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On Statistical Convergence of Order α in Neutrosophic Normed Spaces

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Abstract. In this paper, we introduce the notion of statistical convergent of order α in the neutrosophic normed spaces. We investigate a few properties of the newly introduced notion and examine the relationship with statistical convergence in the neutrosophic normed spaces. Finally, we introduce the concept of statistical Cauchy sequence of order α and show that statistical Cauchy sequences of order α are equivalent to statistical convergent of order α sequences in the neutrosophic normed spaces.

Keywords: $\alpha-$density; statistical convergent of order α; neutrosophic normed space.

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1. **Introduction**

In 1951, statistical convergence was introduced independently by Fast [12] and Steinhaus [33] to provide deeper insights into summability theory. Subsequently, researchers like Fridy [15], Salat [29], Connor [8], and others explored it from the sequence space perspective. This led to investigations by Hazarika and Esi [17], Altinok et al. [1], Mursaleen [25], Tripathy and Sen [34], Savas and Gurdal [30]. Statistical convergence found applications in number theory, mathematical analysis, probability theory, and other areas.

In 2010, Colak [6] introduced the notion of statistical convergence with order α by means of α-density. Later, it was generalized to τ-statistical convergence with a degree of order α [7] and lacunary statistical convergence with a degree of order α [31]. For additional information regarding statistical convergence with a degree of order α and its generalizations, one may refer to $[5, 10, 11]$, which provide many more references.

The notion of fuzzy sets was proposed by Zadeh [35] in 1965, extending classical set theory. To address membership degree uncertainties, in 1986, Atanassov [4] introduced intuitionistic fuzzy sets, which have since found utility in decision-making contexts. Smarandache [32] introduced neutrosophic sets in 2005 as a generalization of fuzzy and intuitionistic fuzzy sets. Neutrosophic sets consist of truth-membership (T), indeterminacy-membership (F), and falsity-membership (I) functions, suitable for tricomponent outcomes in uncertain scenarios.

Felbin [13] introduced fuzzy normed spaces in 1992, followed by the exploration into Saadati and Park [28] in 2006 conducted research on intuitionistic fuzzy normed spaces. Subsequently, Karakus et al. [18] delved into the realm of statistical convergence within intuitionistic fuzzy normed spaces in 2008. Further investigation into convergence of sequences in neutrosophic normed spaces was pursued by Kirisci and Simsek [22]. The research in this area is still developing, showing analogies to conventional normed spaces.

2. **Definition and Background**

Definition 2.1. [6] Consider $M \subseteq \mathbb{N}$, and let M_n represent the consisting of elements in M that are less than or equal to n. The concept of natural density of order α $(0 < \alpha \leq 1)$ associated with the set M is denoted and defined as follows:

$$\delta^\alpha(M) = \lim_{n\to\infty} \frac{|M_n|}{n^\alpha},$$

provided that the limit exists. Here, the symbol $|M_n|$ denotes the cardinality or number of elements in the set M_n.

Definition 2.2. [6] A sequence (y_k) is considered statistically convergent of order α $(0 < \alpha \leq 1)$ to l if, for every $\zeta > 0$, the following condition holds:

$$\delta^\alpha(M(\zeta)) = 0,$$

where $M(\zeta) = \{k \in \mathbb{N} : |y_k - \ell| \geq \zeta\}$. In this context, l is referred to as the statistical limit of order α for the sequence (y_k), denoted as $y_k \xrightarrow{st^\alpha} l$.

In particular, if $\alpha = 1$, then Definition 2.1 and Definition 2.2 reduces to the definitions of natural density [14] and statistical convergence [15] respectively.

Definition 2.3. [24] A binary operation $\circ$ that takes two values from the interval $[0, 1]$ and produces a result within the same interval is referred to as a continuous triangular norm, given that it satisfies the following set of criteria:

(1) The operation $\circ$ possesses the properties of associativity and commutativity.
(2) The operation $\circ$ remains continuous.
(3) For any s belonging to the interval $[0, 1]$, the operation $s \circ 1$ results in s.

(4) If p is less than or equal to q and r is less than or equal to s, where p, q, r, and s fall within the range of values from 0 to 1, then it follows that the inequality $p \circ r \leq q \circ s$ is satisfied.

Definition 2.4. [24] An operation $\bullet$ that takes two values from the interval $[0, 1]$ and returns a value within the same interval is termed a continuous triangular co-norm, provided it meets the following set of criteria:

(1) The operation $\bullet$ possesses the properties of associativity and commutativity.
(2) The operation $\bullet$ maintains continuity.
(3) For all $s \in [0, 1]$, the operation $s \bullet 0$ results in s.
(4) If values p, q, r, and s are such that $p \leq r$ and $q \leq s$, with p, q, r, and s belonging to the interval $[0, 1]$, then the outcome of the operation $p \bullet q$ is no greater than the result of $r \bullet s$.

Definition 2.5. [32] Consider a universe of discourse denoted as X. The subset K_{NS} of X is defined as:

$$K_{NS} = \{< \nu, \mathfrak{R}_K(\nu), \mathfrak{T}_K(\nu), \mathfrak{W}_K(\nu) >: \nu \in X\}$$

This defined set is known as a neutrosophic set. In this context, $\mathfrak{R}_K(\nu)$, $\mathfrak{T}_K(\nu)$, and $\mathfrak{W}_K(\nu)$ are functions mapping X to the interval $[0, 1]$, Expressing the levels of truth-membership, uncertainty-membership, and falsehood-membership, respectively. It is important to satisfy the constraint $0 \leq \mathfrak{R}_K(\nu) + \mathfrak{T}_K(\nu) + \mathfrak{W}_K(\nu) \leq 3$.

Definition 2.6. [22] Imagine a vector space denoted as F, and consider a normed space $\{< \nu, \mathfrak{R}(\nu), \mathfrak{T}(\nu), \mathfrak{W}(\nu) >: \nu \in F\}$. In this normed space, $\mathfrak{R}$, $\mathfrak{T}$, and $\mathfrak{W}$ are functions mapping $F \times \mathbb{R}^+$ to the interval $[0, 1]$. Furthermore, consider $\circ$ to represent a continuous triangular norm operation, and $\bullet$ to signify a continuous triangular co-norm operation.. If the four-tuple $V = (F, \mathfrak{N}, \circ, \bullet)$ satisfies the following conditions, for any $\nu, v \in F$ and $\tau, \mu > 0$, and for every $\sigma \neq 0$:

(1) The values $\mathfrak{R}(\nu, \tau)$, $\mathfrak{T}(\nu, \tau)$, and $\mathfrak{W}(\nu, \tau)$ are restricted to the range $[0, 1]$.
(2) The sum of $\mathfrak{R}(\nu, \tau)$, $\mathfrak{T}(\nu, \tau)$, and $\mathfrak{W}(\nu, \tau)$ is bounded by $[0, 3]$.
(3) $\mathfrak{R}(\nu, \tau) = 1$ $(for\ \tau > 0)$ iff $\nu = 0$.
(4) $\mathfrak{R}(\sigma\nu, \tau) = \mathfrak{R}(\nu, \frac{\tau}{|\sigma|})$.
(5) $\mathfrak{R}(\nu, \tau) \circ \mathfrak{R}(v, \mu) \leq \mathfrak{R}(\nu + v, \tau + \mu)$.
(6) The function $\mathfrak{R}(\nu, .)$ is both continuous and non-decreasing.
(7) As τ approaches infinity, $\lim_{\tau\to\infty} \mathfrak{R}(\nu, \tau) = 1$.
(8) $\mathfrak{T}(\nu, \tau) = 0$ $(for\ \tau > 0)$ iff $\nu = 0$.
(9) $\mathfrak{T}(\sigma\nu, \tau) = \mathfrak{T}(\nu, \frac{\tau}{|\sigma|})$.
(10) $\mathfrak{T}(\nu, \tau) \bullet \mathfrak{T}(\nu, \mu) \geq \mathfrak{T}(\nu + v, \tau + \mu)$.

(11) The function $\mathfrak{T}(\nu, .)$ is continuous and non-increasing.
(12) As τ goes to infinity, $\lim_{\tau\to\infty} \mathfrak{T}(\nu, \tau) = 0$.
(13) $\mathfrak{W}(\nu, \tau) = 0$ $(for\ \tau > 0)$ iff $\nu = 0$.
(14) $\mathfrak{W}(\sigma\nu, \tau) = \mathfrak{W}(\nu, \frac{\tau}{|\sigma|})$.
(15) $\mathfrak{W}(\nu, \tau) \bullet \mathfrak{W}(v, \mu) \geq \gamma(\nu + v, \tau + \mu)$.
(16) The function $\mathfrak{W}(\nu, .)$ is continuous and non-increasing.
(17) As τ approaches infinity, $\lim_{\tau\to\infty} \mathfrak{W}(\nu, \tau) = 0$.
(18) If $\tau \leq 0$, then we have $\mathfrak{R}(\nu, \tau) = 0, \mathfrak{T}(\nu, \tau) = 1$, and $\mathfrak{W}(\nu, \tau) = 1$.

Furthermore, $\mathfrak{N} = (\mathfrak{R}, \mathfrak{T}, \mathfrak{W})$ forms a neutrosophic norm (NN).

Example 2.7. [22] Let's consider a normed space denoted as $(F, ||\cdot||)$. In this context, take any two values, denoted as s and t, from the interval $[0, 1]$. We establish the triangular norm operation $\circ$ as the product of s and t, denoted by $s \circ t = st$, and the triangular co-norm operation $\bullet$ as s added to t minus their product, given by $s \bullet t = s + t - st$. Now, let's delve into the construction of the functions $\mathfrak{R}(u, \tau)$, $\mathfrak{T}(u, \tau)$, and $\mathfrak{W}(u, \tau)$ for any τ that surpasses $||u||$:

$$\mathfrak{R}(u, \tau) = \frac{\tau}{\tau + ||u||}, \quad \mathfrak{T}(u, \tau) = \frac{||u||}{\tau + ||u||}, \quad \mathfrak{W}(u, \tau) = \frac{||u||}{\tau}$$

These definitions stand valid for all elements $u \in F$ and $\tau > 0$. However, if τ is less than or equal to $||u||$, then we redefine the functions as follows: $\mathfrak{R}(u, \tau) = 0$, $\mathfrak{T}(u, \tau) = 1$, and $\mathfrak{W}(u, \tau) = 1$. With these specified conditions, we can confidently assert that the arrangement $(F, \mathfrak{N}, \circ, \bullet)$ establishes itself as a neutrosophic normed space (NNS).

Definition 2.8. [22] Let V represent a NNS (Neutrosophic Normed Space). We define a sequence (y_k) to exhibit statistical convergence towards l with respect to the (NN) if, for any $0 < \zeta < 1$, the set $M_\zeta = k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(y_k - \ell, \tau) \geq \zeta\ ,\ \mathfrak{W}(y_k - \ell, \tau) \geq \zeta$ satisfies the property that $\delta(M_\zeta) = 0$.

This notion can be denoted symbolically as $st - \mathfrak{N} - \lim y_k = l$ or equivalently $y_k \to l(st - \mathfrak{N})$.

Example 2.9. Consider a normed space denoted as $(F, ||\cdot||)$. Let any pair of values, denoted as s and t, lie within the interval $[0, 1]$. We establish the continuous triangular norm as $s \circ t = st$, and the continuous triangular co-norm as $s \bullet t$ is defined as the minimum of the sum of s and t or 1. Drawing inspiration from the functions $\mathfrak{R}$, $\mathfrak{T}$, and $\mathfrak{W}$ showcased in Example 2.7, and under the constraint of $\tau > 0$, we have: Let's explore the sequence (y_k) defined as follows within a centered environment:

$$y_k = \begin{cases} 2, & \text{if } k = p^4 \text{ where } p \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}.$$

As a result, we establish the convergence $y_k \to 0(st - \mathfrak{N})$.

Rationale: For any $0 < \zeta < 1$, we delve into the composition of the set M:

$$M = \{k \leq n : \mathfrak{R}(y_k - 0, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(y_k - 0, \tau) \geq \zeta \text{ , } \mathfrak{W}(y_k - 0, \tau) \geq \zeta\}.$$

This exploration uncovers:

$$\begin{aligned} M &= \{k \leq n : \frac{\tau}{\tau + |y_k|} \leq 1 - \zeta \text{ or } \frac{|y_k|}{\tau + |y_k|} \geq \zeta \text{ , } \frac{|y_k|}{\tau} \geq \zeta\} \\ &= \{k \leq n : |y_k| \geq \frac{\tau\zeta}{1-\zeta} \text{ or } |y_k| \geq \tau\zeta\} \\ &= \{k \leq n : y_k = 2\}. \end{aligned}$$

Thus, we conclude that $\delta(M) = \lim_{n\to\infty} \frac{|M|}{n} \leq \lim_{n\to\infty} \frac{\sqrt[4]{n}}{n} = 0$. As a result, the sequence $y_k \to 0(st - \mathfrak{N})$ holds true.

Definition 2.10. [22] Consider the sequence (y_k) within a NNS V. We deem (y_k) to be statistically Cauchy if, for any $0 < \zeta < 1$, there exists an associated natural number $N = N(\zeta)$ satisfying the condition:

$$\delta(MC_\zeta) = 0, \text{ where}$$
$$MC_\zeta = \{k \in \mathbb{N} : \mathfrak{R}(y_k - s_N, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(y_k - s_N, \tau) \geq \zeta, \mathfrak{W}(y_k - s_N, \tau) \geq \zeta\}.$$

3. **Key Findings**

Definition 3.1. Within the domain of NNS, denoted as V, consider a value $0 < \alpha \leq 1$. A sequence (y_k) is classified as statistical convergence with a degree of order α to the value l concerning the neutrosophic norm (NN). This categorization is established when, for all $0 < \zeta < 1$, the following condition holds:

$\delta^\alpha(M_\zeta) = 0$, where $M_\zeta = \{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(y_k - \ell, \tau) \geq \zeta \text{ , } \mathfrak{W}(y_k - \ell, \tau) \geq \zeta\}$.

In this context, we symbolize the statement as $st^\alpha - \mathfrak{N} - \lim y_k = l$ or $y_k \to l(st^\alpha - \mathfrak{N})$.

Of particular note, should we select $\alpha = 1$, the above definition aligns with the notion of statistical convergence in NNS, as previously explored by Kirisci and Simsek [22].

Example 3.2. Consider a normed space denoted by $(F, ||\cdot||)$. For all s and t within the range of $[0, 1]$, The continuous triangular norm is characterized by the equation $s \circ t = st$, while the continuous triangular co-norm is expressed as the operation $s \bullet t$ as taking the minimum of either the sum of s and t or 1. Utilizing the functions $\mathfrak{R}$, $\mathfrak{T}$, and $\mathfrak{W}$ presented in Example 2.7 for all $\tau > 0$, we confidently affirm the characterization of V as a NNS.

Let us now introduce the sequence (y_k):

$$y_k = \begin{cases} 1, & k = p^3 \ (p \in \mathbb{N}) \\ 0, & \text{otherwise} \end{cases}.$$

Then, $y_k \to 0(st^\alpha - \mathfrak{N}),\ for\ \alpha \in (\frac{1}{3}, 1]$.

Justification: Any $0 < \zeta < 1$, we have

$$M = \{k \leq n : \mathfrak{R}(y_k - 0, \tau) \leq 1 - \zeta\ or\ \mathfrak{T}(y_k - 0, \tau) \geq \zeta\ ,\ \mathfrak{W}(y_k - 0, \tau) \geq \zeta\}.$$

This implies that,

$$\begin{aligned} M &= \{k \leq n : \frac{\tau}{\tau + |y_k|} \leq 1 - \zeta\ or\ \frac{|y_k|}{\tau + |y_k|} \geq \zeta\ \frac{|y_k|}{\tau} \geq \zeta\} \\ &= \{k \leq n : |y_k| \geq \frac{\tau\zeta}{1-\zeta}\ \ or\ \ |y_k| \geq \tau\zeta\} \\ &= \{k \leq n : y_k = 1\}. \end{aligned}$$

Then, $\delta^\alpha(M) = \lim_{n\to\infty} \frac{|M|}{n^\alpha} \leq \lim_{n\to\infty} \frac{\sqrt[3]{n}}{n^\alpha} = 0,\ for\ \alpha \in (\frac{1}{3}, 1]$. Hence, $y_k \to 0(st^\alpha - \mathfrak{N})$.

Lemma 3.3. *Considering V as a NNS (Neutrosophic Normed Space), and for all $0 < \zeta < 1$, and $0 < \alpha \leq 1$, as well as $\tau > 0$, we unveil the equivalence of the following propositions:*

(i) $y_k \to l(st^\alpha - \mathfrak{N})$;

(ii) $\delta^\alpha(\{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \tau) \leq 1 - \zeta\}) = \delta^\alpha(\{k \in \mathbb{N} : \mathfrak{T}(y_k - \ell, \tau) \geq \zeta\}) = \delta^\alpha(\{k \in \mathbb{N} : \mathfrak{W}(y_k - \ell, \tau) \geq \zeta\}) = 0$;

(iii) $\delta^\alpha(\{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \tau) > 1 - \zeta\ or\ \mathfrak{T}(y_k - \ell, \tau) < \zeta\ ,\ \mathfrak{W}(y_k - \ell, \tau) < \zeta\}) = 1$;

(iv) $\delta^\alpha(\{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \tau) > 1 - \zeta\}) = \delta^\alpha(\{k \in \mathbb{N} : \mathfrak{T}(y_k - \ell, \tau) < \zeta\}) = \delta^\alpha(\{k \in \mathbb{N} : \mathfrak{W}(y_k - \ell, \tau) < \zeta\}) = 1$;

(v) $\mathfrak{R}(y_k - \ell, \tau) \to 1(st^\alpha - \mathfrak{N})\ \ or\ \ \mathfrak{T}(y_k - \ell, \tau) \to 0(st^\alpha - \mathfrak{N})\ ,\ \mathfrak{W}(y_k - \ell, \tau) \to 0(st^\alpha - \mathfrak{N})$.

Theorem 3.4. *Let V be a NNS and let (y_k) be a sequence such that $y_k \to l(st^\alpha - \mathfrak{N})$. Then l is uniquely determined.*

Proof. Assume, for the sake of contradiction, that $y_k \to l_1(st^\alpha - \mathfrak{N})$ and $y_k \to l_2(st^\alpha - \mathfrak{N})$ where $l_1 \neq l_2$. We select a constant $0 < \zeta < 1$ and designate $\mu > 0$ such that $(1-\zeta) \circ (1-\zeta) > 1 - \mu$ and $\zeta \bullet \zeta < \mu$. In the context of any positive τ, we introduce the subsequent sets:

$$\begin{aligned} K_{\mathfrak{R}_1}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell_1, \frac{\tau}{2}) \leq 1 - \zeta\} \\ K_{\mathfrak{R}_2}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell_2, \frac{\tau}{2}) \leq 1 - \zeta\} \\ K_{\mathfrak{T}_1}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{T}(y_k - \ell_1, \frac{\tau}{2}) \geq \zeta\} \\ K_{\mathfrak{T}_2}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{T}(y_k - \ell_2, \frac{\tau}{2}) \geq \zeta\} \\ K_{\mathfrak{W}_1}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{W}(y_k - \ell_1, \frac{\tau}{2}) \geq \zeta\} \\ K_{\mathfrak{W}_2}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{W}(y_k - \ell_2, \frac{\tau}{2}) \geq \zeta\}. \end{aligned}$$

Since $y_k \to l_1(st^\alpha - \mathfrak{N})$, we apply Lemma 3.3 to conclude that for any $\tau > 0$,

$$\delta^\alpha(K_{\mathfrak{R}_1}(\zeta, \tau)) = \delta^\alpha(K_{\mathfrak{T}_1}(\zeta, \tau)) = \delta^\alpha(K_{\mathfrak{W}_1}(\zeta, \tau)) = 0.$$

Similarly, since $y_k \to l_2(st^{\alpha} - \mathfrak{N})$, we again apply Lemma 3.3 to deduce that for any $\tau > 0$,

$$\delta^{\alpha}(K_{\mathfrak{R}_2}(\zeta,\tau)) = \delta^{\alpha}(K_{\mathfrak{T}_2}(\zeta,\tau)) = \delta^{\alpha}(K_{\mathfrak{W}_2}(\zeta,\tau)) = 0.$$

Let's define $K(\zeta,\tau) = (K_{\mathfrak{R}_1}(\zeta,\tau) \cup K_{\mathfrak{R}_2}(\zeta,\tau)) \cap (K_{\mathfrak{T}_1}(\zeta,\tau) \cup K_{\mathfrak{T}_2}(\zeta,\tau)) \cap (K_{\mathfrak{W}_1}(\zeta,\tau) \cup K_{\mathfrak{W}_2}(\zeta,\tau))$. Consequently, $\delta^{\alpha}(K(\zeta,\tau)) = 0$, which implies $\delta^{\alpha}(\mathbb{N} \setminus K(\zeta,\tau)) = 1$ and hence The set of natural numbers excluding those in $K(\zeta,\tau)$ is non-empty. Let's pick p is an element of $\mathbb{N}$ that is not in $K(\zeta,\tau)$. We proceed with three cases:

(i) If $p \in (\mathbb{N} \setminus (K_{\mathfrak{R}_1}(\zeta,\tau)) \cup (\mathbb{N} \setminus (K_{\mathfrak{R}_2}(\zeta,\tau))$;
(ii) If $p \in (\mathbb{N} \setminus (K_{\mathfrak{T}_1}(\zeta,\tau)) \cup (\mathbb{N} \setminus (K_{\mathfrak{T}_2}(\zeta,\tau))$;
(iii) If $p \in (\mathbb{N} \setminus (K_{\mathfrak{W}_1}(\zeta,\tau)) \cup (\mathbb{N} \setminus (K_{\mathfrak{W}_2}(\zeta,\tau))$.

For Case (i), we possess:

$$\mathfrak{R}(l_1 - l_2, \tau) \geq \mathfrak{R}(y_k - \ell_1, \frac{\tau}{2}) \circ \mathfrak{R}(y_k - \ell_2, \frac{\tau}{2}) > (1-\zeta) \circ (1-\zeta) > 1 - \mu. \tag{1}$$

Since μ can take any value,, Equation (1) implies that for any $\tau > 0$, $\mathfrak{R}(l_1 - l_2, \tau) = 1$, leading to $l_1 = l_2$.

For Case (ii), we have:

$$\mathfrak{T}(l_1 - l_2, \tau) \leq \mathfrak{T}(y_k - \ell_1, \frac{\tau}{2}) \bullet \mathfrak{T}(y_k - \ell_2, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu. \tag{2}$$

Since μ can take any value, Equation (2) implies that for any $\tau > 0$, $\mathfrak{T}(l_1 - l_2, \tau) = 0$, leading to $l_1 = l_2$.

For Case (iii), we have:

$$\mathfrak{W}(l_1 - l_2, \tau) \leq \mathfrak{W}(y_k - \ell_1, \frac{\tau}{2}) \bullet \mathfrak{W}(y_k - \ell_2, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu. \tag{3}$$

Since μ can take any value, Equation (3) implies that for any $\tau > 0$, $\mathfrak{W}(l_1 - l_2, \tau) = 0$, leading to $l_1 = l_2$.

In all cases, we consistently arrive at $l_1 = l_2$, which contradicts our assumption. Hence, the assumption that $l_1 \neq l_2$ must be false, and therefore $l_1 = l_2$. This concludes the proof. □

Remark 3.5. The notion of statistical convergence with a parameter α within neutrosophic normed spaces is precisely defined for values of $0 < \alpha \leq 1$. However, for $\alpha > 1$, a counterexample can be provided as follows:

Example 3.6. Consider a normed space $(F, ||\cdot||)$. For any $s, t \in [0,1]$, we introduce the definitions of the continuous triangular norm as $s \circ t = st$ and the continuous triangular conorm as The operation $s \cdot t$ is defined as the minimum of $s + t$ and 1. By adopting the $\mathfrak{R}, \mathfrak{T}, \mathfrak{W}$ functions illustrated in Example 2.7, valid for all $\tau > 0$, the space V is established as a NNS. Let us now examine the sequence $x = (y_k)$, defined as:

$$y_k = \begin{cases} 1, & k = 2n \\ 0, & k \neq 2n \end{cases},$$

where $n \in \mathbb{N}$.

For $\alpha > 1$, we observe that

$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \in \mathbb{N} : \mathfrak{R}(y_k - 0, \tau) \leq 1 - \zeta \, or \, \mathfrak{T}(y_k - 0, \tau) \geq \zeta, \mathfrak{W}(y_k - 0, \tau) \geq \zeta\}| \leq \lim_{n\to\infty} \frac{n}{2n^\alpha}$ is zero,

and similarly,

$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \in \mathbb{N} : \mathfrak{R}(y_k - 1, \tau) \leq 1 - \zeta \, or \, \mathfrak{T}(y_k - 1, \tau) \geq \zeta, \mathfrak{W}(y_k - 1, \tau) \geq \zeta\}| \leq \lim_{n\to\infty} \frac{n}{2n^\alpha}$ is zero.

This implies that the statements $y_k \to 0(st^\alpha - \mathfrak{N})$ and $y_k \to 1(st^\alpha - \mathfrak{N})$ hold simultaneously, leading to a contradiction.

Theorem 3.7. *Let us consider two sequences, denoted as (t_k) and (s_k), within the NNS V. such that $t_k \to t_1(st^\alpha - \mathfrak{N})$ and $x_k \to t_2(st^\alpha - \mathfrak{N})$. Then,*
(i) $y_k + x_k \to t_1 + t_2(st^\alpha - \mathfrak{N})$ and (ii) $r.t_k \to r.t_1(st^\alpha - \mathfrak{N})$ where $r \in \mathbb{R}$.

Proof. (i) Suppose $t_k \to t_1(st^\alpha - \mathfrak{N})$and $s_k \to t_2(st^\alpha - \mathfrak{N})$. Now by definition for any $0 < \zeta < 1$,

$\delta^\alpha(M) = 0,$ then we have $M = \{k \in \mathbb{N} : \mathfrak{R}(t_k - t_1, \frac{\tau}{2}) \leq 1 - \zeta$ or $\mathfrak{T}(t_k - t_1, \frac{\tau}{2}) \geq \zeta, \mathfrak{W}(t_k - t_1, \frac{\tau}{2}) \geq \zeta\}$

and

$\delta^\alpha(M') = 0,$ then we have $M' = \{k \in \mathbb{N} : \mathfrak{R}(s_k - t_2, \frac{\tau}{2}) \leq 1 - \zeta or \mathfrak{T}(s_k - t_2, \frac{\tau}{2}) \geq \zeta, \mathfrak{W}(s_k - t_2, \frac{\tau}{2}) \geq \zeta\}$.

Now as the inclusion

$$(\mathbb{N} \setminus M) \cap (\mathbb{N} \setminus M') \subseteq \{k \in \mathbb{N} : \mathfrak{R}(t_k + s_k - t_1 - t_2, \tau) > 1 - \zeta \; or \; \mathfrak{T}(t_k + s_k - t_1 - t_2, \tau) < \zeta, \\ \mathfrak{W}(t_k + s_k - t_1 - t_2, \tau) < \zeta\}$$

holds, so we must have

$$M'' = \{k \in \mathbb{N} : \mathfrak{R}(t_k + s_k - t_1 - t_2, \tau) \leq 1 - \zeta \; or \; \mathfrak{T}(t_k + s_k - t_1 - t_2, \tau) \geq \zeta, \\ \mathfrak{W}(t_k + s_k - t_1 - t_2, \tau) \geq \zeta\} \subseteq M \cup M'$$

and consequently, $\delta^\alpha(M'') = 0$ i.e., $t_k + s_k \to t_1 + t_2(st^\alpha - \mathfrak{N})$.

(ii) If $r = 0$, then there is nothing to prove. So let us assume $r \neq 0$. Then since $t_k \to t_1(st^\alpha - \mathfrak{N})$, we have for any $0 < \zeta < 1$, $\delta^\alpha(M_\zeta) = 0$, where

$$M = \{k \in \mathbb{N} : \mathfrak{R}(t_k - t_1, \tfrac{\tau}{|r|}) \leq 1 - \zeta \; or \; \mathfrak{T}(t_k - t_1, \tfrac{\tau}{|r|}) \geq \zeta \, , \, \mathfrak{W}(t_k - t_1, \tfrac{\tau}{|r|}) \geq \zeta\}.$$

Now let M' denote the set

$$\{k \in \mathbb{N} : \mathfrak{R}(r.t_k - r.t_1, \tau) \leq 1 - \zeta \; or \; \mathfrak{T}(r.t_k - r.t_1, \tau) \geq \zeta \, , \, \mathfrak{W}(r.t_k - r.t_1, \tau) \geq \zeta\}.$$

Then $M' \subseteq M$ holds and consequently, $\delta^{\alpha}(M') = 0$. Hence, $r.t_k \to r.l_1(st^{\alpha} - \mathfrak{N})$. □

Theorem 3.8. *Let us consider two sequences, denoted as (t_k) and (s_k), within the NNS V. such that $t_k \to l(\mathfrak{N})$ and $\delta^{\alpha}(\{k \in \mathbb{N} : t_k \neq s_k\}) = 0$. Then, $s_k \to l(st^{\alpha} - \mathfrak{N})$.*

Proof. Suppose $\delta^{\alpha}(\{k \in \mathbb{N} : t_k \neq s_k\}) = 0$ holds and $t_k \to l(\mathfrak{N})$. So, as per the definition, for any $0 < \zeta < 1$, the set $M_{\zeta} = \{k \in \mathbb{N} : \mathfrak{R}(t_k - \ell, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(t_k - \ell, \tau) \geq \zeta, \mathfrak{W}(t_k - \ell, \tau) \geq \zeta\}$ has a maximum number of finite elements. Consequently, $\delta^{\alpha}(M_{\zeta}) = 0$. Now, since the inclusion

$$M'_{\zeta} = \{k \in \mathbb{N} : \mathfrak{R}(s_k - \ell, \tau) \leq 1 - \zeta or \mathfrak{T}(s_k - \ell, \tau) \geq \zeta, \mathfrak{W}(s_k - \ell, \tau) \geq \zeta\} \subseteq M_{\zeta} \cap \{k \in \mathbb{N} : t_k \neq s_k\}$$

holds, so we must have, $\delta^{\alpha}(M') = 0$. Hence, $y_k \to l(st^{\alpha} - \mathfrak{N})$. □

Definition 3.9. Let (y_k) be a sequence in a NNS V. We say that (y_k) is a statistical Cauchy sequence of order α if, For any $0 < \zeta < 1$, there is a positive integer $N = N(\zeta)$. such that the α-order statistical deviation of the set MC_{ζ} satisfies $\delta^{\alpha}(MC_{\zeta}) = 0$, where:

$$MC_{\zeta} = \{k \in \mathbb{N} : \mathfrak{R}(y_k - s_N, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(y_k - s_N, \tau) \geq \zeta, \mathfrak{W}(y_k - s_N, \tau) \geq \zeta\}.$$

Theorem 3.10. *Consider a NNS V. Then, for the sequence (y_k) within this NNS V, it holds that (y_k) is a statistical convergent sequence of order α if and only if it is a statistical Cauchy sequence of order α.*

Proof. Assuming that y_k converges towards $l(st^{\alpha} - \mathfrak{N})$, let's select a value of $\mu > 0$ for a given $0 < \zeta < 1$ in such a manner that $(1 - \zeta) \circ (1 - \zeta) > 1 - \mu$, and also $\zeta \bullet \zeta < \mu$. Now, according to the definition, for any $0 < \zeta < 1$, we find that $\delta^{\alpha}(\mathbb{N} \setminus M_{\zeta}) = 1$, where

$$M_{\zeta} = \{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \frac{\tau}{2}) \leq 1 - \zeta \; or \; \mathfrak{T}(y_k - \ell, \frac{\tau}{2}) \geq \zeta \; , \; \mathfrak{W}(y_k - \ell, \frac{\tau}{2}) \geq \zeta\}.$$

Hence, the set the element of an element $\mathbb{N}$ that is not in M_{ζ} contains at least one element. Let us consider $N \in \mathbb{N} \setminus M_{\zeta}$. Subsequently, we obtain,

$$\mathfrak{R}(s_N - \ell, \frac{\tau}{2}) > 1 - \zeta \; or \; \mathfrak{T}(s_N - \ell, \frac{\tau}{2}) < \zeta \; , \; \mathfrak{W}(s_N - \ell, \frac{\tau}{2}) < \zeta.$$

Now suppose

$$MC_{\zeta} = \{k \in \mathbb{N} : \mathfrak{R}(y_k - s_N, \tau) \leq 1 - \mu \; or \; \mathfrak{T}(y_k - s_N, \tau) \geq \mu \; , \; \mathfrak{W}(y_k - s_N, \tau) \geq \mu\}.$$

We assert that MC_{ζ} is a subset of M_{ζ} because if this inclusion does not hold, it implies that there must be some $N_0 \in MC_{\zeta} \setminus M_{\zeta}$ which immediately yields $\mathfrak{R}(s_{N_0} - s_N, \tau) \leq 1 - \mu$ and $\mathfrak{R}(s_{N_0} - \ell, \frac{\tau}{2}) > 1 - \zeta$. Especially, $\mathfrak{R}(s_N - \ell, \frac{\tau}{2}) > 1 - \zeta$. But then,

$$1 - \mu \geq \mathfrak{R}(s_{N_0} - s_N, \tau) \geq \mathfrak{R}(s_{N_0} - \ell, \frac{\tau}{2}) \circ \mathfrak{R}(s_N - \ell, \frac{\tau}{2}) > (1 - \zeta) \circ (1 - \zeta) > 1 - \mu,$$

this leads to a contradiction. Moreover, we can observe that, $\mathfrak{T}(s_{N_0} - s_N, \tau) \geq \mu$ and $\mathfrak{T}(s_{N_0} - \ell, \frac{\tau}{2}) < \zeta$. Especially, $\mathfrak{T}(s_N - \ell, \frac{\tau}{2}) < \zeta$. But then,

$$\mu \leq \mathfrak{T}(s_{N_0} - s_N, \tau) \leq \mathfrak{T}(s_{N_0} - \ell, \frac{\tau}{2}) \bullet \mathfrak{T}(s_N - \ell, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu,$$

this leads to a contradiction. Moreover, we can observe that, $\mathfrak{W}(s_{N_0} - s_N, \tau) \geq \mu$ and $\mathfrak{W}(s_{N_0} - \ell, \frac{\tau}{2}) < \zeta$. Especially, $\mathfrak{W}(s_N - \ell, \frac{\tau}{2}) < \zeta$. But then,

$$\mu \leq \mathfrak{W}(s_{N_0} - s_N, \tau) \leq \mathfrak{W}(s_{N_0} - \ell, \frac{\tau}{2}) \bullet \mathfrak{W}(s_N - \ell, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu,$$

This situation presents a contradiction. Therefore, all potential scenarios are in conflict with the presence of an element $N_0 \in MC_\zeta \setminus M_\zeta$. As a result, it becomes evident that $MC_\zeta \subseteq M_\zeta$, and consequently, $\delta^\alpha(MC_\zeta) = 0$. This ultimately establishes that (y_k) is, in fact, a statistical Cauchy sequence of order α.

To establish the converse aspect of the argument, we start by assuming that (y_k) is a statistical Cauchy sequence of order α but does not qualify as a statistical convergent sequence of the same order α. Given a specific value $0 < \zeta < 1$, we can select a positive parameter $\mu > 0$ such that the composition $(1 - \zeta) \circ (1 - \zeta) > 1 - \mu$ holds, and simultaneously $\zeta \bullet \zeta < \mu$.

Due to the fact that (y_k) does not satisfy the criteria for being statistically convergent of order α, we proceed with the assumption.

$$\mathfrak{R}(y_k - s_N, \tau) \geq \mathfrak{R}(y_k - \ell, \frac{\tau}{2}) \circ \mathfrak{R}(s_N - \ell, \frac{\tau}{2}) > (1 - \zeta) \circ (1 - \zeta) > 1 - \mu,$$
$$\mathfrak{T}(y_k - s_N, \tau) \leq \mathfrak{T}(y_k - \ell, \frac{\tau}{2}) \bullet \mathfrak{T}(s_N - \ell, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu,$$
$$\mathfrak{W}(y_k - s_N, \tau) \leq \mathfrak{W}(y_k - \ell, \frac{\tau}{2}) \bullet \mathfrak{W}(s_N - \ell, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu,$$

This property is valid for $P(\zeta, \mu)$ holds true for values of k within the range of k from 1 to N. $N : \mathfrak{T}(y_k - s_N, \tau) \leq 1 - \mu$.

As a result, $\delta^\alpha(P(\zeta, \mu)) = 1$, which contradicts the assumption that (y_k) is a statistical Cauchy sequence of order α. Therefore, it is evident that (y_k) must indeed be a statistical convergent sequence of order α. This conclusion serves to finalize the entirety of the proof. □

4. Conclusions

In this study, our primary focus was on exploring fundamental characteristics of statistical convergence with a degree of α. We also established a relationship between this type of convergence and the Statistical convergence in neutrosophic normed spaces has been newly introduced by Kirisci and Simsek. Additionally, we introduced the concept of statistical Cauchy sequences of order α and demonstrated that, in a neutrosophic normed space, every sequence that converges statistically with a degree of α. also qualifies as a statistical Cauchy sequence of the same order, and vice versa.

In the future, there's potential to extend this research to encompass multisequences, enabling a deeper exploration of the resulting sequence space's structure.

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