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Distributive Properties of $Q-$neutrosophic Soft Quasigroups

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Abstract. The $Q-$neutrosophic soft quasigroup is a mathematical innovation for dealing with indeterminate occurrences. The characterization of quasigroups using the concept of $Q-$neutrosophic soft set is an evolving area of study that, in recent times, has attracted pools of researchers. Different researchers have defined the idea of a Q-neutrosophic soft set under associative structures like groups, fields, rings, and modules. The distributive and symmetric properties of the $Q-$neutrosophic soft quasigroup are examined in this study, which extends the idea of a $Q-$neutrosophic soft set to a non-associative behaviour known as a quasigroup. Our findings were quite revealing. In particular, after defining $Q-$neutrosophic soft quasigroup in relation to the three binary operations of product, right, and left division operations, it was found that these operations are distributive over one another. Additionally, these binary operations are distributive over the operations of intersection, union, AND, and OR. It was obtained that, $Q-$neutrosophic soft quasigroup does not obey the key laws, and that the quasigroup is self-distributive with respect to the product, left, and right divisions. The effort which is novel, has advanced the course of study in this emerging field.

Keywords: Quasigroup; Distributive; Symmetric properties; Soft set; Q-neutrosophic soft set

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1. Introduction

Definition 1.1. Suppose that $\hat{G}$ is a non-empty set and the binary operation $(\odot)$ is define on $\hat{G}$ such that $r \odot w \in \hat{G}$ for all $r, w \in \hat{G}$ and if there exist $\alpha, \beta \in \hat{G}$, the pair $(\hat{G}, \odot)$ is called a *groupoid*. If the equations:

$$\alpha \odot r = \beta \text{ and } w \odot \alpha = \beta$$

has unique solutions $r, w \in \hat{G}$ for all $\alpha, \beta \in \hat{G}$, then $(\hat{G}, \odot)$ is called quasigroup. Suppose there is a unique element $1 \in \hat{G}$ called the identity element such that $1 \odot r = r \odot 1 = r$ for all $r \in \hat{G}$, then $(\hat{G}, \odot)$ is a loop.

In this research, we sometime write rw instead of $r \odot w$, when the operation $\odot$ is a multiplication in $\hat{G}$. Suppose that r is a fixed element in a quasigroup $(\hat{G}, \odot)$. Then, the left and right translation maps for all $r \in \hat{G}$, written as L_r and R_r respectively are defined by $wL_r = r \odot w \quad \text{and} \quad wR_r = w \odot r$. It is shown that a groupoid $(\hat{G}, \odot)$ is a quasigroup if the left and right translation maps are bijective. Hence, the inverse mappings L_r^{-1} and R_r^{-1} also exist.Thus,

$$r \backslash w = wL_r^{-1} \quad \text{and} \quad r/w = rR_w^{-1}$$

Zadeh launched fuzzy set concept for the first time in [15]; Atanassov extended on it in [20] with the definition of intuitionistic fuzzy set. Fuzzification of quasigroup was first introduced in 1998, [9] by Dudek. In 1999, Dudek and Jun [10] extended the results in [9] to fuzzy subquasigroup under t-norm. In 2000, Kyung et.al. [12] studied intuitionistic fuzzy subquasigroups as a way of generalizing the results obtained in [9]. In 2005, intuitionistic fuzzy subquasigroups were further studied by Dudek [13]. In, 2008 Muhammad and Dudek presented fuzzy subquasigroups with different types of (α, β)- fuzzy quasigroups. Although, these two notions has some limitations and difficulties when dealing with uncertainty and incomplete data stated in [16]. A soft set theory was presented by Molodtsov in [16] as an analytical instrument for addressing uncertainty in order to address some of the aforementioned issues. Over the years, many experts in the field of algebra have applied this mathematical concept to an algebraic structure and studied it through the structural characteristic of the algebraic structure. For example, the algebraic properties of soft sets under a quasigroup were introduced by Oyem. et.al. [18, 19]. It is well known that one of the most beautiful properties of soft set theory is that its parameter set has a capacity to accommodate a wide range of information in terms of decision-making in real-life problems. Although, the characterization of membership degrees present in neutrosophic set are not applicable in the study of soft set theory. Therefore, soft set theory is not applicable when solving problems involving indeterminate data.

To deal with indeterminate real-world data, a mathematical concept called neutrosophy was launched. This mathematical concept was launched in 1998 by Smarandache [23, 24]. It is well known that this unique idea is the only application of classical set theory that has been generalized in the literature to address issues with uncertainty and indeterminacy. The culture of a neutrosophic set is characterized via three independent membership degrees called the true, indeterminate, and falsity which are respectively denoted as $\mathcal{T}^{`}$, $\mathcal{I}^{`}$, and $\mathcal{F}^{`}$. The concept of the neutrosophic set and its method of determining the indeterminate in real-life data are applicable in different fields of study. For example, the authors in [26] used the concept to study the inspection assignment form for product quality control, while characterizations of separation axioms in neutrosophic topological spaces were studied in [27]. The concept of neutrosophy culture is also applicable in the area of operation research in management. In particular, the authors in [28] presented a study on neutrosophic methods of operation research in the management of corporate work.

Recently, the study of a neutrosophic set combined with the concept of soft set theory has received tremendous attention in the field of mathematics [1–4, 17, 22, 30]. This is because the combination of these two mathematical concepts provides a generalized structure for dealing with uncertainties and indeterminacy present in real-world problems. For example, the $Q-$neutrosophic soft set ($Q-$NS) set is an expanded model of the neutrosophic soft set described by two universal sets. Hence, it has the capacity to handle the two universal sets and its indeterminate membership at the same time.

Since the notion of a $Q-$neutrosophic soft set of two universal sets was defined in [6], different authors have applied it to associative behavior such as fields, groups, rings, and modules [6, 7, 21, 25]. In addition. the concept of a $Q-$neutrosophic soft set is long overdue to be extended to the structure of a quasigroup where associative property is not assumed.

This work characterizes the distributive properties of the $Q-$neutrosophic soft set under a non-associative algebra termed quasigroup. In particular, the distributive properties of quasigroups have a very interesting characteristic in the classical study of quasigroup theory. It is well known that quasigroups are not inherently distributive across their binary operations [14]. This serves as motivation to investigate the distributive properties of the $Q-$neutrosophic soft quasigroup.

2. **Preliminaries**

Definition 2.1. [14] Let $(\hat{G}, \odot)$ be quasigroup and $P \leq \hat{G}$. Then, P is called subgroupoid (subquasigroup) of $\hat{G}$ if $(P, \odot)$ is a quasigroup. Let V and K be non empty subsets of $\hat{G}$, then the product $V \odot K = \{v \odot k \mid v \in K, k \in K\}$, the right division $V/K = \{v/k \mid v \in V, k \in K\}$ and left division $V\backslash K = \{v\backslash k \mid v \in V, k \in K\}$

Definition 2.2. [14] Let $(\hat{G}, \odot)$ be a quasigroup. $(\hat{G}, \odot)$ is a left distributive if $f \odot (w^1 \odot z) = (f\odot) \odot (f \odot z)$ and a right distributive if $(f \odot w^1) \odot z = (f \odot z) \odot (w^1 \odot z)$. Whenever right and left distributive properties hold in $(\hat{G}, \odot)$, it is called a distributive quasigroup.

Definition 2.3. A groupoid (quasigroup) $(\hat{G}, \odot)$ is

(1) right symmetric if $(\alpha \odot \beta) \odot \beta = \alpha$ for all $\alpha, \beta \in \hat{G}$
(2) left symmetric if $\beta \odot (\beta \odot \alpha) = \alpha$ for all $\alpha, \beta \in \hat{G}$

Definition 2.4. A quasigroup $(\hat{G}, \odot)$ is said to obeys key-laws if it satisfies both Definitions 2.3

Definition 2.5. Let W^1 be a set, if it is a poset in which any two elements have supremum and infimum. Then, it is called a lattice for $\sup\{k^*, m^*\}$ and $\inf\{k^*, m^*\}$ are respectively denoted as $k^* \vee m^*$ and $k^* \wedge m^*$. It called a distributive lattice if $k^* \wedge (m^* \vee n^*) = (k^* \wedge m^*) \vee (k^* \wedge n^*)$ for any $k^*, m^*, n^* \in L$

Definition 2.6. [1] Given the two $Q-$NS sets $(\Lambda^Q, \mathfrak{A}^1)$ and $(\Theta^Q, \mathfrak{B}^1)$. Then, the intersection, AND, union and OR operations are defined as follows:

(1) $(\Lambda^Q, \mathfrak{A}^1) \cap (\Theta^Q, \mathfrak{B}^1) = (\triangle_1^Q, \mathfrak{C}^1)$ is a $Q-$NS set, where $\mathfrak{C}^1 = \mathfrak{A}^1 \cap \mathfrak{B}^1$

$$T_{\triangle_1^Q(\alpha)}(w^1, u^1) = \min\{T_{\Lambda^Q(\alpha)}(w^1, u^1), T_{\Theta^Q(\alpha)}(w^1, u^1)\}$$
$$I_{\triangle_1^Q(\alpha)}(w^1, u^1) = \max\{I_{\Lambda^Q(\alpha)}(w^1, u^1), I_{\Theta^Q(\alpha)}(w^1, u^1)\}$$
$$F_{\triangle_1^Q(\alpha)}(w^1, u^1) = \max\{F_{\Lambda^Q(\alpha)}(w^1, u^1), F_{\Theta^Q(\alpha)}(w^1, u^1)\}$$

(2) $(\Lambda^Q, \mathfrak{A}^1) \cup (\Theta^Q, \mathfrak{B}^1) = (\triangle_1^Q, \mathfrak{C}^1)$ is a $Q-$NS, where $\mathfrak{C}^1 = \mathfrak{A}^1 \cup \mathfrak{B}^1$

$$T_{\triangle_1^Q(a)}(w^1, u^1) = \begin{cases} T_{\Lambda^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{A}^1 - \mathfrak{B}^1 \\ T_{\Theta^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{B}^1 - \mathfrak{A}^1 \\ \max\{T_{\Lambda^Q(\alpha)}(w^1, u^1), T_{\Theta^Q(\alpha)}(w^1, u^1)\}, & \text{if } \alpha \in \mathfrak{B}^1 \cap \mathfrak{A}^1 \end{cases}$$

$$I_{\triangle_1^Q(a)}(w^1, u^1) = \begin{cases} I_{\Lambda^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{A}^1 - \mathfrak{B}^1 \\ I_{\Theta^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{B}^1 - \mathfrak{A}^1 \\ \min\{I_{\Lambda^Q(\alpha)}(w^1, u^1), I_{\Theta^Q(\alpha)}(w^1, u^1)\}, & \text{if } \alpha \in \mathfrak{B}^1 \cap \mathfrak{A}^1 \end{cases}$$

$$F_{\triangle_1^Q(a)}(w^1, u^1) = \begin{cases} F_{\Lambda^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{A}^1 - \mathfrak{B}^1 \\ F_{\Theta^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{B}^1 - \mathfrak{A}^1 \\ \min\{F_{\Lambda^Q(\alpha)}(w^1, u^1), F_{\Theta^Q(\alpha)}(w^1, u^1)\}, & \text{if } \alpha \in \mathfrak{B}^1 \cap \mathfrak{A}^1 \end{cases}$$

(3) $(\Lambda^Q, \mathfrak{A}^1) \wedge (\Theta^Q, \mathfrak{B}^1) = (\triangle_1^Q, \mathfrak{C}^1)$ is a $Q-$NS set, where $\triangle_{Q(\alpha,\beta)} = \Lambda_{Q(\alpha)} \cap \Theta_{Q(\beta)}$ and $(\alpha, \beta) \in \mathfrak{A}^1 \times \mathfrak{B}^1, w^1 \in W^1$, and $u^1 \in Q$.

$$T_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \min\{T_{\Lambda^Q(\alpha)}(w^1, u^1), T_{\Theta^Q(\beta)}(w^1, u^1)\}$$
$$I_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \max\{I_{\Lambda^Q(\alpha)}(w^1, u^1), I_{\Theta^Q(\beta)}(w^1, u^1)\}$$
$$F_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \max\{F_{\Lambda^Q(\alpha)}(w^1, u^1), F_{\Theta^Q(\beta)}(w^1, u^1)\}$$

(4) $(\Lambda^Q, \mathfrak{A}^1) \vee (\Theta^Q, \mathfrak{B}^1) = (\triangle_1^Q, \mathfrak{C}^1)$ is a $Q-$NS set, where $\triangle_{Q(\alpha,\beta)} = \Lambda_{Q(\alpha)} \cup \Theta_{Q(\beta)}$ and $(\alpha, \beta) \in \mathfrak{A}^1 \times \mathfrak{B}^1, w^1 \in W^1$, and $u^1 \in Q$.

$$T_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \max\{T_{\Lambda^Q(\alpha)}(w^1, u^1), T_{\Theta^Q(\beta)}(w^1, u^1)\}$$
$$I_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \min\{I_{\Lambda^Q(\alpha)}(w^1, u^1), I_{\Theta^Q(\beta)}(w^1, u^1)\}$$
$$F_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \min\{F_{\Lambda^Q(\alpha)}(w^1, u^1), F_{\Theta^Q(\beta)}(w^1, u^1)\}$$

Definition 2.7. [16] Let W^1 be set, a pair $(F, \mathfrak{A}^1)$ is called a soft set if $F : \mathfrak{A}^1 \to P(W^1)$, where $P(W^1)$ is power set of W^1 and $\mathfrak{A}^1$ is a set of parameters.

Definition 2.8. [24] Let W^1 be a set. A neutrosophic set (NS) is described as
$\Phi = \{\langle w^1, (T_\Phi(w^1), I_\Phi(w^1), F_\Phi(w^1))\rangle : w^1 \in W^1\}$ such that $T_\Phi, I_\Phi, F_\Phi : W^1 \to]^-0, 1^+[$.

Definition 2.9. [6] A $Q-$neutrosophic set Π_1^Q in W^1 is described in the form
$\Pi_1^Q = \{\langle (w^1, u^1), T_{\Phi^Q}(w^1, u^1), I_{\Phi^Q}(w^1, u^1), F_{\Phi^Q}(w^1, u^1)\rangle : w^1 \in W^1, u^1 \in Q\}$, where $T_{\Phi^Q}, I_{\Phi^Q}, F_{\Phi^Q} : W^1 \times Q \to]^-0, 1^+[$ are the membership degrees.

Definition 2.10. [17] Let W^1 be a set and $\mathfrak{A}^1$ be a parameter sets. A (NS) set $(\Phi, \mathfrak{A}^1)$ is described as $(\Phi, \mathfrak{A}^1) = \{\langle w^1, (T_\Phi(w^1), I_\Phi(w^1), F_\Phi(w^1))\rangle : w^1 \in W^1\}$.

Definition 2.11. [6] Let k be any positive integer, I be a unit interval $[0, 1]$, W^1 be a universe of discourse and Q be a non-empty sets. A $Q-$neutrosophic set Π_1^Q in W^1 and Q is described as

$$\Pi_1^Q = \{< (w^1, u^1), T_{\Phi_1}^{Q_i}(w^1, u^1), I_{\Phi_1}^{Q_i}(w^1, u^1), F_{\Phi_1}^{Q_i}(w^1, u^1) >: w^1 \in W^1, u^1 \in Q \, \forall \, i = 1, 2, 3, ..., k\},$$

where $T_{\hat{\Phi_1}^{Q_i}}, I_{\hat{\Phi_1}^{Q_i}}, F_{\hat{\Phi_1}^{Q_i}} : W^1 \times Q \to I^k \; \forall i = 1, 2, ..., k$ are membership degrees.

Definition 2.12. [1] Suppose that W^1 is a universal set and Q is a non-empty set. Let $\mathfrak{A}^1 \subset E$ be a set of parameters. A pair $(\Pi_1^Q, \mathfrak{A}^1)$ is called a $(Q-$NSS$)$ over W^1 and Q, where $\Pi_1^Q : \mathfrak{A} \to \rho^l QNS(W^1)$ is a map such that $\Pi_1^Q(a) = \emptyset$ if $a \notin \mathfrak{A}^1$. It is denoted by $(\Pi_1^Q, \mathfrak{A}^1) = \{(a, \Pi_1^Q(a)) : a \in \mathfrak{A}^1, \Pi_1^Q(a) \in \rho^l QNS(W^1)\}$

Definition 2.13. The direct product $(\Pi_1^Q, \mathfrak{A}^1) \times (\Psi_1^Q, \mathfrak{B}^1)$ of $(\Pi_1^Q, \mathfrak{A}^1)$ and $(\Psi_1^Q, \mathfrak{B}^1)$ is a $Q-$NS set $(\Pi_1^Q, \mathfrak{C}^1)$ under $\hat{W_1^1} \times \hat{W_2^1}$ such that $\mathfrak{A}^1 \times \mathfrak{B}^1 = \mathfrak{C}^1$.

$$\Pi_1^Q(\alpha, \beta) = \Big\{ \langle ((w_1^1, w_2^1), u^1), T_{\Phi_1^Q(\alpha,\beta)}((w_1^1, w_2^1), u^1)), I_{\Phi_1^Q(\alpha,\beta)}((w_1^1, w_2^1), q), F_{\Phi_1^Q(\alpha,\beta)}((w_1^1, w_2^1), q)\rangle : (w_1^1, w_2^1) \in \hat{Q}_1 \times \hat{Q}_2, u^1 \in Q \Big\}$$

The membership degrees are defined as

$$T_{\Pi_1^Q(\alpha,\beta)}((w_1^1,w_2^1),u^1) = \min\{T_{\Pi_1^Q(\alpha)}(w_1^1,u^1), T_{\Psi_1^Q(\beta)}(w_2^1,u^1)\},$$
$$I_{\Pi_1^Q(\alpha,\beta)}((w_1^1,w_2^1),u^1) = \max\{I_{\Pi_1^Q(\alpha)}(w_1^1,u^1), I_{\Psi_1^Q(\beta)}(w_2^1,u^1)\},$$
$$F_{\Pi_1^Q(\alpha,\beta)}((w_1^1,w_2^1),u^1) = \max\{F_{\Pi_1^Q(\alpha)}(w_1^1,u^1), F_{\Psi_1^Q(\beta)}(w_2^1,u^1))\}.$$

3. **Main Results**

Definition 3.1. Let $(\nabla_1^Q, \mathfrak{A}^1)$ be a $Q-$NS set defined over a quasigroup $(\hat{G}, \odot/, \backslash)$. Then $(\nabla_1^Q, \mathfrak{A}^1)$ is called a $Q-$NS quasigroup over a quasigroup $\hat{G}$ if for all $\alpha \in \mathfrak{A}^1, u^1 \in Q, \nabla_1^Q(\alpha)$ is a $Q-$NS quasigroup given a map $\nabla_1^Q(a) : \hat{G} \times Q \to [0,1]^3$

Definition 3.2. Let $(\nabla_1^Q, \mathfrak{A}^1)$ be a $Q-$NS set defined under a quasigroup $(\hat{G}, \odot/, \backslash)$. Then $(\nabla_1^Q, \mathfrak{A}^1)$ is called a $Q-$neutrosphic soft quasigroup if for all $a \in \mathfrak{A}^1, w^1, t^1 \in \hat{G}, u^1 \in Q$ satisfies the following

(1) $T_{\nabla_1^Q(a)}((w^1 * t^1), u^1) \geq \min\{T_{\nabla_1^Q(a)}(w^1, u), T_{\nabla_1^Q(a)}(t^1, u^1)\}$
(2) $I_{\nabla_1^Q(a)}((w^1 * t^1), u^1) \leq \max\{I_{\nabla_1^Q(a)}(w^1, u), I_{\nabla_1^Q(a)}(t^1, u^1)\}$
(3) $F_{\nabla_1^Q(a)}((w^1 * t^1), u^1) \leq \max\{F_{\nabla_1^Q(a)}(w^1, u), F_{\nabla_1^Q(a)}(t^1, u^1)\}$

where $* \in \{\odot, /, \backslash\}$

Example 3.3. Let $\hat{G} = \{i,j,k,l,m,n,o\}$ be quasigroup of order 7 and $\mathfrak{A}^1$ be a subset of E called the parameter sets. Given the quasigroup in Cayley table below.

TABLE 1. Quasigroup of order 7

$\odot$	i	j	k	l	m	n	o
i	i	m	o	n	j	l	k
j	m	j	n	o	i	k	l
k	o	n	k	m	l	j	i
l	n	o	m	k	l	i	j
m	j	i	l	k	m	o	n
n	l	k	j	i	o	n	m
o	k	l	i	j	n	m	o

Define a $Q-$NS set $(\nabla_1^Q, \mathfrak{A}^1)$, for all $u^1 \in Q$ and $w^1, t^1, z^1 \in \hat{G}$ such that $z^1 = w^1 * t^1 \in \hat{G}$. Let $\mathfrak{A}^1$ be the set parameters and $n \in \mathbb{N}$ a set of natural numbers.

$$T_{\nabla_1^Q(a)}((w^1 * t^1), u^1) = \begin{cases} 1 - \frac{1}{2n}, & \text{if } z^1 = \{j,l,k,m,n,o\} \\ 1, & \text{otherwise.} \end{cases}$$

$$I_{\nabla_1^Q(a)}((w^1 * t^1), u^1) = \begin{cases} 0, & \text{if } z^1 = \{j,l,m,k,n,o\} \\ 1 - \frac{1}{2n}, & \text{otherwise.} \end{cases}$$

$$F_{\nabla_1^Q(a)}((w^1 * t^1), u^1) = \begin{cases} 0, & \text{if } z^1 = \{j,k,m,l,n,o\} \\ 1 - \frac{1}{2n}, & \text{otherwise.} \end{cases}$$

Considering

the operation " $\odot$ ", then, $T_{\nabla_1^Q(a)}((w^1 * t^1), u^1) \geq \min\{T_{\nabla_1^Q(a)}(w^1, u^1), T_{\nabla_1^Q(a)}(t^1, u^1)\}$. Put $w^1 = j, t^1 = m$, then we have

$$\begin{aligned} T_{\nabla_1^Q(a)}(j \odot m, u^1) &= T_{\nabla_1^Q(a)}(i, u^1) \\ &\Rightarrow \text{RHS} = 1 \in [0,1] \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned} &\min\{T_{\nabla_1^Q(a)}(j, u^1), T_{\nabla_1^Q(a)}(m, u^1)\} = \\ &\min\{(1 - \frac{1}{2n}, u^1), (1 - \frac{1}{2n}, u^1)\} = 1 - \frac{1}{2n} = 0.5 \in [0,1] \text{ for } n = 1 \end{aligned} \tag{2}$$

Hence, from the definition 3.2, we have that $1 \geq \min\{1-\frac{1}{2n}, 1-\frac{1}{2n}\} \Rightarrow 1 \geq 1-\frac{1}{2n}$ for all $n \in \mathbb{N}$. It holds for true membership degree. The results for right and left division operations "/", and "\″ can also be verify in similar way. Also, the results for indeterminate and falsity membership degrees are similar with the result obtained for true membership degree. Hence, $(\nabla_1^Q, \mathfrak{A}^1)$ is a $Q-$neutrosophic soft quasigroup over the quasigroup $(\hat{G}, \odot, /, \backslash)$

Definition 3.4. Given the two $Q-$neutrosophic soft quasigroups $(\nabla_1^Q, \mathfrak{A}^1)$ and $(\Psi, \mathfrak{B}^1)$ over a quasigroup $(\hat{G}, \odot, /, \backslash)$, and let Q be a non empty set. Then,

(1) The product $(\nabla_1^Q, \mathfrak{A}^1) \odot (\Psi_1^Q, \mathfrak{B}^1)$ of $(\nabla_1^Q, \mathfrak{A}^1)$ and $(\Psi, \mathfrak{B}^1)$ is a $Q-$NSS $(\Pi_1^Q, \mathfrak{C}^1)$ over $(\hat{G}, \odot)$ such that $\mathfrak{C}^1 = \mathfrak{A}^1 \cap \mathfrak{B}^1$.

$$\Pi_1^Q(\alpha \odot \beta) = \left\{ \langle ((w^1, u^1), T_{\nabla_1^Q(a,b)}((w^1, u^1), I_{\nabla_1^Q(a,b)}(w^1, u^1), F_{\nabla_1^Q(a,b)}((t, u^1)\rangle : w^1, f^1 \in \hat{G}, u^1 \in Q \right\}$$

where

$$\begin{aligned} T_{\Pi_1^Q(\alpha \odot \beta)}(w^1 \odot f^1, u^1) &\geq \min\{T_{\nabla_1^Q(\alpha)}(w^1, u^1), T_{\Psi_1^Q(\beta)}(f^1, u^1)\}, \\ I_{\Pi_1^Q(\alpha \odot \beta)}(w^1 \odot f^1, u^1) &\leq \max\{I_{\nabla_1^Q(\alpha)}(w^1, u^1), I_{\Psi_1^Q(\beta)}(f^1, u^1)\}, \\ F_{\Pi_1^Q(\alpha \odot \beta)}(w^1 \odot f^1, u^1) &\leq \max\{F_{\nabla_1^Q(\alpha)}(w^1, u^1), F_{\Psi_1^Q(\beta)}(f^1, u^1))\}, \end{aligned}$$

(2) The right division $(\nabla_1^Q, \mathfrak{A}^1)/(\Psi_1^Q, \mathfrak{B}^1)$ of $(\nabla_1^Q, \mathfrak{A}^1)$ and $(\Psi_1^Q, \mathfrak{A}^1)$ is a $Q-NSS$ $(\Pi_1^Q, \mathfrak{C}^1)$ over $(\hat{G}, /)$ such that $\mathfrak{A}^1 \cap \mathfrak{B}^1 = \mathfrak{C}^1$. Thus,

$$\Pi_1^Q(\alpha/\beta) = \left\{ \langle (w^1/f^1, u^1), T_{\nabla_1^Q(\alpha/\beta)}((w^1/f^1, u^1), I_{\nabla_1^Q(\alpha/\beta)}((w^1/f^1, u^1), F_{\nabla_1^Q(\alpha/\beta)}(w^1/f^1, u^1)\rangle : w^1, f^1 \in \hat{G}, u^1 \in Q \right\}$$

where

$$\begin{aligned} T_{\Pi_1^Q(\alpha/\beta)}(w^1/f^1,u^1) &\geq \max\{T_{\nabla_1^Q(\alpha)}(w^1,u^1),T_{\Psi_1^Q(\beta)}(f^1,u^1)\},\\ I_{\Pi_1^Q(\alpha/\beta)}(w^1/f^1,u^1) &\leq \min\{I_{\nabla_1^Q(\alpha)}(w^1,u^1),I_{\Psi_1^Q(\beta)}(f^1,u^1)\},\\ I_{\Pi_1^Q(\alpha/\beta)}(w^1/f^1,u^1) &\leq \min\{I_{\nabla_1^Q(\alpha)}(w^1,u^1),I_{\Psi_1^Q(\beta)}(f^1,u^1)\} \end{aligned}$$

(3) The left division $(\nabla_1^Q,\mathfrak{A}^1)\backslash(\Psi_1^Q,\mathfrak{B}^1)$ of $(\nabla_1^Q,\mathfrak{A}^1)$ and $(\Psi,\mathfrak{B}^1)$ is a $Q-NSS$ $(\Pi_1^Q,\mathfrak{C}^1)$ over $\hat{G}$ such that $\mathfrak{A}^1\cap\mathfrak{A}^1=\mathfrak{C}^1$. Thus,

$$\Pi_{Q(\alpha\backslash\beta)}=\Big\{\langle(w^1/f^1,u^1),T_{\nabla_1^Q(\alpha\backslash\beta)}((w^1/f^1,u^1),I_{\nabla_1^Q(\alpha\backslash\beta)}((w^1/f^1,u^1),F_{\nabla_1^Q(\alpha\backslash\beta)}(w^1/f^1,u^1)\rangle: w^1,f^1\in\hat{G},u^1\in Q\Big\}$$

where

$$\begin{aligned} T_{\Pi_1^Q(\alpha\backslash\beta)}(w^1\backslash f^1,u^1) &\geq \max\{T_{\Psi_1^Q(\alpha)}(w^1,u^1),T_{\nabla_1^Q(\beta)}(f^1,u^1)\},\\ I_{\Pi_1^Q(\alpha\backslash\beta)}(w^1\backslash f^1,u^1) &\leq \min\{I_{\Psi_1^Q(\alpha)}(w^1,u^1),I_{\nabla_1^Q(\beta)}(f^1,u^1)\},\\ F_{\Pi_1^Q(\alpha\backslash\beta)}(w^1\backslash f^1,u^1) &\leq \min\{F_{\Psi_1^Q(\alpha)}(w^1,u^1),F_{\nabla_1^Q(\beta)}(f^1,u^1)\} \end{aligned}$$

Theorem 3.5. *Let $(\Lambda_1^Q,\mathfrak{A}^1),(\Theta_1^Q,\mathfrak{B}^1)$ and $(\triangle_1^Q,\mathfrak{C}^1)$ be $Q-$neutrosophic soft quasigroups over quasigroup $(\hat{G},\odot)$. Then, the following holds*

(1) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\cap(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\cap\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(2) $\big((\Lambda_1^Q,\mathfrak{A}^1)\cap(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\cap\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(3) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(4) $\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\wedge\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(5) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\cup(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\cup\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(6) $\big((\Lambda_1^Q,\mathfrak{A}^1)\cup(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\cup\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(7) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\vee\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(8) $\big((\Lambda_1^Q,\mathfrak{A}^1)\vee(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\vee\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

Proof:

(1) We shall show that $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\cap(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\cap\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

Let

$$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)=(\Psi_1^Q,\mathfrak{C}^1)\text{ such that}$$

$$T_{\Psi_1^Q(a\odot b)}(w\odot t,u^1)=\min\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot b)\in\mathfrak{A}^1\cap\mathfrak{B}^1} \tag{3}$$

for all $w,t\in\hat{G}$ and $u^1\in Q$. And, let

$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)=(\mho_1^Q,\mathfrak{F}^1)$ such that for all $w,t\in\hat{G},u^1\in Q$ we have

$$T_{\mho_1^Q(a\odot c)}(w\odot t,u^1)=\min\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\triangle_1^Q(c)}(t,u^1)\}}_{(a\odot c)=f\in\mathfrak{A}^1\cap\mathfrak{C}^1}\tag{4}$$

Let $(\Psi_1^Q,\mathfrak{E}^1)\cap(\mho_1^Q,\mathfrak{F}^1)=(\Pi_1^Q,\mathfrak{G}^1)$ such that $g\in(\mathfrak{E}^1\cap\mathfrak{F}^1)=\mathfrak{G}^1$ for all $g\in\mathfrak{G}^1$

Combining equations (3) and (4), for all $w,t\in\hat{G}$ and $u^1\in Q$ we have

$$\begin{aligned}T_{\Pi_1^Q(g)}(wt,u^1) &= \min\underbrace{\{T_{\Psi_1^Q(g)}(wt,u^1),T_{\mho_1^Q(g)}(wt,u^1)\}}_{(g)\in\mathfrak{E}^1\cap\mathfrak{F}^1.}\\ &= \min\{\underbrace{T_{\Psi_1^Q(a\odot b)}(wt,u^1),T_{\mho_1^Q(a\odot c)}(wt,u^1)}_{a\in((\mathfrak{A}^1\cap\mathfrak{B}^1)\cap(\mathfrak{A}^1\cap\mathfrak{C}^1))}\end{aligned}\tag{5}$$

Considering the LHS: Let

$(\Theta_1^Q,\mathfrak{B}^1)\cap(\triangle_1^Q,\mathfrak{C}^1)=(\Phi_1^Q,\mathfrak{D}^1)$ such that for all $w,t\in\hat{G}$ and $u^1\in Q, wehave$

$$T_{\Phi_1^Q(d)}(wt,u^1)=\min\underbrace{\{T_{\Theta_1^Q(d)}(wt,u^1),T_{\triangle_1^Q(d)}(wt,u^1)\}}_{d\in(\mathfrak{B}^1\cap\mathfrak{C}^1)}\tag{6}$$

Also, let

$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Phi_1^Q,\mathfrak{D}^1)=(\Xi_1^Q,\mathfrak{H}^1)$ such that for all $w,t\in\hat{G}$ and $u^1\in Q,$ we have

$$\begin{aligned}T_{\Xi_1^Q(a\odot d)}(w\odot t,u^1)=\min &\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\Phi_1^Q(d)}(t,u^1)\}}_{\text{this implies that for all} a\in\mathfrak{A}^1,d\in\mathfrak{D}^1,a\odot d\in(\mathfrak{A}^1\cap\mathfrak{D}^1)}\\ &=\min\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1),\min\{T_{\Theta_1^Q(d)}(t,u^1),T_{\triangle_1^Q(d)}(t,u^1)\}\}}\\ &=\min\underbrace{\{\min\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\Lambda_1^Q(a)}(w,u^1)\},\min\{T_{\Theta_1^Q(d)}(t,u^1),T_{\triangle_1^Q(d)}(t,u^1)\}\}}\\ &=\min\underbrace{\{\min\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\Theta_1^Q(d)}(t,u^1)\},\min\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\triangle_1^Q(d)}(t,u^1)\}\}}\\ &=\min\underbrace{\{T_{\Psi_1^Q(a\odot d)}(w\odot t,u^1),T_{\mho_1^Q(a\odot d)}(w\odot t,u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\cap(\mathfrak{A}^1\cap\mathfrak{C}^1))}\end{aligned}\tag{7}$$

Comparing (5) and (7), we have $(\Pi_1^Q,\mathfrak{G}^1)=(\Xi_1^Q,\mathfrak{H}^1)$ for the true membership degree.

Next, is to verify for indeterminate membership degree.

Let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Psi_1^Q, \mathfrak{E}^1) \text{ such that}$$
$$I_{\Psi_1^Q(a\odot b)}(w \odot t, u^1) = \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(b)}(t, u^1)\}}_{(a\odot b)=t\in\mathfrak{A}^1\cap\mathfrak{B}^1} \tag{8}$$

for all $w, t \in \hat{G}$ and $u^1 \in Q$. And, let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\mho_1^Q, \mathfrak{F}^1) \text{ such that for all } w, t \in \hat{G}, \text{ and } u^1 \in Q \text{ we have}$$
$$I_{\mho_1^Q(a\odot c)}(w \odot t, u^1) = \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(c)}(t, u^1)\}}_{(a\odot c)\in\mathfrak{A}^1\cap\mathfrak{C}^1} \tag{9}$$

Let $(\Psi_1^Q, \mathfrak{E}^1) \cap (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{G}^1)$ such that $g \in (\mathfrak{E}^1 \cap \mathfrak{F}^1) = \mathfrak{G}^1$ for all $g \in \mathfrak{G}^1$

Combining equations (8) and (9), for all $w, t \in \hat{G}$ and $u^1 \in Q$ we have

$$\begin{aligned} I_{\Pi_1^Q(g)}(wt, u^1) &= \max \underbrace{\{I_{\Psi_1^Q(g)}(wt, u^1), I_{\mho_1^Q(g)}(wt, u^1)\}}_{(g)\in\mathfrak{E}\cap\mathfrak{F}.} \\ &= \max \{ \underbrace{I_{\Psi_1^Q(a\odot b)}(wt, u^1), I_{\mho_1^Q(a\odot c)}(wt, u^1)}_{a\in((\mathfrak{A}^1\cap\mathfrak{B}^1)\cap(\mathfrak{A}^1\cap\mathfrak{C}^1))} \end{aligned} \tag{10}$$

Considering the LHS: Let

$$(\Theta_1^Q, \mathfrak{B}^1) \cap (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1) \text{ such that for all } w, t \in \hat{G} \text{ and } u^1 \in Q, \text{we have}$$
$$I_{\Phi_1^Q(d)}(wt, u^1) = \max \underbrace{\{I_{\Theta_1^Q(d)}(wt, u^1), I_{\triangle_1^Q(d)}(wt, u^1)\}}_{d\in(\mathfrak{B}^1\cap\mathfrak{C}^1)} \tag{11}$$

Also, let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1) \text{ such that for all } w, t \in \hat{G} \text{ and } u^1 \in Q, \text{ we have}$$
$$\begin{aligned} I_{\Xi_1^Q(a\odot d)}(w \odot t, u^1) = \max &\underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{this implies that for all} a\in\mathfrak{A}^1, d\in\mathfrak{D}^1, a\odot d\in(\mathfrak{A}^1\cap\mathfrak{D}^1)} \\ = \max &\underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), \max\{I_{\Theta_1^Q(d)}(t, u^1), I_{\triangle_1^Q(d)}(t, u^1)\}\}} \\ = \max &\underbrace{\{ \max\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Lambda_1^Q(a)}(w, u^1)\}, \max\{I_{\Theta_1^Q(d)}(t, u^1), I_{\triangle_1^Q(d)}(t, u^1)\}\}} \\ = \max &\underbrace{\{ \max\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(d)}(t, u^1)\}, \max\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(d)}(t, u^1)\}\}} \\ = \max &\underbrace{\{I_{\Psi_1^Q(a\odot d)}(w \odot t, u^1), I_{\mho_1^Q(a\odot d)}(w \odot t, u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\cap(\mathfrak{A}^1\cap\mathfrak{C}^1))} \end{aligned} \tag{12}$$

Comparing (10) and (12) to get that $(\Pi_1^Q, \mathfrak{G}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ for the indeterminate membership degree.

Next, verifying the falsity membership degree is similar with the result obtain for indeterminate membership degree .

(2) It has a similar argument with (1)

(3) We shall show that $(\Lambda_1^Q, \mathbb{A}) \odot \big((\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1)\big) = \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big) \wedge \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big)$

Let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Psi_1^Q, \mathfrak{E}^1) \text{ such that}$$
$$T_{\Psi_1^Q(a\odot b)}(w \odot t, u^1) = \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(b)}(t, u^1)\}}_{(a\odot b) = t \in \mathfrak{A}^1 \cap \mathfrak{B}^1} \tag{13}$$

for all $w, t \in \hat{G}$ and $u^1 \in Q$.

And, let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\mho_1^Q, \mathfrak{F}^1) \text{ such that for all } w \in \hat{G}, u^1 \in Q \text{ wt have}$$
$$T_{\mho_1^Q(a\odot c)}(w \odot t, u^1) = \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}}_{(a\odot c) \in \mathfrak{A}^1 \cap \mathfrak{C}^1} \tag{14}$$

Now, from equations (13) and (14) we have $(\Psi_1^Q, \mathfrak{E}^1) \wedge (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{E}^1 \times \mathfrak{F}^1)$ $(e^*, f^*) \in (\mathfrak{E}^1 \times \mathfrak{F}^1)$ where $t^* = a \odot b$ and $f^* = a \odot c$ are parameter sets

Hence,

$$\begin{aligned}
T_{\Pi_1^Q(g)}(w \odot t, u^1) &= T_{\Pi_1^Q(t^*, f^*)}((w \odot t), u^1) \\
&= \min \underbrace{\{T_{\Psi_1^Q(t)}((w \odot t), u^1), T_{\mho_1^Q(f)}((w \odot t), u^1)\}}_{(t^*, f^*) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\
&= \min \underbrace{\{T_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), T_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(t^*, f^*) \in \mathfrak{E} \times \mathfrak{F}.} \qquad (15)
\end{aligned}$$

Note that t^* and f^* are set of parameters.

Substituting (13) and (14) into (15), give

$$\min \underbrace{\{T_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), T_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(t^*, f^*) \in \mathfrak{E}^1 \times \mathfrak{F}^1.}$$
$$\geq \min \left\{ \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(b)}(t, u^1)\}}_{a\odot b \in \mathfrak{A}^1 \cap \mathfrak{B}^1.}, \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}}_{a \odot c \in \mathfrak{A}^1 \cap \mathfrak{C}^1.} \right\}$$
$$= \min \underbrace{\{T_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), T_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \times (\mathfrak{A}^1 \cap \mathfrak{C}^1).} \tag{16}$$

Considering the LHS:

Let $(\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that $(\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{B}^1 \times \mathfrak{C}^1)$ for all $w, t \in \hat{G}, u^1 \in Q$ and $(b, c) \in \mathfrak{B}^1 \times \mathfrak{C}^1$. Then, this follows

$$T_{\Phi_1^Q(b,c)}(wt, u^1) = \min\{T_{\Theta_1^Q(b)}(wt, u^1), T_{\triangle_1^Q(c)}(wt, u^1)\} \tag{17}$$

And, let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}, u^1 \in Q$, we have

$$T_{\Xi_1^Q(a\odot d)}(wt, u^1) = \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a\odot d \in \mathfrak{A}\cap\mathfrak{D} \text{ where } d=(b,c)\in\mathfrak{B}^1\times\mathfrak{C}^1} \tag{18}$$

Then, putting (17) into (18), give

$$\begin{aligned}
&T_{\Xi_1^Q(h)}(wt, u^1) = T_{\Xi_1^Q(a,d)}(wt, u^1) \\
&= \min\{\underbrace{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a\odot d\in\mathfrak{A}^1\cap\mathfrak{D}^1 \text{ where } wt=t\in\hat{G}} \\
&= \min\underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(b,c)}(wt, u^1)\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.} \\
&= \min\underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), \min\{T_{\Theta_1^Q(b)}(t, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.} \\
&= \min\underbrace{\{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Lambda_1^Q(a)}(w, u^1)\}, \min\{T_{\Theta_1^Q(b)}(t, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\}}_{(a\odot(b,c)\in\mathfrak{A}^1\cap(\mathfrak{B}^1\times\mathfrak{C}^1).} \\
&= \min\{\underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(b)}(t, u^1)\}}_{(a\odot(b))\in\mathfrak{A}^1\cap\mathfrak{B}^1.}, \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\}}_{(a\odot c))\in\mathfrak{A}^1\cap\mathfrak{C}^1.} \\
&\min\underbrace{\{T_{\Psi_1^Q(a\odot b)}((w\odot t), u^1), T_{\mho_1^Q(a\odot c)}((w\odot t), u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\times(\mathfrak{A}^1\cap\mathfrak{C}^1).}
\end{aligned} \tag{19}$$

For indeterminacy membership degree.

Let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Psi_1^Q, \mathfrak{E}^1) \text{ such that}$$
$$I_{\Psi_1^Q(a\odot b)}(w\odot t, u^1) = \max\underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(b)}(t, u^1)\}}_{(a\odot b)=t\in\mathfrak{A}^1\cap\mathfrak{B}^1} \tag{20}$$

for all $w \in \hat{G}$ and $u^1 \in Q$. And, let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\mho_1^Q, \mathfrak{F}^1) \text{ such that for all } w, t \in \hat{G}, u^1 \in Q \text{ we have}$$
$$I_{\mho_1^Q(a\odot c)}(w\odot t, u^1) = \max\underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(c)}(t, u^1)\}}_{(a\odot c)\in\mathfrak{A}^1\cap\mathfrak{C}^1} \tag{21}$$

Now, from equations (20) and (21) we have that $(\Psi_1^Q, \mathfrak{E}^1) \wedge (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{E}^1 \times \mathfrak{F}^1)$ for all $(t^*, f^*) \in (\mathfrak{E}^1 \times \mathfrak{F}^1)$ where $t^* = a \odot b$ and $f^* = a \odot c$ are set of parameters. Then,

$$\begin{aligned} I_{\Pi_1^Q(g)}(w \odot t, u^1) &= I_{\Pi_1^Q(t,f)}((w \odot t), u^1) \\ &= \max \underbrace{\{I_{\Psi_1^Q(t)}((w \odot t), u^1), I_{\mho_1^Q(f)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ &= \max \underbrace{\{I_{\Psi_1^Q(a \odot b)}((w \odot t), u^1), I_{\mho_1^Q(a \odot c)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \end{aligned} \tag{22}$$

Substituting (20) and (21) into (22), give

$$\begin{aligned} &\max \underbrace{\{I_{\Psi_1^Q(a \odot b)}((w \odot t), u^1), I_{\mho_1^Q(a \odot c)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ &\geq \max \left\{ \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(b)}(t, u^1)\}}_{a \odot b \in \mathfrak{A}^1 \cap \mathfrak{B}^1.}, \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(c)}(t, u^1)\}}_{a \odot c \in \mathfrak{A}^1 \cap \mathfrak{C}^1.} \right\} \\ &= \max \underbrace{\{I_{\Psi_1^Q(a \odot b)}((w \odot t), u^1), I_{\mho_1^Q(a \odot c)}((w \odot t), u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \times (\mathfrak{A}^1 \cap \mathfrak{C}^1).} \end{aligned} \tag{23}$$

Considering the LHS:
Let $(\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that $(\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{B}^1 \times \mathfrak{C}^1)$ for all $w, t \in \hat{G}, u^1 \in Q$ and $(b, c) \in \mathfrak{B}^1 \times \mathfrak{C}^1$, It follows that

$$I_{\Phi_1^Q(b,c)}(wt, u^1) = \max\{I_{\Theta_1^Q(b)}(wt, u^1), I_{\triangle_1^Q(c)}(wt, u^1)\} \tag{24}$$

And, let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}, u^1 \in Q$,

$$I_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) = \underbrace{\max\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a \odot d \in \mathfrak{A}^1 \cap \mathfrak{D}^1 \text{ where } d = (b,c) \in \mathfrak{B}^1 \times \mathfrak{C}^1} \tag{25}$$

Then, putting (24) into (25), we have

$$\begin{aligned}
&I_{\Xi_1^Q(h)}(wt,u^1) = I_{\Xi_1^Q(a\odot d)}(wt,u^1)\\
&= \max\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Phi_1^Q(d)}(t,u^1)\}}_{\text{for all } a\odot d\in\mathfrak{A}^1\cap\mathfrak{D}^1 \text{ where } w\odot t=t\in\hat{G}}\\
&= \max\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Phi_1^Q(b,c)}(wt,u^1)\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&= \max\Big\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1), \max\{I_{\Theta_1^Q(b)}(t,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}\Big\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&= \max\Big\{\underbrace{\max\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Lambda_1^Q(a)}(w,u^1)\}, \max\{I_{\Theta_1^Q(b)}(t,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}\Big\}}_{(a\odot(b,c))\in\mathfrak{A}^1\cap(\mathfrak{B}^1\times\mathfrak{C}^1).}\\
&= \max\Big\{\underbrace{\max\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot(b))\in\mathfrak{A}^1\cap\mathfrak{B}^1.}, \underbrace{\max\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}\Big\}}_{(a\odot c))\in\mathfrak{A}^1\cap\mathfrak{C}^1.}\\
&\max\underbrace{\{I_{\Psi_1^Q(a\odot b)}((w\odot t),u^1), I_{\mho_1^Q(a\odot c)}((w\odot t),u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\times(\mathfrak{A}^1\cap\mathfrak{C}^1).}
\end{aligned} \tag{26}$$

The proof of falsity membership degree has a similar argument with the proof of indeterminate membership

Therefore, we shown that $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big) = \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(4) The proof is similar with 3

(5) We shall show that $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\cup(\triangle_1^Q,\mathfrak{C}^1)\big) = \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\cup \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

Let

$$\begin{aligned}
&(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1) = (\Psi_1^Q,\mathfrak{C}^1) \text{ such that}\\
&T_{\Psi_1^Q(a\odot b)}(w\odot t,u^1) = \min\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot b)=s\in\mathfrak{A}^1\cap\mathfrak{B}^1}
\end{aligned} \tag{27}$$

for all $w,t\in\hat{G}$ and $u^1\in Q$. And, let

$$\begin{aligned}
&(\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1) = (\mho_1^Q,\mathfrak{F}^1) \text{ such that for all } w,t\in Q, u^1\in Q \text{ we have}\\
&T_{\mho_1^Q(a\odot c)}(w\odot t,u^1) = \max\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\triangle_1^Q(c)}(t,u^1)\}}_{(f=a\odot c)\in\mathfrak{A}^1\cap\mathfrak{C}^1}
\end{aligned} \tag{28}$$

Now, from equations (27) and (28), we get $(\Psi_1^Q, \mathfrak{E}^1) \cup (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{G}^1)$. Then, for all $w, t \in \hat{G}, u^1 \in Q$, let $d \in \mathfrak{D}^1$ such that $(a \odot b) - (a \odot c) = g \in \mathfrak{G}^1 = \mathfrak{E}^1 \cup \mathfrak{F}^1$,

$$T_{\Pi_1^Q(g)}(w \odot t, u^1) = \begin{cases} T_{\Psi_1^Q(g)}(w \odot t, u^1), & \text{if } g \in \mathfrak{E}^1 - \mathfrak{F}^1 \\ T_{\mho_1^Q(g)}(w \odot t, u^1), & \text{if } g \in \mathfrak{F} - \mathfrak{E} \\ \max\{T_{\Psi_1^Q(g)}(w \odot t, u^1), T_{\mho_1^Q(g)}(w \odot t, u^1)\}, & \text{if } g \in \mathfrak{E}^1 \cap \mathfrak{F}^1 \end{cases} \tag{29}$$

$$= \begin{cases} \min\{T_{\Lambda_1^Q(g)}(w, u^1), T_{\Theta_1^Q(g)}(t, u^1)\}, & \text{if } g \in ((\mathfrak{A}^1 \cap \mathfrak{B}^1) - (\mathfrak{A}^1 \cap \mathfrak{C}^1)) \\ \min\{T_{\Lambda_1^Q(g)}(w, u^1), T_{\mho_1^Q(g)}(t, u^1)\}, & \text{if } g \in ((\mathfrak{A}^1 \cap \mathfrak{C}^1) - (\mathfrak{A}^1 \cap \mathfrak{B}^1)) \\ \max\left\{ \min\{T_{\Lambda_1^Q(g)}(w, u^1), T_{\Theta_1^Q(g)}(t, u^1)\}, \right. & \\ \left. \min\{T_{\Lambda_1^Q(g)}(w, u^1), T_{\triangle_1^Q(g)}(t, u^1)\} \right\}, & \text{if } g \in (\mathfrak{A}^1 \cap \mathfrak{B}^1) \cap (\mathfrak{A}^1 \cap \mathfrak{C}^1) \end{cases} \tag{30}$$

Considering the LHS

Let $(\Theta_1^Q, \mathfrak{B}^1) \cup (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that for all $w, t \in \hat{G}$ and $u^1 \in Q$, we have

$$T_{\Phi_1^Q(d)}(w \odot t, u^1) = \begin{cases} T_{\Theta_1^Q(d)}(w \odot t, u^1), & \text{if } d \in \mathfrak{B}^1 - \mathfrak{C}^1 \\ T_{\triangle_1^Q(d)}(w \odot t, u^1), & \text{if } d \in \mathfrak{C}^1 - \mathfrak{B}^1 \\ \max\{T_{\Theta_1^Q(d)}(w \odot t, u^1), T_{\triangle_1^Q(d)}(w \odot t, u^1)\}, & \text{if } d \in \mathfrak{B}^1 \cap \mathfrak{C}^1 \end{cases} \tag{31}$$

Let

$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1))$ such that for all $w, t \in \hat{G}, u^1 \in Q$ we have

$$T_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) = \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{(s = a \odot d) \in \mathfrak{A}^1 \cap \mathfrak{D}^1} \tag{32}$$

$$T_{\Xi_1^Q(t)}(w \odot t, u^1) = \begin{cases} \min\{T_{\Lambda_1^Q(s)}(w, u^1), \Theta_1^Q(t)(t, u^1)\}, & \text{if } s \in \mathfrak{A}^1 \cap (\mathfrak{B}^1 - \mathfrak{C}^1) \\ \min\{T_{\Lambda_1^Q(s)}(w, u^1), T_{\triangle_1^Q(t)}(t, u^1)\}, & \text{if } s \in \mathfrak{A}^1 \cap (\mathfrak{C}^1 - \mathfrak{B}^1) \\ \max\left\{ \{T_{\Lambda_1^Q(s)}(w, u^1)\}, \right. & \\ \left. \min\{T_{\Theta_1^Q(s)}(t, u^1), T_{\triangle_1^Q(s)}(t, u^1)\} \right\}, & \text{if } s \in \mathfrak{A}^1 \cap (\mathfrak{B}^1 \cap \mathfrak{C}^1) \end{cases} \tag{33}$$

$$= \begin{cases} \min\{T_{\Lambda_1^Q(s)}(w, u^1), \Theta_1^Q(t)(t, u^1)\}, & \text{if } s \in (\mathfrak{A}^1 \cap \mathfrak{B}^1) - (\mathfrak{A}^1 \cap \mathfrak{C}^1) \\ \min\{T_{\Lambda_1^Q(s)}(w, u^1), T_{\triangle_1^Q(s)}(t, u^1)\}, & \text{if } s \in (\mathfrak{A}^1 \cap \mathfrak{C}^1) - (\mathfrak{A}^1 \cap \mathfrak{B}^1) \\ \max\left\{ \min\{T_{\Lambda_1^Q(s)}(w, u^1), T_{\Theta_1^Q(s)}(t, u^1)\}, \right. & \\ \left. \min\{T_{\Lambda_1^Q(s)}(w, u^1), T_{\triangle_1^Q(s)}(t, u^1)\} \right\}, & \text{if } s \in (\mathfrak{A}^1 \cap \mathfrak{B}^1) \cap (\mathfrak{A}^1 \cap \mathfrak{C}^1) \end{cases} \tag{34}$$

Comparing equation (30) and (34), we shown that $(\Xi_1^Q, \mathfrak{H}^1) = (\Pi_1^Q, \mathfrak{G}^1)$

Next, the result for indeterminate membership degree is as follows

Let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Psi_1^Q, \mathfrak{E}^1) \text{ then}$$
$$I_{\Psi_1^Q(a\odot b)}(w \odot t, u^1) = \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(b)}(t, u^1)\}}_{(a\odot b)=t\in\mathfrak{A}^1\cap\mathfrak{B}^1} \tag{35}$$

for all $w, t \in \hat{G}$ and $u^1 \in Q$. And, let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\mho_1^Q, \mathfrak{F}^1) \text{ such that for all } w, t \in \hat{G}, u^1 \in Q \text{ we have}$$
$$I_{\mho_1^Q(a\odot c)}(w \odot t, u^1) = \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(c)}(t, u^1)\}}_{(f=a\odot c)\in\mathfrak{A}^1\cap\mathfrak{C}^1} \tag{36}$$

Using equations (35) and (36), we get $(\Psi_1^Q, \mathfrak{E}^1) \cup (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{G}^1)$. Then, for all $w, t \in Q, u^1 \in Q$, let $d \in \mathfrak{D}^1$ such that $(a \odot b) - (a \odot c) = g \in \mathfrak{G}^1 = \mathfrak{E}^1 \cup \mathfrak{F}^1$, then

$$I_{\Pi_1^Q(g)}(w \odot t, u^1) = \begin{cases} I_{\Psi_1^Q(g)}(w \odot t, u^1), & \text{if } g \in \mathfrak{E}^1 - \mathfrak{F}^1 \\ I_{\mho_1^Q(g)}(w \odot t, u^1), & \text{if } g \in \mathfrak{F}^1 - \mathfrak{E}^1 \\ \min\{I_{\Psi_1^Q(g)}(w \odot t, u^1), I_{\mho_1^Q(g)}(w \odot t, u^1)\}, & \text{if } g \in \mathfrak{E}^1 \cap \mathfrak{F}^1 \end{cases} \tag{37}$$

$$= \begin{cases} \max\{I_{\Lambda_1^Q(g)}(w, u^1), I_{\Theta_1^Q(g)}(t, u^1)\}, & \text{if } g \in ((\mathfrak{A}^1 \cap \mathfrak{B}^1) - (\mathfrak{A}^1 \cap \mathfrak{C}^1)) \\ \max\{I_{\Lambda_1^Q(g)}(w, u^1), I_{\mho_1^Q(g)}(t, u^1)\}, & \text{if } g \in ((\mathfrak{A}^1 \cap \mathfrak{C}^1) - (\mathfrak{A}^1 \cap \mathfrak{B}^1)) \\ \min\Big\{ \max\{I_{\Lambda_1^Q(g)}(w, u^1), I_{\Theta_1^Q(g)}(t, u^1)\}, & \\ \max\{I_{\Lambda_1^Q(g)}(w, u^1), I_{\triangle_1^Q(g)}(t, u^1)\}\Big\}, & \text{if } g \in (\mathfrak{A}^1 \cap \mathfrak{B}^1) \cap (\mathfrak{A}^1 \cap \mathfrak{C}^1) \end{cases} \tag{38}$$

Considering the LHS

Let $(\Theta_1^Q, \mathfrak{B}^1) \cup (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that for all $w, t \in \hat{G}$ and $u^1 \in Q$, we have

$$I_{\Phi_1^Q(d)}(w \odot t, u^1) = \begin{cases} I_{\Theta_1^Q(d)}(w \odot t, u^1), & \text{if } d \in \mathfrak{B}^1 - \mathfrak{C}^1 \\ I_{\triangle_1^Q(d)}(w \odot t, u^1), & \text{if } d \in \mathfrak{C}^1 - \mathfrak{B}^1 \\ \min\{I_{\Theta_1^Q(d)}(w \odot t, u^1), I_{\triangle_1^Q(d)}(w \odot t, u^1)\}, & \text{if } d \in \mathfrak{B}^1 \cap \mathfrak{C}^1 \end{cases} \tag{39}$$

Let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)) \text{ such that for all } w, t \in \hat{G}, u^1 \in Q \text{ we have}$$
$$I_{\Xi_1^Q(a\odot d)}(w \odot t, u^1) = \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Phi_1^Q(d)}(t, u^1)\}}_{(s=a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1} \tag{40}$$

$$I_{\Xi_1^Q(t)}(w\odot t,u^1)=\begin{cases}\max\{I_{\Lambda_1^Q(s)}(w,u^1),\Theta_1^Q(s)(t,u^1)\}, & \text{if } s\in\mathfrak{A}^1\cap(\mathfrak{B}^1-\mathfrak{C}^1)\\ \max\{I_{\Lambda_1^Q(s)}(w,u^1),I_{\triangle_1^Q(s)}(t,u^1)\}, & \text{if } s\in\mathfrak{A}^1\cap(\mathfrak{C}^1-\mathfrak{B}^1)\\ \min\Big\{\{I_{\Lambda_1^Q(s)}(w,u^1)\}, & \\ \max\{I_{\Theta_1^Q(s)}(t,u^1),I_{\triangle_1^Q(s)}(t,u^1)\}\Big\}, & \text{if } s\in\mathfrak{A}^1\cap(\mathfrak{B}^1\cap\mathfrak{C}^1)\end{cases}\tag{41}$$

$$=\begin{cases}\max\{I_{\Lambda_1^Q(s)}(w,u^1),\Theta_1^Q(s)(t,u^1)\}, & \text{if } s\in(\mathfrak{A}^1\cap\mathfrak{B}^1)-(\mathfrak{A}^1\cap\mathfrak{C}^1)\\ \max\{I_{\Lambda_1^Q(s)}(w,u^1),I_{\triangle_1^Q(t)}(t,u^1)\}, & \text{if } s\in(\mathfrak{A}^1\cap\mathfrak{C}^1)-(\mathfrak{A}^1\cap\mathfrak{B}^1)\\ \min\Big\{\max\{I_{\Lambda_1^Q(s)}(w,u^1),I_{\Theta_1^Q(s)}(t,u^1)\}, & \\ \max\{I_{\Lambda_1^Q(s)}(w,u^1),I_{\triangle_1^Q(s)}(s,u^1)\}\Big\}, & \text{if } s\in(\mathfrak{A}^1\cap\mathfrak{B}^1)\cap(\mathfrak{A}^1\cap\mathfrak{C}^1)\end{cases}\tag{42}$$

Comparing equation (38) and (42), we shown that $(\Xi_1^Q,\mathfrak{H}^1)=(\Pi_1^Q,\mathfrak{G}^1)$

Next, the result for falsity membership degree is similarly with the argument of indeterminate membership. Hence, we shown that $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\cup(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\cup\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(6) The proof is similar with (5)

(7) We shall show that $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\vee\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

Considering the RHS

Let

$$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)=(\Psi_1^Q,\mathfrak{E}^1)\text{ such that}$$
$$T_{\Psi_1^Q(a\odot b)}(w\odot t,u^1)=\min\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot b)=t\in\mathfrak{A}^1\cap\mathfrak{B}^1}\tag{43}$$

for all $t,w\in\hat{G}$ and $u^1\in Q$. And, let

$$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)=(\mho_1^Q,\mathfrak{F}^1)\text{ such that for all } w,t\in\hat{G},u^1\in Q\text{ we have}$$
$$T_{\mho_1^Q(a\odot c)}(w\odot t,u^1)=\min\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\triangle_1^Q(c)}(t,u^1)\}}_{(a\odot c)\in\mathfrak{A}^1\cap\mathfrak{C}^1}\tag{44}$$

Now, combining equations (43) and (44) give $(\Psi_1^Q,\mathfrak{E}^1)\vee(\mho_1^Q,\mathfrak{F}^1)=(\Pi_1^Q,\mathfrak{E}^1\times\mathfrak{F}^1)$ for all $(t^*,f^*)\in(\mathfrak{E}\times\mathfrak{F})$ where $t^*=a\odot b$ and $f^*=a\odot c$ are set of parameters. Then, equations (43) and (44) gives

$$\begin{aligned} T_{\Pi_1^Q(g)}(w \odot t, u^1) &= T_{\Pi_1^Q(t^*,f^*)}((w \odot t), u^1) \\ &= \min \underbrace{\{T_{\Psi_1^Q(t^*)}((w \odot t), u^1), T_{\mho_1^Q(f^*)}((w \odot t), u^1)\}}_{(t^*, f^*) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ &= \max \underbrace{\{T_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), T_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(t^* = a\odot b, f^* = a\odot c) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \end{aligned} \tag{45}$$

Putting (43) and (44) into (45), we have

$$\begin{aligned} &\max \underbrace{\{T_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), T_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ \geq \max &\left\{ \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(b)}(t, u^1)\}}_{a\odot b \in \mathfrak{A}^1 \cap \mathfrak{B}^1.}, \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}}_{a\odot c \in \mathfrak{A}^1 \cap \mathfrak{C}^1.} \right\} \\ &= \max \underbrace{\{T_{\Psi_1^Q(t)}((w \odot t), u^1), T_{\mho_1^Q(f)}((w \odot t), u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \times (\mathfrak{A}^1 \cap \mathfrak{C}^1).} \end{aligned} \tag{46}$$

Considering the LHS:
Let $(\Theta_1^Q, \mathfrak{B}^1) \vee (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that $(\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{B}^1 \times \mathfrak{C}^1)$. For all $w, t \in \hat{G}, u^1 \in Q$ and $(b, c) \in \mathfrak{B}^1 \times \mathfrak{C}^1$ we have

$$T_{\Phi_1^Q(b,c)}(wt, u^1) = \max\{T_{\Theta_1^Q(b)}(wt, u^1), T_{\triangle_1^Q(c)}(wt, u^1)\} \tag{47}$$

Also, let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}, u^1 \in Q$, we have

$$T_{\Xi_1^Q(a\odot d)}(w \odot t, u^1) = \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a\odot d \in \mathfrak{A}^1 \cap \mathfrak{D}^1 \text{ where } d = (b,c) \in \mathfrak{B}^1 \times \mathfrak{C}^1} \tag{48}$$

Then, putting (47) into (48), we have

$$T_{\Xi_1^Q(h)}(wt,u^1) = T_{\Xi_1^Q(a,d)}(wt,u^1)$$
$$= \min\{\underbrace{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Phi_1^Q(d)}(t,u^1)\}}_{\text{for all } a\odot d\in\mathfrak{A}^1\cap\mathfrak{D}^1 \text{ where } w\odot t=t\in\hat{G}}$$
$$= \min\{\underbrace{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Phi_1^Q(b,c)}(wt,u^1)\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}$$
$$= \min\big\{\underbrace{T_{\Lambda_1^Q(a)}(w,u^1), \max\{T_{\Theta_1^Q(b)}(t,u^1), T_{\triangle_1^Q(c)}(t,u^1)\}\big\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}$$
$$= \min\big\{\underbrace{\max\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Lambda_1^Q(a)}(w,u^1)\}, \max\{T_{\Theta_1^Q(b)}(t,u^1), T_{\triangle_1^Q(c)}(t,u^1)\}\big\}}_{(a\odot(b,c)\in\mathfrak{A}^1\cap(\mathfrak{B}^1\times\mathfrak{C}^1).}$$
$$= \max\big\{\underbrace{\min\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot(b))\in\mathfrak{A}^1\cap\mathfrak{B}^1.}, \underbrace{\min\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\triangle_1^Q(c)}(t,u^1)\}\big\}}_{(a\odot c))\in\mathfrak{A}^1\cap\mathfrak{C}^1.}$$
$$\max\underbrace{\{T_{\Psi_1^Q(a\odot b)}((w\odot t),u^1), T_{\mho_1^Q(a\odot c)}((w\odot t),u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\times(\mathfrak{A}^1\cap\mathfrak{C}^1).} \tag{49}$$

Considering the result for indeterminate membership.

Let

$$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1) = (\Psi_1^Q,\mathfrak{C}^1) \text{ such that}$$
$$I_{\Psi_1^Q(a\odot b)}(w\odot t,u^1) = \max\underbrace{\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot b)=t\in\mathfrak{A}^1\cap\mathfrak{B}^1} \tag{50}$$

for all $t,w\in\hat{G}$ and $u^1\in Q$. And, let

$$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1) = (\mho_1^Q,\mathfrak{F}^1) \text{ such that for all } t,w\in Q, u^1\in Q \text{ we get}$$
$$I_{\mho_1^Q(a\odot c)}(w\odot t,u^1) = \max\underbrace{\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}}_{(a\odot c)\in\mathfrak{A}^1\cap\mathfrak{C}^1} \tag{51}$$

Now, from equations (50) and (51) we have $(\Psi_1^Q,\mathfrak{C}^1)\vee(\mho_1^Q,\mathfrak{F}^1) = (\Pi_1^Q,\mathfrak{C}^1\times\mathfrak{F}^1)$ for all $(t^*,f^*)\in(\mathfrak{C}^1\times\mathfrak{F}^1)$ where $t^*=a\odot b$ and $f=a\odot c$ are parameters. Then, this follows

$$\begin{aligned} I_{\Pi_1^Q(g)}(w \odot t, u^1) &= I_{\Pi_1^Q(t,f)}((w \odot t), u^1) \\ &= \min \underbrace{\{I_{\Psi_1^Q(t)}((w \odot t), u^1), I_{\mho_1^Q(f)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ &= \min \underbrace{\{I_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), I_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(t = a\odot b, f = a \odot c) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \end{aligned} \tag{52}$$

Substitute (50) and (51) in (52), to get

$$\begin{aligned} &\min \underbrace{\{I_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), I_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E} \times \mathfrak{F}.} \\ &\geq \min \left\{ \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(b)}(t, u^1)\}}_{a\odot b \in \mathfrak{A}^1 \cap \mathfrak{B}^1.}, \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(c)}(t, u^1)\}}_{a \odot c \in \mathfrak{A} \cap \mathfrak{C}.} \right\} \\ &= \min \underbrace{\{I_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), I_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \times (\mathfrak{A}^1 \cap \mathfrak{C}^1).} \end{aligned} \tag{53}$$

Considering the LHS:
Let $(\Theta_1^Q, \mathfrak{B}^1) \vee (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that $(\Theta_1^Q, \mathfrak{B}^1) \vee (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{B}^1 \times \mathfrak{C}^1)$ for all $w, t \in \hat{G}, u^1 \in Q$ and $(b, c) \in \mathfrak{B}^1 \times \mathfrak{C}^1$. It is follows that,

$$I_{\Phi_1^Q(b,c)}(wt, u^1) = \min\{I_{\Theta_1^Q(b)}(wt, u^1), I_{\triangle_1^Q(c)}(wt, u^1)\} \tag{54}$$

Also, let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}, u^1 \in Q$,

$$I_{\Xi_1^Q(a\odot d)}(w \odot t, u^1) = \underbrace{\max\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a\odot d \in \mathfrak{A}^1 \cap \mathfrak{D}^1 \text{ where } d = (b,c) \in \mathfrak{B}^1 \times \mathfrak{C}^1} \tag{55}$$

are set of parameters. Then, putting (54) in (55), we get

$$\begin{aligned}
&I_{\Xi_1^Q(h)}(w\odot t,u^1)=I_{\Xi_1^Q(a\odot d)}(wt,u^1)\\
&=\max\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1),I_{\Phi_1^Q(d)}(t,u^1)\}}_{\text{for all } a\odot d\in\mathfrak{A}^1\cap\mathfrak{D}^1 \text{ where } wt=t\in\hat{G}}\\
&=\max\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1),I_{\Phi_1^Q(b,c)}(w\odot t,u^1)\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&=\max\big\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1),\min\{I_{\Theta_1^Q(b)}(t,u^1),I_{\triangle_1^Q(c)}(t,u^1)\}\big\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&=\max\big\{\underbrace{\min\{I_{\Lambda_1^Q(a)}(w,u^1),I_{\Lambda_1^Q(a)}(w,u^1)\},\min\{I_{\Theta_1^Q(b)}(t,u^1),I_{\triangle_1^Q(c)}(t,u^1)\}\big\}}_{(a\odot(b,c)\in\mathfrak{A}^1\cap(\mathfrak{B}^1\times\mathfrak{C}^1).}\\
&=\min\big\{\underbrace{\max\{I_{\Lambda_1^Q(a)}(w,u^1),I_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot(b))\in\mathfrak{A}^1\cap\mathfrak{B}^1.},\underbrace{\max\{I_{\Lambda_1^Q(a)}(w,u^1),I_{\triangle_1^Q(c)}(t,u^1)\}\big\}}_{(a\odot c))\in\mathfrak{A}^1\cap\mathfrak{C}^1.}\\
&\min\underbrace{\{I_{\Psi_1^Q(a\odot b)}((w\odot t),u^1),I_{\mho_1^Q(a\odot c)}((w\odot t),u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\times(\mathfrak{A}^1\cap\mathfrak{C}^1).}
\end{aligned}\tag{56}$$

Next, the result for falsity membership degree is similar with the one obtained for indeterminate membership

Therefore, $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(8) Similar with the result obtained for 7

Theorem 3.6. *Let $(\Lambda_1^Q,\mathfrak{A}^1),(\Theta_1^Q,\mathfrak{B}^1)$ and $(\triangle_1^Q,\mathfrak{C}^1)$ be Q-neutrosophic soft quasigroups over quasigroup $(\hat{G},\odot)$. Then, the following holds*

(1) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\odot\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(2) $\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(3) $(\Lambda_1^Q,\mathfrak{A}^1)/\big((\Theta_1^Q,\mathfrak{B}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)/\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(4) $\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)/(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)/\big((\Theta_1^Q,\mathfrak{B}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(5) $(\Lambda_1^Q,\mathfrak{A}^1)\backslash\big((\Theta_1^Q,\mathfrak{B}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\Theta_1^Q,\mathfrak{B}^1)\big)\backslash\big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)$
(6) $\big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\Theta_1^Q,\mathfrak{B}^1)\big)\backslash(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)\backslash\big((\Theta_1^Q,\mathfrak{B}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)$
(7) $\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\Theta_1^Q,\mathfrak{B}^1)\neq(\Lambda_1^Q,\mathfrak{A}^1)$
(8) $(\Theta_1^Q,\mathfrak{B}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\Lambda_1^Q,\mathfrak{A}^1)\big)\neq(\Lambda_1^Q,\mathfrak{A}^1)$

Proof:

(1) We want to show that $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\odot\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$ Considering the LHS

Let $(\Theta_1^Q, \mathfrak{B}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that for all $w, t \in \hat{G}$ and $u \in Q$ we have

$$T_{\Phi_1^Q(b \odot c)}((w \odot t), u^1) = \min\{T_{\Theta_1^Q(b)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\} \tag{57}$$

And, let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}$ and $u^1 \in Q$, we have

$$T_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) = \underbrace{\max\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a \odot d \in \mathfrak{A}^1 \cap \mathfrak{D}^1 \text{ where } d = (b \odot c) \in \mathfrak{B}^1 \cap \mathfrak{C}^1} \tag{58}$$

.

Substituting (57) into (58), to get

$$\begin{aligned}
T_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) &= T_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) \\
&= \min\{ \underbrace{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a \odot d \in \mathfrak{A}^1 \cap \mathfrak{D}^1 \text{ and let } w \odot t = w \in \hat{G}} \\
&= \min\{\underbrace{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(b \odot c)}(w \odot t, u^1)\}}_{(a \odot d) \in \mathfrak{A}^1 \cap \mathfrak{D}^1.} \\
&= \min \Big\{ \underbrace{T_{\Lambda_1^Q(a)}(w, u^1), \min\{T_{\Theta_1^Q(b)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\Big\}}_{(a \odot d) \in \mathfrak{A}^1 \cap \mathfrak{D}^1.} \\
&= \min \Big\{ \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Lambda_1^Q(a)}(t, u^1)\}, \min\{T_{\Theta_1^Q(b)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\Big\}}_{(a \odot (b \odot c) \in \mathfrak{A}^1 \cap (\mathfrak{B}^1 \times \mathfrak{C}^1).} \\
&= \min \Big\{ \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(b)}(w, u^1)\}}_{(a \odot (b)) \in \mathfrak{A}^1 \cap \mathfrak{B}^1.}, \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\Big\}}_{(a \odot c)) \in \mathfrak{A}^1 \cap \mathfrak{C}^1.} \\
&\min \underbrace{\{T_{\Psi_1^Q(a \odot b)}((w, u^1), T_{\mho_1^Q(a \odot c)}(t, u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \cap (\mathfrak{A}^1 \cap \mathfrak{C}^1).} && (59) \\
&= \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big) \odot \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big) && (60)
\end{aligned}$$

Similarly, we show for indeterminate and falsity membership degrees.

(2) Follow from 1

(3) Apply Definition 3.4 along side with 1 and 2

(4) Similar with 3

(5) Similar with 4

(6) Similar with 5

(7) Proof by contradiction. Suppose that $\big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Lambda_1^Q, \mathfrak{A}^1)$, then we have $\big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big) = (\Lambda_1^Q, \mathfrak{A}^1)/(\Theta_1^Q, \mathfrak{B}^1)$.
Let $z_1 = t_1 \odot w_1, z_2 = t_2 \odot w_2$ for all $z_1, z_2 \in \hat{G}$ and $u^1 \in Q$.

Let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Phi_1^Q, \mathfrak{C}^1)$ such that

$$T_{\Phi_1^Q(a\odot b)}((z_1 \odot z_2), u^1) = \min\{T_{\Lambda_1^Q(a)}(z_1, u^1), T_{\Theta_1^Q(b)}(z_2, u^1)\}$$
$$= \min\left\{ \min\{T_{\Lambda_1^Q(a)}(t_1, u^1), T_{\Lambda_1^Q(a)}(w_1, u^1)\}, \min\{T_{\Theta_1^Q(b)}(t_2, u^1), T_{\Theta_1^Q(b)}(w_2, u^1)\} \right\}$$
$$\min\left\{ \min\{T_{\Lambda_1^Q(a)}(t_1, u^1), T_{\Theta_1^Q(b)}(t_2, u^1)\}, \min\{T_{\Lambda_1^Q(a)}(w_1, u^1), T_{\Theta_1^Q(b)}(w_2, u^1)\} \right\}$$
$$\min\left\{T_{\Phi_1^Q(a\odot b)}(t_1 \odot t_2, u^1), T_{\Phi_1^Q(a\odot b)}(w_1 \odot w_2, u^1)\right\} \tag{61}$$

Considering the RHS, let $(\Lambda_1^Q, \mathfrak{A}^1)/(\Theta_1^Q, \mathfrak{B}^1) = (\Phi_1^Q, \mathfrak{C}^1)$. Then,

$$T_{\Phi_1^Q(a/b)}((z_1/z_2), u^1) = \max\{T_{\Lambda_1^Q(a)}(z_1, u^1), T_{\Theta_1^Q(b)}(z_2, u^1)\}$$
$$= \max\left\{ \min\{T_{\Lambda_1^Q(a)}(t_1, u^1), T_{\Lambda_1^Q(a)}(w_1, u^1)\}, \min\{T_{\Theta_1^Q(b)}(t_2, u^1), T_{\Theta_1^Q(b)}(w_2, u^1)\} \right\}$$
$$\max\left\{ \min\{T_{\Lambda_1^Q(a)}(t_1, u^1), T_{\Theta_1^Q(b)}(t_2, u^1)\}, \min\{T_{\Lambda_1^Q(a)}(w_1, u^1), T_{\Theta_1^Q(b)}(w_2, u^1)\} \right\}$$
$$\max\left\{T_{\Phi_1^Q(a\odot b)}(t_1 \odot t_2, u^1), T_{\Phi_1^Q(a\odot b)}(w_1 \odot w_2, u^1)\right\} \tag{62}$$

Hence, $\max\left\{T_{\Phi_1^Q(a\odot b)}(t_1 \odot t_2, u^1), T_{\Phi_1^Q(a\odot b)}(w_1 \odot w_2, u^1)\right\} \neq \min\left\{T_{\Phi_1^Q(a\odot b)}(t_1 \odot t_2, u^1), T_{\Phi_1^Q(a\odot b)}(w_1 \odot w_2, u^1)\right\}$. The results for indeterminate and falsity membership degrees are similarly obtained.

(8) Similar with 7

Theorem 3.7. *Let $(\Lambda_1^Q, \mathfrak{A}^1), (\Theta_1^Q, \mathfrak{B}^1)$ and $(\triangle_1^Q, \mathfrak{C}^1)$ be Q-neutrosophic soft quasigroups over quasigroup $(\hat{G}, \odot)$. Then, the following holds*

(1) $(\Lambda_1^Q, \mathfrak{A}^1) \odot \big((\Theta_1^Q, \mathfrak{B}^1)/(\triangle_1^Q, \mathfrak{C}^1)\big) = \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big)/\big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big)$
(2) $\big((\Lambda_1^Q, \mathfrak{A}^1)/(\Theta_1^Q, \mathfrak{B}^1)\big) \odot (\triangle_1^Q, \mathfrak{C}^1) = \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big)/\big((\Theta_1^Q, \mathfrak{B}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big)$
(3) $(\Lambda_1^Q, \mathfrak{A}^1) \odot \big((\Theta_1^Q, \mathfrak{B}^1)\backslash(\triangle_1^Q, \mathfrak{C}^1)\big) = \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big)\backslash\big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big)$
(4) $\big((\Lambda_1^Q, \mathfrak{A}^1)\backslash(\Theta_1^Q, \mathfrak{B}^1)\big) \odot (\triangle_1^Q, \mathfrak{C}^1) = \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big)\backslash\big((\Theta_1^Q, \mathfrak{B}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big)$
(5) $(\Lambda_1^Q, \mathfrak{A}^1)/\big((\Theta_1^Q, \mathfrak{B}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big) = \big((\Lambda_1^Q, \mathfrak{A}^1)/(\Theta_1^Q, \mathfrak{B}^1)\big) \odot \big((\Lambda_1^Q, \mathfrak{A}^1)/(\triangle_1^Q, \mathfrak{C}^1)\big)$
(6) $\big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big)/(\triangle_1^Q, \mathfrak{C}^1) = \big((\Lambda_1^Q, \mathfrak{A}^1)/(\triangle_1^Q, \mathfrak{C}^1)\big) \odot \big((\Theta_1^Q, \mathfrak{B}^1)/(\triangle_1^Q, \mathfrak{C}^1)\big)$
(7) $(\Lambda_1^Q, \mathfrak{A}^1)\backslash\big((\Theta_1^Q, \mathfrak{B}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big) = \big((\Lambda_1^Q, \mathfrak{A}^1)\backslash(\Theta_1^Q, \mathfrak{B}^1)\big) \odot \big((\Lambda_1^Q, \mathfrak{A}^1)\backslash(\triangle_1^Q, \mathfrak{C}^1)\big)$
(8) $\big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big)\backslash(\triangle_1^Q, \mathfrak{C}^1) = \big((\Lambda_1^Q, \mathfrak{A}^1)\backslash(\triangle_1^Q, \mathfrak{C}^1)\big) \odot \big((\Theta_1^Q, \mathfrak{B}^1)\backslash(\triangle_1^Q, \mathfrak{C}^1)\big)$

Proof: Similar with Theorem 3.5.

Theorem 3.8. *Let $(\Lambda_1^Q, \mathfrak{A}^1), (\Theta_1^Q, \mathfrak{B}^1)$ and $(\triangle_1^Q, \mathfrak{C}^1)$ be Q-neutrosophic soft quasigroups over quasigroup $(\hat{G}, \odot)$. Then, the following holds*

(1) $(\Lambda_1^Q, \mathfrak{A}^1)/\big((\Theta_1^Q, \mathfrak{B}^1) \cap (\triangle_1^Q, \mathfrak{C}^1)\big) = \big((\Lambda_1^Q, \mathfrak{A}^1)/(\Theta_1^Q, \mathfrak{B}^1)\big) \cap \big((\Lambda_1^Q, \mathfrak{A}^1)/(\triangle_1^Q, \mathfrak{C}^1)\big)$
(2) $\big((\Theta_1^Q, \mathfrak{B}^1) \cap (\triangle_1^Q, \mathfrak{C}^1)\big)/(\Lambda_1^Q, \mathfrak{A}^1) = \big((\Theta_1^Q, \mathfrak{B}^1)/(\Lambda_1^Q, \mathfrak{A}^1)\big) \cap \big((\triangle_1^Q, \mathfrak{C}^1)/(\Lambda_1^Q, \mathfrak{A}^1)\big)$

(3) $(\Lambda_1^Q,\mathfrak{A}^1)/\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(4) $\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)/(\Lambda_1^Q,\mathfrak{A}^1)=\big((\Theta_1^Q,\mathfrak{B}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)\wedge\big((\triangle_1^Q,\mathfrak{C}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)$
(5) $(\Lambda_1^Q,\mathfrak{A}^1)/\big((\Theta_1^Q,\mathfrak{B}^1)\cup(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)\cup\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(6) $\big((\Theta_1^Q,\mathfrak{B}^1)\cup(\triangle_1^Q,\mathfrak{C}^1)\big)/(\Lambda_1^Q,\mathfrak{A}^1)=\big((\Theta_1^Q,\mathfrak{B}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)\cup\big((\triangle_1^Q,\mathfrak{C}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)$
(7) $(\Lambda_1^Q,\mathfrak{A}^1)/\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)\vee\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(8) $\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)/(\Lambda_1^Q,\mathfrak{A}^1)=\big((\Theta_1^Q,\mathfrak{B}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)\vee\big((\triangle_1^Q,\mathfrak{C}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)$

Proof: Similar with Theorem 3.5

Corollary 3.9. *Let* $(\Lambda_1^Q,\mathfrak{A}^1),(\Theta_1^Q,\mathfrak{B}^1)$ *and* $(\triangle_1^Q,\mathfrak{C}^1)$ *be Q-neutrosophic soft sets X. Then, the following holds*

(1) $(\Lambda_1^Q,\mathfrak{A}^1)\wedge\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\Theta_1^Q,\mathfrak{B}^1)\big)\vee\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)$
(2) $\big((\Lambda_1^Q,\mathfrak{A}^1\vee(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)\vee\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)$
(3) $(\Lambda_1^Q,\mathfrak{A}^1)\vee\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\vee(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge\big((\Lambda_1^Q,\mathfrak{A}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)$
(4) $\big((\Lambda_1^Q,\mathfrak{A}^1\wedge(\Theta_1^Q,\mathfrak{B}^1)\big)\vee(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)\wedge\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)$

Proof:
Let $(\Lambda_1^Q,\mathfrak{A}^1)\wedge(\Theta_1^Q,\mathfrak{B}^1)=(\Delta_1^Q,\mathfrak{F}^1)$ and $(\Lambda_1^Q,\mathfrak{A}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)=(\Xi_1^Q,\mathfrak{G}^1)$.
Let

$$(\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)=(\Psi_1^Q,\mathfrak{D}^1)\text{ such that}$$
$$T_{\Psi_1^Q(d)}(w,u^1)T_{\Psi_1^Q(b,c)}(w,u^1)=\max\underbrace{\{T_{\Theta_1^Q(b)}(w,u^1),T_{\triangle_1^Q(c)}(w,u^1)\}}_{(b,c)=t\in\mathfrak{B}^1\times\mathfrak{C}^1}\tag{63}$$

for all $w\in X$, $u^1\in Q$ and $(b,c)\in(\mathfrak{B}^1\times\mathfrak{C}^1)$
Let

$$(\Lambda_1^Q,\mathfrak{A}^1)\wedge(\Psi_1^Q,\mathfrak{D}^1)=(\Pi_1^Q,\mathfrak{E}^1)\text{ such that}$$
$$T_{\Pi_1^Q(t)}(w,u^1)=T_{\Pi_1^Q(a,d)}(w,u^1)=\min\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\Psi_1^Q(d)}(w,u^1)\}}_{(a,d)=t\in\mathfrak{A}^1\times\mathfrak{D}^1}\tag{64}$$

for all $w\in X$ and $u^1\in Q$ and $(a,d)\in\mathfrak{A}^1\times\mathfrak{D}^1)$.

Substituting 63 into 64, wt have

$$T_{\Pi_1^Q(t)}(w,u^1) = \min \underbrace{\left\{ T_{\Lambda_1^Q(a)}(w,u^1), \max \underbrace{\{T_{\Theta_1^Q(b)}(w,u^1), T_{\triangle_1^Q(c)}(w,u^1)\}}_{(b,c)=t\in\mathfrak{B}^1\times\mathfrak{C}^1} \right\}}_{(a,d)=t\in\mathfrak{A}^1\times\mathfrak{D}^1}$$

$$= \min \underbrace{\left\{ \max\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Lambda_1^Q(a)}(w,u^1)\}, \max\{T_{\Theta_1^Q(b)}(w,u^1), T_{\triangle_1^Q(c)}(w,u^1)\} \right\}}_{(a,d)=\in\mathfrak{A}^1\times\mathfrak{D}^1}$$

$$\max\left\{ \min \underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Theta_1^Q(b)}(w,u^1)\}}_{(a,b)=\in\mathfrak{A}^1\times\mathfrak{B}^1}, \min \underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\triangle_1^Q(c)}(w,u^1)\}}_{(a,c)=\in\mathfrak{A}^1\times\mathfrak{C}^1} \right\}$$

$$= \max\{T_{\Delta_1^Q(a,b)}(w,u^1), T_{\Xi_1^Q(a,c)}(w,u^1)\} \tag{65}$$

Hence, $(\Lambda_1^Q,\mathfrak{A}^1)\wedge\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\Theta_1^Q,\mathfrak{B}^1)\big)\vee\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)$. It hold for true membership degree. Also, the proofs for indeterminate and falsity membership degrees are similar.

The results for 2 , 3 and 4 are similar to 1

4. Conclusion

In this paper, the notion of $Q-$neutrosophic soft set is extended to a non-associative algebraic structure. In particular, we focus on presenting the distributive properties of $Q-$neutrosophic soft quasigroup. Regarding the three binary operations of the quasigroup, a $Q-$neutrosophic soft set is defined under the structure of quasigroup. These three operations were used to demonstrate its characteristic in relation to the intersection, union, AND, and OR operations. A fascinating finding of the study is that the $Q-$neutrosophic soft quasigroup is self-distributive under the three binary operations and also distributive over each another. The three binary operations are distributive over intersection, union, AND and OR operations. It was further shown that $Q-$neutrosophic soft quasigroup does not adhere to left and right symmetric properties. Thus, the notion does not obey key laws. It was established that $Q-$neutrosophic soft set is a distributive lattice. In future research, Definitions 3.1, 3.2 and 3.4 will be used to examine the algebraic properties of $Q-$neutrosophic soft set under a class of qausigroup known as entropy, unipotent, and idempotent quasigroups.

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