# Fermatean Neutrosophic Matrices and Their Basic Operations

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**Abstract:** This paper aims to define a special case of neutrosophic matrices referred to as Fermatean neutrosophic matrices (FNMs). FNMs were introduced as generalization of Fermatean fuzzy matrices, intuitionistic fuzzy matrices, and Pythagorean neutrosophic matrices. In FNMs, some properties were discussed in connection with the well-known standard operations ($\oplus$, $\otimes$, $\wedge$ and $\vee$). In addition, the scalar multiplication (nA) and exponentiation ($A^n$) of a Fermatean neutrosophic matrix A were proposed, and their basic properties were investigated. Lastly, a new operation denoted by @ was defined on Fermatean neutrosophic matrices, and some of the properties of these matrices were examined.

**Keywords:** Fermatean Neutrosophic Matrices; Pythagorean neutrosophic sets; Fermatean Neutrosophic sets; Pythagorean Neutrosophic Matrices; Scalar multiplication; Exponentiation operations.

## 1. Introduction

The Fuzzy set theory [1] was introduced by Zadeh in 1965. This theory provides a way to deal with vague concepts by assigning a degree of membership to each element of a set. Fuzzy set theory was extensively examined and put together by academics and technology professionals, with broadened usage in fuzzy logic, fuzzy topology, fuzzy control systems, etc. Additionally, theories like fuzzy probability, soft set concepts, and rough set concepts are employed to tackle similar issues. Atanassov's intuitionistic fuzzy sets (IFS) described in [2] are suitable in this circumstance. Only imperfect information taking into account both the truth-membership and falsity-membership values may be handled by intuitionistic fuzzy sets. The ambiguous and contradictory information that is present in belief systems aren't dealt with by it. In 1995, neutrosophic sets are a mathematical notion established by Smarandache [3] that can be used to solve issues involving imperfect, ambiguous, and inconsistent data. Numerous scholars have looked at the applications of Neutrosophic sets and their extensions to ambiguous real-world situations. In the paper authored by Senapati et al. [4], the concept of Fermatean fuzzy sets is introduced, which is characterized by a restriction on the sum of the cubes of membership and non-membership degrees that is not to exceed a value of 1. This characteristic gives FFS a wider range of applicability compared to both IFSs and PFSs. They then produce certain operations for Fermatean Fuzzy Sets. Several studies on Fermatean fuzzy sets were applied in different fields later on. Ganie [5] developed distance and knowledge measurements with Fermatean fuzzy sets. Xu and Shen[6] suggested a technique for Fermatean fuzzy sets to identify patterns that utilizes similarity metrics. Zhou[7] demonstrated a Fermatean Fuzzy ELECTRE Technique for MCDM. Yang[8] illustrates the calculus of Continuities, Derivatives, and Differentials in Fermatean Fuzzy Processes. Barraza [9] included a Fermatean fuzzy matrix which is utilized while collaborating on municipal construction projects. Aydemir [10] and Mishra[11] proposed fermatean fuzzy TOPSIS and WASPAS in MCDM.

By extending Fermatean fuzzy sets, Sweety and R. Jansi [12] presented the concept of Fermatean neutrosophic sets. Fermatean neutrosophic sets are specific types of neutrosophic sets that are used to model uncertainty, indeterminacy, and incomplete information in decision-making processes. These sets have found real applications under uncertainty in decision-making [13-14] and graph theory [15-17, 18].

The theory of fuzzy matrix proposed by Thomason in his paper [19] has been extended and generalized in many ways. To overcome the limitations of fuzzy matrices .The idea of intuitionistic fuzzy matrix (IFM) was proposed by Khan et al. [20, 21] and Im et al. [22] model more complex decision-making problems than fuzzy matrix. In such a situation, IFM fails to produce a reasonable solution. To address this situation, in 2018, Silambarasan and Sriram [23] designed Pythagorean fuzzy matrices (PFMs) and studied their algebraic operations. Recently, major contributions on the extension of IFMs have been published ( see Table 1).

**Table 1.** Review on the Extensions of Intuitionistic fuzzy matrices.

| References | Extensions of Intuitionistic fuzzy matrices | Year |
|---|---|---|
| [24] | Single valued neutrosophic matrices | 2018 |
| [25] | Spherical fuzzy matrices | 2020 |
| [26] | Picture fuzzy matrices | 2020 |
| [27] | T-Spherical Fuzzy Matrices | 2022 |
| [28] | Multi-valued neutrosophic fuzzy matrix (MVNFM) | 2022 |
| [29] | Fermatean fuzzy matrices | 2022 |
| proposed | Fermatean neutrosophic matrices | 2023 |

Based on literature review that reflects no research has been carried out on Fermatean neutrosophic matrix and to merge this gap, we established a new special class of neutrosophic matrices.

These following concepts constitute the components of this paper. Basic concepts of existing work have been given in introduction. Some fundamental definitions of IFMs and PFMs are presented in the Preliminaries section. In third section, Fermatean neutrosophic matrices and their fundamental operations are described. Also a new operation (@) on Fermatean neutrosophic matrices is presented and some algebraic characteristics are discussed. In the final segment, the work is concluded.

## 2. Preliminaries

This section of the article presents some fundamental ideas concerning the Pythagorean fuzzy matrix (PyFM), intuitionistic fuzzy matrix (IFM), Fermatean fuzzy matrices (FFM), Pythagorean neutrosophic sets and Fermatean neutrosophic sets.

**Definition 2.1 [20]**

The definition of an intuitionistic fuzzy matrix (IFM) $\mathcal{R}$ with dimensions $m \times n$ is given by.

$$\mathcal{R} = \left[X_{ij}, \langle T_{ijp} \quad , F_{ijp} \rangle\right]_{m\times n}$$

Where $T_{ijp}, F_{ijp} \in [0,1]$ are referred to as truth, and falsity of in $\mathcal{R}$, which maintaining the condition $0 \leq T_{ijp} + F_{ijp} \leq 1$, to simplify matters, we express it as R=$\left[X_{ij}, \mathcal{R}_{ij}\right]_{m\times n}$ or $\left[\mathcal{R}_{ij}\right]_{m\times n}$ where

$$\mathcal{R}_{ij}=\langle T_{ijp}, F_{ijp}\rangle$$

**Example 2.1.** Let $\mathcal{R}$ be a $2\times 2$ IFM.

$\mathcal{R} = \begin{bmatrix} (0.2,\ 0.3) & (0.5,\ 0.1) \\ (0.3, 0.5) & (0.6,\ 0.2) \end{bmatrix}$ is not a FM, but $\mathcal{R}$ is a IFM

**Definition 2.2 [23]**

The definition of Pythagorean fuzzy matrix (PyFM) $\mathcal{R}$ with dimensions $m \times n$ is given by.

$$\mathcal{R} = \left[X_{ij}, \langle T_{ij\mathcal{R}}, F_{ij\mathcal{R}}\rangle\right]_{m\times n}$$

Where $T_{ij\mathcal{R}}, F_{ij\mathcal{R}} \in [0,1]$ are referred to as the degrees of the truth and the falsity of in $\mathcal{R}$, which maintaining the condition $0 \le T_{ij\mathcal{R}}{}^2 + F_{ij\mathcal{R}}{}^2 \le 2$, to simplify matters, we express it as R=$\left[X_{ij}, \mathcal{R}_{ij}\right]_{m\times n}$ or $\left[\mathcal{R}_{ij}\right]_{m\times n}$ where

$$\mathcal{R}_{ij}=\langle T_{ij\mathcal{R}},, F_{ij\mathcal{R}}\rangle$$

**Example 2.1**. Let $\mathcal{R}$ be a $2\times 2$ PyFM

$\mathcal{R} = \begin{bmatrix} (0.5, 0.3) & (0.3,\ 0.6) \\ (0.7, 0.3) & [(0.5,\ 0.2) \end{bmatrix}$ is not a IFM, but $\mathcal{R}$ is a PyFM.

**Definition 2.3 [29]**

The definition of Fermatean fuzzy matrix (FFM) $\mathcal{R}$ with dimensions $m \times n$ is given by.

$$\mathcal{R} = \left[X_{ij}, \langle T_{ij\mathcal{R}}\ , F_{ij\mathcal{R}}\rangle\right]_{m\times n}$$

Where $T_{ij\mathcal{R}}, F_{ij\mathcal{R}} \in [0,1]$ are referred to as truth, and falsity of in $\mathcal{R}$, which maintaining the condition $0 \le T_{ij\mathcal{R}}{}^3 + F_{ij\mathcal{R}}{}^3 \le 2$, to simplify matters, we express it as R=$\left[X_{ij}, \mathcal{R}_{ij}\right]_{m\times n}$ or $\left[\mathcal{R}_{ij}\right]_{m\times n}$ where

$$\mathcal{R}_{ij}=\langle T_{ij\mathcal{R}}, F_{ij\mathcal{R}}\rangle$$

**Example 2.3**. Let $\mathcal{R}$ be a $2\times 2$ FFM.

$\mathcal{R} = \begin{bmatrix} (0.7,\ 0.7) & (0.3,\ 0.1) \\ (0.7,\ 0.8) & (0.5,\ 0.2) \end{bmatrix}$ is not a IFM and not a PyFM , but $\mathcal{R}$ is a FFM

**Definition 2.4 [30]**

The concept of Pythagorean neutrosophic sets (PyN sets) $\tilde{P}$ on $\mathfrak{U}$ is an object that can be expressed as

$\tilde{P} = \{(\ \mathfrak{r},\ T_{\tilde{P}}(\mathfrak{r}),\ I_{\tilde{P}}(\mathfrak{r}),\ F_{\tilde{P}}(\mathfrak{r})\text{):}\ \mathfrak{r} \in \mathfrak{U}\}$, where $ó_{\tilde{P}}\ (\mathfrak{r}) \in [0,1]$ represents the membership degree of $\mathfrak{r}$ in $\mathfrak{U}$, $I_{\tilde{P}}(\mathfrak{r}) \in [0,1]$ is the indeterminacy degree of $\mathfrak{r}$ in $\mathfrak{U}$ and $F_{\tilde{P}}(\mathfrak{r}) \in [0,1]$ denotes the non-membership degree of $\mathfrak{r}$ in $\mathfrak{U}$, and these three are satisfying the relation;

$$0 \le (T_{\tilde{P}}(\mathfrak{r}))^{\ 2} + (I_{\tilde{P}}(\mathfrak{r}))^{\ 2} + (F_{\tilde{P}}(\mathfrak{r}))^{\ 2} \le 2$$

Here $T_{\tilde{P}}(\mathfrak{r})$ and $I_{\tilde{P}}(\mathfrak{r})$ are dependent component and $F_{\tilde{P}}(\mathfrak{r})$ is an independent component.

**Definition 2.5 [12]**

The concept of Fermatean neutrosophic sets (FN sets) $\tilde{N}$ on $\mathfrak{U}$ is an object that can be represented mathematically by

$\tilde{N}$ = {( $\mathfrak{r}$, $T_{\tilde{N}}(\mathfrak{r})$, $I_{\tilde{N}}(\mathfrak{r})$, $F_{\tilde{N}}(\mathfrak{r})$): $\mathfrak{r} \in \mathfrak{U}$}, where $T_{\tilde{N}}(\mathfrak{r}) \in [0,1]$ represents the membership degree of $\mathfrak{r}$ in $\mathfrak{U}$, $I_{\tilde{N}}(\mathfrak{r}) \in [0,1]$ is the indeterminacy degree of $\mathfrak{r}$ in $\mathfrak{U}$ and $F_{\tilde{N}}(\mathfrak{r}) \in [0,1]$ denotes the non-membership degree of $\mathfrak{r}$ in $\mathfrak{U}$, and these three are satisfying the relation;

$$0 \leq (T(\mathfrak{r}))^3 + (F_{\tilde{N}}(\mathfrak{r}))^3 \leq 1 \text{ and } 0 \leq (I_{\tilde{N}}(\mathfrak{r}))^3 \leq 1;$$

Then, $$0 \leq (T(\mathfrak{r}))^3 + (I_{\tilde{N}}(\mathfrak{r}))^3 + (F_{\tilde{N}}(\mathfrak{r}))^3 \leq 2$$

Here $T_{\tilde{N}}(\mathfrak{r})$ and $F_{\tilde{N}}(\mathfrak{r})$ are dependent component and $I_{\tilde{N}}(\mathfrak{r})$ is an independent component.

**Definition 2.6 [12]**

Let's suppose that the universe of discourse is termed by $\mathfrak{A}$ and the unit interval [0,1]. Let K and L two Fermatean neutrosophic sets expressed mathematically by

$K$ = {( $\mathfrak{r}$, $T_{\tilde{P}}(\mathfrak{r})$, $I_{\tilde{P}}(\mathfrak{r})$, $F_{\tilde{P}}(\mathfrak{r})$)/ $\mathfrak{r} \in \mathfrak{A}$ } and L = {( $\mathfrak{r}$, $T_{\tilde{N}}(\mathfrak{r})$, $I_{\tilde{N}}(\mathfrak{r})$, $F_{\tilde{N}}(\mathfrak{r})$/$\mathfrak{r} \in \mathfrak{A}$ }. Then

a. $K^c$ = {( $\mathfrak{r}$, $F_{\tilde{P}}(\mathfrak{r})$, $1 - I_{\tilde{P}}(\mathfrak{r})$, $T_{\tilde{P}}(\mathfrak{r})$): $\mathfrak{r} \in \mathfrak{A}$ }

b. K∪ L ={( $\mathfrak{r}$, max($T_{\tilde{P}}(\mathfrak{r})$, $T_{\tilde{N}}(\mathfrak{r})$), min($I_{\tilde{P}}(\mathfrak{r})$, $I_{\tilde{N}}(\mathfrak{r})$), min ($F_{\tilde{P}}(\mathfrak{r})$, $F_{\tilde{N}}(\mathfrak{r})$): $\mathfrak{r} \in \mathfrak{A}$ }

c. K∩L ={( $\mathfrak{r}$, min($T_{\tilde{P}}(\mathfrak{r})$, $T_{\tilde{N}}(\mathfrak{r})$), max($I_{\tilde{P}}(\mathfrak{r})$, $I_{\tilde{N}}(\mathfrak{r})$), max($F_{\tilde{P}}(\mathfrak{r})$, $F_{\tilde{N}}(\mathfrak{r})$): $\mathfrak{r} \in \mathfrak{A}$ }

## 3. Fermatean Neutrosophic Matrices

Before introducing the concept of Fermatean neutrosophic matrices, we briefly presented the definition of Pythagorean neutrosophic matrix (PyNM) which will be used in this section

**Definition 3.1: Pythagorean neutrosophic matrix**

The definition of Pythagorean neutrosophic matrix (PyNM) $\mathcal{R}$ with dimensions $m \times n$ is given by.

$$\mathcal{R} = [X_{ij}, \langle T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \rangle]_{m \times n}$$

Where $T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \in [0,1]$ are referred to as the degrees of the truth, the indeterminacy, and the falsity of in $\mathcal{R}$, which maintaining the condition $0 \leq {T_{ij\mathcal{R}}}^2 + {I_{ij\mathcal{R}}}^2 + {F_{ij\mathcal{R}}}^2 \leq 2$, to simplify matters, we express it as R=$[X_{ij}, \mathcal{R}_{ij}]_{m \times n}$ or $[\mathcal{R}_{ij}]_{m \times n}$ where

$$\mathcal{R}_{ij} = \langle T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \rangle$$

**Example 3.1**. Let $\mathcal{R}$ be a $2 \times 2$ PyNM

$\mathcal{R} = \begin{bmatrix} (0.5, 0.7, 0.3) & (0.3, 0.4, 0.6) \\ (0.7, 0.4, 0.3) & (0.5, 0.4, 0.2) \end{bmatrix}$ is not a IFM, but $\mathcal{R}$ is a PyNM.

Each element in an PyNM is expressed by an ordered pair $T_{\tilde{P}}$ (r), $I_{\tilde{P}}$ (r), $F_{\tilde{P}}$ (r) with $T_{\tilde{P}}$ (r), $I_{\tilde{P}}$ (r) and $F_{\tilde{P}}$ (r)∈ [0, 1] and 0 ≤ $T_{\tilde{P}}$ $(r)^2$+ $I_{\tilde{P}}$ $(r)^2$+ $F_{\tilde{P}}$ $(r)^2$≤ 2 and 0 ≤ $T_{\tilde{P}}$ $(r)^2$+ $F_{\tilde{P}}$ $(r)^2$≤ 1. It became obvious to observe this $(0.7)^2 + (0.7)^2 + (0.8)^2$= = 1.62 ≤ 2 and $(0.7)^2$+ $(0.8)^2$ =1.13 > 1, and therefore, Its description was beyond the scope of PyNM. In this study, the Fermatean neutrosophic matrix (FNM) and related algebraic operations to characterize an assessment of this sort are provided. A pair of ordered elements can be utilized to represent each element in a FNM, $T_{\tilde{P}}$ (r), $I_{\tilde{P}}$ (r), $F_{\tilde{P}}$ (r) with $T_{\tilde{P}}$ (r), $I_{\tilde{P}}$ (r), $F_{\tilde{P}}$ (r) ∈ [0, 1] and 0 ≤ $T_{\tilde{P}}$ $(r)^3$+ $I_{\tilde{P}}$ $(r)^3$+ $F_{\tilde{P}}$ $(r)^3$≤ 2 and 0 ≤ $T_{\tilde{P}}$ $(r)^3$+ $F_{\tilde{P}}$ $(r)^3$≤ 1.Also, we can get $(0.7)^3$+ $(0.8)^3$ =0.855 < 1 and $(0.7)^3 + (0.7)^3 + (0.8)^3$= 1.198 ≤ 2, which enables the FNM to be utilized for managing it.

By limiting the measure of truth , indeterminacy and falsity membership but preserving their total in the range [0, $\sqrt[2]{2}$], Fermatean neutrosophic matrices algebraic operations are provided in.

**Definition 3.2:**

The definition of Fermatean neutrosophic matrix (FNM) $\mathcal{R}$ with dimensions $m \times n$ is given by.

$$\mathcal{R} = [X_{ij}, \langle T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \rangle]_{m \times n}$$

Where $T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \in [0,1]$ are referred to as the degrees of the truth, the indeterminacy, and the falsity of in $\mathcal{R}$, which maintaining the condition 0 ≤ $T_{ij\mathcal{R}}3$ + $F_{ij\mathcal{R}}3$ + $F_{ij\mathcal{R}}3$ ≤ 2, to simplify matters, we express it as R=$[X_{ij}, \mathcal{R}_{ij}]_{m \times n}$ or $[\mathcal{R}_{ij}]_{m \times n}$ where

$$\mathcal{R}_{ij} = \langle T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \rangle$$

**Example 3.2.** Let $\mathcal{R}$ be a $2 \times 2$ FNM.

$\mathcal{R} = \begin{bmatrix} (0.7, 0.7, 0.7) & (0.3, 0.4, 0.1) \\ (0.3, 0.4, 0.2) & [(0.5, 0.4, 0.2) \end{bmatrix}$ is not a FFM and not , but $\mathcal{R}$ is a FNM.

In Fermatean neutrosophic matrix FNMs the cube sum of the triplet of membership, non-membership, and indeterminacy of the Fermatean neutrosophic element is is less than or equal to 2 , while in PyNMs the square sum of these triplet is bounded by 2

On the other hand, if we drop the indeterminacy degree from each triplet of a Fermatean neutrosophic matrix in example 3.2 then FNM, is reduced to Fermatean fuzzy matrix

**Definition 3.3.**

Suppose $\mathbb{N}$ and $\wp$ are two Fermatean neutrosophic matrices, then

- $\mathbb{N} < \wp$ iff $\forall$ i,j$\in \mathfrak{A}$, $\tilde{T}_{a_{ij}} \le \tilde{T}_{b_{ij}}$, $\tilde{I}_{a_{ij}} \le \tilde{I}_{b_{ij}}$ or $\tilde{I}_{a_{ij}} \ge \tilde{I}_{b_{ij}}$, $\tilde{F}_{a_{ij}} \ge \tilde{F}_{b_{ij}}$:
- $\mathbb{N}^C = \left(\langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle\right)_{mxn}$
- $\mathbb{N} \vee \wp = \left(\langle \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}) \ , \min \ (\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$
- $\mathbb{N} \wedge \wp) = \left(\langle \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$
- $\mathbb{N} \oplus \wp = \left(\langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}} \ \tilde{I}_{b_{ij}} \ , \tilde{F}_{a_{ij}} \ \tilde{F}_{b_{ij}} \rangle\right)_{\boldsymbol{mxn}}$
- $\mathbb{N} \otimes \wp = \left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}} \rangle\right)_{\boldsymbol{mxn}}$

**Definition 3.4**

A scalar multiplication operation on FNM, n $\mathbb{N}$ and is defined as follow:

$$\text{n } \mathbb{N} = \langle\left(\sqrt[3]{1 - [1 - \tilde{T}^3_{a_{ij}}]^n}, \left[\tilde{I}_{a_{ij}} \ \right]^n, \left[\tilde{F}_{a_{ij}}\right]^n\right)\rangle_{mxn}$$

**Definition 3.5**

An exponentiation operation on FNM, $\mathbb{N}$ and is defined as follow:

$$\mathbb{N}^n = \langle\left(\left[\tilde{T}_{a_{ij}}\right]^n, \sqrt[3]{1 - [1 - \tilde{I}^3_{a_{ij}}]^n}, \sqrt[3]{1 - [1 - \tilde{F}^3_{a_{ij}}]^n}\right)\rangle_{mxn}$$

Let $FN_{mxn}$ be the set of all the Fermatean neutrosophic matrices.

The following theorem relation between algebraic sum, and algebraic product of FNMs.

**Theorem 3.1**.If $\mathbb{N}$, $\wp \in FN_{mxn}$, then $\mathbb{N} \otimes \wp \le \mathbb{N} \oplus \wp$.

Proof

Let $\mathbb{N} \oplus \wp = \left(\langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \ \tilde{I}_{a_{ij}} \ \tilde{I}_{b_{ij}} \ , \tilde{F}_{a_{ij}} \ \tilde{F}_{b_{ij}} \rangle\right)_{\boldsymbol{mxn}}$

$$\mathbb{N} \otimes \wp = \left(\langle \tilde{T}_{a_{ij}} \ \tilde{T}_{b_{ij}} \ , \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}} \rangle\right)_{\boldsymbol{mxn}}$$

Suppose that

$$\tilde{T}_{a_{ij}} \ \tilde{T}_{b_{ij}} \le \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}$$

Cubing both sides

$\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}} \le \tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}$

$\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}} - \tilde{T}^3_{b_{ij}} + \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}} \le 0$

$\tilde{T}^3_{a_{ij}}\left(\tilde{T}^3_{b_{ij}} - 1\right) - \tilde{T}^3_{b_{ij}}\left(1 - \tilde{T}^3_{a_{ij}}\right) \le 0$

$-\tilde{T}^3_{a_{ij}}\left(1 - \tilde{T}^3_{b_{ij}}\right) - \tilde{T}^3_{b_{ij}}\left(1 - \tilde{T}^3_{a_{ij}}\right) \le 0$

$\tilde{T}^3_{a_{ij}}\left(1 - \tilde{T}^3_{b_{ij}}\right) + \tilde{T}^3_{b_{ij}}\left(1 - \tilde{T}^3_{a_{ij}}\right) \ge 0$

It is clear that $0 \le \tilde{T}^3_{a_{ij}} \le 1$ and $0 \le \tilde{T}^3_{b_{ij}} \le 1$

And

$\tilde{I}_{a_{ij}}\ \tilde{I}_{b_{ij}} \le \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}$

Ie, $\tilde{I}_{a_{ij}}\ \tilde{I}_{b_{ij}} - \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}} \ge 0$

Ie,$\tilde{I}^3_{a_{ij}}\left(1 - \tilde{I}^3_{b_{ij}}\right) + \tilde{I}^3_{b_{ij}}(1 - \tilde{I}^3_{a_{ij}}) \ge 0$

It is clear that $0 \le \tilde{I}^3_{a_{ij}} \le 1$ and $0 \le \tilde{I}^3_{b_{ij}} \le 1$

And

$\tilde{F}_{a_{ij}}\ \tilde{F}_{b_{ij}} \le \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}$

Ie, $\tilde{F}_{a_{ij}}\ \tilde{F}_{b_{ij}} - \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}} \ge 0$

Ie,$\tilde{F}^3_{a_{ij}}\left(1 - \tilde{F}^3_{b_{ij}}\right) + \tilde{F}^3_{b_{ij}}(1 - \tilde{F}^3_{a_{ij}}) \ge 0$

It is clear that $0 \le \tilde{F}^3_{a_{ij}} \le 1$ and $0 \le \tilde{F}^3_{b_{ij}} \le 1$

Hence, $\mathbb{N} \otimes \wp \le \mathbb{N} \oplus \wp$.

**Theorem 3.2**. For any Fermatean neutrosophic matrix $\mathbb{N}$,

(i) $\mathbb{N} \oplus \mathbb{N} \ge \mathbb{N}$

(ii) $\mathbb{N} \otimes \mathbb{N} \le \mathbb{N}$.

Proof.

(i) Let $\mathbb{N} \oplus \mathbb{N} = (\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle) \oplus (\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle)$

$$\mathbb{N} \oplus \mathbb{N} = \left(\langle \sqrt[3]{2\ \tilde{T}^3_{a_{ij}} - \left(\tilde{T}^3_{a_{ij}}\right)^2}, \left(\tilde{I}_{a_{ij}}\right)^2, \left(\tilde{F}_{a_{ij}}\right)^2 \rangle\right)_{mxn}$$

$$\sqrt[3]{2\ \tilde{T}_{a_{ij}}^{3}-\left(\tilde{T}_{a_{ij}}^{3}\right)^{2}}=\sqrt[3]{\tilde{T}_{a_{ij}}^{3}+\tilde{T}_{a_{ij}}^{3}\left(1-\tilde{T}_{a_{ij}}^{3}\right)}\geq \tilde{T}_{a_{ij}}^{3},\forall\ \ i,j\in\ \ \mathfrak{A}$$

And $\left(\tilde{I}_{a_{ij}}\right)^{2}\leq\ \tilde{I}_{a_{ij}},$

And $\left(\tilde{F}_{a_{ij}}\right)^{2}\leq\ \tilde{F}_{a_{ij}},$

Hence $\mathbb{N}\oplus\mathbb{N}\geq\mathbb{N}$

(ii) Let $\mathbb{N}\otimes\mathbb{N}=(\langle\tilde{T}_{a_{ij}},\tilde{I}_{a_{ij}},\tilde{F}_{a_{ij}}\rangle)\otimes(\langle\tilde{T}_{a_{ij}},\tilde{I}_{a_{ij}},\tilde{F}_{a_{ij}}\rangle)$

$$\mathbb{N}\ \otimes\mathbb{N}=\left(\langle\left(\tilde{T}_{a_{ij}}\right)^{2},\sqrt[3]{2\ \tilde{I}_{a_{ij}}^{3}-\left(\tilde{I}_{a_{ij}}^{3}\right)^{2}},\sqrt[3]{2\ \tilde{F}_{a_{ij}}^{3}-\left(\tilde{F}_{a_{ij}}^{3}\right)^{2}}\rangle\right)_{mxn}$$

$$\sqrt[3]{2\ \tilde{I}_{a_{ij}}^{3}-\left(\tilde{I}_{a_{ij}}^{3}\right)^{2}}=\sqrt[3]{\tilde{I}_{a_{ij}}^{3}+\tilde{I}_{a_{ij}}^{3}\left(1-\tilde{I}_{a_{ij}}^{3}\right)}\leq\tilde{I}_{a_{ij}}^{3}\ ,\forall\ \ \mathfrak{r}\in\ \ \mathfrak{A}$$

$$\sqrt[3]{2\ \tilde{F}_{a_{ij}}^{3}-\left(\tilde{F}_{a_{ij}}^{3}\right)^{2}}=\sqrt[3]{\tilde{F}_{a_{ij}}^{3}+\tilde{F}_{a_{ij}}^{3}\left(1-\tilde{F}_{a_{ij}}^{3}\right)}\leq\tilde{F}_{a_{ij}}^{3}\ ,\forall\ \ \mathfrak{r}\in\ \ \mathfrak{A}$$

And $\left(\tilde{T}_{a_{ij}}\right)^{2}\geq\ \tilde{T}_{a_{ij}},$

Hence $\mathbb{N}\otimes\mathbb{N}\leq\mathbb{N}$.

**Theorem 3.3**. If $\mathbb{N}$, $\wp$, M $\in FN_{mxn}$, then

(i) $\mathbb{N}\oplus\wp=\wp\oplus\mathbb{N}$,

(ii) $\mathbb{N}\otimes\wp=\wp\otimes\mathbb{N}$,

(iii) $(\mathbb{N}\oplus\wp)\oplus M=\mathbb{N}\oplus(\wp\oplus M)$,

(iv) $(\mathbb{N}\otimes\wp)\otimes M=\mathbb{N}\otimes(\wp\otimes M)$.

Proof. (i) Assume $\mathbb{N}\oplus\wp=\left(\langle\sqrt[3]{\tilde{T}_{a_{ij}}^{3}+\tilde{T}_{b_{ij}}^{3}-\tilde{T}_{a_{ij}}^{3}\tilde{T}_{b_{ij}}^{3}},\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}},\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\rangle\right)_{mxn}$

$$=\left(\langle\sqrt[3]{\tilde{T}_{b_{ij}}^{3}+\tilde{T}_{a_{ij}}^{3}-\tilde{T}_{b_{ij}}^{3}\tilde{T}_{a_{ij}}^{3}},\tilde{I}_{b_{ij}}\tilde{I}_{a_{ij}},\tilde{F}_{b_{ij}}\tilde{F}_{a_{ij}}\rangle\right)_{mxn}$$

$=\wp\oplus\mathbb{N}$

(iii) $\mathbb{N}\ \otimes\wp=\left(\langle\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}},\sqrt[3]{\tilde{I}_{a_{ij}}^{3}+\tilde{I}_{b_{ij}}^{3}-\tilde{I}_{a_{ij}}^{3}\tilde{I}_{b_{ij}}^{3}},\sqrt[3]{\tilde{F}_{a_{ij}}^{3}+\tilde{F}_{b_{ij}}^{3}-\tilde{F}_{a_{ij}}^{3}\tilde{F}_{b_{ij}}^{3}}\rangle\right)_{mxn}$

$$=\left(\langle\tilde{T}_{b_{ij}}\tilde{T}_{a_{ij}},\sqrt[3]{\tilde{I}_{b_{ij}}^{3}+\tilde{I}_{a_{ij}}^{3}-\tilde{I}_{b_{ij}}^{3}\tilde{I}_{a_{ij}}^{3}},\sqrt[3]{\tilde{F}_{b_{ij}}^{3}+\tilde{F}_{a_{ij}}^{3}-\tilde{F}_{b_{ij}}^{3}\tilde{F}_{a_{ij}}^{3}}\rangle\right)_{mxn}$$

$=\wp\ \otimes\mathbb{N}$

(v) $(\mathbb{N}\oplus\wp)\oplus M=\mathbb{N}\oplus(\wp\oplus M)$, Let $(\mathbb{N} \oplus \wp) \oplus \mathrm{M}$

$$=\left(\langle(\sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}) \oplus (\tilde{T}_{c_{ij}}, \tilde{I}_{c_{ij}}, \tilde{F}_{c_{ij}})\rangle\right)_{mxn}$$

$$=\begin{bmatrix} \sqrt[3]{\left(\sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3}\right) 2 + \tilde{T}_{c_{ij}}^3 - \left(\sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3}\right)^2 \tilde{T}_{c_{ij}}^3}, \\ \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\tilde{I}_{c_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\tilde{F}_{c_{ij}} \end{bmatrix}_{mxn}$$

$$= \begin{bmatrix} \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 + \tilde{T}_{c_{ij}}^3 - \tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3\tilde{T}_{c_{ij}}^3 - \tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3 + \tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3} \\ \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\tilde{I}_{c_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\tilde{F}_{c_{ij}} \end{bmatrix}_{mxn}$$

Let $\mathbb{N} \oplus (\wp \oplus \mathrm{M}) =$

$$= \left((\tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}}) \oplus \langle \sqrt[3]{\tilde{T}_{b_{ij}}^3 + \tilde{T}_{c_{ij}}^3 - \tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3}, \tilde{I}_{b_{ij}}\tilde{I}_{c_{ij}}, \tilde{F}_{b_{ij}}\tilde{F}_{c_{ij}})\rangle\right)$$

$$=\begin{bmatrix} \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \left(\sqrt[3]{\tilde{T}_{b_{ij}}^3 + \tilde{T}_{c_{ij}}^3 - \tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3}\right) 2 - \tilde{T}_{a_{ij}}^3 \left(\sqrt[3]{\tilde{T}_{b_{ij}}^3 + \tilde{T}_{c_{ij}}^3 - \tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3}\right)^2}, \\ \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\tilde{I}_{c_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\tilde{F}_{c_{ij}} \end{bmatrix}_{mxn}$$

$$= \begin{matrix} \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 + \tilde{T}_{c_{ij}}^3 - \tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3\tilde{T}_{c_{ij}}^3 - \tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3 + \tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3} \\ \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\tilde{I}_{c_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\tilde{F}_{c_{ij}} \end{matrix}$$

Hence $(\mathbb{N} \oplus \wp) \oplus \mathrm{M} = \mathbb{N} \oplus (\wp \oplus \mathrm{M})$,

iv) $(\mathbb{N} \otimes \wp) \otimes \mathrm{M} = \mathbb{N} \otimes (\wp \otimes \mathrm{M})$

$(\mathbb{N} \otimes \wp)$

$$= \left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3\tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3\tilde{F}_{b_{ij}}^3}\rangle\right)_{mxn}$$

$(\mathbb{N} \otimes \wp) \otimes \mathrm{M}$

$$=\left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3\tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3\tilde{F}_{b_{ij}}^3}\rangle \otimes \langle \tilde{F}_{a_{ij}}^3\tilde{F}_{b_{ij}}^3\tilde{F}_{c_{ij}}^3\rangle\right)_{mxn}$$

$$=\left(\begin{matrix} \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\tilde{T}_{c_{ij}} \\ \langle \sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}+\tilde{I}^3_{c_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{c_{ij}}-\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}+\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}}, \rangle \\ \sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}+\tilde{F}^3_{c_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{c_{ij}}-\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}+\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}} \end{matrix}\right)_{mxn}$$

Let $\mathbb{N} \otimes (\wp \otimes \mathrm{M})$

$(\wp \otimes \mathrm{M})$

$$= \left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{c_{ij}}, \sqrt[3]{\tilde{I}^3_{b_{ij}}+\tilde{I}^3_{c_{ij}}-\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}}, \sqrt[3]{\tilde{F}^3_{b_{ij}}+\tilde{F}^3_{c_{ij}}-\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}}\rangle\right)_{\boldsymbol{mxn}}$$

$\mathbb{N} \otimes (\wp \otimes \mathrm{M}) =$

$$\left(\begin{matrix} \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\tilde{T}_{c_{ij}}, \\ \langle \sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}+\tilde{I}^3_{c_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{c_{ij}}-\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}+\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}}, \rangle \\ \sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}+\tilde{F}^3_{c_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{c_{ij}}-\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}+\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}} \end{matrix}\right)_{mxn}$$

Hence $(\mathbb{N} \otimes \wp) \otimes \mathrm{M} = \mathbb{N} \otimes (\wp \otimes \mathrm{M})$

**Theorem 3.4.** If $\mathbb{N}, \wp \in FN_{mxn}$, then

(i) $\mathbb{N} \oplus (\mathbb{N} \otimes \wp) \geq \mathbb{N}$,

(ii) $\mathbb{N} \otimes (\mathbb{N} \oplus \wp) \leq \mathbb{N}$.

Proof. (i) Let $\mathbb{N} \oplus (\mathbb{N} \otimes \wp) =$

$$(\mathbb{N} \otimes \wp) = \left(\langle\ \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}\rangle\right)_{\boldsymbol{mxn}}$$

$\mathbb{N} \oplus (\mathbb{N} \otimes \wp) =$

$$= \left(\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}}\rangle\right) \oplus$$

$$\left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}\rangle\right)$$

$$=\begin{bmatrix} \sqrt[3]{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}-\tilde{T}^3_{a_{ij}}\left[\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}\right]}, \tilde{I}_{a_{ij}}\left[\sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}},\right] \\ \tilde{F}_{a_{ij}}\left[\sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}\right] \end{bmatrix}_{mxn}$$

$=$

$$\begin{bmatrix} \sqrt[3]{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}-\tilde{T}^3_{a_{ij}}\left[\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}\right]}, \tilde{I}_{a_{ij}}\sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]\left[1-\tilde{I}^3_{b_{ij}}\right]} \\ \tilde{F}_{a_{ij}}\left[\sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]\left[1-\tilde{F}^3_{b_{ij}}\right]}\right] \end{bmatrix}_{mxn}$$

$\geq \mathbb{N}$

Hence

$\mathbb{N} \oplus (\mathbb{N} \otimes \wp) \geq \mathbb{N}$.

ii)$\mathbb{N} \oplus \wp = \left(\langle \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3}, \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}} \rangle\right)_{mxn}$

$\mathbb{N} \otimes (\mathbb{N} \oplus \wp) = \left(\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle\right) \otimes$

$$\left(\langle \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3}, \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}} \rangle\right)$$

$$=\left[\sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3 \left[1 - \tilde{T}_{a_{ij}}^3\right]}, \tilde{I}_{a_{ij}} \left(\sqrt[3]{1 - \left[1 - \tilde{I}_{a_{ij}}^3\right]\left[1 - \tilde{I}_{b_{ij}}^3\right]}\right)\right], \quad \tilde{F}_{a_{ij}} \left[\sqrt[3]{1 - \left[1 - \tilde{F}_{a_{ij}}^3\right]\left[1 - \tilde{F}_{b_{ij}}^3\right]}\right]$$

$$=\begin{bmatrix} \tilde{T}_{a_{ij}} \sqrt[3]{1 - \left[1 - \tilde{T}_{a_{ij}}^3\right]\left[1 - \tilde{T}_{b_{ij}}^3\right]}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3 \left[1 - \tilde{I}_{a_{ij}}^3\right]}, \\ \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3 \left[1 - \tilde{F}_{a_{ij}}^3\right]}, \end{bmatrix}_{mxn}$$

$\leq \mathbb{N}$.

Hence $\mathbb{N} \otimes (\mathbb{N} \oplus \wp) \leq \mathbb{N}$

**Theorem 3.5.** If $\mathbb{A}$, $\wp \in FN_{mxn}$,Then

(i) $\mathbb{N} \vee \wp = \mathbb{N} \vee \wp$

(ii) $\mathbb{N} \wedge \wp = \mathbb{N} \wedge \wp$

(iii) $\mathbb{N} \vee \wp = \wp \vee \mathbb{N}$

$\mathbb{N} \vee \wp = \left(\langle \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$

$\wp \vee \mathbb{N} = \left(\langle \max(\tilde{T}_{b_{ij}}, \tilde{T}_{a_{ij}}), \min(\tilde{I}_{b_{ij}}, \tilde{I}_{a_{ij}}), \min(\tilde{F}_{b_{ij}}, \tilde{F}_{a_{ij}}) \rangle\right)_{mxn}$

$= \left(\langle \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$

$= \mathbb{N} \vee \wp$

(i) $\mathbb{N} \wedge \wp = \wp \wedge \mathbb{N}$

$\mathbb{N} \wedge \wp = \left(\langle \min\left(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}\right), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$

$\wp \wedge \mathbb{N} = \left(\langle \min(\tilde{T}_{b_{ij}}, \tilde{T}_{a_{ij}}), \max(\tilde{I}_{b_{ij}}, \tilde{I}_{a_{ij}}), \max(\tilde{F}_{b_{ij}}, \tilde{F}_{a_{ij}}) \rangle\right)_{mxn}$

$$=\left( \langle \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}),\ \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}),\ \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}})\ \rangle \right)_{mxn}$$

$$= \wp \vee \mathbb{N}$$

**Theorem 3.6.** Let $\mathbb{N}, \wp\ , \mathbb{C} \in FN_{mxn}$, then

(i). $\mathbb{N} \oplus (\wp \vee \mathbb{C}) = (\mathbb{N} \oplus \wp) \vee (\mathbb{N} \oplus \mathbb{C})$,

(ii). $\mathbb{N} \otimes (\wp \vee \mathbb{C}) = (\mathbb{N} \otimes \wp) \vee (\mathbb{N} \otimes \mathbb{C})$,

(iii). $\mathbb{N} \oplus (\wp \wedge \mathbb{C}) = (\mathbb{N} \oplus \wp) \wedge (\mathbb{N} \oplus \mathbb{C})$,

(iv). $\mathbb{N} \otimes (\wp \wedge \mathbb{C}) = (\mathbb{N} \otimes \wp) \wedge (\mathbb{N} \otimes \mathbb{C})$,

Proof.

(i) Let $\mathbb{N} \oplus (\wp \vee \mathbb{C})$

$$= \left( \langle \begin{matrix} \sqrt[3]{\tilde{T}^3_{a_{ij}} + \max\left(\tilde{T}^3_{b_{ij}} \tilde{T}^3_{c_{ij}}\right) - \tilde{T}^3_{a_{ij}} . \max\left(\tilde{T}^3_{b_{ij}} \tilde{T}^3_{c_{ij}}\right)}, \\ \tilde{I}_{a_{ij}}, \max\left(\tilde{I}^3_{b_{ij}} \tilde{I}^3_{c_{ij}}\right) \\ \tilde{F}_{a_{ij}}, \max\left(\tilde{F}^3_{b_{ij}} \tilde{F}^3_{c_{ij}}\right) \end{matrix} \rangle \right)_{mxn}$$

$$= \left( \langle \begin{matrix} \sqrt[3]{\max\left(\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}}, \tilde{T}^3_{a_{ij}} + \tilde{T}^3_{c_{ij}}\right) - \max\left(\tilde{T}^3_{a_{ij}} \tilde{T}^3_{b_{ij}}, \tilde{T}^3_{a_{ij}} \tilde{T}^3_{c_{ij}}\right)} \\ , \min\left(\tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{I}_{a_{ij}} \tilde{I}_{c_{ij}}\right) \\ , \min\left(\tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}}, \tilde{F}_{a_{ij}} \tilde{F}_{c_{ij}}\right) \end{matrix} \rangle \right)_{mxn}$$

$$= \left( \langle \begin{matrix} \sqrt[3]{\max\left(\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}} \tilde{T}^3_{b_{ij}}, \tilde{T}^3_{a_{ij}} + \tilde{T}^3_{c_{ij}} - \tilde{T}^3_{a_{ij}} \tilde{T}^3_{c_{ij}}\right)} \\ , \min\left(\tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{I}_{a_{ij}} \tilde{I}_{c_{ij}}\right) \\ , \min\left(\tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}}, \tilde{F}_{a_{ij}} \tilde{F}_{c_{ij}}\right) \end{matrix} \rangle \right)_{mxn}$$

$$= (\mathbb{N} \oplus \wp) \vee (\mathbb{N} \oplus \mathbb{C})$$

**Theorem 3.7**. Let $\mathbb{N}, \wp \in FN_{mxn}$, then

(i) $(\mathbb{N} \wedge \wp) \oplus (\mathbb{N} \vee \wp) = \mathbb{N} \oplus \wp$,

(ii) $(\mathbb{N} \wedge \wp) \otimes (\mathbb{N} \vee \wp) = \mathbb{N} \otimes \wp$,

(iii) $(\mathbb{N} \oplus \wp) \wedge (\mathbb{N} \otimes \wp) = \mathbb{N} \otimes \wp$,

(iv) $(\mathbb{N} \oplus \wp) \vee (\mathbb{N} \otimes \wp) = \mathbb{N} \oplus \wp$.

**Proof.** In the following, we shall prove (i), and (ii) − (iv) can be proved similarly.

(v) $(\mathbb{N} \wedge \wp) \oplus (\mathbb{N} \vee \wp) = \mathbb{N} \oplus \wp$,

Let $(\mathbb{N} \wedge \wp) \oplus (\mathbb{N} \vee \wp)$
=

$$\begin{bmatrix} \sqrt[3]{\min\left(\tilde{T}^3_{a_{ij}}, \tilde{T}^3_{b_{ij}}\right) + \max\left(\tilde{T}^3_{a_{ij}}, \tilde{T}^3_{b_{ij}}\right) - min\left(\tilde{T}^3_{a_{ij}}, \tilde{T}^3_{b_{ij}}\right).max\left(\tilde{T}^3_{a_{ij}}, \tilde{T}^3_{b_{ij}}\right)}, \\ \max\left(\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right), \min\left(\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right), \\ \max\left(\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right), \min\left(\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right) \end{bmatrix}_{mxn}$$

$= \left(\langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\rangle\right)_{mxn}$

$= \mathbb{N} \oplus \wp$

**Theorem 3.8.** If $\mathbb{N}$, $\wp \in FN_{mxn}$, then

(i) $(\mathbb{N} \oplus \wp)^C = \mathbb{N}^C \otimes \wp^C$,

(ii) $(\mathbb{N} \otimes \wp)^C = \mathbb{N}^C \oplus \wp^C$,

(iii) $(\mathbb{N} \oplus \wp)^C \leq \mathbb{N}^C \oplus \wp^C$,

(iv) $(\mathbb{N} \otimes \wp)^C \geq \mathbb{N}^C \otimes \wp^C$.

**Proof**.

(i) $(\mathbb{N} \oplus \wp)^C = \mathbb{N}^C \otimes \wp^C$

$(\mathbb{N} \oplus \wp)^C$

$=\left(\langle \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}\rangle\right)_{\boldsymbol{mxn}}$

$=(\langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}}\rangle \otimes (\langle \tilde{F}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{T}_{b_{ij}}\rangle$

$= \mathbb{N}^C \otimes \wp^C$

(ii) $(\mathbb{N} \otimes \wp)^C = \mathbb{N}^C \oplus \wp^C$,

$= \left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}\rangle\right)^c$

$=\left(\langle \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}\rangle\right)_{\boldsymbol{mxn}}$

$=(\langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}}\rangle \oplus \left(\langle \tilde{F}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{T}_{b_{ij}}\rangle\right)$

$= \mathbb{N}^C \oplus \wp^C$

(iii) $(\mathbb{N} \oplus \wp)^C \leq \mathbb{N}^C \oplus \wp^C$,

$(\mathbb{N} \oplus \wp)^C$

$$= \left(\langle \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3}, \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}} \rangle\right)_{mxn}^{c}$$

$$= \left(\langle \tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3} \rangle\right)_{mxn}$$

$\mathbb{N}^C \oplus \wp^C$

$$= \left(\tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3}\right)_{mxn}$$

$$= \left(\sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3}, \quad \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{T}_{a_{ij}} \tilde{T}_{b_{ij}}\right)_{mxn}$$

$$\tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}} \leq \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3},$$

$$\sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3} \geq \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}$$

$$\sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3} \geq \tilde{T}_{a_{ij}} \tilde{T}_{b_{ij}}$$

Hence $(\mathbb{N} \oplus \wp)^C \leq \mathbb{N}^C \oplus \wp^C$,

(ii) $(\mathbb{N} \otimes \wp)^C \geq \mathbb{N}^C \otimes \wp^C$.

$$\mathbb{N} \otimes \wp = \left(\langle \tilde{T}_{a_{ij}} \tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3} \rangle\right)_{mxn}$$

$$(\mathbb{N} \otimes \wp)^C = \left(\langle \tilde{T}_{a_{ij}} \tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3} \rangle\right)_{mxn}^{c}$$

$$= \left(\langle \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3}, \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{T}_{a_{ij}} \tilde{T}_{b_{ij}} \rangle\right)_{mxn}$$

$$\mathbb{N}^C \otimes \wp^C = (\langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle \otimes \left(\langle \tilde{F}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{T}_{b_{ij}} \rangle\right)$$

Since

$$\sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3} \geq \tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}}$$

$$\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}} \leq \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}$$

$$\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}} \leq \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}$$

Hence $(\mathbb{N} \otimes \wp)^C \geq \mathbb{N}^C \otimes \wp^C$

**Theorem 3.9** . Let $\mathbb{N}, \wp \in FN_{mxn}$, then

(i). $(\mathbb{N}^C)^C = \mathbb{N}$,

(ii). $(\mathbb{N} \vee \wp)^C = \mathbb{N}^C \wedge \wp^C$,

(iii). $(\mathbb{N} \wedge \wp)^C = \mathbb{N}^C \vee \wp^C$.

Proof.

(i) $((\mathbb{N}^C)^C = \mathbb{N}$,

$\mathbb{N} = (\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle$

$\mathbb{N}^C = (\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle)^c = \langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle$

$(\mathbb{N}^C)^C = (\langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle^c = \langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle$

(ii) $(\mathbb{N} \vee \wp)^C = \mathbb{N}^C \wedge \wp^C$,

$$\mathbb{N} \vee \wp = \left(\langle \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$$

$$(\mathbb{N} \vee \wp)^C = \left(\langle \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}^{c}$$

$$= \left(\langle \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}) \rangle\right)_{mxn}$$

$\mathbb{N}^C = \langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle$

$\wp^C = \langle \tilde{F}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{T}_{b_{ij}} \rangle$

$$\mathbb{N}^C \wedge \wp^C = \left(\langle \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}) \rangle\right)_{mxn}$$

Hence $(\mathbb{N} \vee \wp)^C = \mathbb{N}^C \wedge \wp^C$.

(iii) $(\mathbb{N} \wedge \wp)^C = \mathbb{N}^C \vee \wp^C$.

$$\mathbb{N} \wedge \wp = \left(\langle \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$$

$$(\mathbb{N} \wedge \wp)^C = \left(\langle \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}^{c}$$

$$= \left(\langle \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}) \rangle\right)_{mxn}$$

$\mathbb{N}^C = \langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle$

$\wp^C = \langle \tilde{F}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{T}_{b_{ij}} \rangle$

$$\mathbb{N}^C \vee \wp^C = \left( \langle \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}) \rangle \right)_{mxn}$$

$$= (\mathbb{N} \wedge \wp)^C$$

Within this section, we will show that the operations of scalar multiplication and exponentiation presented in definitions 3.3, 3.4, and 3.5 are well-defined for Fermatean neutrosophic matrices.

**Theorem 3.10** For $\mathbb{N}$, $\wp \in FN_{mxn}$ and n, n1, n2 > 0, we have

(i) $n(\mathbb{N} \oplus \wp) = n\mathbb{N} \oplus n\wp$,

(ii) $n1\,\mathbb{N} \oplus n2\,\mathbb{N} = (n1+n2)\,\mathbb{N}$,

(iii) $(\mathbb{N} \otimes \wp)^n = \mathbb{N}^n \otimes \wp^n$ ,

(iv) $\mathbb{N}^{n_1} \otimes \mathbb{N}^{n_2} = \mathbb{N}^{n_1+n_2}$

Proof. We will consider two Fermatean neutrosophic matrices $\mathbb{N}$ and $\wp$ and positive real numbers n, $n_1$, and $n_2$ . As stated in the definition, the following can be obtained.

(i) $n(\mathbb{N} \oplus \wp) = n\mathbb{N} \oplus n\wp$,

$$= n\left( \langle \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}} \rangle \right)_{mxn}$$

$$= \left( \langle \sqrt[3]{1 - \left[1 - \tilde{T}_{a_{ij}}^3\right][1 - \tilde{T}_{b_{ij}}^3]^n}, \left[\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]^n \rangle \right)_{mxn}$$

$$= \left( \langle \sqrt[3]{1 - \left[1 - \tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3\right]^n}, \left[\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]^n \rangle \right)_{mxn}$$

$n\mathbb{N} \oplus n\wp$

$$= \left( \langle \left( \sqrt[3]{1 - [1 - \tilde{T}_{a_{ij}}^3]^n}, \left[\tilde{I}_{a_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\right]^n \right) \oplus \left( \sqrt[3]{1 - [1 - \tilde{T}_{b_{ij}}^3]^n}, \left[\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{b_{ij}}\right]^n \right) \rangle \right)_{mxn}$$

$$= \begin{bmatrix} \sqrt[3]{\left(1 - \left[1 - \tilde{T}_{a_{ij}}^3\right]^n + 1 - \left[1 - \tilde{T}_{b_{ij}}^3\right]^n\right) - \left(1 - \left[1 - \tilde{T}_{a_{ij}}^3\right]^n\right)\left(1 - \left[1 - \tilde{T}_{b_{ij}}^3\right]^n\right)} \\ \left[\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]^n \end{bmatrix}_{mxn}$$

$$= \left( \langle \sqrt[3]{1 - \left[1 - \tilde{T}_{a_{ij}}^3\right][1 - \tilde{T}_{b_{ij}}^3]^n}, \left[\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]^n \rangle \right)_{mxn}$$

$$= \left( \langle \sqrt[3]{1 - \left[1 - \tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3\right]^n}, \left[\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]^n \rangle \right)_{mxn}$$

$= n(\mathbb{N} \oplus \wp)$

(ii) Let $n1\,\mathbb{N} \oplus n2\,\mathbb{N} = (n1+n2)\,\mathbb{N}$,

$n1\,\mathbb{N} \oplus n2\,\mathbb{N}$

$$= \left( \langle \left( \sqrt[3]{1 - [1 - \tilde{T}_{a_{ij}}^3]^{n_1}}, \left[\tilde{I}_{a_{ij}}\right]^{n_1}, \left[\tilde{F}_{a_{ij}}\right]^{n_1} \right) \oplus \left( \sqrt[3]{1 - [1 - \tilde{T}_{a_{ij}}^3]^{n_2}}, \left[\tilde{I}_{a_{ij}}\right]^{n_2}, \left[\tilde{F}_{a_{ij}}\right]^{n_2} \right) \rangle \right)_{mxn}$$

$$= \left[ \begin{matrix} \sqrt[3]{\left(1-\left[1-\tilde{T}^3_{a_{ij}}\right]^{n_1}+1-\left[1-\tilde{T}^3_{a_{ij}}\right]^{n_2}\right)-\left(1-\left[1-\tilde{T}^3_{a_{ij}}\right]^{n_1}\right)(1-\left[1-\tilde{T}^3_{a_{ij}}\right]^{n_2})} \\ \left[\tilde{I}_{a_{ij}}\right]^{n_1}\left[\tilde{I}_{a_{ij}}\right]^{n_2}, \left[\tilde{F}_{a_{ij}}\right]^{n_1}\left[\tilde{F}_{a_{ij}}\right]^{n_2} \end{matrix} \right]_{mxn}$$

$$= \left( \langle \sqrt[3]{1-[1-\tilde{T}^3_{a_{ij}}]^{n_1+n_2}}, \left[\tilde{I}^3_{a_{ij}}\right]^{n_1+n_2}, \left[\tilde{F}_{a_{ij}}\right]^{n_1+n_2} \rangle \right)_{mxn}$$

(iii)

$$(\mathbb{N} \otimes \wp)^n = \left( \langle \begin{matrix} \left(\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\right)^n, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}\right]^n}, \\ \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}\right]^n} \end{matrix} \rangle \right)_{\boldsymbol{mxn}}$$

$$= \left( \langle \begin{matrix} \left(\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\right)^n, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^n\left[1-\tilde{I}^3_{b_{ij}}\right]^n}, \\ \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^n\left[1-\tilde{F}^3_{b_{ij}}\right]^n} \end{matrix} \rangle \right)_{\boldsymbol{mxn}}$$

$\mathbb{N}^n \otimes \wp^n$

$$= \left[ \begin{matrix} \left(\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\right)^n, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^n+1-\left[1-\tilde{I}^3_{b_{ij}}\right]^n-\left(1-\left[1-\tilde{I}^3_{a_{ij}}\right]^n\right)\left(1-\left[1-\tilde{I}^3_{b_{ij}}\right]^n\right)} \\ \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^n+1-\left[1-\tilde{F}^3_{b_{ij}}\right]^n-\left(1-\left[1-\tilde{F}^3_{a_{ij}}\right]^n\right)\left(1-\left[1-\tilde{F}^3_{b_{ij}}\right]^n\right)} \end{matrix} \right]_{mxn}$$

$$= \left( \langle \begin{matrix} \left(\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\right)^n, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^n\left[1-\tilde{I}^3_{b_{ij}}\right]^n}, \\ \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^n\left[1-\tilde{F}^3_{b_{ij}}\right]^n} \end{matrix} \rangle \right)_{mxn}$$

$= (\mathbb{N} \otimes \wp)^n$

iv) $\mathbb{N}^{n_1} \otimes \mathbb{N}^{n_2} = \mathbb{N}^{(n_1+n_2)}$

$\mathbb{N}^{n_1} \otimes \mathbb{N}^{n_2} =$

$$\left[ \begin{matrix} \left(\tilde{T}_{a_{ij}}\right)^{n_1+n_2}, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{n_1}+1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{n_2}-\left(1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{n_1}\right)\left(1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{n_2}\right)} \\ \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{n_1}+1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{n_2}-\left(1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{n_1}\right)\left(1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{n_2}\right)} \end{matrix} \right]_{mxn}$$

$$=\left(\langle \begin{matrix} \left(\tilde{T}_{a_{ij}}\right)^{n_1+n_2}, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{n_1+n_2}}, \\ \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{n_1+n_2}} \end{matrix} \rangle\right) = \mathbb{N}^{(n_1+n_2)}$$

As a result, it is proven.

**Theorem 3.11.** Given that $\mathbb{N}$ and $\wp$ are matrices of size m×n and n>0, the following holds.

(i) $n\mathbb{N} \leq n\wp$,

(ii) $\mathbb{N}^n \leq \wp^n$

Proof:

Suppose $\mathbb{N} \leq \wp$

$\Rightarrow \tilde{T}_{a_{ij}} \leq \tilde{T}_{b_{ij}}$ and $\tilde{I}_{a_{ij}} \geq \tilde{I}_{b_{ij}}$ and $\tilde{F}_{a_{ij}} \geq \tilde{F}_{b_{ij}}$ for all i,j.

$$\Rightarrow \sqrt[3]{1-[1-\tilde{T}^3_{a_{ij}}]^n} \leq \sqrt[3]{1-[1-\tilde{T}^3_{b_{ij}}]^n}$$

$\left[\tilde{I}^3_{a_{ij}}\right]^n \geq \left[\tilde{I}^3_{b_{ij}}\right]^n$ and $\left[\tilde{F}^3_{a_{ij}}\right]^n \geq \left[\tilde{F}^3_{a_{ij}}\right]^n$ ,for all i,j.

ii) also $\left[\tilde{F}^3_{a_{ij}}\right]^n \geq \left[\tilde{F}^3_{b_{ij}}\right]^n$,for all i,j.

$\sqrt[3]{1-[1-\tilde{I}^3_{a_{ij}}]^n} \leq \sqrt[3]{1-[1-\tilde{I}^3_{b_{ij}}]^n}$,

$\sqrt[3]{1-[1-\tilde{F}^3_{a_{ij}}]^n} \leq \sqrt[3]{1-[1-\tilde{F}^3_{b_{ij}}]^n}$ for all i,j.

**Theorem 3.12.** Given that $\mathbb{N}$ and $\wp$ are matrices of size m×n and n>0, the following holds.

(i) $n(\mathbb{N} \wedge \wp) = n\ \mathbb{N} \wedge n\wp$,

(ii) $n(\mathbb{N} \vee \wp) = n\ \mathbb{N} \vee n\wp$

Proof

(i) $n(\mathbb{N} \wedge \wp)$

$$= \left(\langle \sqrt[3]{1-\left[1-\min\left(\tilde{T}^3_{a_{ij}}, \tilde{T}^3_{b_{ij}}\right)\right]^n}, \max\left(\left|\tilde{I}_{a_{ij}}\right|^n, \left|\tilde{I}_{b_{ij}}\right|^n\right), \max\left(\left|\tilde{F}_{a_{ij}}\right|^n, \left|\tilde{F}_{b_{ij}}\right|^n\right)\rangle\right)_{mxn}$$

$$= \left[\sqrt[3]{1-\left[\max\left(1-\tilde{T}^3_{a_{ij}}, 1-\tilde{T}^3_{b_{ij}}\right)\right]^n}, \max\left(\left|\tilde{I}_{a_{ij}}\right|^n, \left|\tilde{I}_{b_{ij}}\right|^n\right), \max\left(\left|\tilde{F}_{a_{ij}}\right|^n, \left|\tilde{F}_{b_{ij}}\right|^n\right)\right]_{mxn}$$

$$= \left[\sqrt[3]{1-(\max\left(\left[1-\tilde{T}^3_{a_{ij}}\right]^n, \left[1-\tilde{T}^3_{b_{ij}}\right]^n\right)}, \max\left(\left|\tilde{I}_{a_{ij}}\right|^n, \left|\tilde{I}_{b_{ij}}\right|^n\right), \max\left(\left|\tilde{F}_{a_{ij}}\right|^n, \left|\tilde{F}_{b_{ij}}\right|^n\right)\right]_{mxn}$$

$$= \left[\max\left(\sqrt[3]{1-\left[1-\tilde{T}^3_{a_{ij}}\right]^n}, \sqrt[3]{1-\left[1-\tilde{T}^3_{b_{ij}}\right]^n}\right), \max\left(\left|\tilde{I}_{a_{ij}}\right|^n, \left|\tilde{I}_{b_{ij}}\right|^n\right), \max\left(\left|\tilde{F}_{a_{ij}}\right|^n, \left|\tilde{F}_{b_{ij}}\right|^n\right)\right]_{mxn}$$

= n ℕ ∧ n℘

Hence, n(ℕ ∧ ℘) = n ℕ ∧ n℘,

,

Similarly we can prove n(ℕ ∨ ℘) = nℕ ∨ n℘

**Theorem 3.13**. For ℕ, ℘ $\in FN_{mxn}$ and n > 0, the following holds

(i) $(ℕ \wedge ℘)^n = ℕ^n \wedge ℘^n$,

(ii) $(ℕ \vee ℘)^n = ℕ^n \vee ℘^n$

Let $(ℕ \wedge ℘)^n$

$$= \begin{bmatrix} \min\left(\left|\tilde{T}_{a_{ij}}\right|^n, \left|\tilde{T}_{b_{ij}}\right|^n\right), \sqrt[3]{1-\max\left(\left[1-\tilde{I}^3_{a_{ij}}\right]^{\mathrm{n}}, \left[1-\tilde{I}^3_{b_{ij}}\right]^{\mathrm{n}}\right)}, \\ \sqrt[3]{1-\max\left(\left[1-\tilde{F}^3_{a_{ij}}\right]^{\mathrm{n}}, \left[1-\tilde{F}^3_{b_{ij}}\right]^{\mathrm{n}}\right)} \end{bmatrix}_{mxn}$$

$$= \begin{bmatrix} \min\left(\left|\tilde{T}_{a_{ij}}\right|^n, \left|\tilde{T}_{b_{ij}}\right|^n\right), \sqrt[3]{1-\left[1-\min\left(\left[1-\tilde{I}^3_{a_{ij}}\right]^{\mathrm{n}}, \left[1-\tilde{I}^3_{b_{ij}}\right]^{\mathrm{n}}\right)\right]}, \\ \sqrt[3]{1-\left[1-\min\left(\left[1-\tilde{F}^3_{a_{ij}}\right]^{\mathrm{n}}, \left[1-\tilde{F}^3_{b_{ij}}\right]^{\mathrm{n}}\right)\right]} \end{bmatrix}_{mxn}$$

$$= \begin{bmatrix} \min\left(\left|\tilde{T}_{a_{ij}}\right|^n, \left|\tilde{T}_{b_{ij}}\right|^n\right), \\ max\left(\sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{I}^3_{b_{ij}}\right]^{\mathrm{n}}}\right) \\ \max\left(\sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{F}^3_{b_{ij}}\right]^{\mathrm{n}}}\right) \end{bmatrix}_{mxn}$$

$$ℕ^n \wedge ℘^n = \left[\left(\left|\tilde{T}_{a_{ij}}\right|^n, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{I}^3_{b_{ij}}\right]^{\mathrm{n}}}\right) \wedge \left(\left|\tilde{T}_{b_{ij}}\right|^n, \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{F}^3_{b_{ij}}\right]^{\mathrm{n}}}\right)\right]$$

$$= \begin{bmatrix} \min\left(\left|\tilde{T}_{a_{ij}}\right|^n, \left|\tilde{T}_{b_{ij}}\right|^n\right), \\ max\left(\sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{I}^3_{b_{ij}}\right]^{\mathrm{n}}}\right) \\ \max\left(\sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{F}^3_{b_{ij}}\right]^{\mathrm{n}}}\right) \end{bmatrix}_{mxn}$$

$= (ℕ \wedge ℘)^n$

Hence $(ℕ \wedge ℘)^n = ℕ^n \wedge ℘^n$

Similarly we can prove $(ℕ \vee ℘)^n = ℕ^n \vee ℘^n$

**Theorem 3.14.**

For $\mathbb{N}$, $\wp \in FN_{mxn}$ and n > 0, we have $(\mathbb{N} \oplus \wp)^n \neq \mathbb{N}^n \oplus \wp^n$

Proof: Let $(\mathbb{N} \oplus \wp)^n$

$$\mathbb{N} \oplus \wp = \left(\langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}} \rangle\right)_{mxn}$$

$$(\mathbb{N} \oplus \wp)^n = \left(\langle \left(\sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}\right)^n, \sqrt[3]{1 - \left[1 - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}\right]^n}, \sqrt[3]{1 - \left[1 - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}\right]^n} \rangle\right)_{mxn}$$

$$\mathbb{N}^n = \left(\langle \left|\tilde{T}_{a_{ij}}\right|^n, \sqrt[3]{1 - \left[1 - \tilde{I}^3_{a_{ij}}\right]^n}, \sqrt[3]{1 - \left[1 - \tilde{F}^3_{a_{ij}}\right]^n} \rangle\right)_{mxn}$$

$$\wp^n = \left(\langle \left|\tilde{T}^3_{b_{ij}}\right|^n, \sqrt[3]{1 - \left[1 - \tilde{I}^3_{b_{ij}}\right]^n}, \sqrt[3]{1 - \left[1 - \tilde{F}^3_{b_{ij}}\right]^n} \rangle\right)_{mxn}$$

$$\mathbb{N}^n \oplus \wp^n = \begin{bmatrix} \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \left(\sqrt[3]{1 - \left[1 - \tilde{I}^3_{a_{ij}}\right]^n}\right)^n \left(\sqrt[3]{1 - \left[1 - \tilde{I}^3_{b_{ij}}\right]^n}\right)^n, \\ \left(\sqrt[3]{1 - \left[1 - \tilde{F}^3_{a_{ij}}\right]^n}\right)^n \left(\sqrt[3]{1 - \left[1 - \tilde{F}^3_{b_{ij}}\right]^n}\right)^n \end{bmatrix}_{mxn}$$

Hence $(\mathbb{N} \oplus \wp)^n \neq \mathbb{N}^n \oplus \wp^n$

## 4. A new operation (@) on Fermatean neutrosophic matrices

Motivated by the existing operations presented in [29].A novel operation on Fermatean neutrosophic matrices, symbolized by (@), is presented along with the demonstration of its favorable characteristics.

**Definition 4.1** If $\mathbb{N} = \langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle$ and $\wp = \langle \tilde{T}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{F}_{b_{ij}} \rangle$ are two Fermatean Neutrosophic Matrices ,then the new operation of FNM is defined by

$$\mathbb{N} @ \wp = \left[\langle \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}}}{2}} \rangle\right]_{mxn}$$

**Note :**

It is obvious that for any two Fermatean neutrosophic matrices $\mathbb{N}$ and $\wp$, the matrix $\mathbb{N} @ \wp$ is also a Fermatean neutrosophic matrix.

i.e., $0 \leq \frac{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}}}{2} + \frac{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}}}{2} + \frac{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}}}{2}$

$$\leq \frac{\tilde{T}^3_{a_{ij}} + \tilde{I}^3_{a_{ij}} + \tilde{F}^3_{a_{ij}}}{2} + \frac{\tilde{T}^3_{b_{ij}} + \tilde{I}^3_{b_{ij}} + \tilde{F}^3_{b_{ij}}}{2} \leq \frac{1}{2} + \frac{1}{2} = 1$$

**Theorem 4.1.** If $\mathbb{N}$ is a Fermatean Neutrosophic Matrix, then $\mathbb{N} @ \mathbb{N} = \mathbb{N}$.

Proof: Let $\mathbb{N} @ \mathbb{N} = \left(\langle \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{a_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{a_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{a_{ij}}}{2}} \rangle\right)$

$$= \left(\langle \left(\sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{a_{ij}}}{2}}\right)^2, \left(\sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{a_{ij}}}{2}}\right)^2, \left(\sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{a_{ij}}}{2}}\right)^2 \rangle\right)_{mxn}$$

$$= \left(\langle \left(\sqrt[3]{\frac{2\tilde{T}^3_{a_{ij}}}{2}}\right)^2, \left(\sqrt[3]{\frac{2\tilde{I}^3_{a_{ij}}}{2}}\right)^2, \left(\sqrt[3]{\frac{2\tilde{F}^3_{a_{ij}}}{2}}\right)^2 \rangle\right) = \left(\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} )\rangle\right)_{mxn}$$

Since $\tilde{T}^3_{a_{ij}} \le \tilde{T}_{a_{ij}}, \tilde{I}^3_{a_{ij}} \le \tilde{I}_{a_{ij}}, \tilde{F}^3_{a_{ij}} \le \tilde{F}_{a_{ij}}$.

Note 4.2 . if x, y ∈ [0, 1], then $xy \le \frac{x + y}{2}$ , $\frac{x + y}{2} \le x + y - xy$.

**Theorem 4.2.** Let $\mathbb{N}$, ℘ ∈ $FN_{mxn}$, then

(i) $(\mathbb{N} \oplus ℘) \vee (\mathbb{N} @ ℘) = \mathbb{N} \oplus ℘$,

(ii) $(\mathbb{N} \otimes ℘) \wedge (\mathbb{N} @ ℘) = \mathbb{N} \otimes ℘$,

(iii) $(\mathbb{N} \oplus ℘) \wedge (\mathbb{N} @ ℘) = \mathbb{N} @ ℘$,

(iv) $(\mathbb{N} \otimes ℘) \vee (\mathbb{N} @ ℘) = \mathbb{N} @ ℘$

Proof:

(i) $(\mathbb{N} \oplus ℘) \vee (\mathbb{N} @ ℘) = \mathbb{N} \oplus ℘$

Let $(\mathbb{N} \oplus ℘) \vee (\mathbb{N} @ ℘)$

$$= \begin{bmatrix} max\left(\sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}\right), \\ min\left(\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}\right), min\left(\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}}\right) \end{bmatrix}_{mxn}$$

$$= \left[\sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]_{mxn}$$

$= \mathbb{N} \oplus ℘$

(ii) $(\mathbb{N} \otimes ℘) \wedge (\mathbb{N} @ ℘) = \mathbb{N} \otimes ℘$

Let $(\mathbb{N} \otimes ℘) \wedge (\mathbb{N} @ ℘)$

=

$$=\left[\left\langle \begin{array}{c} min\left(\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}, \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}\right), max\left(\sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}\right) \\ max\left(\sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}}\right) \end{array}\right\rangle\right]_{mxn}$$

$$= \left[\langle\left(\langle\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}\rangle\right)\rangle\right]_{mxn}$$

$= \mathbb{N} \otimes \wp$

(iii) $(\mathbb{N} \oplus \wp) \wedge (\mathbb{N} \ @ \ \wp) = \mathbb{N} \ @ \ \wp$

Let $(\mathbb{N} \oplus \wp) \wedge (\mathbb{N} \ @ \ \wp)$

$$= \left[\begin{array}{c} min\left(\sqrt[3]{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}-\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}\right), \\ max\left(\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}\right), \\ max\left(\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}}\right) \end{array}\right]_{mxn}$$

$$= \left(\langle \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}}\rangle\right)_{mxn} = (\mathbb{N} \ @ \ \wp)$$

(iv) $(\mathbb{N} \otimes \wp) \vee (\mathbb{N} \ @ \ \wp) = \mathbb{N} \ @ \ \wp$

Let $(\mathbb{N} \otimes \wp) \vee (\mathbb{N} \ @ \ \wp)$

$$=\left[\left\langle \begin{array}{c} max\left(\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}, \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}\right), min\left(\sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}\right) \\ min\left(\sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}}\right) \end{array}\right\rangle\right]_{mxn}$$

$$= \left( \left\langle \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}} \right\rangle \right)_{mxn} = (\mathbb{N} \ @ \ \wp)$$

Hence, $(\mathbb{N} \otimes \wp) \vee (\mathbb{N} \ @ \ \wp) = \mathbb{N} \ @ \ \wp$

## 5. Conclusion

This research aims to enrich the domain of neutrosophic matrix theory and neutrosophic logic by investigating Fermatean neutrosophic matrices (FNMs), a novel approach to handling uncertainty. The paper study some fundamental algebraic operations on Fermatean neutrosophic matrices. The Fermatean neutrosophic matrices considered as a generalization of Fermatean fuzzy matrices, intuitionistic fuzzy matrices, Pythagorean fuzzy matrices, and Pythagorean neutrosophic matrices. The paper shows that the properties of Fermatean neutrosophic matrices are consistent with the properties of the standard operations. To conclude, a novel operation (@) on Fermatean neutrosophic matrices is defined and distributive rules are examined. In the upcoming, it will be significant to probe how the suggested aggregating operators of FNMs might be used in decision-making problem, Information fusion, and Operations research under neutrosophic environment.

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