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On The Symbolic 2-plithogenic Fermat's Non-Linear Diophantine Equation

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Abstract:

This paper is dedicated to find all symbolic 2-plithogenic integer solutions for the symbolic 2-plithogenic Fermat's Diophantine equation $X^n + Y^n = Z^n$ for $n \geq 3$. We prove that it has exactly 27 solutions, and we find all possible solutions.

Keywords: symbolic 2-plithogenic integer, Fermat's Diophantine equation, symbolic 2-plithogenic ring.

Introduction and basic definitions.

The theory of Diophantine equations is considered as an important and central theory in commutative algebra.

In our days, many developments of algebraic structures have helped us with general cases of Diophantine equations, for example, neutrosophic rings and their generalizations [1-4] have led to many new related Diophantine equations such as neutrosophic Pell's equation [5], refined neutrosophic Diophantine equation [6] and n-refined equations [7]. The main application of generalized versions of number theory is cryptography algorithms, see [15-19].

The concept of symbolic 2-plithogenic rings was defined in [8], then it was studied and generalized by many authors, see [9-14].

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The Fermat's triple is defined as a solution of the Diophantine non-linear equation $X^n + Y^n = Z^n$ in the ring R, with $n \geq 3$.

We refer to that for the special case of $n = 2$, the Fermat's triple is called a Pythagoras triple.

It is useful for the reader to ensure that neutrosophic and plithogenic number theory is useful in cryptography [16-20].

In this work, we find all solutions of $X^n + Y^n = Z^n$ in the symbolic ring of integers $2 - SP_Z$.

Definition.

Let Z be the ring of integers, the corresponding symbolic 2-plithogenicc ring of integers is defined as follow:

$2 - SP_Z = \{x + yP_1 + zP_2; x, y, z \in Z, {P_i}^2 = P_i, P_1 \times P_2 = P_2 \times P_1 = P_2\}$.

Theorem.

For $X = l_0 + l_1P_1 + l_2P_2 \in 2 - SP_Z$, then;

$X^n = {l_0}^n + P_1[(l_0 + l_1)^n - {l_0}^n] + P_2[(l_0 + l_1 + l_2)^n - (l_0 + l_1)^n]$.

Remark.

In Z, we have three Fermat's triples:

$(0,1,1), (1,0,1), (0,0,0)$ for all $n \geq 3$.

Main results

Theorem.

Let $2 - SP_Z$ be the symbolic 2-plithogenic ring of integers, then it has exactly 27 Fermat's triples.

Proof.

Let (T, S, K) be a Fermat's triple of $2 - SP_Z$

with $T = t_0 + t_1P_1 + t_2P_2, S = s_0 + s_1P_1 + s_2P_2, K = k_0 + k_1P_1 + k_2P_2$,

the equation $T^n + S^n = K^n$ is equivalent to:

$$\begin{cases} {t_0}^n + {s_0}^n = {k_0}^n \\ (t_0 + t_1)^n + (s_0 + s_1)^n = (k_0 + k_1)^n \\ (t_0 + t_1 + t_2)^n + (s_0 + s_1 + s_2)^n = (k_0 + k_1 + k_2)^n \end{cases}$$

Thus $A_1=(t_0,s_0,k_0), A_2=(t_0+t_1,s_0+s_1,k_0+k_1), A_3=(t_0+t_1+t_2,s_0+s_1+s_2,k_0+k_1+k_2)$ are three triples in Z.

So that, $A_1,A_2,A_3\in\{(0,1,1),(1,0,1),(0,0,0)\}$, thus three exists 27 solutions of the symbolic 2-plithogenic Fermat's Diophantine equation.

Now, we discus all possible cases:

Case1.

$$\begin{cases} t_0=s_0=k_0=0 \\ t_0+t_1=s_0+s_1=k_0+k_1=0 \\ t_0+t_1+t_2=s_0+s_1+s_2=k_0+k_1+k_2=0 \end{cases}$$

Thus $F_1=(0,0,0)$

Case2.

$$\begin{cases} t_0=s_0=k_0=0 \\ t_0+t_1=s_0+s_1=k_0+k_1=0 \\ t_0+t_1+t_2=0, s_0+s_1+s_2=k_0+k_1+k_2=1 \end{cases}$$

Thus $F_2=(0,P_2,P_2)$

Case3.

$$\begin{cases} t_0=s_0=k_0=0 \\ t_0+t_1=s_0+{s_1}^n=k_0+k_1=0 \\ t_0+t_1+t_2=k_0+k_1+k_2=1, s_0+s_1+s_2=0 \end{cases}$$

Thus $F_3=(P_2,0,P_2)$

Case4.

$$\begin{cases} t_0=s_0=k_0=0 \\ t_0+t_1=k_0+k_1=1, s_0+s_1=0 \\ t_0+t_1+t_2=k_0+k_1+k_2=s_0+s_1+s_2=0 \end{cases}$$

Thus $F_4=(P_1-P_2,0,P_1-P_2)$

Case5.

$$\begin{cases} t_0=s_0=k_0=0 \\ t_0+t_1=k_0+k_1=1, s_0+s_1=0 \\ t_0+t_1+t_2=k_0+k_1+k_2=1, s_0+s_1+s_2=0 \end{cases}$$

Thus $F_5=(P_1,0,P_1)$

Case6.

$$\begin{cases} t_0=s_0=k_0=0 \\ t_0+t_1=k_0+k_1=1, s_0+s_1=0 \\ t_0+t_1+t_2=0, k_0+k_1+k_2=s_0+s_1+s_2=1 \end{cases}$$

Thus $F_6 = (P_1 - P_2, P_2, P_1)$

Case7.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_7 = (0, P_1 - P_2, P_1 - P_2)$

Case8.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_8 = (P_2, P_1 - P_2, P_1)$

Case9.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_9 = (0, P_1, P_1)$

Case10.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{10} = (1 - P_1, 0{,}1 - P_1)$

Case11.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{11} = (1 - P_1, 0{,}1 - P_1 + P_2)$

Case12.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_{12} = (1 - P_1, P_2, 1 - P_1 + P_2)$

Case13.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{13} = (1 - P_1, 0,1 - P_2)$

Case14.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{14} = (1,0,1)$

Case15.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{15} = (1 - P_1, P_2, 1)$

Case16.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{16} = (1 - P_1, P_1 - P_2, 1 - P_1)$

Case17.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{17} = (1 - P_1 + P_2, P_1 - P_2, 1)$

Case18.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_{18} = (1 - P_1, P_1, 1)$

Case19.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{19} = (0,1 - P_1, 1 - P_1)$

Case20.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{20} = (P_2, 1 - P_1, 1 - P_1 + P_2)$

Case21.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{21} = (0,1 - P_1 + P_2, 1 - P_1 + P_2)$

Case22.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{22} = (P_1 - P_2, 1 - P_1, 1 - P_2)$

Case23.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{23} = (P_1, 1 - P_1, 1)$

Case24.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_{22} = (P_1 - P_2, 1 - P_1 + P_2, 1)$

Case25.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{25} = (0,1 - P_2, 1 - P_2)$

Case26.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{26} = (P_2, 1 - P_2, 1)$

Case27.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_{27} = (0,1,1)$.

Conclusion

In this paper, we have studied the solutions of symbolic 2-plithogenic Fermat's non-linear Diophantine equation, where we have proved that it has exactly 27 solutions, and we presented the all-27 possible solutions.

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