NSS Neutrosophic Sets and Systems, Vol. 58, 2023

University of New Mexico

A Total Ordering on n - Valued Refined Neutrosophic Sets using Dictionary Ranking based on Total ordering on n - Valued Neutrosophic Tuplets

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Abstract. The notion of fuzzy subsets was first introduced by Zadeh in 1965, and was later extended to intuitionistic fuzzy subsets by Atanassov in 1983. Since the inception of fuzzy set theory, we have encountered a number of generalizations of sets, one of which is neutrosophic sets introduced by Smarandache [15]. Later neutrosophic sets was generalized into interval valued neutrosophic, triangular valued neutrosophic, trapezoidal valued neutrosophic and n - valued refined neutrosophic sets in the literature [19, 31, 33, 35]. Further, the ordering on single-valued neutrosophic triplets and interval valued neutrosophic triplets have been proposed by Smarandache in [16] and they are further extended to total ordering on interval valued neutrosophic triplets in [32].The total ordering of n - valued neutrosophic tuplets is very significant in multi-criteria decision making (MCDM) involving n - valued neutrosophic tuplets. Hence, in this paper, different methods for ordering n - valued neutrosophic tuplets (NVNT) are developed with the goal of achieving a total ordering on n - valued neutrosophic tuplets and the applicability of the proposed methods is shown by illustrative examples in MCDM problems involving n - valued neutrosophic tuplets. Further, a total ordering algorithm for n - valued refined neutrosophic sets by following dictionary ranking method at the final stage is developed using those proposed total ordering methods on n - valued neutrosophic tuplets.

Keywords: n - Valued Refined Neutrosophic Sets, Dictionary, Neutrosophic Tuplets, Uncertainty

1. Introduction

Our daily life is full of uncertain situations and we need to make better decisions based on their volatility. Despite this, Zadeh established the concept of fuzzy sets in 1965 to handle such ambiguity [18]. Though this idea of fuzzy sets was reluctantly acknowledged initially, researchers believed that analyzing this concept might bring a tremendous revolution in the future with real-life MCDM and MADM problems with with uncertainty or vagueness. Hence, a great progress has been made in the research of fuzzy set generalisation, resulting in numerous forms of fuzzy sets such as intuitionistic fuzzy sets, neutrosophic sets, picture fuzzy sets, bipolar fuzzy sets, and so on [3–5, 15, 20]. These versions of fuzzy sets were widely used in a variety of real-world issues. The multi-criteria decision making (MCDM) problem is a rising topic of research due to its importance in most real-world challenges [12, 14, 17, 19].

The neutrosophic sets are introduced by Florentine smarandache [15], as a generalization of intuitionistic fuzzy sets. In intuitionistic fuzzy sets, we usually consider membership, non membership values. But, in neutrosophic sets, we consider membership, non membership values and an indeterminacy value which differentiates neutrosophic sets from intuitionistic fuzzy sets. Later, n - valued redefined neutrosophic sets were introduced by Florentine smarandache as further generalization and some MCDM problems have been studied in real-world scenarios using n - valued redefined neutrosophic sets in [33]. To solve such MCDM problems, we need total ordering on n - valued neutrosophic tuplets. For each fuzzy MCDM problem, there are several total ordering on fuzzy numbers in the literature [7–9, 11]. Furthermore, the decision maker selects the total ordering strategy that best suits his needs. For a fuzzy MCDM, the total order does not have to be unique. The ranking of single valued neutrosophic triplets has been analysed in [13,16] and further extended to total ordering on interval valued neutrosophic triplets in [32].

Dictionary ordering is usually followed to rank totally the elements of $\mathbb{Y}^X$ using the total order $<$ on Y for any countable set X. In detail, to compare $(a_1, a_2, \ldots a_n, \ldots)$ and $(b_1, b_2, \ldots b_n, \ldots)$, we first compare a_1 and b_1 using total order $<$ on Y. If $a_1 < b_1$(or $b_1 < a_1$) then $(a_1, a_2, \ldots a_n, \ldots) < (b_1, b_2, \ldots b_n, \ldots)$(or $(b_1, b_2, \ldots b_n, \ldots) < (a_1, a_2, \ldots a_n, \ldots)$). If $a_1 = b_1$, then we follow the same procedure for comparing a_2 and b_2 using total order $<$ on Y and so on. The same method only indirectly followed in any ranking of fuzzy numbers, intuitionistic fuzzy numbers, single valued neutrosophic numbers and interval valued neutrosophic numbers using score functions [7, 8, 11, 16, 32].

In this paper, we aim to achieve a total ordering on n - valued neutrosophic tuplets and n - valued refined neutrosophic sets. To derive total ordering on n - valued refined neutrosophic sets, we need a total ordering on n - valued neutrosophic tuplets. First, we derive total ordering on n - valued neutrosophic tuplets for which we introduce two algorithms. In the first stage of the both algorithms, we convert n - valued neutrosophic tuplets into single valued neutrosophic triplets and then we try to rank them. In the next stage, first method follows a reverse dictionary order and second method follows a method of ranking based on the fluctuations on truth, falsity and indeterminacy values. To rank the n -valued neutrosophic sets, we develop a total ordering algorithm for n - valued refined neutrosophic sets by following dictionary ranking method at the final stage using those proposed total ordering methods on n - valued neutrosophic tuplets.

2. Preliminaries

This section contains all of the necessary definitions to move deeper into the concept of total ordering on n - valued neutrosophic tuplets.

Definition 2.1. [16] Let $\mathcal{M} = \{(T, I, F), \text{where } T, I, F \in [0,1], 0 \leq T+I+F \leq 3\}$ be the set of single valued neutrosophic triplet (SVNT) numbers. Let $N = (T, I, F) \in \mathcal{M}$ be a generic SVNT number, where T denotes grade of membership ; I denotes indeterminacy grade ; F denotes grade of non-membership.

Definition 2.2. [32] A SVNT membership score $S^+ : \mathcal{M} \to [0,1]$ is defined by

$$S^+(T, I, F) = \frac{2 + (T-F)(2-I) - I}{4}.$$

Definition 2.3. [32] A SVNT non-membership score $S^- : \mathcal{M} \to [0,1]$ is defined by

$$S^-(T, I, F) = \frac{2 + (F-T)(2-I) - I}{4}.$$

Definition 2.4. [32] A SVNT average score $C : \mathcal{M} \to [0,1]$ is defined by

$$C(T, I, F) = \frac{T+F}{2}$$

Definition 2.5. [33] Let (T, I, F) be a n - valued neutrosophic triplet number, where T can be split into many types of truths as $T_1, T_2 \dots T_p$, I can be split into many types of indeterminacies as $I_1, I_2 \dots I_q$ and F can be split into many types of falsities as $F_1, F_2 \dots F_r$ where $T_i, I_j, F_k \in [0,1]$ for $i \in \{1, \dots p\}, j \in \{1, \dots q\}$ and $k \in \{1, \dots r\}$ and $p + q + r = n$. Therefore we have $0 \leq \sum_{i=1}^{p} T_i + \sum_{j=1}^{q} I_j + \sum_{k=1}^{r} F_k \leq n$.

Definition 2.6. Let $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ and $N_2 = (T_1', T_2', \dots T_p', I_1', I_2', \dots I_q', F_1', F_2', \dots F_r')$ be two n - valued neutrosophic triplet numbers, where $p + q + r = n$. Then we define $N_1 + N_2 = (T_1 + T_1', T_2 + T_2', \dots T_p + T_p', I_1 + I_1', I_2 + I_2', \dots I_q + I_q', F_1 + F_1', F_2 + F_2', \dots F_r + F_r')$ and $\alpha N_1 = (\alpha T_1, \dots \alpha T_p, \alpha I_1, \dots \alpha I_q, \alpha F_1, \dots \alpha F_r)$ where $\alpha \in \mathbb{R}$.

3. A Total order on n-Valued Neutrosophic tuplets

In this section, we present a ranking technique for n - valued neutrosophic tuplets that inherits total ordering.

3.1. *Ranking algorithm for n - valued neutrosophic tuplets*

Let $A = (T, I, F)$ and $B = (T', I', F')$ be two n - valued neutrosophic tuplets such that $A \neq B$, where T can be split into many types of truths in ascending order as $T_1, T_2 \ldots T_{p_1}$, I can be split into many types of indeterminacies in ascending order as $I_1, I_2 \ldots I_{q_1}$ and F can be split into many types of falsities in ascending order as $F_1, F_2 \ldots F_{r_1}$ where $T_i, I_j, F_k \in [0,1]$ and $p_1+q_1+r_1 = n$. Therefore we have $0 \leq \sum_{i=1}^{p_1} T_i+\sum_{j=1}^{q_1} I_j+\sum_{k=1}^{r_1} F_k \leq n$. Similarly T' can be split into many types of truths in ascending order as $T'_1, T'_2 \ldots T'_{p_2}$, I' can be split into many types of indeterminacies in ascending order as $I'_1, I'_2 \ldots I'_{q_2}$ and F' can be split into many types of falsities in ascending order as $F'_1, F'_2 \ldots F'_{r_2}$ where $T'_i, I'_j, F'_k \in [0,1]$ and $p_2 + q_2 + r_2 = n$. Therefore we have $0 \leq \sum_{i=1}^{p_2} T_i + \sum_{j=1}^{q_2} I_j + \sum_{k=1}^{r_2} F_k \leq n$.

Step 1: Choose $k = lcm\{p_1, p_2, q_1, q_2, r_1, r_2\}$. Now convert both n - valued neutrosophic tuplets A and B by rewriting T, I, F and T', I', F' as follows;

Suppose $k = x_1p_1$ then $T = (\underbrace{T_1, \ldots T_1}_{x_1 \text{ times}}, \underbrace{T_2, \ldots T_2}_{x_1 \text{ times}}, \ldots \underbrace{T_{p_1}, \ldots T_{p_1}}_{x_1 \text{ times}}) = (T_1, T_2, \ldots T_k)$

Suppose $k = y_1q_1$ then $I = (\underbrace{I_1, \ldots I_1}_{y_1 \text{ times}}, \underbrace{I_2, \ldots I_2}_{y_1 \text{ times}}, \ldots \underbrace{I_{q_1}, \ldots I_{q_1}}_{y_1 \text{ times}}) = (I_1, I_2, \ldots I_k)$

Suppose $k = z_1r_1$ then $F = (\underbrace{F_1, \ldots F_1}_{z_1 \text{ times}}, \underbrace{F_2, \ldots F_2}_{z_1 \text{ times}}, \ldots \underbrace{F_{r_1}, \ldots F_{r_1}}_{z_1 \text{ times}}) = (F_1, F_2, \ldots F_k)$

Suppose $k = x_2p_2$ then $T' = (\underbrace{T'_1, \ldots T'_1}_{x_2 \text{ times}}, \underbrace{T'_2, \ldots T'_2}_{x_2 \text{ times}}, \ldots \underbrace{T'_{p_2}, \ldots T_{p_2}}_{x_2 \text{ times}}) = (T'_1, T'_2, \ldots T'_k)$

Suppose $k = y_2q_2$ then $I' = (\underbrace{I'_1, \ldots I'_1}_{y_2 \text{ times}}, \underbrace{I'_2, \ldots I'_2}_{y_2 \text{ times}}, \ldots \underbrace{I'_{q_2}, \ldots I'_{q_2}}_{y_2 \text{ times}}) = (I'_1, I'_2, \ldots I'_k)$

Suppose $k = z_2r_2$ then $F = (\underbrace{F'_1, \ldots F'_1}_{z_2 \text{ times}}, \underbrace{F'_2, \ldots F'_2}_{z_2 \text{ times}}, \ldots \underbrace{F'_{r_2}, \ldots F'_{r_2}}_{z_2 \text{ times}}) = (F'_1, F'_2, \ldots F'_k)$

Now we have $A = (T_1, \ldots T_k, I_1, \ldots I_k, F_1, \ldots F_k)$ and $B = (T'_1, \ldots T'_k, I'_1, \ldots I'_k, F'_1, \ldots F'_k)$ as a $3k$ valued neutrosophic tuplets where truth, falsity and indeterminacy values are k tuple.

Let $(T_0, I_0, F_0) = (\frac{\sum_{i=1}^{k} T_i}{k}, \frac{\sum_{i=1}^{k} I_i}{k}, \frac{\sum_{i=1}^{k} F_i}{k})$ and $(T'_0, I'_0, F'_0) = (\frac{\sum_{i=1}^{k} T'_i}{k}, \frac{\sum_{i=1}^{k} I'_i}{k}, \frac{\sum_{i=1}^{k} F'_i}{k})$.

Step 2: We compare (T_0, I_0, F_0) and (T'_0, I'_0, F'_0) using score functions. Apply neutrosophic membership score function S^+. Suppose $S^+(T_0, I_0, F_0) > S^+(T'_0, I'_0, F'_0)$, then we have $A > B$. Suppose $S^+(T_0, I_0, F_0) < S^+(T'_0, I'_0, F'_0)$, then we have $A < B$. Suppose $S^+(T_0, I_0, F_0) = S^+(T'_0, I'_0, F'_0)$, go to step 3.

Step 3: Apply neutrosophic non-membership score function S^-. Suppose $S^-(T_0, I_0, F_0) > S^-(T'_0, I'_0, F'_0)$, then we have $A < B$. Suppose $S^-(T_0, I_0, F_0) < S^-(T'_0, I'_0, F'_0)$, then we have $A > B$. Suppose $S^-(T_0, I_0, F_0) = S^-(T'_0, I'_0, F'_0)$, go to step 4.

Step 4: Apply neutrosophic average function C. Suppose $C(T_0, I_0, F_0) > C(T'_0, I'_0, F'_0)$, then we have $A > B$. Suppose $C(T_0, I_0, F_0) < C(T'_0, I'_0, F'_0)$, then we have $A < B$. Suppose $C(T_0, I_0, F_0) = C(T'_0, I'_0, F'_0)$, then go to step 5.

Step 5: Now we compare (T_m, I_m, F_m) and (T'_m, I'_m, F'_m) for $m = k$ by considering

$(T_0, I_0, F_0) = (T_m, I_m, F_m)$ and $(T'_0, I'_0, F'_0) = (T'_m, I'_m, F'_m)$ using steps 2, 3 and 4. If we are not still able to differentiate A and B, then we compare for $m = m-1$ by applying step 5 till ranking A and B.

Theorem 3.1. *Proposed ranking algorithm inherits a total order on set of all n - valued neutrosophic tuplets.*

Proof. We show that for any two n - valued neutrosophic sets (T, I, F) and (T', I', F'), either $(T, I, F) < (T', I', F')$ or $(T, I, F) > (T', I', F')$ or $(T, I, F) = (T', I', F')$. Let $A = (T, I, F) = (T_1, T_2 \ldots T_{p_1}, I_1, I_2 \ldots I_{q_1}, F_1, F_2 \ldots F_{r_1})$ and $B = (T', I', F') = (T'_1, T'_2 \ldots T'_{p_2}, I'_1,$ $I'_2 \ldots I'_{q_2}, F'_1, F'_2 \ldots F'_{r_2})$ be two n - valued neutrosophic tuplets such that $A \neq B$, where $p_1 + q_1 + r_1 = p_2 + q_2 + r_2 = n$. Now we show that either $A < B$ or $B < A$. By applying step 1, we have $A = (T_1, \ldots T_k, I_1, \ldots I_k, F_1, \ldots F_k,)$ and $B = (T'_1, \ldots T'_k, I'_1, \ldots I'_k, F'_1, \ldots F'_k,)$ where $k = lcm\{p_1, q_1, r_1, p_2, q_2, r_2\}$.

Now,
let $(T_0, I_0, F_0) = (\sum_{i=1}^{k} T_i, \sum_{i=1}^{k} I_i, \sum_{i=1}^{k} F_i)$ and $(T'_0, I'_0, F'_0) = (\sum_{i=1}^{k} T'_i, \sum_{i=1}^{k} I'_i, \sum_{i=1}^{k} F'_i)$. First we apply membership score function S^+. Suppose $S^+\ (T_0, I_0, F_0) > S^+\ (T'_0, I'_0, F'_0)$ (or $S^+(T_0, I_0, F_0) < S^+(T'_0, I'_0, F'_0)$, then we have $A > B$(or $A < B$), which is done. When $S^+\ (T_0, I_0, F_0) = S^+\ (T'_0, I'_0, F'_0)$, we have to go to next step. So, suppose $\frac{2+(T_0-F_0)(2-I_0)-I_0}{4} = \frac{2+(T_0-F_0)(2-I_0)-I_0}{4}$, equivalently, if $(T_0 - F_0)(2 - I_0) - I_0 = (T'_0 - F'_0)(2 - I'_0) - I'_0$, we apply non-membership score function. Hence, if $S^-\ (T_0, I_0, F_0) > S^-\ (T'_0, I'_0, F'_0)$ $(S^-\ (T_0, I_0, F_0) < S^-\ (T'_0, I'_0, F'_0))$, then $A < B (A > B)$, which is done. When $S^-\ (T_0, I_0, F_0) = S^-\ (T'_0, I'_0, F'_0)$, equivalently, if $(F_0 - T_0)(2 - I_0) - I_0 = (F'_0 - T'_0)(2 - I'_0) - I'_0$, we have to go to next step by using average score function. Hence, suppose $C\ (T_0, I_0, F_0) > C\ (T'_0, I'_0, F'_0)$ (or $C\ (T_0, I_0, F_0)$ $< C\ (T'_0, I'_0, F'_0)$), then we have $A > B$(or $A < B$), which is done. When $C\ (T_0, I_0, F_0) = C\ (T'_0, I'_0, F'_0)$, we have $T_0 + F_0 = T'_0 + F'_0$. At this stage, we have triplets (T_0, I_0, F_0) and (T'_0, I'_0, F'_0) satisfying following system of 3 equations.

$$(T_0 - F_0)(2 - I_0) - I_0 = (T'_0 - F'_0)(2 - I'_0) - I'_0 \tag{1}$$

$$(F_0 - T_0)(2 - I_0) - I_0 = (F'_0 - T'_0)(2 - I'_0) - I'_0 \tag{2}$$

$$T_0 + F_0 = T'_0 + F'_0 \tag{3}$$

Now, we solve this system of equations. By adding equations 1 and 2, we get $I_0 = I'_0$ which makes equation 1 into

$$T_0 - F_0 = T'_0 - F'_0$$

now, by adding the above equation with equation 3, we get $F_0 = F_0'$ and $T_0 = T_0'$.
Thus, we get

$$(T_0, I_0, F_0) = (T_0', I_0', F_0').$$

As a result, we have

$$T_1 + T_2 + \dots T_k = T_1' + T_2' + \dots T_k' \tag{4}$$

$$I_1 + I_2 + \dots I_k = I_1' + I_2' + \dots I_k' \tag{5}$$

$$F_1 + F_2 + \dots F_k = F_1' + F_2' + \dots F_k' \tag{6}$$

Let us compare (T_k, I_k, F_k) and (T_k', I_k', F_k'). First we apply membership score function S^+. Suppose S^+ $(T_k, I_k, F_k) > S^+$ (T_k', I_k', F_k')(or $S^+(T_k, I_k, F_k) < S^+(T_k', I_k', F_k')$, then we have $A > B$(or $A < B$), which is done. When S^+ $(T_k, I_k, F_k) = S^+$ (T_k', I_k', F_k'), we have to go to next step. So, suppose $\frac{2+(T_k-F_k)(2-I_k)-I_k}{4} = \frac{2+(T_k'-F_k')(2-I_k')-I_k'}{4}$, equivalently, if $(T_k - F_k)(2 - I_k) - I_k = (T_k' - F_k')(2 - I_k) - I_k$, we apply non-membership score function. Hence, if S^- $(T_k, I_k, F_k) > S^-$ (T_k', I_k', F_k') (S^- $(T_k, I_k, F_k) < S^-$ (T_k', I_k', F_k'), then $A < B(A > B)$, which is done. When S^- $(T_k, I_k, F_k) = S^-$ $T_k', I_k', F_k')$, equivalently, if $(F_k - T_k)(2 - I_k) - I_k = (F_k' - T_k')(2 - I_k') - I_k'$, we have to go to average score function. Hence, suppose C (T_k, I_k, F_k) $> C$ (T_k', I_k', F_k') (or C $(T_k, I_k, F_k) < C$ $(T_k', I_k', F_k'))$, then we have $A > B$ (or $A < B$), which is done. When C $(T_k, I_k, F_k) = C$ (T_k', I_k', F_k'), we have $T_k + F_k = T_k' + F_k'$. At this stage, we have triplets (T_k, I_k, F_k) and (T_k', I_k', F_k') satisfying following system of 3 equations.

$$(T_k - F_k)(2 - I_k) - I_k = (T_k' - F_k')(2 - I_k') - I_k' \tag{7}$$

$$(F_k - T_k)(2 - I_k) - I_k = (F_k' - T_k)(2 - I_k) - I_k \tag{8}$$

$$T_k + F_k = T_k' + F_k' \tag{9}$$

Now, we solve this system of equations. By adding equations 7 and 8, we get $I_k = I_k'$ which makes equation 7 into

$$T_k - F_k = T_k' - F_k'$$

now, by adding the above equation with equation 9, we get $F_k = F_k'$ and $T_k = T_k'$.
Thus, we get

$$(T_k, I_k, F_k) = (T_k', I_k', F_k').$$

Similarly, by continuing the above process for $m = k - 1, \dots 2, 1$, till we get $A < B$ or $B < A$. If we have $(T_m, I_m, F_m) = (T_m', I_m', F_m')$ for $m = \{k, k - 1, k - 2 \dots 2, 1\}$. By solving with equations 3.7, 3.8, and 3.9, we get $A = B$, a contradiction. Thus we have proved the proposed ranking algorithm inherits a total order on set of all n - valued neutrosophic tuplets. □

The following statement's proofs are direct applications of definitions, hence proofs are omitted.

Proposition 3.2. *Let* $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ *and* $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ *be two* n *- valued neutrosophic tuplets, where* $p + q + r = n$.

(1) *If* $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n$ *and* $\sum_{n=1}^{q} I_n > \sum_{n=1}^{q} I'_n$*, then we get* $N_1 < N_2$.

(2) *If* $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n$ *and* $\sum_{n=1}^{q_1} I_n < \sum_{n=1}^{q_2} I'_n$*, then* $N_1 > N_2$.

Proposition 3.3. *Let* $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ *and* $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ *be two* n *- valued neutrosophic tuplets, where* $p + q + r = n$.

(1) *If* $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{q} I_n = \sum_{n=1}^{q} I'_n$ *and* $\sum_{n=1}^{r} F_n > \sum_{n=1}^{r} F'_n$*, then we get* $N_1 < N_2$.

(2) *If* $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{q_1} I_n = \sum_{n=1}^{q_2} I'_n$ *and* $\sum_{n=1}^{r} F_n < \sum_{n=1}^{r} F'_n$*, then we get* $N_1 > N_2$.

Proposition 3.4. *Let* $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ *and* $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ *be two* n *- valued neutrosophic tuplets, where* $p + q + r = n$.

(1) *If* $\sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n, \sum_{n=1}^{q} I_n = \sum_{n=1}^{q} I'_n$ *and* $\sum_{n=1}^{p} T_n > \sum_{n=1}^{p} T'_n$*, then we get* $N_1 > N_2$.

(2) *If* $\sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n, \sum_{n=1}^{q} I_n = \sum_{n=1}^{q} I'_n$ *and* $\sum_{n=1}^{p} T_n < \sum_{n=1}^{p} T'_n$*, then we get* $N_1 < N_2$.

Remark 3.5. Let $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ and $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ be two n - valued neutrosophic tuplets, where $p + q + r = n$. We suppose that $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n$, $\sum_{n=1}^{q} I_n = 0$ and $\sum_{n=1}^{q} I'_n = q$ in which collective membership and non membership grades of N_1 and N_2 are equal, whereas N_1 has no indeterminacy and N_2 has full indeterminacy. Then we get $N_1 > N_2$ which favours our intuition.

Remark 3.6. Let $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ and $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ be two n - valued neutrosophic tuplets, where $p + q + r = n$. We suppose that $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{q} I_n = \sum_{n=1}^{q} I'_n$, $\sum_{n=1}^{r} F_n = 0$ and $\sum_{n=1}^{r} F'_n = r$ in which collective membership and indeterminacy grades of N_1 and N_2 are equal, whereas N_1 has no non membership grade and N_2 has full non membership grade. Then we get $N_1 > N_2$ which favours our intuition.

Remark 3.7. Let $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ and $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ be two n - valued neutrosophic tuplets, where $p + q + r = n$. We suppose that $\sum_{n=1}^{q} I_n = \sum_{n=1}^{q} I'_n, \sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n$, $\sum_{n=1}^{p} T_n = 0$ and $\sum_{n=1}^{p} T'_n = p$ in which collective hesitancy and non membership grades of N_1 and N_2 are equal, whereas N_1 has no membership grade and N_2 has full membership grade. Then we get $N_1 < N_2$ which favours our intuition.

Remark 3.8. We know that n-valued neutrosophic tuplets are generalization of single valued neutrosophic triplets and hence we can apply our ranking method also to them which will be a total ordering on single valued neutrosophic triplets.

4. Numerical examples

Let us consider the following example as a brief example for the proposed total ordering algorithm. Assume that $(T; I; F) = (0.8, 0.7, 0.9; 0.4; 0.2, 0.7, 0.6)$ and $(T'; I'; F') = (0.9, 0.7; 0.2, 0.8; 0.2, 0.4, 0.6)$ be 7 - valued neutrosophic tuplets.

First we rearrange these two 7 - valued neutrosophic tuplets in ascending order as follows $(T; I; F) = (0.7, 0.8, 0.9; 0.4; 0.2, 0.6, 0.7)$ and $(T'; I'; F') = (0.7, 0.9; 0.2, 0.8; 0.2, 0.4, 0.6)$

Now $k = lcm\{3, 1, 3, 2, 2, 3\} = 6$. Since T has 3 elements and $k = 6 = 2(3)$, we rewrite $T = (0.7, 0.8, 0.9)$ as $T = (0.7, 0.7, 0.8, 0.8, 0.9, 0.9)$. In similar manner, we rewrite I, F, T', I', F' as follows;

$I = (0.4, 0.4, 0.4, 0.4, 0.4, 0.4), F = (0.2, 0.2, 0.6, 0.6, 0.7, 0.7), T' = (0.7, 0.7, 0.7, 0.9, 0.9, 0.9)$, $I' = (0.2, 0.2, 0.2, 0.6, 0.6, 0.6)$. Now we take $(a, b, c) = (\frac{\sum_{i=1}^{6} T_i}{6}, \frac{\sum_{i=1}^{6} I_i}{6}, \frac{\sum_{i=1}^{6} F_i}{6})$ and $(d, e, f) = (\frac{\sum_{i=1}^{6} T'_i}{6}, \frac{\sum_{i=1}^{6} I'_i}{6}, \frac{\sum_{i=1}^{6} F'_i}{6})$, which implies $(a, b, c) = (0.8, 0.4, 0.5)$ and $(d, e, f) = (0.8, 0.4, 0.5)$. By applying steps 2, 3 and 4, we cannot rank (T, I, F) and (T', I', F'). Now we go to step 5. In step 5, we take (T_6, I_6, F_6) and (T'_6, I'_6, F'_6). Since $S^+(0.9, 0.4, 0.7) = 0.48 > 0.3 = S^+(0.9, 0.8, 0.9)$, we get the ranking as $(T, I, F) > (T', I', F')$.

5. Application to MCDM Problem

Consider the following MCDM problem based on 6 - valued neutrosophic numbers. Now we have to find the ranking between the alternatives A_1, A_2, A_3, A_4 with respect to criteria C_1, C_2, C_3. The ratings of the alternatives with respect to the criteria are given in the form of6 - valued neutrosophic number as shown in table 1. We are given that the respective weights of the criteria C_1, C_2, C_3 are $0.3, 0.3, 0.4$.

Now we rearrange the truth, indeterminacy, and false membership grades in the table 1 as ascending order which is given in table 2. Next we multiply corresponding weights of the criteria into the decision table which results table 3.

	C_1 (Criteria 1)	C_2 (Criteria 2)	C_3 (Criteria 3)
A_1 (Alternative 1)	(0.3;0.6,0.2;0.1,0.5,0.2)	(0.4,0.2;0.3;0.7,0.8,0.3)	(0.6,0.5;0.1,0.6;0.2,0.4)
A_2 (Alternative 2)	(0.9,0.7,0.8;0,0.2;0.2)	(0.5,0.1;0.2,0.6;0.4,0.5)	(0.1,0.4;0.5,0.6,0.1;0.5)
A_3 (Alternative 3)	(0.3,0.6,0.4;0.2;0.1,0.4)	(0.7,0.1;0.1,0.2;0.4,0.8)	(0.3,0.5,0.7;0.2,0.5;0.2)
A_4 (Alternative 4)	(0.7,0.3,0.2;0.2,0.3;0.1)	(0.5,0.3.0.1;0.2,0.6;0.4)	(0.5,0.9;0.6,1,0.1;0.6)

Table 1. MCDM decision matrix

	C_1 (Criteria 1)	C_2 (Criteria 2)	C_3 (Criteria 3)
A_1 (Alternative 1)	(0.3;0.2,0.6;0.1,0.2,0.5)	(0.2,0.4;0.3;0.3,0.7,0.8)	(0.5,0.6;0.1,0.6;0.2,0.4)
A_2 (Alternative 2)	(0.7,0.8,0.9;0,0.2;0.2)	(0.1,0.5;0.2,0.6;0.4,0.5)	(0.1,0.4;0.1,0.5,0.6;0.5)
A_3 (Alternative 3)	(0.3,0.4,0.6;0.2;0.1,0.4)	(0.1,0.7;0.1,0.2;0.4,0.8)	(0.3,0.5,0.7;0.2,0.5;0.2)
A_4 (Alternative 4)	(0.2,0.3,0.7;0.2,0.3;0.1)	(0.1,0.3.0.5;0.2,0.6;0.4)	(0.5,0.9;0.1,0.6,1;0.6)

Table 2. MCDM decision matrix in an rearranged form

	C_1 (Criteria 1)	C_2 (Criteria 2)	C_3 (Criteria 3)
A_1 (Alternative 1)	(.09;.06,.18;.03,.06,.15)	(.06,.12;.09;.09,.21,.24)	(.2,.24;.04,.26;.08,.16)
A_2 (Alternative 2)	(.21,.24,.27;0,.06;.06)	(.03,.15;.06,.18;.12,.15)	(.04,.16;.04,.2,.24;.2)
A_3 (Alternative 3)	(.09,.12,.18;.06;.03,.12)	(.03,.21;.03,.06;.12,.24)	(.12,.2,.28;.08,.2;.08)
A_4 (Alternative 4)	(.06,.09,.21;.06,.09;.03)	(.03,.09,.15;.06,.18;.12)	(.2,.36;.04,.24,.4;.18)

Table 3. Weighted MCDM decision matrix

From table 3, we find $k = lcm\{1, 2, 3\} = 6$. Thus we rewrite each entries of MCDM table as follows

$A_1C_1 = (.09, .09, .09, .09, .09, .09; .06, .06, .06, .18, .18, .18; .03, .03, .06, .06, .15, .15)$

$A_1C_2 = (.2, .2, .2, .24, .24, .24; .04, .04, .04, .26, .26, .26; .08, .08, .08, .16, .16, .16)$

$A_1C_3 = (.06, .06, .06, .12, .12, .12; .09, .09, .09, .09, .09, .09; .09, .09, .21, .21, .24, .24)$

$A_2C_1 = (.21, .21, .24, .24, .27, .27; 0, 0, 0, .3, .3, .3; .06, .06, .06, .06, .06, .06)$

$A_2C_2 = (.03, .03, .03, .15, .15, .15; .06, .06, .06, .18, .18, .18; .12, .12, .12, .15, .15, .15)$

$A_2C_3 = (.04, .04, .04, .16, .16, .16; .04, .04, .2, .2, .24, .24; .2, .2, .2, .2, .2, .2)$

$A_3C_1 = (.09, .09, .12, .12, .18, .18; .06, .06, .06, .06, .06, .06; .03, .03, .03, .12, .12, .12)$

$A_3C_2 = (.03, .03, .03, .21, .21, .21; .03, .03, .03, .06, .06, .06; .12, .12, .12, .24, .24, .24)$

$A_3C_3 = (.12, .12, .2, .2, .28, .28; .08, .08, .08, .2, .2, .2; .08, .08, .08, .08, .08, .08)$

$A_4C_1 = (.06, .06, .09, .09, .21, .21; .06, .06, .06, .09, .09, .09; .03, .03, .03, .03, .03, .03)$

$A_4C_2 = (.03, .03, .09, .09, .15, .15; .06, .06, .06, .18, .18, .18; .12, .12, .12, .12, .12, .12)$

$A_4C_3 = (.2, .2, .2, .36, .36, .36; .04, .04, .24, .24, .4, .4; .18, .18, .18, .18, .18, .18)$. Now we have the following weighted arithmetic neutrosophic scores for each A_i as $A_i = A_iC_1 + A_iC_2 + A_iC_3, i =$

1 to 4. $A_1 = (.35, .35, .35, .45, .45, .45; .19, .19, .19, .53, .53, .53; .21, .21, .36, .36, .54, .54)$
$A_2 = .27, .27, .3, .54, .57, .57; .09, .09, .27, .69, .72, .72; .4, .4, .4, .42, .42, .42)$
$A_3 = (.24, .24, .36, .54, .66, .66; .18, .18, .18, .33, .33, .33; .24, .24, .24, .45, .45, .45)$
$A_4 = (.3, .3, .4, .54, .72, .72; .15, .15, .36, .5, .66, .66; .33, .33, .33, .33, .33, .33)$
Now, by applying proposed ranking algorithm, we have $(a_1, b_1, c_1) = (\frac{.35+.35+.35+.45+.45+.45}{6}, \frac{.19+.19+.19+.53+.53+.53}{6}, \frac{21+.21+.36+.36+.54+.54}{6}) = (0.4, 0.36, 0.37)$. Similarly, we find $(a_2, b_2, c_2) = (0.42, 0.43, 0.37), (a_3, b_3, c_3) = (0.45, 0.26, 0.35)$ and $(a_4, b_4, c_4) = (0.5, 0.41, 0.33)$. Now $S^+(a_1, b_1, c_1) = 0.422, S^+(a_2, b_2, c_2) = 0.412, S^+(a_3, b_3, c_3) = 0.479$ and $S^+(a_4, b_4, c_4) = 0.465$. Therefore, we get the ranking as $A_3 > A_4 > A_1 > A_2$.

5.1. *Limitations of the proposed method*

In the proposed ranking method, summation of the collective membership, non membership, indeterminacy grades are first taken into account and then highest to lowest membership, non membership, indeterminacy grades are used to rank in the next stages. In some cases, when there is a fluctuation between membership, non membership, and indeterminacy grades, the proposed ranking method may rank differently to intuition of some decision maker. For example, take the following two 7 - valued neutrosophic tuplets $A = (0.3, 0.34, 0.36, 0.6; 0.15, 0.25; 0.3), B = (0.4; 0.15, 0.25; 0.15, 0.25, 0.35, 0.45)$.
Now, we rewrite $A = (0.3, 0.34, 0.36, 0.6; 0.15, 0.15, 0.25, 0.25; 0.3, 0.3, 0.3, 0.3), B = (0.4, 0.4, 0.4, 0.4; 0.15, 0.15, 0.25, 0.25; 0.15, 0.25, 0.35, 0.45)$. Then $(a, b, c) = (\frac{\sum_{i=1}^{k} T_i}{k}, \frac{\sum_{i=1}^{k} I_i}{k}, \frac{\sum_{i=1}^{k} F_i}{k}) = (0.4, 0.2, 0.3)$ and $(d, e, f) = (\frac{\sum_{i=1}^{k} T'_i}{k}, \frac{\sum_{i=1}^{k} I'_i}{k}, \frac{\sum_{i=1}^{k} F'_i}{k}) = (0.4, 0.2, 0.3)$. Therefore, we go to next step, which implies $S^+(T_4, I_4, F_4) = S^+(0.6, 0.25, 0.3) = 0.569 > 0.416 = S^+(0.4, 0.25, 0.45) = S^+(T'_4, I'_4, F'_4)$. Thus, we get the ranking as $A > B$.
But, we have the membership value as a single element 0.4 in B where as there is a fluctuation between membership grades in A and three of them are lesser than the membership grade of B. But, non membership and hesitancy information are same for A and B. Since there is more fluctuation in A, we expect the ranking as $A < B$ intuitively. To overcome this, we have given the improved ranking algorithm in the next section.

6. Improved ranking algorithm for n - valued neutrosophic tuplets

In this section, we present an improved ranking algorithm for n-valued neutrosophic tuplets that inherits total ordering.

Let $A = (T, I, F) = (T_1, T_2 \dots T_{p_1}, I_1, I_2 \dots I_{q_1}, F_1, F_2 \dots F_{r_1})$ and $B = (T', I', F') = (T'_1, T'_2 \dots T'_{p_2}, I'_1, I'_2 \dots I'_{q_2}, F'_1, F'_2 \dots F'_{r_2})$ be two n - valued neutrosophic tuplets such that $A \neq B$, where $p_1 + q_1 + r_1 = p_2 + q_2 + r_2 = n$.

Step 1: We follow step 1 to step 4 in the previous ranking algorithm in section 3.1. If A and

B are not ranked at this stage and if step 4 fails to rank, then we go to step 2.

Step 2: Now let neutrosophic triplets $(a_m, b_m, c_m) = (T_m - (\frac{\sum_{i=1}^{m-1} T_i}{m-1}), I_m - (\frac{\sum_{i=1}^{m-1} I_i}{m-1}), F_m - (\frac{\sum_{i=1}^{m-1} F_i}{m-1}))$ and $(d_m, e_m, f_m) = (T'_m - (\frac{\sum_{i=1}^{m-1} T'_i}{m-1}), I'_m - (\frac{\sum_{i=1}^{m-1} I'_i}{m-1}), F'_m - (\frac{\sum_{i=1}^{m-1} F'_i}{m-1}))$. For $m = k$, by applying step 2, 3, and 4 of proposed algorithm in 3.1by considering $(T_0, I_0, F_0) = (a_m, b_m, c_m)$, we will have either $(a_m, b_m, c_m) < (d_m, e_m, f_m)$ or $(d_m, e_m, f_m) < (a_m, b_m, c_m)$ and hence either $A < B$ or $B < A$. If step 4 fails to rank, then we go to step 3.

Step 3: By successive application of step 2 for $m = m - 1$, we will have either $A < B$ or $B < A$.

Theorem 6.1. *Proposed ranking algorithm inherits a total order on set of all n valued neutrosophic tuplets.*

Proof. Let $A = (T, I, F) = (T_1, T_2 \dots T_{p_1}, I_1, I_2 \dots I_{q_1}, F_1, F_2 \dots F_{r_1})$ and $B = (T', I', F') = (T'_1, T'_2 \dots T'_{p_2}, I'_1, I'_2 \dots I'_{q_2}, F'_1, F'_2 \dots F'_{r_2})$ be two n - valued neutrosophic tuplets such that $A \neq B$, where $p_1 + q_1 + r_1 = p_2 + q_2 + r_2 = n$. Now we show that either $A < B$ or $B < A$.

By applying step 1 in the previous ranking algorithm in section 3.1, we have $A = (T_1, \dots T_k, I_1, \dots I_k, F_1, \dots F_k,)$ and $B = (T'_1, \dots T'_k, I'_1, \dots I'_k, F'_1, \dots F'_k,)$ where $k = lcm\{p_1, q_1, r_1, p_2, q_2, r_2\}$.

By applying step 2 to step 4 in the previous ranking algorithm in section 3.1, we get either $A < B$ or $B < A$. If A and B are not ranked at this stage and if step 4 fails to rank, then we go to step 2 of the improved ranking algorithm.

$$T_1 + T_2 + \dots T_k = T'_1 + T'_2 + \dots T'_k \tag{10}$$

$$I_1 + I_2 + \dots I_k = I'_1 + I'_2 + \dots I'_k \tag{11}$$

$$F_1 + F_2 + \dots F_k = F'_1 + F'_2 + \dots F'_k \tag{12}$$

We apply step 2 of proposed algorithm for $m = k$ by letting neutrosophic triplets $(a_m, b_m, c_m) = (T_m - (\frac{\sum_{i=1}^{m-1} T_i}{m-1}), I_m - (\frac{\sum_{i=1}^{m-1} I_i}{m-1}), F_m - (\frac{\sum_{i=1}^{m-1} F_i}{m-1}))$ and $(d_m, e_m, f_m) = (T'_m - (\frac{\sum_{i=1}^{m-1} T'_i}{m-1}), I'_m - (\frac{\sum_{i=1}^{m-1} I'_i}{m-1}), F'_m - (\frac{\sum_{i=1}^{m-1} F'_i}{m-1}))$. we will have either $(a_m, b_m, c_m) < (d_m, e_m, f_m)$ or $(d_m, e_m, f_m) < (a_m, b_m, c_m)$ and hence either $A < B$ or $B < A$. Otherwise, we go to step 3. At this stage, we have $(a_m, b_m, c_m) = (d_m, e_m, f_m)$ and hence,

$$T_k - (\frac{\sum_{i=1}^{k-1} T_i}{k-1}) = T'_k - (\frac{\sum_{i=1}^{k-1} T'_i}{k-1}) \tag{13}$$

$$I_k - (\frac{\sum_{i=1}^{k-1} I_i}{k-1}) = I'_k - (\frac{\sum_{i=1}^{k-1} I'_i}{k-1}) \tag{14}$$

$$F_k - (\frac{\sum_{i=1}^{k-1} F_i}{k-1}) = F'_k - (\frac{\sum_{i=1}^{k-1} F'_i}{k-1}) \tag{15}$$

From equations 10, 11, 12, 13, 14 and 15, we get $T_k = T'_k, I_k = I'_k$ and $F_k = F'_k$.

Now we apply step 3 of proposed improved algorithm for $m = m - 1$. So we have to apply step 2 by considering $m = k - 1$ and hence we get either $A < B$ or $B < A$. Otherwise, we go to step 3. At this stage, we have $(a_m, b_m, c_m) = (d_m, e_m, f_m)$ for $m = k - 1$ and hence at this stage, we have $(T_{k-1} - (\frac{\sum_{i=1}^{k-2} T_i}{k-2}), I_{k-1} - (\frac{\sum_{i=1}^{k-2} I_i}{k-2}), F_{k-1} - (\frac{\sum_{i=1}^{k-2} F_i}{k-2}))$ and $(T'_{k-1} - (\frac{\sum_{i=1}^{k-2} T'_i}{k-2}), I'_{k-1} - (\frac{\sum_{i=1}^{k-2} I'_i}{k-2}), F'_{k-1} - (\frac{\sum_{i=1}^{k-2} F'_i}{k-2}))$, then continue the same process. As a result, we get $T_{k-1} = T'_{k-1}, I_{k-1} = I'_{k-1}$ and $F_{k-1} = F'_{k-1}$. By repeating step 3 for $m = m - 1$ again and again, we will have either $A < B$ or $B < A$ or otherwise $T_k = T'_k, I_k = I'_k$ and $F_k = F'_k$, for every $m = k, k - 1, \ldots, 1$, a contradiction to $A \neq B$. Thus we have shown that proposed improved ordering algorithm is a total order on n - valued neutrosophic tuplets. □

Remark 6.2. For example, take the following two 7 - valuedneutrosophic tuplets $A = (0.3, 0.34, 0.36, 0.6; 0.15, 0.25; 0.3), B = (0.4; 0.15, 0.25; 0.15, 0.25, 0.35, 0.45)$.

Now this can be rewritten as $A = (0.3, 0.34, 0.36, 0.6; 0.15, 0.15, 0.25, 0.25; 0.3, 0.3, 0.3, 0.3), B = (0.4, 0.4, 0.4, 0.4; 0.15, 0.15, 0.25, 0.25; 0.15, 0.25, 0.35, 0.45)$. Then $(T_0, I_0, F_0) = (\frac{\sum_{i=1}^{k} T_i}{k}, \frac{\sum_{i=1}^{k} I_i}{k}, \frac{\sum_{i=1}^{k} F_i}{k}) = (0.4, 0.2, 0.3)$ and $(T'_0, I'_0, F'_0) = (\frac{\sum_{i=1}^{k} T'_i}{k}, \frac{\sum_{i=1}^{k} I'_i}{k}, \frac{\sum_{i=1}^{k} F'_i}{k}) = (0.4, 0.2, 0.3)$. Therefore we go to step 2 of the improved algorithm. So we have to apply step 2, followed by step 3 and step 4 of algorithm in section 3.1 if needed by letting $(a_4, b_4, c_4) = (T_4 - \frac{T_1+T_2+T_3}{3}, I_4 - \frac{I_1+I_2+I_3}{3}, F_4 - \frac{F_1+F_2+F_3}{3}$ and $(d_m, e_m, f_m) = (T'_4 - \frac{T'_1+T'_2+T'_3}{3}, I'_4 - \frac{I'_1+I'_2+I'_3}{3}, F'_4 - \frac{F'_1+F'_2+F'_3}{3})$.

Now by step 2 of algorithm in section 3.1, $S^+(T_4 - \frac{T_1+T_2+T_3}{3}, I_4 - \frac{I_1+I_2+I_3}{3}, F_4 - \frac{F_1+F_2+F_3}{3}) = S^+(0.27, 0.07, 0) = 0.61 > 0.39 = S^+(0, 0.07, 0.2) = S^+(T'_4 - \frac{T'_1+T'_2+T'_3}{3}, I'_4 - \frac{I'_1+I'_2+I'_3}{3}, F'_4 - \frac{F'_1+F'_2+F'_3}{3})$. Thus we get the ranking as $A < B$. As we stated in remark 5.1, since there is more fluctuation in A, as an intuition we expect the ranking as $A < B$ which coincide with our ranking.

7. Comparision between proposed ranking method and improved ranking method via MCDM problem

Consider the following MCDM problem based on 5 - valued neutrosophic numbers. Now we rank alternatives A_1, A_2, A_3, A_4 with respect to criteria C_1, C_2, C_3. The ratings of the alternatives with respect to the criteria are given in the form of5 - valued neutrosophic number as shown in table 4. We assume that the respective weights of the criteria C_1, C_2, C_3 are $0.3, 0.3, 0.4$.

Next we multiply corresponding weights of the criteria into the table 4 and we rewrite

	C_1 (Criteria 1)	C_2 (Criteria 2)	C_3 (Criteria 3)
A_1 (Alternative 1)	(0.3,0.4;0.2,0.6;0.6)	(0.5,0.51;0.3,0.2;0.7)	(0.4;0.3,0.6;0.2,0.3)
A_2 (Alternative 2)	(0.2,0.5;0.4;0.8,0.4)	(0.6,0.4;0.25;0.4,1)	(0.1,0.7;0.45;0.1,0.4)
A_3 (Alternative 3)	(0.6,0.1;0.35,0.45;0.6)	(0.3,0.7;0.1,0.4;0.7)	(0.2,0.6;0.2,0.7;0.25)
A_4 (Alternative 4)	(0.35;0.2,0.6;0.3,0.9)	(0.1,0.9;0.25;0.6,0.8)	(0.4;0.3,0.6;0.15,0.35)

TABLE 4. MCDM decision matrix

according to our algorithm by taking $k = 2$ which results table 5.

	C_1	C_2	C_3
A_1	(0.09,0.12;0.06,0.18;0.18,0.18)	(0.15,0.153;0.06,0.09;0.0.21,0.21)	(0.16,0.16;0.12,0.24;0.08,0.12)
A_2	(0.06,0.15;0.12,0.12;0.12,0.24)	(0.12,0.18;0.075,0.075;0.12,0.3)	(0.04,0.28;0.18,0.18;0.04,0.16)
A_3	(0.18,0.03;0.105,0.135;0.18,0.18)	(0.09,0.21;0.03,0.12;0.21,0.21)	(0.08,0.24;0.08,0.28;0.1,0.1)
A_4	(0.105,0.105;0.06,0.18;0.09,0.27)	(0.03,0.27;0.075,0.075;0.18,0.24)	(0.16,0.16;0.12,0.24;0.06,0.14)

TABLE 5. weighted MCDM decision matrix in an rearranged form

Now we have the following weighted arithmetic neutrosophic scores for each $A_i, i = 1 \,\text{to}\, 4$. $A_1 = A_1C_1 + A_1C_2 + A_1C_3 = (0.4, 0.43; 0.24, 0.51; 0.47, 0.51)$. Similarly we get $A_2 = (0.28, 0.55; 0.375, 0.375; 0.4, 0.58)$, $A_3 = (0.35, 0.48; 0.215, 0.535; 0.489, 0.489)$, $A_4 = (0.294, 0.534; 0.255, 0.495; 0.33, 0.65)$. Now we go to next step, $(a_1, b_1, c_1) = (\frac{0.4+0.43}{2}, \frac{0.24+0.51}{2}, \frac{0.47+0.51}{2}) = (0.415, 0.375, 0.49)$. Similarly we find $(a_2, b_2, c_2) = (0.415, 0.375, 0.49), (a_3, b_3, c_3) = (0.415, 0.375, 0.49)$ and $(a_4, b_4, c_4) = (0.415, 0.375, 0.49)$. Now $(a_1, b_1, c_1) = (a_2, b_2, c_2) = (a_3, b_3, c_3) = (a_4, b_4, c_4)$. Therefore we go to next step.

	C_1 (Criteria 1)	C_2 (Criteria 2)	C_3 (Criteria 3)
A_1 (Alternative 1)	(0.1;0.4;0)	(0;0.1;0)	(0;0.3;0.1)
A_2 (Alternative 2)	(0.3;0;0.4)	(0.2;0;0.6)	(0.6;0;0.3)
A_3 (Alternative 3)	(0.5;0.10;0)	(0.4;0.3;0)	(0.4;0.5;0)
A_4 (Alternative 4)	(0;0.4;0.6)	(0.8;0;0.2)	(0;0.3;0.2)

TABLE 6. fluctuation of MCDM decision matrix

Now we have $A_1 = A_1C_1 + A_1C_2 + A_1C_3 = (0.03, 0.27, 0.04)$. Similarly we find $A_2 = (0.39, 0, 0.42), A_3 = (0.43, 0.32, 0)$ and $A_4 = (0.24, 0.24, 0.32)$. Therefore $S^+(A_1) = 0.429$, $S^+(A_2) = 0.485, S^+(A_3) = 0.6$ and $S^+(A_4) = 0.41$. We get the ranking as $A_3 < A_2 < A_1 < A_4$. And as a comparison purpose suppose we apply previous proposed ranking algorithm we get the ranking as $A_3 > A_2 > A_1 > A_4$ for this problem.

	C_1 (Criteria 1)	C_2 (Criteria 2)	C_3 (Criteria 3)
A_1 (Alternative 1)	(0.03;0.12;0)	(0;0.03;0)	(0;0.12;0.04)
A_2 (Alternative 2)	(0.09;0;0.12)	(0.06;0;0.18)	(0.24;0;0.12)
A_3 (Alternative 3)	(0.15;0.03;0)	(0.12;0.09;0)	(0.16;0.2;0)
A_4 (Alternative 4)	(0;0.12;0.18)	(0.24;0;0.06)	(0;0.12;0.08)

TABLE 7. Weighted fluctuation of MCDM decision matrix

Remark 7.1. From our proposed ranking methods, we have shown that we can rank any two n - valued neutrosophic tuplets. As an extension, we can rank any two m_1 valued and n_1 valued neutrosophic tuplets where $m_1 \neq n_1$. In detail, suppose that $A = (T, I, F) = (T_1, T_2 \dots T_{p_1}, I_1, I_2 \dots I_{q_1}, F_1, F_2 \dots F_{r_1})$ be a m_1 - valued neutrosophic triplet and $B = (T', I', F') = (T'_1, T'_2 \dots T'_{p_2}, I'_1, I'_2 \dots I'_{q_2}, F'_1, F'_2 \dots F'_{r_2})$ be n_1 - valued neutrosophic tuplets, where $p_1 + q_1 + r_1 = m_1, p_2 + q_2 + r_2 = n_1$. To rank A and B, we rewrite using $n = lcm\{m_1, n_1\}$ as follows. Suppose $n = x_1 m_1$, then we rewrite (T, I, F) as $A = (\underbrace{T_1, \dots T_1}_{x_1 \text{ times}}, \dots, \underbrace{T_{p_1}, \dots T_{p_1}}_{x_1 \text{ times}}, \underbrace{I_1, \dots I_1}_{x_1 \text{ times}}, \dots \underbrace{I_{q_1}, \dots I_{q_1}}_{x_1 \text{ times}}, \underbrace{F_1, \dots F_1}_{x_1 \text{ times}}, \dots \underbrace{F_{r_1}, \dots F_{r_1}}_{x_1 \text{ times}})$
Suppose $n = y_1 n_1$, then we rewrite (T', I', F') as follows
$B = (\underbrace{T_1, \dots T_1}_{y_1 \text{ times}}, \dots, \underbrace{T_{p_2}, \dots T_{p_2}}_{y_1 \text{ times}}, \underbrace{I_1, \dots I_1}_{y_1 \text{ times}}, \dots \underbrace{I_{q_2}, \dots I_{q_2}}_{y_1 \text{ times}}, \underbrace{F_1, \dots F_1}_{y_1 \text{ times}}, \dots \underbrace{F_{r_2}, \dots F_{r_2}}_{y_1 \text{ times}})$
Now in this stage, we have two n - valued neutrosophic tuplets A and B which can be ordered by our proposed algorithms.

8. Total ordering on n - valued refined neutrosophic sets

In this section, we derive an algorithm to rank any two n-valued refined neutrosophic sets by using the proposed total ordering method on n - valued neutrosophic tuplets.

Let $X = \{x_1, x_2, \dots x_m\}$ be a universe of discourse. Let N_1 and N_2 be two arbitrary n-valued refined neutrosophic sets. Hence $N_1 = (N_1(x_1), N_1(x_2), \dots N_1(x_m)), N_2 = (N_2(x_1), N_2(x_2), \dots N_2(x_m))$ are ordered m-tuples of n - valued neutrosophic tuplets. Now to prove the total ordering, if $N_1 \neq N_2$, then we need to show that either $N_1 > N_2$ or $N_1 < N_2$. We assume that all the elements of X are equally important. Let m_1, m_2 be the number of elements in x for which $N_1(x) > N_2(x)$ and $N_1(x) < N_2(x)$ respectively using the proposed total ordering algorithm for n - valued neutrosophic tuplets.

Step 1: If $m_1 > m_2\,(m_1 < m_2)$, then $N_1 > N_2\,(N_1 < N_2)$. If $m_1 = m_2$, then go to step 2.

Step 2: Apply dictionary order on m-tuples using proposed total ordering algorithm for n - valued neutrosophic tuplets. That is, if $N_1(x_1) > N_2(x_1)$ by proposed total ordering method, then $N_1 > N_2$. If $N_1(x_1) = N_2(x_1)$, then go to next step.

Step 3: If $N_1(x_{j+1}) > N_2(x_{j+1})\,(N_1(x_{j+1}) < N_2(x_{j+1}))$ for $j = 1$, then $N_1 > N_2\,(N_1 < N_2)$.

If $N_1(x_{j+1}) = N_2(x_{j+1})$, then go to step 4.

Step 4: Repeat the step 3 for $j = j + 1$ up to $j = m - 1$ till we reach $N_1 < N_2$ or $N_1 > N_2$.

Remark 8.1. This proposed algorithm derives total ordering algorithm on n - valued neutrosophic sets. Let $X = \{x_1, x_2, \ldots x_m\}$ be an universe of discourse. To prove that, take any two distinct n - valued neutrosophic sets with $N_1 = (N_1(x_1), N_1(x_2), \ldots N_1(x_m)), N_2 = (N_2(x_1), N_2(x_2), \ldots N_2(x_m))$ are ordered m-tuples of n - valued neutrosophic tuplets. Let m_1, m_2 be the number of elements in X for which $N_1(x) > N_2(x)$ and $N_1(x) < N_2(x)$ respectively using the proposed total ordering algorithm for n - valued neutrosophic tuplets. By step 1, if $m_1 > m_2\,(m_1 < m_2)$, then $N_1 > N_2\,(N_1 < N_2)$ and hence the ordering is done. If $m_1 = m_2$, we go to step 2. We apply dictionary order on m-tuples of n - valued neutrosophic tuplets using proposed total ordering algorithm for n - valued neutrosophic tuplets. If $N_1(x_1) > N_2(x_1)$ by proposed total ordering method, then $N_1 > N_2$. If $N_1(x_1) = N_2(x_1)$, then we go to next step. By applying step 3 and step 4, we get if $N_1(x_{j+1}) > N_2(x_{j+1})\,(N_1(x_{j+1}) < N_2(x_{j+1}))$ for some j. Otherwise we get $N_1(x_i) = N_2(x_i)$ for every $i \in \{1, \ldots m\}$ which implies $N_1 = N_2$, a contradiction to $N_1 \neq N_2$. Thus we have proved the total ordering.

Example 8.2. Let $X = \{x_1, x_2, x_3\}$ be a universe of discourse. Let us take three n - valued refined neutrosophic sets $(n = 5)$ N_1, N_2 and N_3, where

$N_1 = \{((x_1, (0.3, 0.4; 0.2, 0.6; 0.6)), (x_2, (0.4, 0.6; 0.3, 0.2; 0.7)), (x_3, (0.3; 0.3, 0.6; 0.2, 0.3))\}$

$N_2 = \{((x_1, (0.2, 0.4; 0.2, 0.6; 0.6)), (x_2, (0.2, 0.5; 0.4; 0.8, 0.4)), (x_3, (0.6, 0.4; 0.25; 0.4, 1))\}$

$N_3 = \{((x_1, (0.6, 0.8; 0.2, 0.6; 0.4)), (x_2, (0.4, 0.6; 0.3, 0.2; 0.7)), (x_3, (0.1; 0.4, 0.5; 0.5, 0.6))\}$

Now we find the ordering between N_1, N_2 and N_3. Now we compare the n - valued neutrosophic sets N_1 and N_2. Now we get $S^+(N_1(x_1)) = 0.3, S^+(N_2(x_1)) = 0.28$ using proposed total ordering method, which implies $N_1(x_1)) > N_2(x_1)$. In similar manner, we find that $N_1(x_2) > N_2(x_2)$, $N_1(x_2) > N_2(x_2)$. Thus we find that $m_1 = 3 > 0 = m_2$. By step 1, we get $N_2 < N_1$.

Now we compare N_3 and N_1. We get $S^+(N_1(x_1)) = 0.3, S^+(N_3(x_1)) = 0.52$ using proposed total ordering method, which implies $N_3(x_1)) > N_1(x_1)$. In similar manner, we find that $N_3(x_2) = N_1(x_2)$, $N_3(x_3) < N_1(x_3)$. Hence we find that $m_1 = 1 = m_2$. Since step 1 fails to rank them, we go to step 2. By dictionary order, we compare $N_1(x_1)$ and $N_3(x_1)$. Thus we get $N_1 < N_3$. Finally our ordering for these three n - valued $(n = 5)$ refined neutrosophic sets is $N_3 > N_1 > N_2$.

9. Conclusion and future scope

We have proposed two ranking algorithms for total ordering n - valued neutrosophic tuplets. The proposed first ranking method accounts summation of the collective membership, non membership, indeterminacy grades in the first stage and then highest to lowest membership, non membership, indeterminacy grades are used to rank in the next stages. In some cases, when there is a fluctuation between highest and lowest membership, non membership, and indeterminacy grades, the proposed ranking method may rank differently to decision maker's intuition. To overcome this, we have proposed improved ranking method which also first accounts summation of the collective membership, non membership, indeterminacy grades. But, it considers the fluctuation between the membership values, non membership values and indeterminacy values in the next stages. Further, the score functions used in the both the ranking approaches takes into account not only membership, non-membership, and indeterminacy values, but also the portion of membership and non-membership value that is contained within the hesitance value. Through the proposed ranking algorithms for total ordering n - valued neutrosophic tuplets using score functions, we develop a total ordering algorithm for n - valued refined neutrosophic sets using dictionary order at the final stage. In near future, a total order on n - valued refined neutrosophic sets may be developed by defining more number of score and accuracy functions on n - valued neutrosophic tuplets.

Acknowledgement: We would like to acknowledge Dr. Florentin Smarandache, University of Mexico for raising the problem of research that defining a total ordering on n - valued refined neutrosophic sets.

Funding: This research was funded by Council of Scientific and Industrial Research (CSIR-HRDG) India, grant number 09/895(0014)/2019-EMR-I.

Conflicts of Interest: The authors declare that there is no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results. As mentioned earlier the corresponding author thank the Council of Scientific and Industrial Research (CSIR-HRDG) India, for funding the research work under CSIR-SRF.

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Received: June 1, 2023. Accepted: Sep 29, 2023