vee (F_\zeta(k_1) \vee F_\zeta(k_2)) \\
&= (F_\xi(k_1) \vee F_\zeta(k_1)) \vee (F_\xi(k_2) \vee F_\zeta(k_2)) \\
&= (F_\xi \cap F_\zeta)(k_1) \vee (F_\xi \cap F_\zeta)(k_2).
\end{aligned}$$

So $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_s$.

(ii) Let $\mathscr{M}_\xi,\ \mathscr{M}_\zeta \in \mathscr{M}_l$. Now for any $k_1, k_2 \in \mathscr{M}$,

$$\begin{aligned}
(T_\xi \cap T_\zeta)(k_1k_2) &= T_\xi(k_1k_2) \vee T_\zeta(k_1k_2) \leq T_\xi(k_2) \vee T_\zeta(k_2) = (T_\xi \cap T_\zeta)(k_2), \\
(I_\xi \cap I_\zeta)(k_1k_2) &= I_\xi(k_1k_2) \wedge I_\zeta(k_1k_2) \geq I_\xi(k_2) \wedge I_\zeta(k_2) = (I_\xi \cap I_\zeta)(k_2), \\
(F_\xi \cap F_\zeta)(k_1k_2) &= F_\xi(k_1k_2) \vee F_\zeta(k_1k_2) \leq F_\xi(k_2) \vee F_\zeta(k_2) = (F_\xi \cap F_\zeta)(k_2).
\end{aligned}$$

So $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_l$.

The proof of (iii) and (iv) is left to the reader. □

Lemma 3.6. *If $\mathscr{M}$ is having left identity e, then $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) = \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.*

Proof. Let $k_1 \in \mathscr{M}$. Then $k_1 = ek_1$. Now,

$$(\chi_{\mathscr{M}}(T)_\zeta \circ \chi_{\mathscr{M}}(T)_\zeta)(k_1) = \bigwedge_{k_1 = x_1x_2} \{\chi_{\mathscr{M}}(T)_\zeta(x_1) \vee \chi_{\mathscr{M}}(T)_\zeta(x_2)\} \leq \chi_{\mathscr{M}}(T)_\zeta(e) \vee \chi_{\mathscr{M}}(T)_\zeta(k_1) = 0$$

which implies $(\chi_{\mathscr{M}}(T)_\zeta \circ \chi_{\mathscr{M}}(T)_\zeta)(k_1) = 0 = \chi_{\mathscr{M}}(T)_\zeta(k_1)$.

$$(\chi_{\mathscr{M}}(I)_\zeta \circ \chi_{\mathscr{M}}(I)_\zeta)(k_1) = \bigvee_{k_1 = x_1x_2} \{\chi_{\mathscr{M}}(I)_\zeta(x_1) \wedge \chi_{\mathscr{M}}(I)_\zeta(x_2)\} \geq \chi_{\mathscr{M}}(I)_\zeta(e) \wedge \chi_{\mathscr{M}}(I)_\zeta(k_1) = -1$$

which implies $(\chi_{\mathscr{M}}(I)_\zeta \circ \chi_{\mathscr{M}}(I)_\zeta)(k_1) = -1 = \chi_{\mathscr{M}}(I)_\zeta(k_1)$.

$$(\chi_{\mathscr{M}}(F)_\zeta \circ \chi_{\mathscr{M}}(F)_\zeta)(k_1) = \bigwedge_{k_1=x_1x_2} \{\chi_{\mathscr{M}}(F)_\zeta(x_1) \vee \chi_{\mathscr{M}}(F)_\zeta(x_2)\} \leq \chi_{\mathscr{M}}(F)_\zeta(e) \vee \chi_{\mathscr{M}}(F)_\zeta(k_1) = 0$$

which implies $(\chi_{\mathscr{M}}(F)_\zeta \circ \chi_{\mathscr{M}}(F)_\zeta)(k_1) = 0 = \chi_{\mathscr{M}}(F)_\zeta(k_1)$.

Therefore $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) = \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)$. □

Lemma 3.7. *If $\mathscr{M}$ has left identity e, then for any $\mathscr{M}_\zeta \in \mathscr{M}$, we have $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi = \mathscr{M}_\xi$ for every $\mathscr{M}_\xi \in \mathscr{M}_l$.*

Proof. Let $k_1 \in \mathscr{M}$. Then $k_1 = ek_1$. Now,

$$(\chi_{\mathscr{M}}(T)_\zeta \circ (T)_\xi)(k_1) = \bigwedge_{k_1=x_1x_2} \{(\chi_{\mathscr{M}}(T)_\zeta)(x_1) \vee (T)_\xi(x_2)\} \leq (\chi_{\mathscr{M}}(T)_\zeta)(e) \vee (T)_\xi(k_1) = (T)_\xi(k_1),$$

$$(\chi_{\mathscr{M}}(I)_\zeta \circ (I)_\xi)(k_1) = \bigvee_{k_1=x_1x_2} \{(\chi_{\mathscr{M}}(I)_\zeta)(x_1) \wedge (I)_\xi(x_2)\} \geq (\chi_{\mathscr{M}}(I)_\zeta)(e) \wedge (I)_\xi(k_1) = (I)_\xi(k_1),$$

$$(\chi_{\mathscr{M}}(F)_\zeta \circ (F)_\xi)(k_1) = \bigwedge_{k_1=x_1x_2} \{(\chi_{\mathscr{M}}(F)_\zeta)(x_1) \vee (F)_\xi(x_2)\} \leq (\chi_{\mathscr{M}}(F)_\zeta)(e) \vee (F)_\xi(k_1) = (F)_\xi(k_1).$$

So $\mathscr{M}_\xi \subseteq \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi$. By Theorem 3.4, $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ and hence $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi = \mathscr{M}_\xi$. □

Proposition 3.8. *Suppose $\mathscr{M}$ is having left identity. If $\mathscr{M}_\xi$, $\mathscr{M}_\zeta \in \mathscr{M}_l$, then for any $\mathscr{M}_{\mathscr{R}}, \mathscr{M}_{\mathscr{Q}} \in \mathscr{M}$, $\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}} = \mathscr{M}_\zeta \odot \mathscr{M}_{\mathscr{Q}}$ implies $\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_\xi = \mathscr{M}_{\mathscr{Q}} \odot \mathscr{M}_\zeta$.*

Proof. Since $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_l$, we have by Lemma 3.7, $\chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{S}}) \odot \mathscr{M}_\xi = \mathscr{M}_\xi$ and $\chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{U}}) \odot \mathscr{M}_\zeta = \mathscr{M}_\zeta$ for $\mathscr{M}_{\mathscr{S}}, \mathscr{M}_{\mathscr{U}} \in \mathscr{M}$. Now, for any $\mathscr{M}_{\mathscr{M}} \in \mathscr{M}$, $\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_\xi = (\chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{M}}) \odot \mathscr{M}_{\mathscr{R}}) \odot \mathscr{M}_\xi = (\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}}) \odot \chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{M}}) = (\mathscr{M}_\zeta \odot \mathscr{M}_{\mathscr{Q}}) \odot \chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{M}}) = (\chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{M}}) \odot \mathscr{M}_{\mathscr{Q}}) \odot \mathscr{M}_\zeta = \mathscr{M}_{\mathscr{Q}} \odot \mathscr{M}_\zeta$.
□

Corollary 3.9. *For any $\mathscr{M}_\xi, \mathscr{M}_\zeta, \mathscr{M}_{\mathscr{R}} \in \mathscr{M}$, the listed claims are equivalent:*

(i) $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot \mathscr{M}_{\mathscr{R}} = \mathscr{M}_\zeta \odot (\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}})$,

(ii) $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot \mathscr{M}_{\mathscr{R}} = \mathscr{M}_\zeta \odot (\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_\xi)$.

Proposition 3.10. *Let $\mathscr{M}_\xi \in \mathscr{M}_l$. If $\mathscr{M}_\xi \in \mathscr{M}_d$, then $\mathscr{M}_\xi \in \mathscr{M}_i$.*

Proof. Let $\mathscr{M}_\xi \in \mathscr{M}_l$ and $\mathscr{M}_\xi \in \mathscr{M}_d$. Then for any $\mathscr{M}_\zeta \in \mathscr{M}$, $\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) = (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi = \mathscr{M}_\xi$, so $\mathscr{M}_\xi \in \mathscr{M}_i$. □

Remark 3.11. If $\mathscr{M}$ has left identity, then $\mathscr{M}_l = \mathscr{M}_r$.

Theorem 3.12. *Suppose $\mathscr{M}$ has left identity and $\mathscr{M}_\xi \in \mathscr{M}_d$. Then the listed claims holds:*

(i) $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \in \mathscr{M}_d$ *for* $\mathscr{M}_\zeta \in \mathscr{M}$,

(ii) Every $\mathscr{M}_\zeta \in \mathscr{M}_l$ *commutes with* $\mathscr{M}_\xi$.

Proof. (i) It is clear from Corollary 3.2 and Lemma 3.6.

(ii) Let $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}$. Now, $\mathscr{M}_\xi \odot \mathscr{M}_\zeta = (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot \mathscr{M}_\zeta = (\mathscr{M}_\zeta \odot \mathscr{M}_\xi) \odot \mathscr{M}_\xi \subseteq (\mathscr{M}_\zeta \odot \chi_{\mathscr{M}}(\mathscr{M}_\xi)) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\zeta \odot \mathscr{M}_\xi$. Also, $\mathscr{M}_\zeta \odot \mathscr{M}_\xi = \mathscr{M}_\zeta \odot (\mathscr{M}_\xi \odot \mathscr{M}_\xi) = \mathscr{M}_\xi \odot (\mathscr{M}_\zeta \odot \mathscr{M}_\xi) \subseteq \mathscr{M}_\xi \odot (\mathscr{M}_\zeta \odot \chi_{\mathscr{M}}(\mathscr{M}_\xi)) \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\zeta$. □

Lemma 3.13. *If $\mathscr{M}$ has left identity and $\mathscr{M}_\xi \in \mathscr{M}_r$, then $\mathscr{M}_\xi \in \mathscr{M}_i$.*

Proof. Let $\mathscr{M}_\xi \in \mathscr{M}_r$. Then for $\mathscr{M}_\zeta \in \mathscr{M}$, $\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$. By Lemma 3.6, $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi = (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$. So $\mathscr{M}_\xi \in \mathscr{M}_l$ and hence $\mathscr{M}_\xi \in \mathscr{M}_i$. □

Remark 3.14. Suppose $\mathscr{M}$ has left identity. If $\mathscr{M}_\xi \in \mathscr{M}_r$, then $\mathscr{M}_\xi \cup (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi)$ and $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi)$ are neutrosophic $\varkappa$- ideals for $\mathscr{M}_\zeta \in \mathscr{M}$.

Theorem 3.15. *Suppose $\mathscr{M}_\xi \in \mathscr{M}_l$ with left identity. Then $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta))$ and $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi)$ are neutrosophic $\varkappa$-ideals for $\mathscr{M}_\zeta \in \mathscr{M}$.*

Proof. Now,

$$\begin{aligned}
(\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta))) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) &= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup ((\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \\
&= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup ((\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi) \\
&= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \\
&= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup \mathscr{M}_\xi \\
&= \mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)).
\end{aligned}$$

Thus $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \in \mathscr{M}_r$ and hence $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \in \mathscr{M}_i$ by Lemma 3.13.

Now, for any $\mathscr{M}_\zeta \in \mathscr{M}$,

$$\begin{aligned}
(\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi)) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) &= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup ((\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \\
&= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup ((\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \mathscr{M}_\xi) \\
&\subseteq (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \\
&= (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \cup (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \\
&\subseteq (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \cup \mathscr{M}_\xi \\
&= \mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi).
\end{aligned}$$

Thus $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \in \mathscr{M}_r$ and so $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \in \mathscr{M}_i$ by Lemma 3.13. □

Theorem 3.16. *Suppose $\phi \neq U \subseteq \mathscr{M}$. Then the below claims are equivalent:*

(i) U is bi-ideal,

(ii) For any $\mathscr{M}_\xi \in \mathscr{M}$, $\chi_U(\mathscr{M}_\xi) \in \mathscr{M}_b$.

Proof. This is similar to Theorem 3.1 in [20]. □

Lemma 3.17. *Let $\mathscr{M}_\xi \in \mathscr{M}_s$. Then the listed claims are equivalent:*

(i) $\mathscr{M}_\xi \in \mathscr{M}_b$,

(ii) $(\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.

Proof. Assume $\mathscr{M}_\xi \in \mathscr{M}_b$ and let $k_1 \in \mathscr{M}$. Suppose $\exists\ x_1, x_2 \in \mathscr{M} \ni k_1 = x_1x_2$. Then

$$\begin{aligned}
(((T)_\xi \circ \chi_{\mathscr{M}}(T)_\zeta) \circ (T)_\xi)(k_1) &= \bigwedge_{k_1=x_1x_2} \{((T)_\xi \circ \chi_{\mathscr{M}}(T)_\zeta)(x_1) \vee (T)_\xi(x_2)\} \\
&= \bigwedge_{k_1=x_1x_2} \{ \bigwedge_{x_1=x_3x_4} \{(T)_\xi(x_3) \vee \chi_{\mathscr{M}}(T)_\zeta(x_4)\} \vee (T)_\xi(x_2)\} \\
&= \bigwedge_{k_1=x_3x_4x_2} \{((T)_\xi(x_3) \vee (-1)) \vee (T)_\xi(x_2)\} \\
&= \bigwedge_{k_1=x_3x_4x_2} \{(T)_\xi(x_3) \vee (T)_\xi(x_2)\} \\
&\geq \bigwedge_{k_1=x_3x_4x_2} (T)_\xi(x_3x_4x_2) \\
&= (T)_\xi(k_1),
\end{aligned}$$

$$\begin{aligned}
(((I)_\xi \circ \chi_{\mathscr{M}}(I)_\zeta) \circ (I)_\xi)(k_1) &= \bigvee_{k_1=x_1x_2} \{((I)_\xi \circ \chi_{\mathscr{M}}(I)_\zeta)(x_1) \wedge (I)_\xi(x_2)\} \\
&= \bigvee_{k_1=x_1x_2} \{ \bigvee_{x_1=x_3x_4} \{(I)_\xi(x_3) \wedge \chi_{\mathscr{M}}(I)_\zeta(x_4)\} \wedge (I)_\xi(x_2)\} \\
&= \bigvee_{k_1=x_3x_4x_2} \{((I)_\xi(x_3) \wedge 0) \wedge (I)_\xi(x_2)\} \\
&= \bigvee_{k_1=x_3x_4x_2} \{(I)_\xi(x_3) \wedge (I)_\xi(x_2)\} \\
&\leq \bigvee_{k_1=x_3x_4x_2} (I)_\xi(x_3x_4x_2) \\
&= (I)_\xi(k_1), \\
(((F)_\xi \circ \chi_{\mathscr{M}}(F)_\zeta) \circ (F)_\xi)(k_1) &= \bigwedge_{k_1=x_1x_2} \{((F)_\xi \circ \chi_{\mathscr{M}}(F)_\zeta)(x_1) \vee (F)_\xi(x_2)\} \\
&= \bigwedge_{k_1=x_1x_2} \{ \bigwedge_{x_1=x_3x_4} \{(F)_\xi(x_3) \vee \chi_{\mathscr{M}}(F)_\zeta(x_4)\} \vee (F)_\xi(x_2)\} \\
&= \bigwedge_{k_1=x_3x_4x_2} \{((F)_\xi(x_3) \vee (-1)) \vee (F)_\xi(x_2)\} \\
&= \bigwedge_{k_1=x_3x_4x_2} \{(F)_\xi(x_3) \vee (F)_\xi(x_2)\} \\
&\geq \bigwedge_{k_1=x_3x_4x_2} (F)_\xi(x_3x_4x_2) \\
&= (F)_\xi(k_1).
\end{aligned}$$

Suppose there is no $x_1, x_2 \in \mathscr{M} \ni k_1 = x_1x_2$. Then

$$(((T)_\xi \circ \chi_{\mathscr{M}}(T)_\zeta) \circ (T)_\xi)(k_1) = 0 \geq (T)_\xi(k_1),$$

$$(((I)_\xi \circ \chi_{\mathscr{M}}(I)_\zeta) \circ (I)_\xi)(k_1) = -1 \leq (I)_\xi(k_1),$$

$$(((F)_\xi \circ \chi_{\mathscr{M}}(F)_\zeta) \circ (F)_\xi)(k_1) = 0 \geq (F)_\xi(k_1).$$

Therefore $(\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.

Conversely, assume $(\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$. Let $x_1, x_2 \in \mathscr{M}$. Then

$$(T)_\xi(x_1x_2) \leq ((T)_\xi \circ (T)_\xi)(x_1x_2) \leq (T)_\xi(x_1) \vee (T)_\xi(x_2),$$

$$(I)_\xi(x_1x_2) \geq ((I)_\xi \circ (I)_\xi)(x_1x_2) \geq (I)_\xi(x_1) \wedge (I)_\xi(x_2),$$

$$(F)_\xi(x_1x_2) \leq ((F)_\xi \circ (F)_\xi)(x_1x_2) \leq (F)_\xi(x_1) \vee (F)_\xi(x_2).$$

So $\mathscr{M}_\xi \in \mathscr{M}_s$.

Let $x_1, x_2, x_3 \in \mathscr{M}$. Then

$(T)_\xi(x_1x_2x_3) \leq (((T)_\xi \circ \chi_{\mathscr{M}}(T)_\zeta) \circ (T)_\xi)(x_1x_2x_3) \leq ((T)_\xi \circ \chi_{\mathscr{M}}(T)_\zeta)(x_1x_2) \vee (T)_\xi(x_3) \leq \{(T)_\xi(x_1) \vee \chi_{\mathscr{M}}(T)_\zeta(x_2)\} \vee (T)_\xi(x_3) = (T)_\xi(x_1) \vee (T)_\xi(x_3)$,
$(I)_\xi(x_1x_2x_3) \geq (((I)_\xi \circ \chi_{\mathscr{M}}(I)_\zeta) \circ (I)_\xi)(x_1x_2x_3) \geq ((I)_\xi \circ \chi_{\mathscr{M}}(I)_\zeta)(x_1x_2) \wedge (I)_\xi(x_3) \geq \{(I)_\xi(x_1) \wedge$

$\chi_{\mathscr{M}}(I)_\zeta(x_2)\} \wedge (I)_\xi(x_3) = (I)_\xi(x_1) \wedge (I)_\xi(x_3)$,
$(F)_\xi(x_1x_2x_3) \leq (((F)_\xi \circ \chi_{\mathscr{M}}(F)_\zeta) \circ (F)_\xi)(x_1x_2x_3) \leq ((F)_\xi \circ \chi_{\mathscr{M}}(F)_\zeta)(x_1x_2) \vee (F)_\xi(x_3) \leq \{(F)_\xi(x_1) \vee \chi_{\mathscr{M}}(F)_\zeta(x_2)\} \vee (F)_\xi(x_3) = (F)_\xi(x_1) \vee (F)_\xi(x_3)$. Therefore $\mathscr{M}_\xi \in \mathscr{M}_b$. □

Lemma 3.18. *Suppose $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_r$ having left identity. Then $\mathscr{M}_\xi \odot \mathscr{M}_\zeta \in \mathscr{M}_b$ and $\mathscr{M}_\zeta \odot \mathscr{M}_\xi \in \mathscr{M}_b$.*

Proof. By Corollary 3.2, $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\zeta) = (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot (\mathscr{M}_\zeta \odot \mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\zeta$. Hence $\mathscr{M}_\xi \odot \mathscr{M}_\zeta \in \mathscr{M}_s$. Now, by Corollary 3.2 and Lemma 3.6, for any $\mathscr{M}_{\mathscr{R}} \in \mathscr{M}$,

$$\begin{aligned}
((\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{R}})) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\zeta) &= ((\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{R}}) \odot \chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{R}}))) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\zeta)\\
&= ((\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{R}})) \odot (\mathscr{M}_\zeta \odot \chi_{\mathscr{M}}(\mathscr{M}_{\mathscr{R}}))) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\zeta)\\
&\subseteq (\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\zeta)\\
&\subseteq \mathscr{M}_\xi \odot \mathscr{M}_\zeta.
\end{aligned}$$

By Lemma 3.17, $\mathscr{M}_\xi \odot \mathscr{M}_\zeta \in \mathscr{M}_b$. Similarly, $\mathscr{M}_\zeta \odot \mathscr{M}_\xi \in \mathscr{M}_b$. □

Lemma 3.19. *Let $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_b$. Then $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_b$.*

Proof. Let $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_b$ and $k_1, k_2, a \in \mathscr{M}$. Then

$$\begin{aligned}
(T_\xi \cap T_\zeta)(k_1ak_2) &= T_\xi(k_1ak_2) \vee T_\zeta(k_1ak_2)\\
&\leq (T_\xi(k_1) \vee T_\xi(k_2)) \vee (T_\zeta(k_1) \vee T_\zeta(k_2))\\
&= (T_\xi \cap T_\zeta)(k_1) \vee (T_\xi \cap T_\zeta)(k_2),\\
(I_\xi \cap I_\zeta)(k_1ak_2) &= I_\xi(k_1ak_2) \wedge I_\zeta(k_1ak_2)\\
&\geq (I_\xi(k_1) \wedge I_\xi(k_2)) \wedge (I_\zeta(k_1) \wedge I_\zeta(k_2))\\
&= (I_\xi \cap I_\zeta)(k_1) \wedge (I_\xi \cap I_\zeta)(k_2),\\
(F_\xi \cap F_\zeta)(k_1ak_2) &= F_\xi(k_1ak_2) \vee F_\zeta(k_1ak_2)\\
&\leq (F_\xi(k_1) \vee F_\xi(k_2)) \vee (F_\zeta(k_1) \vee F_\zeta(k_2))\\
&= (F_\xi \cap F_\zeta)(k_1) \vee (F_\xi \cap F_\zeta)(k_2).
\end{aligned}$$

Hence $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_b$. □

Theorem 3.20. *Let $\mathscr{M}_\xi \in \mathscr{M}_s$. Then $\mathscr{M}_\xi \in \mathscr{M}_n$ if and only if $(\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.*

Proof. This is similar to Theorem 3.18 in [21]. □

Proposition 3.21. *If every neutrosophic $\varkappa$-left ideal is neutrosophic $\varkappa$-idempotent in $\mathscr{M}$, then the following statements hold:*

(i) $\mathscr{M}_\xi \in \mathscr{M}_b$,

(ii) $\mathscr{M}_\xi \in \mathscr{M}_n$.

Proof. (i) Assume $\mathscr{M}_\xi \in \mathscr{M}_l$. Then $\mathscr{M}_\xi \odot \mathscr{M}_\xi = \mathscr{M}_\xi$. By Corollary 3.2, for any $\mathscr{M}_\zeta \in \mathscr{M}$, $(\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi = (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\xi) = (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi = \mathscr{M}_\xi$. Hence $\mathscr{M}_\xi \in \mathscr{M}_b$.

(ii) For any $\mathscr{M}_\zeta \in \mathscr{M}$, $(\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) = (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi = \mathscr{M}_\xi$. Hence $\mathscr{M}_\xi \in \mathscr{M}_n$. □

Lemma 3.22. *Suppose $\mathscr{M}$ is having left identity e. Then the listed claims are equivalent:*

(i) $\mathscr{M}_\xi \in \mathscr{M}_r$,

(ii) $\mathscr{M}_\xi \in \mathscr{M}_n$.

Proof. Let $r \in \mathscr{M}$. Then $er = r$.

$(i) \Rightarrow (ii)$ Assume $\mathscr{M}_\xi \in \mathscr{M}_r$ and $x_1, x_2, x_3 \in \mathscr{M}$. Then

$$(T)_\xi((x_1x_2)x_3) \leq (T)_\xi(x_1x_2) = (T)_\xi((ex_1)x_2) = (T)_\xi((x_2x_1)e) \leq (T)_\xi(x_2x_1) \leq (T)_\xi(x_2),$$

$$(I)_\xi((x_1x_2)x_3) \geq (I)_\xi(x_1x_2) = (I)_\xi((ex_1)x_2) = (I)_\xi((x_2x_1)e) \geq (I)_\xi(x_2x_1) \geq (I)_\xi(x_2),$$

$$(F)_\xi((x_1x_2)x_3) \leq (F)_\xi(x_1x_2) = (F)_\xi((ex_1)x_2) = (F)_\xi((x_2x_1)e) \leq (F)_\xi(x_2x_1) \leq (F)_\xi(x_2).$$

So $\mathscr{M}_\xi \in \mathscr{M}_n$.

$(ii) \Rightarrow (i)$ Let $\mathscr{M}_\xi \in \mathscr{M}_n$. For any $x_1, x_3 \in \mathscr{M}$, we can have $(T)_\xi(x_1x_3) = (T)_\xi((ex_1)x_3) \leq (T)_\xi(x_1)$, $(I)_\xi(x_1x_3) = (I)_\xi((ex_1)x_3) \geq (I)_\xi(x_1)$, $(F)_\xi(x_1x_3) = (F)_\xi((ex_1)x_3) \leq (F)_\xi(x_1)$. So $\mathscr{M}_\xi \in \mathscr{M}_r$. □

Lemma 3.23. *Let $\mathscr{M}_\xi \in \mathscr{M}_l$ such that $e \in \mathscr{M}$ as left identity. If $\mathscr{M}_\xi \in \mathscr{M}_n$, then $\mathscr{M}_\xi \in \mathscr{M}_b$.*

Proof. Since $\mathscr{M}_\xi \in \mathscr{M}_l$, $(T)_\xi(x_1x_2) \leq (T)_\xi(x_2)$, $(I)_\xi(x_1x_2) \geq (I)_\xi(x_2)$ and $(F)_\xi(x_1x_2) \leq (F)_\xi(x_2)$ for any $x_1, x_2 \in \mathscr{M}$. As $e \in \mathscr{M}$, $ex_1 = x_1 \ \forall \ x_1 \in \mathscr{M}$. Now for any $x_1, x_2 \in \mathscr{M}$, $(T)_\xi(x_1x_2) = (T)_\xi((ex_1)x_2) \leq (T)_\xi(x_1)$, $(I)_\xi(x_1x_2) = (I)_\xi((ex_1)x_2) \geq (I)_\xi(x_1)$, $(F)_\xi(x_1x_2) = (F)_\xi((ex_1)x_2) \leq (F)_\xi(x_1)$ which imply $(T)_\xi(x_1x_2) \leq (T)_\xi(x_1) \vee (T)_\xi(x_2)$, $(I)_\xi(x_1x_2) \geq (I)_\xi(x_1) \wedge (I)_\xi(x_2)$, $(F)_\xi(x_1x_2) \leq (F)_\xi(x_1) \vee (F)_\xi(x_2)$. So $\mathscr{M}_\xi \in \mathscr{M}_s$.

For any $x_1, x_2, x_3 \in \mathscr{M}$,

$$(T)_\xi((x_1x_2)x_3) = (T)_\xi((x_1(ex_2))x_3) = (T)_\xi((e(x_1x_2))x_3) \leq (T)_\xi(x_1x_2) = (T)_\xi((ex_1)x_2) \leq (T)_\xi(x_1),$$

$$(I)_\xi((x_1x_2)x_3) = (I)_\xi((x_1(ex_2))x_3) = (I)_\xi((e(x_1x_2))x_3) \geq (I)_\xi(x_1x_2) = (I)_\xi((ex_1)x_2) \geq (I)_\xi(x_1),$$

$$(F)_\xi((x_1x_2)x_3) = (F)_\xi((x_1(ex_2))x_3) = (F)_\xi((e(x_1x_2))x_3) \leq (F)_\xi(x_1x_2) = (F)_\xi((ex_1)x_2) \leq (F)_\xi(x_1).$$

Also, $(T)_\xi((x_1x_2)x_3) = (T)_\xi((x_3x_2)x_1) = (T)_\xi((x_3(ex_2))x_1) = (T)_\xi((e(x_3x_2))x_1) \le (T)_\xi(x_3x_2) = (T)_\xi((ex_3)x_2) \le (T)_\xi(x_3)$. Hence $(T)_\xi((x_1x_2)x_3) \le (T)_\xi(x_1) \vee (T)_\xi(x_3)$. Now, $(I)_\xi((x_1x_2)x_3) = (I)_\xi((x_3x_2)x_1) = (I)_\xi((x_3(ex_2))x_1) = (I)_\xi((e(x_3x_2))x_1) \ge (I)_\xi(x_3x_2) = (I)_\xi((ex_3)x_2) \ge (I)_\xi(x_3)$. Hence $(I)_\xi((x_1x_2)x_3) \ge (I)_\xi(x_1) \wedge (I)_\xi(x_3)$. Now, $(F)_\xi((x_1x_2)x_3) = (F)_\xi((x_3x_2)x_1) = (F)_\xi((x_3(ex_2))x_1) = (F)_\xi((e(x_3x_2))x_1) \le (F)_\xi(x_3x_2) = (F)_\xi((ex_3)x_2) \le (F)_\xi(x_3)$. Hence $(F)_\xi((x_1x_2)x_3) \le (F)_\xi(x_1) \vee (F)_\xi(x_3)$. Therefore $\mathscr{M}_\xi \in \mathscr{M}_b$. □

Proposition 3.24. *Let $e \in \mathscr{M}$ be left identity. If $\mathscr{M}_\xi \in \mathscr{M}_l$ (resp., $\mathscr{M}_\xi \in \mathscr{M}_r$, $\mathscr{M}_\xi \in \mathscr{M}_i$), then $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_i$.*

Proof. Since $\mathscr{M}_\xi \in \mathscr{M}_l$, then for $\mathscr{M}_\zeta \in \mathscr{M}$, by Theorem 3.4, $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$. By Lemma 3.6 and Corollary 3.2, $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\xi) = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\xi) = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi$. Also by Theorem 3.1, $(\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi$. Thus $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_i$. □

Corollary 3.25. *Let $\mathscr{M}_\xi \in \mathscr{M}$ has left identity. If $\mathscr{M}_\xi \in \mathscr{M}_l$, then $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_b$ and $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_n$.*

Proof. By Proposition 3.24, $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_i$. Now by Lemmas 3.22 and 3.23, $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_b$ and $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_n$. □

Theorem 3.26. *If $\mathscr{M}_\xi \in \mathscr{M}_i$, then $\mathscr{M}_\xi \in \mathscr{M}_b$ and $\mathscr{M}_\xi \in \mathscr{M}_n$.*

Proof. Let $\mathscr{M}_\xi \in \mathscr{M}_i$. Then $\mathscr{M}_\xi$ is neutrosophic $\varkappa$-AG-subgroupoid since $\mathscr{M}_\xi \odot \mathscr{M}_\xi \subseteq \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$. Now, $(\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$ and $(\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ which imply $\mathscr{M}_\xi \in \mathscr{M}_b$ and $\mathscr{M}_\xi \in \mathscr{M}_n$. □

4. **Conclusion**

We presented the ideas of neutrosophic $\varkappa$-ideal structures in an AG-groupoid and proved that the product of two neutrosophic $\varkappa$-right-ideal is a neutrosophic $\varkappa$-bi-ideal, and neutrosophic $\varkappa$-right-ideal is equivalent to neutrosophic $\varkappa$-interior-ideal, under certain condition. In future, we will define neutrosophic $\varkappa$-structures over an ordered AG-groupoid and investigate the features of an ordered AG-groupoid using the results in an AG-groupoid.

Conflict of interest:

The authors declare that they have no conflict of interest.

Acknowledgement:

The authors express their gratitude to the referees for valuable comments and suggestions which improve the article.

References

[1] Zadeh, L. A. Fuzzy sets. Information and Control **1965**, 8, 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Atanassov, K. T. Intuitionistic fuzzy sets. Fuzzy Sets and Systems **1986**, 20, 87-96. https://doi.org/10.1016/S0165-0114(86)80034-3

[3] Rosenfeld, A. Fuzzy groups. Journal of Mathematical Analysis and Applications **1971**, 35, 512–517. https://doi.org/10.1016/0022-247X(71)90199-5

[4] Elavarasan, B.; Muhiuddin, G.; Porselvi, K.; Jun, Y. B. On hybrid k-ideals in semirings. Journal of Intelligent and Fuzzy Systems **2023**, 44, 4681–4691. https://doi.org/10.3233/JIFS-222335

[5] Elavarasan, B.; Muhiuddin, G.; Porselvi, K.; Jun, Y. B. Hybrid structures applied to ideals in near-rings. Complex and Intelligent Systems **2021**, 7(3), 1489–1498. https://doi.org/10.1007/s40747-021-00271-7

[6] Kehayopulu, N.; Tsingelis, M. A note on fuzzy sets in semigroups. Scientiae Mathematicae **1999**, 2, 411 - 413.

[7] Kehayopulu, N.; Tsingelis, M. Fuzzy interior ideals in ordered semigroups. Lobachevskii Journal of Mathematics **2006**, 21, 65 - 71.

[8] Kuroki, N. On fuzzy ideals and fuzzy bi-ideals in semigroups. Fuzzy Sets and Systems **1981**, 5, 203-215.

[9] Molodtsov, D. Soft set theory - First results. Computers and Mathematics with Applications **1999**, 37, 19 - 31.

[10] Mukherjee, T. K. and Sen, M. K. Prime fuzzy ideals in rings. Fuzzy Sets and Systems **1989**, 32, 337 - 341.

[11] Porselvi, K.; Muhiuddin, G.; Elavarasan, B.; Assiry, A. Hybrid Nil Radical of a Ring. Symmetry **2022**, 14(7), 1367. https://doi.org/10.3390/sym14071367

[12] Porselvi, K.; Muhiuddin, G.; Elavarasan, B.; Jun, Y.B.; Catherine Grace John, J. Hybrid Ideals in an AG-Groupoid. New Mathematics and Natural Computation **2023**, 19(1), 289-305. https://doi.org/10.1142/S1793005723500084

[13] Muhiuddin, G.; Elavarasan, B.; Catherine G. John, J.; Porselvi, K.; Al-Kadi, D. Properties of k-hybrid ideals in ternary semiring. Journal of Intelligent and Fuzzy Systems **2021**, 42(6), 5799–5807.

[14] Porselvi, K.; Elavarasan, B. On hybrid interior ideals in semigroups. Problemy Analysis Issues of Analysis **2019**, 8(26)(3), 137-146.

[15] Smarandache, F. A. Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set. Neutrosophic Probability. American Research Press, Rehoboth, NM 1999.

[16] Muhiuddin, G.; Bordbar, H.; Smarandache, F.; Jun, Y. B. Further results on $(\epsilon;\epsilon)$-neutrosophic subalgebras and ideals in BCK/BCI- algebras. Neutrosophic Sets and Systems **2018**, 20, 36-43. http://doi.org/10.5281/zenodo.1235357

[17] Muhiuddin, G.; Kim, S. J.; Jun, Y. B. Implicative N-ideals of BCK-algebras based on neutrosophic N structures. Discrete Mathematics, Algorithms and Applications **2019**, 11(1), 1950011.

[18] Khan, M.; Ani, S.; Smarandache, F.; Jun, Y. B. Neutrosophic $\aleph$-structures and their applications in semigroups. Annals of Fuzzy Mathematics and Informatics **2017**, 14(6), 583–598.

[19] Elavarasan, B.; Smarandache, F.; Jun, Y. B. Neutrosophic $\aleph$-ideals in semigroups. Neutrosophic Sets and Systems **2019**, 28, 273-280. https://zenodo.org/badge/DOI/10.5281/zenodo.3382554

[20] Porselvi, K.; Elavarasan, B.; Smarandache, F.; Jun, Y. B. Neutrosophic $\aleph$-bi-ideals in semigroups. Neutrosophic Sets and Systems **2020**, 35, 422-434. https://doi.org/10.5281/zenodo.3951696

[21] Porselvi, K.; Elavarasan, B.; Smarandache, F. Neutrosophic $\varkappa$-interior ideals in semigroups. Neutrosophic Sets and Systems **2020**, 36, 70-80. DOI: 10.5281/zenodo.4065385

[22] Elavarasan, B.; Porselvi, K.; Jun, Y. B.; Muhiuddin, G. Neutrosophic $\aleph$-filters in semigroups. Neutrosophic Sets and Systems **2022**, 50, 515-531. DOI: 10.5281/zenodo.6774914

[23] Muhiuddin, G.; Porselvi, K.; Elavarasan, B.; Al-Kadi, D. Neutrosophic $\aleph$-Structures in Ordered Semigroups. Computer Modeling in Engineering and Sciences **2022**, 131 (2), 979-999. https://doi.org/10.32604/cmes.2022.018615

[24] Khan, M.; Nouman Aslam Khan, M. On fuzzy Abel Grassmann's groupoids. Advances in fuzzy Mathematics, **2010**, 5(3), 349-360.

[25] Khan, Faiz Muhammad, Khan, Hidayat Ullah, Mukhtar, Safyan, Khan, Asghar and Sarmin, Nor Haniza. Some Innovative Types of Fuzzy Ideals in AG-Groupoids. Journal of Intelligent Systems, **2019**, 28(4), 649-667. https://doi.org/10.1515/jisys-2017-0258

[26] Florentin Smarandache, New Types of Topologies and Neutrosophic Topologies, Neutrosophic Systems with Applications, **2023**, 1, 1–3.

[27] Runu Dhar, Compactness and Neutrosophic Topological Space via Grills, Neutrosophic Systems with Applications, **2023**, 2, 1–7.

[28] Sudeep Dey, Gautam Chandra Ray, Properties of Redefined Neutrosophic Composite Relation, Neutrosophic Systems with Applications, **2023**, 7, 1–12.

Received: June 6, 2023. Accepted: Oct 1, 2023