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On The Algebraic Homomorphisms Between Symbolic 2-plithogenic Rings And 2-cyclic Refined Rings

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Abstract:

The main goal of this research paper is to find an algebraic ring homomorphism between symbolic 2-plithogenic ring and the corresponding 2-cyclic refined ring.

This work presents some applications of the defined homomorphism to explain some algebraic relationships between symbolic 2-plithogenic algebraic structures and 2-cyclic refined structures.

Keywords: 2-cyclic refined ring, symbolic 2-plithogenic ring, symbolic 2-plithogenic matrix, 2-cyclic refined vector spaces.

Introduction and preliminaries

Algebraic homomorphisms play a central role in the classification of rings, where they are considered a very rich material to find the algebraic relationships between different rings.

The symbolic 2-plithogenic rings were defined in [3], they have many interesting properties, since they are a good extension of classical rings, see [1-2, 7-10]. Symbolic 2-plithogenic rings are examples about symbolic n-plithogenic sets and structures founded by Smarandache [4, 11-12].

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On the other hand, another extension of rings was defined and handled by many authors, where n-cyclic refined rings are neutrosophic structures with an algebraic structure similar to the cyclic ring of integers [5-6].

This work is dedicated to find an algebraic relation by using homomorphisms between symbolic 2-plithogenic rings and 2-cyclic refined rings, where these homomorphisms can be used between the matrices defined over these rings, and vectors defined over them.

Many examples will be presented as a sign of the validity of our work.

For the definitions of algebraic relations between symbolic 2-plithogenic elements see [3]. For the definitions of algebraic relations between n—cyclic refiend elements see [6].

Main discussion

Theorem.

Let $R_2(I)$ be the 2-cyclic refined ring, ideals of the ring R, $2-SP_R$ be the symbolic 2-plithogenic ring refined over the ring R, then there exists a ring homomorphism $f: R_2(I) \to 2-SP_R$.

Proof.

We define $f: R_2(I) \to 2-SP_R$ such that:

$f(V_0 + V_1I_1 + V_2I_2) = V_0 + (V_1 + V_2)P_1 - 2V_1P_2$

f is well defined:

Assume that $V_0 + V_1I_1 + V_2I_2 = w_0 + w_1I_1 + w_2I_2$, then $V_i = w_i$ for all $0 \leq i \leq 2$, thus:

$V_0 + (V_1 + V_2)P_1 - 2V_1P_2 = w_0 + (w_1 + w_2)P_1 - 2w_1P_2$,

hence $f(V_0 + V_1I_1 + V_2I_2) = f(w_0 + w_1I_1 + w_2I_2)$.

f preserves addition:

For $V = V_0 + V_1I_1 + V_2W = I_2, w_0 + w_1I_1 + w_2I_2$, we have:

$V+W=(V_0+w_0)+(V_1+w_1)I_1+(V_2+w_2)I_2$

$f(V+W)=(V_0+w_0)+(V_1+w_1+V_2+w_2)P_1-2(V_1+w_1)P_2=$

$[V_0+(V_1+V_2)P_1-2V_1P_2]+[w_0+(w_1+w_2)P_1-2w_1P_2]=f(V)+f(W).$

f preserves multiplication:

$V.W=V_0.w_0+(V_0w_1+V_1w_0+V_1w_2+V_2w_1)I_1+(V_0w_2+V_2w_0+V_2w_2+V_1w_1)I_2$

$$f(V.W)=V_0.w_0+(V_0w_1+V_1w_0+V_1w_2+V_2w_1+V_0w_2+V_2w_0+V_2w_2+V_1w_1)P_1 - 2(V_0w_1+V_1w_0+V_1w_2+V_2w_1)P_2$$

On the other hand, we have:

$f(V).f(W)=[V_0+(V_1+V_2)P_1-2V_1P_2].[w_0+(w_1+w_2)P_1-2w_1P_2]=V_0.w_0+(V_0w_1+V_1w_0+V_1w_2+V_2w_1+V_0w_2+V_2w_0+V_2w_2+V_1w_1)P_1-2(V_0w_1+V_1w_0+V_1w_2+V_2w_1)P_2=f(V.W).$

So that, f is a ring homomorphism.

Theorem.

Let f be the previous homomorphism defined with $f: R_2(I)\to 2-SP_R$, then:

1. $ker(f)=\{yI_1-yI_2; 2y=0, y\in R\}$
2. $ker(f)$ is a zero ring.

Proof.

1. $ker(f)=\{x+yI_1+zI_2; f(x+yI_1+zI_2)=0\}$, hence $x+(y+z)P_1-2yP_2=0$,

   thus $x=0, z=-y, 2y=0$ which implies that:

$ker(f)=\{yI_1-yI_2; 2y=0, y\in R\}$.

2. Let $M=mI_1-mI_2, N=nI_1-nI_2\in ker(f)$, then $2m-2n=0$, thus:

$M.N=(mI_1-mI_2)(nI_1-nI_2)=mnI_2-mnI_1-mnI_1+mnI_2=-2mnI_1+2mnI_2=0-0=0$, which means that $ker(f)$ is a zero ring.

Theorem.

Let R be a field, then $R_2(I)\cong 2-SP_R$

Proof.

According to the previous theorem, we have $2y = 0$ implies that $y = 0$, thus $ker(f) = \{O\}$ and f is injective.

To prove that the homomorphism f is surjective, we take an arbitrary element

$V_0 + V_1P_1 + V_2P_2 \in 2 - SP_R$, then there exists

$W = V_0 + \left(\frac{-V_2}{2}\right) I_1 + \left(V_1 + \frac{V_2}{2}\right) I_2 \in R_2(I)$.

Such that:

$f(W) = V_0 + V_1P_1 + V_2P_2$, so that f is an isomorphism.

Applications to Vector Spaces.

By using the previous relationship between symbolic 2-plithogenic ring and 2-cyclic refined rings, we will be able to show the algebraic relations between symbolic 2-plithogenic vector spaces and 2-cyclic refined vector spaces.

Theorem.

Let F be an algebraic field, T be a vector space over F. Assume that $2 - SP_T = \{t_0 + t_1P_1 + t_2P_2; t_i \in T\}$ is the corresponding symbolic 2-plithogenic vector space over $2 - SP_F$.

$T_2(I) = \{t_0 + t_1P_1 + t_2P_2; t_i \in T\}$ is the corresponding 2-cyclic refined vector space over $F_2(I)$, then there exists a semi module homomorphism between $T_2(I)$ and $2 - SP_T$.

Proof.

According to the previous theorems, there exists a ring homomorphism $f: F_2(I) \to 2 - SP_F$ such that

$f(V_0 + V_1I_1 + V_2I_2) = V_0 + (V_1 + V_2)P_1 - 2V_1P_2$

We define $g: T_2(I) \to 2 - SP_T$ such that:

$g(t_0 + t_1I_1 + t_2I_2) = t_0 + (t_1 + t_2)P_1 - 2t_1P_2$

g is well defined:

If $t_0 + t_1I_1 + t_2I_2 = \acute{t}_0 + \acute{t}_1I_1 + \acute{t}_2I_2$, then $t_i = \acute{t}_i; 0 \leq i \leq 2$ and

$t_0 + (t_1 + t_2)P_1 - 2t_1P_2 = \acute{t}_0 + (\acute{t}_1 + \acute{t}_2)P_1 - 2\acute{t}_1P_2$,

which means that

$g(t_0 + t_1I_1 + t_2I_2) = g(\acute{t}_0 + \acute{t}_1I_1 + \acute{t}_2I_2)$.

g preserves addition:

For $M = t_0 + t_1I_1 + t_2I_2, N = \acute{t}_0 + \acute{t}_1I_1 + \acute{t}_2I_2 \in R_2(I)$.

$M + N = (t_0 + \acute{t}_0) + (t_1 + \acute{t}_1)I_1 + (t_2 + \acute{t}_2)I_2$

$g(M + N) = (t_0 + \acute{t}_0) + (t_1 + \acute{t}_1 + t_2 + \acute{t}_2)P_1 - 2(t_1 + \acute{t}_1)P_2 = [t_0 + (t_1 + t_2)P_1 - 2t_1P_2] + [\acute{t}_0 + (\acute{t}_1 + \acute{t}_2)P_1 - 2\acute{t}_1P_2] = g(M) + g(N)$.

To complete the proof, we must prove that:

$g(qM) = f(q).g(M); q = q_0 + q_1I_1 + q_2I_2 \in F_2(I)$ and $M = t_0 + t_1I_1 + t_2I_2 \in T_2(I)$.

First, we have:

$qM = q_0.t_0 + (q_0t_1 + q_1t_0 + q_1t_2 + q_2t_1)I_1 + (q_0t_2 + q_2t_0 + q_2t_2 + q_1t_1)I_2$

$$g(qM) = q_0.t_0 + (q_0t_1 + q_1t_0 + q_1t_2 + q_2t_1 + q_0t_2 + q_2t_0 + q_2t_2 + q_1t_1)P_1 - 2(q_0t_1 + q_1t_0 + q_1t_2 + q_2t_1)P_2$$

$f(q) = q_0 + (q_1 + q_2)P_1 - 2q_1P_2$

$g(M) = t_0 + (t_1 + t_2)P_1 - 2t_1P_2$

$f(q).g(M) = q_0.t_0 + (q_0t_1 + q_1t_0 + q_1t_2 + q_2t_1 + q_0t_2 + q_2t_0 + q_2t_2 + q_1t_1)P_1 - 2(q_0t_1 + q_1t_0 + q_1t_2 + q_2t_1)P_2 = g(qM)$.

This implies that f is a semi-module homomorphism, and $T_2(I)$ is semi homomorphic to $2 - SP_T$.

Remark.

Consider that $H = \{h_0 + h_1I_1 + h_2I_2;\ h_i \in \acute{H}, \acute{H}\ is\ a\ subspace\ of\ T\}$, then H is a submodule of $T_2(I)$, let us find it direct image according to the semi-homomorphism g.

$Im(H) = g(H) = \{h_0 + (h_1 + h_2)P_1 - 2h_1P_2;\ h_i \in \acute{H}\}$.

We prove that $Im(H)$ is a submodule of $2 - SP_T$.

Let $X = x_0 + (x_1 + x_2)P_1 - 2x_1P_2, Y = y_0 + (y_1 + y_2)P_1 - 2y_1P_2 \in Im(H)$, where $x_i, y_i \in \acute{H}$

$X + Y = (x_0 + y_0) + (x_1 + x_2 + y_1 + y_2)P_1 - 2(x_1 + y_1)P_2 \in Im(H)$

Let $q = q_0 + q_1I_1 + q_2I_2 \in 2 - SP_F$, then:

$$qX = q_0.x_0 + (q_0x_1 + q_0x_2 + q_1x_0 + q_1x_2 + q_1x_1)P_1 + (-2q_0x_1 + q_0x_2 - 2q_1x_1 + q_2x_1 + q_2x_2)P_2$$

$\acute{q} = q_0 + \left(\frac{-q_2}{2}\right) I_1 + \left(q_1 + \frac{q_2}{2}\right) I_2 \in F_2(I)$, then:

$f(\acute{q}) = q$, which implies that:

$qX = f(\acute{q})g(\acute{X})$, $\acute{X} = x_0 + \left(\frac{-x_2}{2}\right) I_1 + \left(x_1 + \frac{x_2}{2}\right) I_2$, hence $qX = g(\acute{q}\acute{X}) \in Im(H)$ is a submodule of $2 - SP_T$.

Remark.

Let us find the kernel of the semi-homomorphism g.

$ker(g) = \{M = t_0 + t_1 I_1 + t_2 I_2 \in T_2(I), g(M) = 0\}$, so that:

$$\begin{cases} t_0 = 0 \\ t_1 + t_2 = 0 \Longrightarrow t_2 = 0 \\ -2t_1 = 0 \Longrightarrow t_1 = 0 \end{cases}$$

Hence, $ker(g) = \{O\}$.

Result:

Let $M = t_0 + t_1 I_1 + t_2 I_2 \in T_2(I)$, assume that $E = \{e_0, \dots, e_k\}$ is a basis of $T_2(I)$ over $F_2(I)$, then

$M = n_0 e_0 + n_1 e_1 + \cdots + n_k e_k$; $n_i \in F_2(I)$.

By taking the direct image of M, we can find $g(M) = g\left(\sum_{i=0}^{k} n_i e_i\right) = \sum_{i=0}^{k} f(n_i) g(e_i)$.

This means that the elements of $Im(T_2(I))$ can be written as a linear combination with respect to the elements of the basis E.

On the other hand, the set $g(E) = \{g(e_0), \dots, g(e_k)\}$ is linearly independent, that is because $\sum_{i=0}^{k} f(n_i) g(e_i) = 0 \Longrightarrow \sum_{i=0}^{k} g(n_i e_i) = 0 \Longrightarrow g\left(\sum_{i=0}^{k} n_i e_i\right) = 0 \Longrightarrow \sum_{i=0}^{k} n_i e_i = 0 \Longrightarrow n_i = 0; 0 \leq i \leq k$.

Applications to matrices.

Let $M = (m_{ij})_{k \times k}$ be a square matrix with 2-cyclic refined entries, then $M = M_0 + M_1 I_1 + M_2 I_2$, where M_0, M_1, M_2 are there $k \times k$ classical matrices.

If we take the direct image of M by the ring homomorphism

$f(M) = M_0 + (M_1 + M_2) P_1 - 2 M_1 P_2$, we get a symbolic 2-plithogenic matrix.

For a 2-cyclic refined real number $q = q_0 + q_1 I_1 + q_2 I_2$, we see that qM is a 2-cyclic refined square real matrix.

We can use the semi-homomorphism g to write:

$g(qM) = f(q)g(M); g(M) = M_0 + (M_1 + M_2)P_1 - 2M_1P_2$

Example.

Consider the following 2-cyclic refined real square matrix:

$$M = \begin{pmatrix} 2 + I_1 - I_2 & 1 + 3I_1 + I_2 \\ I_1 - I_2 & I_1 + I_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} I_1 + \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} I_2$$

$$= M_0 + M_1I_1 + M_2I_2$$

The corresponding symbolic 2-plithogeni matrix $g(M)$ is equal to:

$$g(M) = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix} P_1 + \begin{pmatrix} -2 & -6 \\ -2 & -2 \end{pmatrix} P_2 = \begin{pmatrix} 2 - 2P_2 & 1 + 4P_1 - 6P_2 \\ -2P_2 & 2P_1 - 2P_2 \end{pmatrix}$$

Remark.

Let us study the relation between $\det M$ and $det(g(M))$.

$\det M = \det M_0 + I_1[det(M_0 + M_1 + M_2) - det(M_0 - M_1 + M_2)] + \frac{1}{2}I_1[det(M_0 + M_1 + M_2) - det(M_0 - M_1 + M_2) - 2\det M_0 +]$ see [1].

$det(g(M)) = det[M_0 + (M_1 + M_2)P_1 - 2M_1P_2] = \det M_0 + P_1[det(M_0 + M_1 + M_2) - \det M_0] + P_2[det(M_0 - M_1 + M_2) - det(M_0 + M_1 + M_2)]$ see [2].

Consider the ring homomorphism:

$f: R_2(I) \to 2 - SP_R;\ f(a_0 + a_1I_1 + a_2I_2) = a_0 + (a_1 + a_2)P_1 - 2a_1P_2$, then:

$f(detM) = \det M_0 + P_1[det(M_0 + M_1 + M_2) - \det M_0] + P_2[det(M_0 - M_1 + M_2) - det(M_0 + M_1 + M_2)]$

Applications to modules.

If R is a ring, K_R be a module over R.

Let $2 - SP_R$, $R_2(I)$ be the corresponding symbolic 2-plithogenic ring and 2-cyclic refined respectively.

Let $2 - SP_M$ be the corresponding symbolic 2-plithogenic module over $2 - SP_R$ and $M_2(I)$ be the corresponding 2-cyclic refined module over $R_2(I)$, then by a similar discussion of the case of vector spaces, we can write:

1. $g: M_2(I) \to 2 - SP_M$ such that:

$g(m_0 + m_1I_1 + m_2I_2) = m_0 + (m_0 + m_1)P_1 - 2m_1P_2$ is a semi module homomorphism.

2. If S is a submodule of $M_2(I)$, then $Im(S) = g(S)$ is a submodule of $2 - SP_M$.
3. For $q = q_0 + q_1 I_1 + q_2 I_2 \in R_2(I)$ and $m = m_0 + m_1 I_1 + m_2 I_2 \in M_2(I)$, then $g(q) = 0$ does not imply that $m = 0$, because:

$g(qm) = 0 \Longrightarrow f(q)g(m) = 0$, since R is not a field, then it may has zero divisors, which means that $f(q) = 0$ without $q = 0$.

This is a big difference between the case of 2-cyclic refined vector spaces and 2-cyclic refined modules.

Conclusion

In this paper, we have found an algebraic homomorphism between 2-cyclic refined rings and symbolic 2-plithogenic rings, and we have used this homomorphism to study some algebraic relations between 2-cyclic refined matrices and symbolic 2-plithogenic matrices.

In the future, we aim to find the algebraic relations between other kinds of neutrosophic rings and symbolic n-plithogenic rings.

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