Neutrosophic Sets and Systems, Vol. 58, 2023
University of New Mexico

Finite Difference Method for Neutrosophic Fuzzy Second Order Differential Equation under Generalized Hukuhara Differentiability

Bassem Kamal [1,*], Huda E. Khalid[2], A. A. Salama [3] and Galal I. El-Baghdady [4]

[1] Department of Mathematics and Engineering Physics, Faculty of Engineering, Delta University for science and technology, Egypt
[2] Telafer University, The Administration Assistant for the President of the Telafer University, Telafer, Iraq.
[3] Department of Mathematics and Computer Science, Faculty of Science, Port Saied University, Egypt.
[4] Department of Mathematics and Engineering Physics, Faculty of Engineering, Mansoura University, Egypt.
Emails: [1,*]belmahmoudy@gmail.com , [1,*]Basem.Kamal@deltauniv.edu.eg, [2]dr.huda-ismael@uotelafer.edu.iq, [3]Ahmed_salama_2000@sci.psu.edu.eg; [4]enggalalebrahim@mans.edu.eg

Abstract
In this paper, we solve the boundary value problem of the second-order differential equation under the neutrosophic fuzzy boundary condition. Our solution in this paper will be approximated by using the finite difference method but defuzzification by using strongly generalized Hukuhara differentiability, then the error calculation will be processed by comparing this numerical solution with the analytical solution published in previous work.

Keywords: Fuzzy differential equations, Neutrosophic fuzzy, Hukuhara differentiability, Finite difference method.

1. Introduction

Modeling the real problems as ideal mathematical cases can't help in applying or finding real solutions, and this was the reason for initiating the concept of fuzzy sets and derivatives with Zadeh [1]. Several later works aimed to widen the concept of differentiability, especially Hukuhara [2-6], who developed the concept of difference between fuzzy sets. He then initiated Hukuhara differentiability, and Stefanoni and Bede [4-6] developed the generalized Hukuhara differentiability.

There are several numerical approaches depend on the spectral collocation method to provide an approximate solution for equations pertaining to differential equations [7-10]. On the other hand, many researchers have applied many numerical methods to solve fuzzy problems. The numerical solutions of FIVP by Euler's method were studied by Ma [21]. Abbasbandy and Allahviranloo [18, 19] used the Taylor method and the fourth-order Runge-Kutta method to obtain numerical solutions to FIVPs. Palligkinis [23] applied the Runge-Kutta method to more general problems and proved the convergence of the n-stage Runge-Kutta method. Allahviranloo [20] solved the numerical solution of FDEs by using the predictor-corrector method. The dependency problem in fuzzy computation was discussed by Ahmadi and Hasan [21], where Euler's method based on Zadeh's extension principle to find the numerical solution of FIVPs. Omar and Hasan [22] adopted the same computation method to derive the fourth-order Runge-Kutta method for FIVP.

Uncertainty, vagueness, and incompleteness in data increase the requirement for more generalized fuzzy ideas. That motivated Samarandach [14-17] to initiate a generalized fuzzy concept that not only depends on membership but also depends on three different memberships: Truth, False, and Indeterminacy which can model the more complicated uncertain data, and he called it a neutrosophic fuzzy set.

In this paper, we provide a numerical solution for a neutrosophic fuzzy second-order differential equation under strongly generalized differentiability by using the finite difference method, then compare the results with the analytical solutions of the same numerical examples in our published results [24].

2. Preliminaries

We recollect some relevant basic preliminaries and in particular the work of Samarandach, Attanasov, and our previous work.

Definition 2.1 [16] Neutrosophic fuzzy set

Let X be a universe set and a neutrosophic set A on X is defined as $A = \{(T_A(x), I_A(x), F_A(x)): x\epsilon X\}$ represents the degree of membership $T_A(x)$, the degree of indeterministic $I_A(x)$ and the degree of non-membership $F_A(x)$. Such that $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 2.2 [15] (α, β, γ)-cuts

The (α, β, γ)-cuts are fixed values on the set A where $A_{\alpha,\beta,\gamma} = \{(T_A(x), I_A(x), F_A(x)): x\epsilon X, T_A(x) \ge \alpha, I_A(x) \le \beta, F_A(x) \le \gamma\}$ which define each of $T_A(x), I_A(x), F_A(x)$ in terms of lower and upper functions of (α, β, γ)-cuts.

Definition 2.3 [16, 17] Neutrosophic number

A neutrosophic set A defined on a universal set of real numbers $\boldsymbol{R}$ is said to be a neutrosophic number

i) A is normal if $x_a \in \boldsymbol{R}$, $T_A(x_a) = 1, I_A(x_a) = F_A(x_a) = 0$.

ii) A is a convex set on truth function $T_A(x)$ where $T_A(\lambda x_1 + (1-\lambda)x_2) \ge \min(T_A(x_1), T_A(x_2))$

iii) A is a concave set on indeterministic and falsity functions $I_A(x), F_A(x)$where

$$I_A(\lambda x_1 + (1-\lambda)x_2) \ge \max\big(I_A(x_1), I_A(x_2)\big)$$
$$F_A(\lambda x_1 + (1-\lambda)x_2) \ge \max(F_A(x_1), F_A(x_2))$$

Definition 2.4 [11] Triangular neutrosophic number

Let A be a Generalized triangle neutrosophic number $A_{GTN} = (a, b, c\colon \omega, \eta, \xi)$

$$T_A(x) = \begin{cases} \frac{x-a}{b-a}\omega & a \le x < b \\ \omega & x = b \\ \frac{c-x}{c-b}\omega & b < x \le c \end{cases} \text{ and zero otherwise}$$

$$I_A(x) = \begin{cases} \frac{b-x}{b-a}\eta & a \le x < b \\ \eta & x = b \\ \frac{b-x}{b-c}\eta & b < x \le c \end{cases} \text{ and 1 otherwise}$$

$$F_A(x) = \begin{cases} \frac{b-x}{b-a}\xi & a \le x < b \\ \xi & x = b \\ \frac{b-x}{b-c}\xi & b < x \le c \end{cases} \text{ and 1 otherwise}$$

And we can represent (α, β, γ)-cuts on the generalized triangle neutrosophic number as

$$A_{\alpha,\beta,\gamma} = \left[\underline{A(\alpha)}, \overline{A(\alpha)}\right], \left[\underline{A(\beta)}, \overline{A(\beta)}\right], \left[\underline{A(\gamma)}, \overline{A(\gamma)}\right]$$

$$\left[\underline{A(\alpha)}, \overline{A(\alpha)}\right] = \left[\left(a + \frac{\alpha}{\omega}(b-a)\right), \left(c - \frac{\alpha}{\omega}(c-b)\right)\right]$$

$$\left[\underline{A(\beta)}, \overline{A(\beta)}\right] = \left[\left(a + \frac{1-\beta}{1-\eta}(b-a)\right), \left(c - \frac{1-\beta}{1-\eta}(c-b)\right)\right]$$

$$\left[\underline{A(\gamma)}, \overline{A(\gamma)}\right] = \left[\left(a + \frac{1-\gamma}{1-\xi}(b-a)\right), \left(c - \frac{1-\gamma}{1-\xi}(c-b)\right)\right]$$

Definition 2.5 [12, 13] Trapezoidal neutrosophic number

Let A be a generalized Trapezoidal neutrosophic number $A_{GTRN} = (a, b, c, d\colon \omega, \eta, \xi)$

$$T_A(x) = \begin{cases} \frac{x-a}{b-a}\omega & a \le x \le b \\ \omega & b \le x \le c \\ \frac{d-x}{d-c}\omega & c \le x \le d \end{cases} \text{ and zero otherwise}$$

$$I_A(x) = \begin{cases} \frac{b-x}{b-a}\eta & a \le x \le b \\ \eta & b \le x \le c \\ \frac{c-x}{c-d}\eta & c \le x \le d \end{cases} \text{ and 1 otherwise}$$

$$F_A(x) = \begin{cases} \frac{b-x}{b-a}\xi & a \le x \le b \\ \xi & b \le x \le c \\ \frac{c-x}{c-d}\xi & c \le x \le d \end{cases} \text{ and 1 otherwise}$$

And we can represent (α, β, γ)-cuts on the generalized trapezoidal neutrosophic number as

$$A_{\alpha,\beta,\gamma} = \left[\underline{A(\alpha)}, \overline{A(\alpha)}\right], \left[\underline{A(\beta)}, \overline{A(\beta)}\right], \left[\underline{A(\gamma)}, \overline{A(\gamma)}\right]$$

$$\left[\underline{A(\alpha)}, \overline{A(\alpha)}\right] = \left[\left(a + \frac{\alpha}{\omega}(b-a)\right), \left(d - \frac{\alpha}{\omega}(d-c)\right)\right]$$

$$\left[\underline{A(\beta)}, \overline{A(\beta)}\right] = \left[\left(a + \frac{1-\beta}{1-\eta}(b-a)\right), \left(d - \frac{1-\beta}{1-\eta}(d-c)\right)\right]$$

$$\left[\underline{A(\gamma)}, \overline{A(\gamma)}\right] = \left[\left(a + \frac{1-\gamma}{1-\xi}(b-a)\right), \left(d - \frac{1-\gamma}{1-\xi}(d-c)\right)\right]$$

Definition 2.6 [5, 6]

Let $F:(a,b)\rightarrow \mathcal{F}(R)$, if the next limits

$$\lim_{h\to 0+}\frac{F(x_0+h)\ominus_H F(x_0)}{h},\qquad \lim_{h\to 0+}\frac{F(x_0)\ominus_H F(x_0-h)}{h}$$

exist and equal some elements $F'_H(x_0)\in \mathcal{F}(R)$, then F is Hukuhara differentiable at x_0, and $F'_H(x_0)$ is its derivative at x_0.

Theorem 2.1 [5, 6] Let $F:(a,b)\rightarrow \mathcal{F}(R)$ be a generalized Hukuhara differentiable Function if and only if (a) or (b) are satisfied

(a) $\underline{f'}_\alpha(x)$ is increasing and $\overline{f'}_\alpha(x)$ is decreasing

(b) $\underline{f'}_\alpha(x)$ is decreasing and $\overline{f'}_\alpha(x)$ is increasing

Then,

$$[F'_{gH}(x)]_{\boldsymbol{\alpha}}=\left[min\left(\underline{f'}_\alpha(x),\overline{f'}_\alpha(x)\right),max\left(\underline{f'}_\alpha(x),\overline{f'}_\alpha(x)\right)\right]$$

This concept is closer to the generalized differentiability, but it probably focuses on the cases of the function, and both of these differentiability's change the concept of derivatives, so let us show these changes in the next definition of generalized Hukuhara derivatives.

Definition 2.7 [5, 6] The generalized Hukuhara first derivative of a fuzzy parametric function is defined as;

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)\ominus_{gh} f(x_0)}{h},$$

From the definition, we have two classes:

(i)-differentiable at x_0 : $[f'(x_0)]_\alpha=[\underline{f'}_\alpha(x_0),\overline{f'}_\alpha(x_0)]$

(ii)-differentiable at x_0 : $[f'(x_0)]_\alpha=[\overline{f'}_\alpha(x_0),\underline{f'}_\alpha(x_0)]$

Definition 2.8 [5, 6] The generalized Hukuhara second derivative of the fuzzy function is defined as;

$$f''(x_0)=\lim_{h\to 0}\frac{f'(x_0+h)\ominus_{gh} f'(x_0)}{h}$$

According to the last definitions, we have the following classes:

$f'(x_0)$ is (i)-differentiable if:

$$f''(x_0) = \begin{cases} \left[\underline{f''}_\alpha(x_0), \overline{f''}_\alpha(x_0)\right] \; if\ f\ is\ (i) - differentiable \\ class(1,1) \\ \left[\overline{f''}_\alpha(x_0), \underline{f''}_\alpha(x_0)\right] \; if\ f\ is\ (ii) - differentiable \\ class(2,2) \end{cases}$$

$f'(x_0)$ is (ii)-differentiable if:

$$f''(x_0) = \begin{cases} \left[\overline{f''}_\alpha(x_0), \underline{f''}_\alpha(x_0)\right] \; if\ f\ is\ (i) - differentiable \\ class(1,2) \\ \left[\underline{f''}_\alpha(x_0), \overline{f''}_\alpha(x_0)\right] \; if\ f\ is\ (ii) - differentiable \\ class(2,1) \end{cases}$$

Definition 2. 9 [11] The solution$\tilde{y}(t,\alpha,\beta,\gamma)$ of the neutrosophic fuzzy differential equation is strong only if,

$$\frac{\partial \underline{y}}{\partial \alpha} > 0, \frac{d\bar{y}}{d\alpha} < 0 \text{ but } \frac{\partial \underline{y}}{\partial \beta} < 0, \frac{d\bar{y}}{d\beta} > 0 \text{ and } \frac{\partial \underline{y}}{\partial \gamma} < 0, \frac{d\bar{y}}{d\gamma} > 0.$$

3. Finite difference method under generalized Hukuhara differentiability

The generalized Hukuhara differentiability is first utilized on the neutrosophic fuzzy differential problem, which will transform it into classes of differential equations. Then the solution can be obtained by applying the finite difference method to these classes of the differential equation.

As an example and for a second-order differential equation that has neutrosophic input adopting the generalized Hukuhara differentiability, this problem is split into four classes, each class contains two certain equations. Then by using the normal finite difference method (FDM)$y'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$, $y' = \frac{y_{i+1}-y_{i-1}}{2h}$ on each certain first and second function derivatives.

Under generalized Hukuhara differentiability

$$\widetilde{y''}(t,\alpha,\beta,\gamma) = \ p\widetilde{y'}(t,\alpha,\beta,\gamma) + q\tilde{y}(t,\alpha,\beta,\gamma)$$

$$\tilde{y}(t_0) = \tilde{a} \qquad\qquad \tilde{y}(T) = \tilde{b}$$

Where the parameters are positive constants.

A. Class (1,1)

$$\overline{y''}(t,\alpha,\beta,\gamma) = p.\overline{y'}(t,\alpha,\beta,\gamma) + q.\overline{y}(t,\alpha,\beta,\gamma)$$
$$\overline{y}(t_0,\alpha,\beta,\gamma) = \overline{a} \quad \overline{y}(T,\alpha,\beta,\gamma) = \overline{b}$$
$$\underline{y''}(t,\alpha,\beta,\gamma) = p.\underline{y'}(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma)$$
$$\underline{y}(t_0,\alpha,\beta,\gamma) = \underline{a} \quad \underline{y}(T,\alpha,\beta,\gamma) = \underline{b}$$

By using the traditional finite difference method, we will get the following results:

$\bar{y}(t,\alpha,\beta,\gamma) = v$ and $\underline{y}(t,\alpha,\beta,\gamma) = u$

$$\underline{y''}(t,\alpha,\beta,\gamma) = p.\underline{y'}(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma) \tag{3.1}$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = p\frac{u_{i+1} - u_{i-1}}{2h} + qu_i \tag{3.2}$$

$$u_{i+1} - 2u_i + u_{i-1} = \frac{ph}{2}u_{i+1} - \frac{ph}{2}u_{i-1} + qh^2u_i \tag{3.3}$$

$$\left(1 - \frac{ph}{2}\right)u_{i+1} - (2 + qh^2)u_i + \left(1 + \frac{ph}{2}\right)u_{i-1} = 0. \tag{3.4}$$

Where $h = \Delta x$ and $i = 1,2,3$

$$\begin{bmatrix} -(2+qh^2) & \left(1-\frac{ph}{2}\right) & 0 \\ \left(1+\frac{ph}{2}\right) & -(2+qh^2) & \left(1-\frac{ph}{2}\right) \\ 0 & \left(1+\frac{ph}{2}\right) & -(2+qh^2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\left(1+\frac{ph}{2}\right)u_0 \\ 0 \\ -\left(1-\frac{ph}{2}\right)u_4 \end{bmatrix}$$

Where, $u_0 = \underline{a}$ and $u_4 = \underline{b}$

$$\overline{y''}(t,\alpha,\beta,\gamma) = p.\overline{y'}(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma) \tag{3.5}$$

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = p\frac{v_{i+1} - v_{i-1}}{2h} + qv_i \tag{3.6}$$

$$v_{i+1} - 2v_i + v_{i-1} = \frac{ph}{2}v_{i+1} - \frac{ph}{2}v_{i-1} + qh^2v_i \tag{3.7}$$

$$\left(1 - \frac{ph}{2}\right)v_{i+1} - (2 + qh^2)v_i + \left(1 + \frac{ph}{2}\right)v_{i-1} = 0 \tag{3.8}$$

$$\begin{bmatrix} -(2+qh^2) & \left(1-\frac{ph}{2}\right) & 0 \\ \left(1+\frac{ph}{2}\right) & -(2+qh^2) & \left(1-\frac{ph}{2}\right) \\ 0 & \left(1+\frac{ph}{2}\right) & -(2+qh^2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\left(1+\frac{ph}{2}\right)v_0 \\ 0 \\ -\left(1-\frac{ph}{2}\right)v_4 \end{bmatrix}$$

Where, $v_0 = \bar{a}$ and $v_4 = \bar{b}$

B. Class (1,2)

$$\boldsymbol{\underline{y''}(t,\alpha,\beta,\gamma) = p.\overline{y'}(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma)}$$

$$\boldsymbol{\bar{y}(t_0,\alpha,\beta,\gamma) = \bar{a} \quad \bar{y}(T,\alpha,\beta,\gamma) = \bar{b}}$$

$$\boldsymbol{\overline{y''}(t,\alpha,\beta,\gamma) = p.\underline{y'}(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma)}$$

$$\boldsymbol{\underline{y}(t_0,\alpha,\beta,\gamma) = \underline{a} \quad \underline{y}(T,\alpha,\beta,\gamma) = \underline{b}}$$

Where, $\bar{y}(t,\alpha) = v$ and $\underline{y}(t,\alpha) = u$

$$\underline{y''}(t,\alpha,\beta,\gamma) = p.\overline{y}'(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma) \tag{3.9}$$

$$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}= p\frac{v_{i+1}-v_{i-1}}{2h}+qv_i \tag{3.10}$$

$$u_{i+1}-2u_i+u_{i-1}=\frac{ph}{2}v_{i+1}+qh^2v_i-\frac{ph}{2}v_{i-1} \tag{3.11}$$

$$\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}u_0\\0\\u_4\end{bmatrix}=\begin{bmatrix}qh^2 & \frac{ph}{2} & 0\\ -\frac{ph}{2} & qh^2 & \frac{ph}{2}\\ 0 & -\frac{ph}{2} & qh^2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+\begin{bmatrix}-\frac{ph}{2}v_0\\0\\\frac{ph}{2}v_4\end{bmatrix}, \tag{3.12}$$

$$\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}qh^2 & \frac{ph}{2} & 0\\ -\frac{ph}{2} & qh^2 & \frac{ph}{2}\\ 0 & -\frac{ph}{2} & qh^2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+\begin{bmatrix}-\frac{ph}{2}v_0-u_0\\0\\\frac{ph}{2}v_4-u_4\end{bmatrix}. \tag{3.13}$$

The system of equations in eq. (3.13) can be converted to the next matrix form to get the solution to it.

$$Au=Bv+C, \tag{3.14}$$

$$A^{-1}Au=A^{-1}Bv+A^{-1}C, \tag{3.15}$$

$$u=A^{-1}Bv+A^{-1}C. \tag{3.16}$$

For the second equation;

$$\overline{y}''(t,\alpha,\beta,\gamma)=p.\underline{y}'(t,\alpha,\beta,\gamma)+q.\underline{y}(t,\alpha,\beta,\gamma) \tag{3.17}$$

$$\frac{v_{i+1}-2v_i+v_{i-1}}{h^2}= p\frac{u_{i+1}-u_{i-1}}{2h}+qu_i \tag{3.18}$$

$$v_{i+1}-2v_i+v_{i-1}=\frac{ph}{2}u_{i+1}+qh^2u_i-\frac{ph}{2}u_{i-1} \tag{3.19}$$

$$\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+\begin{bmatrix}v_0\\0\\v_4\end{bmatrix}=\begin{bmatrix}qh^2 & \frac{ph}{2} & 0\\ -\frac{ph}{2} & qh^2 & \frac{ph}{2}\\ 0 & -\frac{ph}{2} & qh^2\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}-\frac{ph}{2}u_0\\0\\\frac{ph}{2}u_4\end{bmatrix} \tag{3.20}$$

$$\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}qh^2 & \frac{ph}{2} & 0\\ -\frac{ph}{2} & qh^2 & \frac{ph}{2}\\ 0 & -\frac{ph}{2} & qh^2\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}-\frac{ph}{2}u_0-v_0\\0\\\frac{ph}{2}u_4-v_4\end{bmatrix} \tag{3.21}$$

$$Av=Bu+D \tag{3.22}$$

$$A^{-1}Av=A^{-1}Bu+A^{-1}D \tag{3.23}$$

$$v=A^{-1}Bu+A^{-1}D \tag{3.24}$$

By substituting the value of *u* in the previous equation, we can get the next solution to each variable.

$$v=A^{-1}B(A^{-1}Bv+A^{-1}C)+A^{-1}D \tag{3.25}$$

$$(I-(A^{-1}B)^2)v=A^{-1}BA^{-1}C+A^{-1}D \tag{3.26}$$

$$v=[v_1 \quad v_2 \quad v_3]^T \tag{3.27}$$

$$u=[u_1 \quad u_2 \quad u_3]^T \tag{3.28}$$

C. Class (2,1)

$$\overline{\boldsymbol{y}}''(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \boldsymbol{p}.\overline{\boldsymbol{y}}'(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) + \boldsymbol{q}.\underline{\boldsymbol{y}}(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),$$

$$\underline{\boldsymbol{y}}(\boldsymbol{t_0},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \underline{\boldsymbol{a}} \qquad \underline{\boldsymbol{y}}(\boldsymbol{T},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \underline{\boldsymbol{b}},$$

$$\underline{\boldsymbol{y}}''(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \boldsymbol{p}.\underline{\boldsymbol{y}}'(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) + \boldsymbol{q}.\overline{\boldsymbol{y}}(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),$$

$$\overline{\boldsymbol{y}}(\boldsymbol{t_0},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \overline{\boldsymbol{a}} \qquad \overline{\boldsymbol{y}}(\boldsymbol{T},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \overline{\boldsymbol{b}}.$$

$$\overline{y}''(t,\alpha,\beta,\gamma) = p.\overline{y}'(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma) \tag{3.29}$$

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = p\frac{v_{i+1} - v_{i-1}}{2h} + qu_i \tag{3.30}$$

$$\left(1 - \frac{ph}{2}\right)v_{i+1} - 2v_i + \left(1 + \frac{ph}{2}\right)v_{i-1} = qh^2u_i \tag{3.31}$$

$$\begin{bmatrix} -2 & \left(1-\frac{ph}{2}\right) & 0 \\ \left(1+\frac{ph}{2}\right) & -2 & \left(1-\frac{ph}{2}\right) \\ 0 & \left(1+\frac{ph}{2}\right) & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = qh^2\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -\left(1+\frac{ph}{2}\right)v_0 \\ 0 \\ -\left(1-\frac{ph}{2}\right)v_4 \end{bmatrix} \tag{3.32}$$

$$Av = qh^2u + B \tag{3.33}$$

$$v = A^{-1}qh^2u + A^{-1}B \tag{3.34}$$

For the second equation

$$\underline{y}''(t,\alpha,\beta,\gamma) = p.\underline{y}'(t,\alpha,\beta,\gamma) + q.\overline{y}(t,\alpha,\beta,\gamma) \tag{3.35}$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = p\frac{u_{i+1} - u_{i-1}}{2h} + qv_i \tag{3.36}$$

$$\left(1 - \frac{ph}{2}\right)u_{i+1} - 2u_i + \left(1 + \frac{ph}{2}\right)u_{i-1} = qh^2v_i \tag{3.37}$$

$$\begin{bmatrix} -2 & \left(1-\frac{ph}{2}\right) & 0 \\ \left(1+\frac{ph}{2}\right) & -2 & \left(1-\frac{ph}{2}\right) \\ 0 & \left(1+\frac{ph}{2}\right) & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = qh^2\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} -\left(1+\frac{ph}{2}\right)u_0 \\ 0 \\ -\left(1-\frac{ph}{2}\right)u_4 \end{bmatrix} \tag{3.38}$$

$$Au = qh^2v + C \tag{3.39}$$

$$u = A^{-1}qh^2v + A^{-1}C \tag{3.40}$$

By substitution the value of v for the previous equation (3.40), yields

$$u = A^{-1}qh^2(A^{-1}qh^2u + A^{-1}B) + A^{-1}C \tag{3.41}$$

$$(1 - (qh^2)^2(A^{-1})^2)u = qh^2(A^{-1})^2B + A^{-1}C \tag{3.42}$$

Then,

$$v = [v_1 \quad v_2 \quad v_3]^T \tag{3.43}$$

$$u = [u_1 \quad u_2 \quad u_3]^T \tag{3.44}$$

D. Class (2,2)

$$\overline{\boldsymbol{y}}''(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \boldsymbol{p}.\underline{\boldsymbol{y}}'(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) + \boldsymbol{q}.\overline{\boldsymbol{y}}(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$$

$$\overline{\boldsymbol{y}}(\boldsymbol{t_0},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \overline{\boldsymbol{a}} \quad \overline{\boldsymbol{y}}(\boldsymbol{T},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \overline{\boldsymbol{b}}$$

$$\underline{\boldsymbol{y}}''(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\alpha}) = \boldsymbol{p}.\overline{\boldsymbol{y}}'(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) + \boldsymbol{q}.\underline{\boldsymbol{y}}(\boldsymbol{t},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$$

$$\underline{\boldsymbol{y}}(\boldsymbol{t_0},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \underline{\boldsymbol{a}} \quad \underline{\boldsymbol{y}}(\boldsymbol{T},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \underline{\boldsymbol{b}}$$

$$\overline{y}''(t,\alpha,\beta,\gamma) = p.\underline{y}'(t,\alpha,\beta,\gamma) + q.\overline{y}(t,\alpha,\beta,\gamma) \tag{3.45}$$

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = p\frac{u_{i+1} - u_{i-1}}{2h} + qv_i \tag{3.46}$$

$$v_{i+1} - (2 + qh^2)v_i + v_{i-1} = \frac{ph}{2}u_{i+1} - \frac{ph}{2}u_{i-1} \tag{3.47}$$

$$\begin{bmatrix} -(2+qh^2) & 1 & 0 \\ 1 & -(2+qh^2) & 1 \\ 0 & 1 & -(2+qh^2) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{ph}{2} & 0 \\ -\frac{ph}{2} & 0 & \frac{ph}{2} \\ 0 & -\frac{ph}{2} & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -\frac{ph}{2}u_0 - v_0 \\ 0 \\ \frac{ph}{2}u_4 - v_4 \end{bmatrix} \tag{3.48}$$

$$Av = Bu + C \tag{3.49}$$

$$A^{-1}Av = A^{-1}Bu + A^{-1}C \tag{3.50}$$

$$v = A^{-1}Bu + A^{-1}C \tag{3.51}$$

For the second equation

$$\underline{y}''(t,\alpha,\beta,\gamma) = p.\overline{y}'(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma) \tag{3.52}$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = p\frac{v_{i+1} - v_{i-1}}{2h} + qu_i \tag{3.53}$$

$$u_{i+1} - (2 + qh^2)u_i + u_{i-1} = \frac{ph}{2}v_{i+1} - \frac{ph}{2}v_{i-1} \tag{3.54}$$

$$\begin{bmatrix} -(2+qh^2) & 1 & 0 \\ 1 & -(2+qh^2) & 1 \\ 0 & 1 & -(2+qh^2) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{ph}{2} & 0 \\ -\frac{ph}{2} & 0 & \frac{ph}{2} \\ 0 & -\frac{ph}{2} & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} -\frac{ph}{2}v_0 - u_0 \\ 0 \\ \frac{ph}{2}v_4 - u_4 \end{bmatrix} \tag{3.55}$$

$$Au = Bv + C \tag{3.56}$$

$$A^{-1}Au = A^{-1}Bv + A^{-1}C \tag{3.57}$$

$$u = A^{-1}Bv + A^{-1}C \tag{3.58}$$

4. Numerical Simulation

In this section, we test the utility of the proposed method by presenting an example, whose analytical solutions are provided in the previous work [24]. To show the efficiency of the proposed technique, we compare the numerical results with the exact solution [24]. The absolute error can be formulated as:

• The absolute error $|e_N|$ of the solution is defined by:

$$|e_N| = |Exact - Approximation|.$$

Example 4.1

$$\widetilde{y''}(t) = 4\widetilde{y}'(t) + 5\widetilde{y}(t)$$

$$\tilde{y}(t_0) = \tilde{a} = (0.8,1,1.4; 0.8,0.2,0.3)$$

$$\tilde{y}(T) = \tilde{b} = (2.6,3,3.1,; 0.8,0.2,0.3)$$

p = 4 & q =5

- **Case (1, 1)**

Analytical Solution

Table (1): *The Analytical Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$

α	$\underline{y}(0.7,\alpha)$	$\overline{y}(0.7,\alpha)$	β	$\underline{y}(0.7,\beta)$	$\overline{y}(0.7,\beta)$	γ	$\underline{y}(0.7,\gamma)$	$\overline{y}(0.7,\gamma)$
0.0	0.884039	1.287469	0.2	1.060290	1.060290	0.3	1.060290	1.060290
0.2	0.928101	1.230675	0.4	1.016227	1.117085	0.5	1.009932	1.125198
0.4	0.972164	1.173880	0.6	0.972164	1.173880	0.7	0.959575	1.190107
0.6	1.016227	1.117085	0.8	0.928101	1.230675	0.9	0.934396	1.255015
0.8	1.060290	1.060290	1.0	0.884039	1.287469	1.0	0.884039	1.287469

Approximation Solution and the related errors

***Table* (2):** *The Approximation Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$*at N = 50.*

α	$\underline{y}(0.7,\alpha)$	$\overline{y}(0.7,\alpha)$	β	$\underline{y}(0.7,\beta)$	$\overline{y}(0.7,\beta)$	γ	$\underline{y}(0.7,\gamma)$	$\overline{y}(0.7,\gamma)$
0.0	0.883563	1.286953	0.2	1.059750	1.059750	0.3	1.060290	1.060290
0.2	0.927609	1.230153	0.4	1.015703	1.116551	0.5	1.009411	1.124665
0.4	0.971656	1.173352	0.6	0.971656	1.173352	0.7	0.959072	1.189581
0.6	1.015703	1.116551	0.8	0.927609	1.230153	0.9	0.908732	1.254496
0.8	1.059750	1.059750	1.0	0.883563	1.286953	1.0	0.884039	1.287469

***Table* (3):** *The Errors Results of Upper and Lower for* (α,β,γ) *at* $t = 0.7$*at N = 50.*

α	$\underline{e}(0.7,\alpha)$	$\bar{e}(0.7,\alpha)$	β	$\underline{e}(0.7,\beta)$	$\bar{e}(0.7,\beta)$	γ	$\underline{e}(0.7,\gamma)$	$\bar{e}(0.7,\gamma)$
0.0	4.76 E-04	5.16 E-04	0.2	5.40 E-04	5.40 E-04	0.3	5.40 E-04	5.40 E-04
0.2	4.92 E-04	522 E-04	0.4	5.24 E-04	5.34 E-04	0.5	5.22 E-04	5.33 E-04
0.4	5.08 E-04	5.28 E-04	0.6	5.08 E-04	5.28 E-04	0.7	5.03 E-04	5.26 E-04
0.6	5.24 E-04	5.34 E-04	0.8	4.92 E-04	5.22 E-04	0.9	4.85 E-04	5.19 E-04
0.8	5.40 E-04	5.40 E-04	1.0	4.76 E-04	5.16 E-04	1.0	4.76 E-04	5.16 E-04

We present the analytical results for $t = 0.7$ in **Table** 1, followed by the approximation results for various values of N in **Tables** 2, 4, and 6, and finally the error between analytical and numerical results in **Tables** 3, 5, and 7.

***Table* (4):** *The Approximation Results of Upper and Lower for* (α,β,γ) *at* $t = 0.7$*at N = 100.*

α	$\underline{y}(0.7,\alpha)$	$\bar{y}(0.7,\alpha)$	β	$\underline{y}(0.7,\beta)$	$\bar{y}(0.7,\beta)$	γ	$\underline{y}(0.7,\gamma)$	$\bar{y}(0.7,\gamma)$
0.0	0.883920	1.287341	0.2	1.060155	1.060155	0.3	1.060155	1.060155
0.2	0.927979	1.230544	0.4	1.016096	1.116951	0.5	1.009802	1.125065
0.4	0.972037	1.173748	0.6	0.972037	1.173748	0.7	0.959449	1.189975
0.6	1.016096	1.116951	0.8	0.927979	1.230544	0.9	0.909096	1.254885
0.8	1.060155	1.060155	1.0	0.883920	1.287341	1.0	0.883920	1.287341

***Table* (5):** *The Errors Results of Upper and Lower for* (α,β,γ) *at* $t = 0.7$*at N = 100.*

α	$\underline{e}(0.7,\alpha)$	$\bar{e}(0.7,\alpha)$	β	$\underline{e}(0.7,\beta)$	$\bar{e}(0.7,\beta)$	γ	$\underline{e}(0.7,\gamma)$	$\bar{e}(0.7,\gamma)$
0.0	1.19 E-04	1.29 E-04	0.2	1.35 E-04	1.35 E-04	0.3	1.35 E-04	1.35 E-04
0.2	1.23 E-04	1.30 E-04	0.4	1.31 E-04	1.33 E-04	0.5	1.30 E-04	1.33 E-04
0.4	1.27 E-04	1.32 E-04	0.6	1.27 E-04	1.32 E-04	0.7	1.26 E-04	1.31 E-04
0.6	1.31 E-04	1.33 E-04	0.8	1.23 E-04	1.31 E-04	0.9	1.21 E-04	1.30 E-04
0.8	1.35 E-04	1.35 E-04	1.0	1.19 E-04	1.29 E-04	1.0	1.19 E-04	1.29 E-04

$N = 200 \,\&\, h = 0.005$

***Table* (6):** *The Approximation Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$*at N = 200.*

α	$\underline{y}(0.7,\alpha)$	$\bar{y}(0.7,\alpha)$	β	$\underline{y}(0.7,\beta)$	$\bar{y}(0.7,\beta)$	γ	$\underline{y}(0.7,\gamma)$	$\bar{y}(0.7,\gamma)$
0.0	0.884009	1.287437	0.2	1.060256	1.060256	0.3	1.060256	1.060256
0.2	0.928071	1.230642	0.4	1.016194	1.117052	0.5	1.009900	1.125165
0.4	0.972133	1.173847	0.6	0.972133	1.173847	0.7	0.959544	1.190074
0.6	1.016194	1.117052	0.8	0.928071	1.230642	0.9	0.909187	1.254983
0.8	1.060256	1.060256	1.0	0.884009	1.287437	1.0	0.884009	1.287437

***Table* (7):** *The Errors Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$*at N = 200.*

α	$\underline{e}(0.7,\alpha)$	$\bar{e}(0.7,\alpha)$	β	$\underline{e}(0.7,\beta)$	$\bar{e}(0.7,\beta)$	γ	$\underline{e}(0.7,\gamma)$	$\bar{e}(0.7,\gamma)$
0.0	2.97 E-05	3.22 E-05	0.2	3.37 E-05	3.37 E-05	0.3	3.37 E-05	3.37 E-05
0.2	3.07 E-05	3.26 E-05	0.4	3.27E-05	3.33 E-05	0.5	3.26 E-05	3.33 E-05
0.4	3.17 E-05	3.30 E-05	0.6	3.17 E-05	3.30 E-05	0.7	3.14 E-05	3.29 E-05
0.6	3.27E-05	3.33 E-05	0.8	3.07 E-05	3.26 E-05	0.9	3.03 E-05	3.24 E-05
0.8	3.37 E-05	3.37 E-05	1.0	2.97 E-05	3.22 E-05	1.0	2.97 E-05	3.22 E-05

Figures 1 through 6 depict the surface of errors seen at a sample size of $N = 200$, throughout the time interval $t \in [0.1: 0.9]$, while considering various values of (α, β, γ).

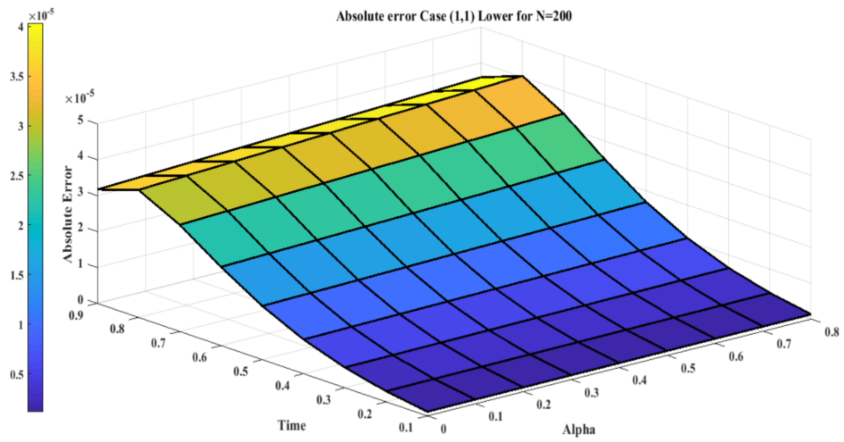

Figure 1: *Surface of Absolute Error for Lower Case (1,1) Alpha*

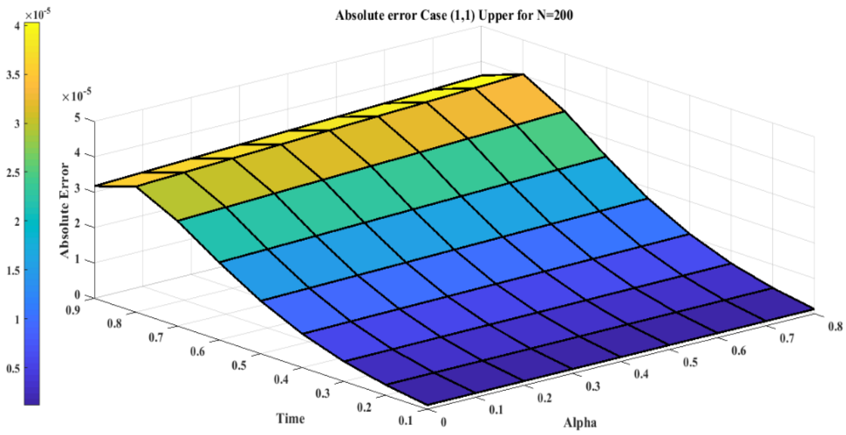

Figure 2: *Surface of Absolute Error for Upper Case (1,1) Alpha*

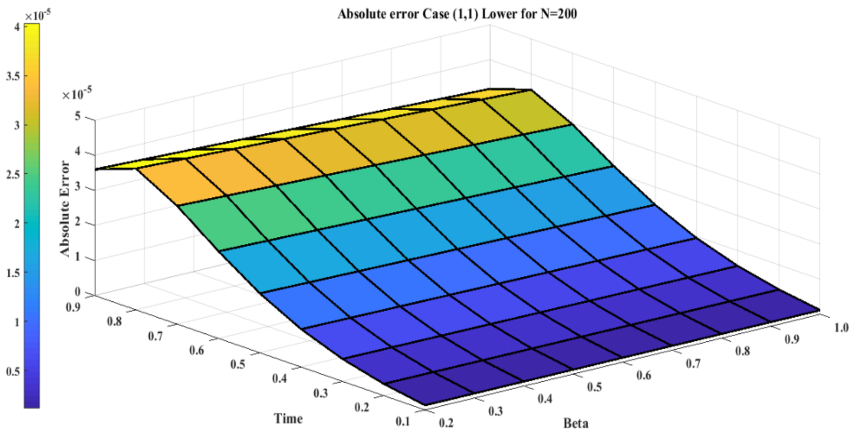

Figure 3: *Surface of Absolute Error for Lower case (1,1) Beta*

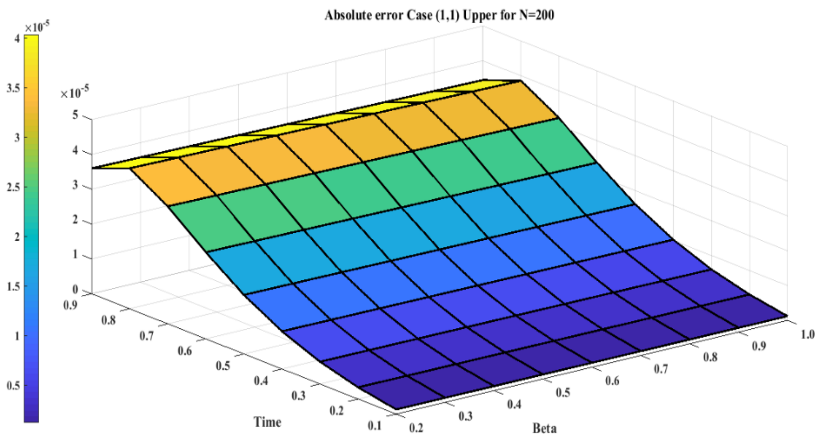

Figure 4: *Surface of Absolute error for Upper Case (1,1) Beta*

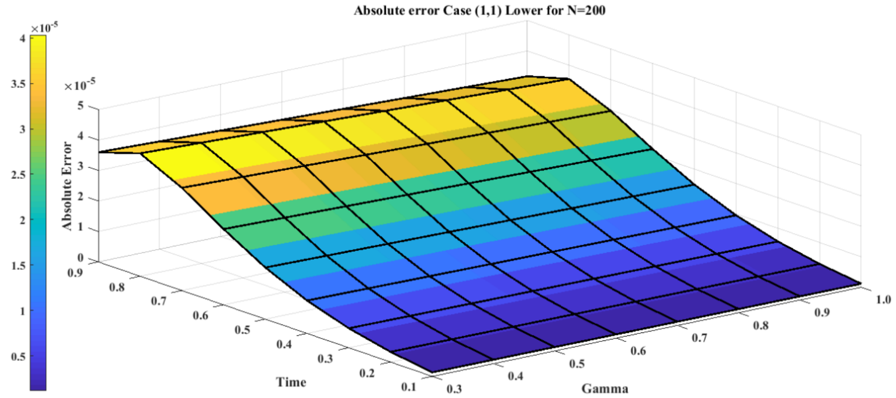

Figure 5: *Surface of Absolute Error for Lower case (1,1) Gamma*

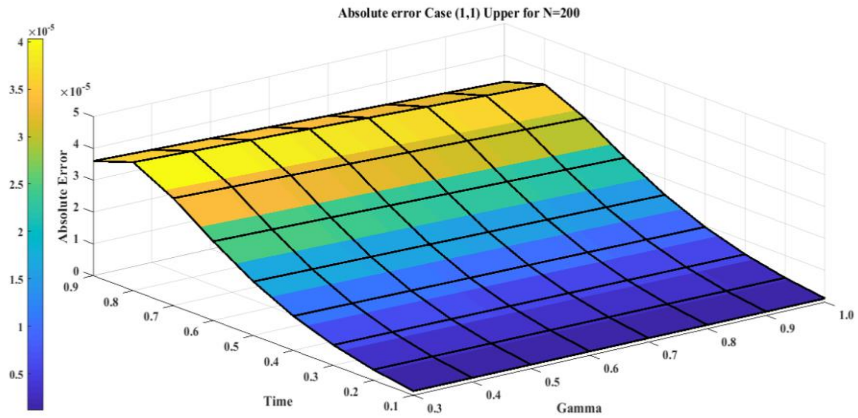

Figure 6: *Surface of Absolute Error for Upper case (1,1) Gamma*

- **Case (1, 2)**

Analytical Solution

***Table* (8)**: *The Analytical Results of Upper and Lower for* (α,β,γ) *at* $t=0.7$

α	$\underline{y}(0.7,\alpha)$	$\overline{y}(0.7,\alpha)$	β	$\underline{y}(0.7,\beta)$	$\overline{y}(0.7,\beta)$	γ	$\underline{y}(0.7,\gamma)$	$\overline{y}(0.7,\gamma)$
0.0	0.762704	1.408804	0.2	1.060290	1.060290	0.3	1.060290	1.060290
0.2	0.837101	1.321675	0.4	0.985893	1.147418	0.5	0.975265	1.159865
0.4	0.911497	1.234547	0.6	0.911497	1.234547	0.7	0.890241	1.259441
0.6	0.985893	1.147418	0.8	0.837101	1.321675	0.9	0.805217	1.359016
0.8	1.060290	1.060290	1.0	0.762704	1.408804	1.0	0.762704	1.408804

Approximation Solution at N = 200 & h = 0.005

***Table* (9)**: *The Approximation Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$ *for* $N = 200$

α	$\underline{y}(0.7, \alpha)$	$\bar{y}(0.7, \alpha)$	β	$\underline{y}(0.7, \beta)$	$\bar{y}(0.7, \beta)$	γ	$\underline{y}(0.7, \gamma)$	$\bar{y}(0.7, \gamma)$
0.0	0.762672	1.408774	0.2	1.060256	1.060256	0.3	1.060256	1.060256
0.2	0.837068	1.321645	0.4	0.985860	1.147386	0.5	0.975232	1.159833
0.4	0.911464	1.234515	0.6	0.911464	1.234515	0.7	0.890208	1.259409
0.6	0.985860	1.147386	0.8	0.837068	1.321645	0.9	0.805184	1.358986
0.8	1.060256	1.060256	1.0	0.762672	1.408774	1.0	0.762672	1.408774

Error at $N = 200$

***Table* (10):** *The Error Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$ *at N = 200.*

α	$\underline{e}(0.7, \alpha)$	$\bar{e}(0.7, \alpha)$	β	$\underline{e}(0.7, \beta)$	$\bar{e}(0.7, \beta)$	γ	$\underline{e}(0.7, \gamma)$	$\bar{e}(0.7, \gamma)$
0.0	3.258E-05	2.935E-05	0.2	3.372E-05	3.372E-05	0.3	3.372E-05	3.372E-05
0.2	3.285E-05	3.044E-05	0.4	3.344E-05	3.263E-05	0.5	3.340E-05	3.247E-05
0.4	3.315E-05	3.153E-05	0.6	3.315E-05	3.153E-05	0.7	3.307E-05	3.122E-05
0.6	3.344E-05	3.263E-05	0.8	3.285E-05	3.044E-05	0.9	3.274E-05	2.997E-05
0.8	3.372E-05	3.372E-05	1.0	3.258E-05	2.935E-05	1.0	3.258E-05	2.935E-05

- **Case (2, 1)**

Analytical Solution

***Table* (11)**: *The Analytical Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$

α	$\underline{y}(0.7, \alpha)$	$\bar{y}(0.7, \alpha)$	β	$\underline{y}(0.7, \beta)$	$\bar{y}(0.7, \beta)$	γ	$\underline{y}(0.7, \gamma)$	$\bar{y}(0.7, \gamma)$
0.0	0.638923	1.532585	0.2	1.060290	1.060290	0.3	1.060290	1.060290
0.2	0.744265	1.414511	0.4	0.954948	1.178364	0.5	0.939899	1.195231
0.4	0.849607	1.296438	0.6	0.849607	1.296438	0.7	0.819509	1.330173
0.6	0.954948	1.178364	0.8	0.744265	1.414511	0.9	0.699119	1.465114
0.8	1.060290	1.060290	1.0	0.638923	1.532585	1.0	0.638923	1.532585

Approximation Solution

***Table* (12):** *The Approximation Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$ *at N = 150.*

α	$\underline{y}(0.7, \alpha)$	$\overline{y}(0.7, \alpha)$	β	$\underline{y}(0.7, \beta)$	$\overline{y}(0.7, \beta)$	γ	$\underline{y}(0.7, \gamma)$	$\overline{y}(0.7, \gamma)$
0.0	0.638851	1.532546	0.2	1.060230	1.060230	0.3	1.060230	1.060230
0.2	0.744196	1.414467	0.4	0.954885	1.178309	0.5	0.939836	1.195178
0.4	0.849541	1.296388	0.6	0.849541	1.296388	0.7	0.819442	1.330125
0.6	0.954885	1.178309	0.8	0.744196	1.414467	0.9	0.699048	1.465073
0.8	1.060230	1.060230	1.0	0.638851	1.532546	1.0	0.638851	1.532546

Error at $N = 150$

***Table* (13):** *The Error Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$ *at* $N = 150$.

α	$\underline{e}(0.7, \alpha)$	$\overline{e}(0.7, \alpha)$	β	$\underline{e}(0.7, \beta)$	$\overline{e}(0.7, \beta)$	γ	$\underline{e}(0.7, \gamma)$	$\overline{e}(0.7, \gamma)$
0.0	7.18 E-05	3.83 E-05	0.2	6.00 E-05	5.60 E-05	0.3	6.00 E-05	5.60 E-05
0.2	6.88 E-05	4.37 E-05	0.4	6.29 E-05	5.45 E-05	0.5	6.33 E-05	5.37 E-05
0.4	6.58 E-05	4.91 E-05	0.6	6.58 E-05	4.91 E-05	0.7	6.67 E-05	4.76 E-05
0.6	6.29 E-05	5.45 E-05	0.8	6.88 E-05	4.37 E-05	0.9	7.00 E-05	4.14 E-05
0.8	6.00 E-05	5.60 E-05	1.0	7.18 E-05	3.83 E-05	1.0	7.18 E-05	3.83 E-05

- **Case (2, 2)**

Analytical Solution

Table (14): *The Analytical Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$

α	$\underline{y}(0.7, \alpha)$	$\overline{y}(0.7, \alpha)$	β	$\underline{y}(0.7, \beta)$	$\overline{y}(0.7, \beta)$	γ	$\underline{y}(0.7, \gamma)$	$\overline{y}(0.7, \gamma)$
0.0	0.880422	1.291087	0.2	1.060290	1.060290	0.3	1.060290	1.060290
0.2	0.925389	1.233387	0.4	1.015323	1.117989	0.5	1.008899	1.126231
0.4	0.970356	1.175688	0.6	0.970356	1.175688	0.7	0.957508	1.192174
0.6	1.015323	1.117989	0.8	0.925389	1.233387	0.9	0.906117	1.258116
0.8	1.060290	1.060290	1.0	0.880422	1.291087	1.0	0.880422	1.291087

Approximation Solution at N = 200 & h = 0.005

***Table* (15):** *The Approximation Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$*at N = 200.*

α	$\underline{y}(0.7, \alpha)$	$\overline{y}(0.7, \alpha)$	β	$\underline{y}(0.7, \beta)$	$\overline{y}(0.7, \beta)$	γ	$\underline{y}(0.7, \gamma)$	$\overline{y}(0.7, \gamma)$
0.0	0.880391	1.291055	0.2	1.060256	1.060256	0.3	1.060256	1.060256
0.2	0.925357	1.233355	0.4	1.015290	1.117956	0.5	1.008866	1.126198
0.4	0.970324	1.175656	0.6	0.970324	1.175656	0.7	0.957476	1.192141
0.6	1.015290	1.117956	0.8	0.925357	1.233355	0.9	0.906086	1.258084
0.8	1.060256	1.060256	1.0	0.880391	1.291055	1.0	0.880391	1.291055

Error at $N = 200$

***Table* (16):** *The Error Results of Upper and Lower for* (α, β, γ) *at* $t = 0.7$*at N = 200.*

α	$\underline{e}(0.7, \alpha)$	$\bar{e}(0.7, \alpha)$	β	$\underline{e}(0.7, \beta)$	$\bar{e}(0.7, \beta)$	γ	$\underline{e}(0.7, \gamma)$	$\bar{e}(0.7, \gamma)$
0.0	3.06 E-05	3.14 E-05	0.2	3.37 E-05	3.37 E-05	0.3	3.37 E-05	3.37 E-05
0.2	3.14 E-05	3.20 E-05	0.4	3.29 E-05	3.31 E-05	0.5	3.28 E-05	3.30 E-05
0.4	3.21 E-05	3.25 E-05	0.6	3.21 E-05	3.25 E-05	0.7	3.19 E-05	3.23 E-05
0.6	3.29 E-05	3.31 E-05	0.8	3.14 E-05	3.20 E-05	0.9	3.10 E-05	3.17 E-05
0.8	3.37 E-05	3.37 E-05	1.0	3.06 E-05	3.14 E-05	1.0	3.06 E-05	3.14 E-05

5. Conclusion:

In this paper, the numerical solution of second order differential equation under the effect of neutrosophic environment in the boundary condition has been presented, this numerical solution by using the finite difference method after using generalized Hukuhara differentiability to view the approximate solution of each class of the four classes of the solution, then comparing it with the analytical solution of the same problem even after applying generalized Hukuhara differentiability, to find that by decreasing the step size the error decreasing and it can be classified as a good error by reaching (10E-6) at $h = 0.005$, in our next work, a new numerical progressed methods will be presented and applied for numerical application to find if it will enhance the error in such uncertain environment.

Acknowledgment

We would like to express my warm thanks to Prof. Dr. Florentine Smarandache [Department of Mathematics, University of New Mexico, USA], for helping us to understand the neutrosophic approach.

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Received: June 3, 2023. Accepted: Oct 1, 2023