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Solving Non-linear Neutrosophic Linear Programming Problems

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Abstract: Non-linear neutrosophic numbers (NLNNs) are different kinds of neutrosophic numbers with at least one non-linear membership function (either of truthiness, falsity or indeterminacy part) of the information. Furthermore, a linear programming problem with non-linear neutrosophic numbers as coefficients/parameters is a special type of programming problem known as a non-linear linear programming problem (NLN-LPP). This paper elaborates on the concepts of non-linear neutrosophic number (NLNN) sets, different forms of non-linear neutrosophic numbers (NLNNs), α, β, γ cuts on non-linear neutrosophic numbers (NLNNs), possibility mean, possibility standard deviation, and possibility variance of non-linear neutrosophic numbers (NLNNs). In this paper, we also propose the solution technique for non-linear neutrosophic linear programming problems (NLN-LPPs) in which all coefficients/parameters are non-linear neutrosophic numbers (NLNNs). In this continuation, we suggest a new modified possibility score function for non-linear NNs in terms of possibility means and possibility standard deviations of non-linear neutrosophic numbers (NLNNs) for better use of all parts of information. This modified score function is used to convert non-linear neutrosophic number (NLNN) coefficients/parameters of non-linear neutrosophic linear programming problem (NLN-LPP) into equivalent crisp values. Thereafter, the equivalent crisp problem is solved with the usual method to obtain the optimal solution of non-linear neutrosophic linear programming problem (NLN-LPP). The proposed solution algorithm is unique and new for solving non-linear neutrosophic linear programming problems. A numerical example is solved with the proposed algorithm to legitimate the research output. A case study is also discussed to show its applicability in solving real-life problems.

Keywords: Linear programming problem; Non-linear neutrosophic numbers (NLNNs), Possibility score function of Non-linear neutrosophic numbers (NLNNs), Possibility mean of Non-linear neutrosophic numbers (NLNNs), Possibility standard deviation of Non-linear neutrosophic numbers (NLNNs).

1. Background of the Problem and Motivation – An Introduction

In 1965, Prof. Zadeh [1] introduced the concept of fuzzy set theory to deal with the uncertainty and ambiguity in information due to human language error and human perceptions. Prof. Zadeh [1] defined a set $A=\left\{x:\left|\mu_A^T(x), 0 \le \mu_A^T(x) \le 1; x \in X\right.\right\}$ with objects x having $\mu_A^T(x)$ degree of acceptance of particular characteristic. This set A is called as fuzzy set with membership function $\mu_A^T(x)$. Since 1965, many researchers have contributed in the area of fuzzy set, fuzzy logic, and its application in solving real-world problems.

Fuzzy sets (FSs) are further classified into two major types – (i) Linear fuzzy set - FS with linear membership function e.g. triangular, trapezoidal, pentagonal (Chakraborty et al. [2]), hexagonal

(Chakraborty et al. [3]), heptagonal (Maity et al. [4]), etc. (ii) Non-linear Fuzzy set – FS with non-linear membership functions e.g. logarithmic, exponential membership function, etc. In general, fuzzy set (FS) theory avoids the involvement of other parts of information. Later, Atanassov [5-6] proposed the intuitionistic fuzzy set (IFS) theory and properties of IFS. Intuitionistic fuzzy set (IFS) – a more generalized FS theory that considers two parts of information i.e. acceptance (truthiness of information) and non-acceptance (falsity of information). Liu and Yuan [7] combined intuitionistic fuzzy set (IFS) and triangular fuzzy number (TFN) to introduce the intuitionistic triangular fuzzy set (ITFS) theory which has triangular membership functions for the truthiness and falsity part of information. Ye [8] extended the TIFS to trapezoidal form to introduce intuitionistic trapezoidal fuzzy set (ITrFS). On the other hand, a linear programming problem is one of the simplest problems of MPPs (Mathematical programming problems) which has linear objective function and linear constraints. LPPs play a vital role in formulating simple real-life problems that arise in Business, Govt. policies, industries, etc. LP problems are easy to solve with the Graphical and Simplex method depending upon the number of decision variables involved. The simplex method is a generalized method for solving any LPP with some manual computational efforts. In contrast with the past, LPPs and NLPPs (non-linear programming problems) are solved quickly and efficiently with the help of computational tools like LINGO©, MATLAB©, etc.

1.1. Fuzzy and neutrosophic programming problems – Literature Review

With time, fuzzy set theory and fuzzy numbers were incorporated in MPPs (LPPs, multiobjective programming problems (MOPPs), Bi-level/Multi-level programming problems (BLPPs/MLPPs), other extension problems, etc.) and many new solution techniques have been developed by researchers for solving MPPs with fuzzy parameters/coefficients. Some of the notable contributions are: Luhandjula [9] developed a new solution technique for fuzzy linear programming problem (FLPP). Arikan and Gunjar [10] proposed a new solution algorithm known as a two-phase approach for MOPPs with fuzzy coefficients. Wu [11] proposed to solve MOPP with fuzzy coefficients using the scalarization technique. For BLPPs/MLPPs, Shih et al. [12] suggested a general solution approach to solve fuzzy multi-level programming problems (FMLPPs). Baky [13] proposed an algorithm for ML-MOPPs through fuzzy goal programming approach. Osman et al. [14] suggested an interactive solution approach for ML-MOPPs with fractional objective functions and fuzzy parameters. Fuzzy set theory is based on only one aspect of information i.e. truthiness and avoids the other two parts of information which are indeterminacy and falsity. On the other hand, intuitionistic fuzzy set (IFS) theory considers two parts of information i.e. truthiness and falsity but ignores a third important part of information i.e. indeterminacy. To disseminate these shortcomings of FS and IFS, Samarandche [15] introduced a new theory known as Neutrosophic set theory (NN set theory) dealing with the object along with three parts of information - truthiness, falsity, and indeterminacy. Later, Samarandche [16-17] specified some properties of neutrosophic set (NS) theory and linear neutrosophic numbers (NNs) including Addition and subtraction of linear NNs, α, β, γ cuts on linear NNs, etc. Wang et al. [27] discovered a new type of NS – Single valued neutrosophic set (SVNS) to apply in real-life problems. Ye [18] introduced trapezoidal interval-valued NNs (IV TrNNs) by combining triangular neutrosophic numbers (TrNNs) and trapezoidal fuzzy numbers (TrFNs). Since the past few years, the combination of neutrosophic set theory (linear NNs) and MPPs (specifically for LPPs) has become a prominent area of research. This is exhibited in a literature survey of recent years, e.g. Hussian et al. [19] used properties of NNs to convert neutrosophic LPP into an equivalent crisp LPP. Abdel-Basset et al. [20] suggested a new ranking function for the solution of neutrosophic LPP. Bera and Mahapatra [21] suggested a real-life application of neutrosophic LPP and developed a simplex method to solve it. Darehmiraki [22] proposed a new parametric ranking function to solve neutrosophic LPPs. Khatter [23] used properties of possibility mean of NNs to solve neutrosophic LPPs. Tamilarasi and Paulraj [24] developed a solution technique for neutrosophic LPPs with triangular NNs and de-neutrosophication of NNs with Melin transform. Similar to FS theory, neutrosophic sets (NN sets) are classified as (i) linear NN sets and (ii) Non-linear NN sets. A linear NN set is a neutrosophic set with all linear membership functions (membership for truthiness, falsity, and indeterminacy) whereas a non-linear NN set is a neutrosophic set with at least one non-linear membership function (either of truthiness, falsity or indeterminacy).On the non-linear neutrosophic numbers (NLNNs), Chakraborty et al. [25] and Javier and Francisco [26] discussed about properties of NLNNs and their applications. Recently, Rabie A et al. [28] suggested a dual artificial variable-free simplex algorithm for neutrosophic linear programming problems. Badr El-Sayed et al. [29] discovered the exterior point simplex method for solving neutrosophic linear programming problems. Badr El-Sayed et al. [30] proposed two phase method approach for solving neutrosophic linear programming problems. Badr El-Sayed et al. [31] proposed an application part of neutrosophic goal programming in the context of sustainable development of Egypt.

1.2. Novelty and Major Contributions

Neutrosophic set theory plays a vital role in dealing with the uncertain and vague information that arises in real-world industrial problems. Many researchers have contributed on neutrosophic set theory and applied new techniques for solving real problems. Some of recent contributions are: Abdel-Basset M et al. [32] suggested important neutrosophic techniques for solving problems in various smart environments. Maissam Jdid and Smarandache [33] described the use of neutrosophic technique in solving two important operation research problems of 'optimal design of warehouses' and 'capital budget allocation'. Abduallah Gamal et al. [34] proposed the use of type -2 neutrosophic number to obtain optimal solution of multi-criteria decision-making problems of autonomous vehicles and distributed resources. During the literature survey on NNs, it is disclosed that contrary to research on linear NNs, only a few researchers contributed on properties of non-linear neutrosophic numbers (NLNNs), arithmetic operations on NLNNs, its application in formulating real problems, etc. These are: Chakraborty et al. [25] discussed different types of Non-linear trapezoidal NNs and their properties. Javier and Francisco [26] discovered the basic properties of NLNNs, a new scoring function, and demonstrated its application to multiple criteria assessment problems of industry. Some typo errors have been pointed out in the work of Javier and Francisco [26] in defining different properties of NLNNs which are rectified in this manuscript. Further, the involvement of non-linear NNs in MPPs (LPPs or other complex MPPs) as coefficients/parameters is hardly ever been researched to date due to the computational complexities of Non-linear NNs, and therefore, no solution methodology has been developed for non-linear neutrosophic linear programming problem (NLN-LPP) till date. This motivates us to extend the use of NLNNs in LPPs, propose a modified score function of NLNNs, and propose a solution technique for NLN-LPPs. In this view, the main contribution of this paper can be summarized:

(i) Proposed a new modified possibility score function for non-linear neutrosophic numbers (NLNNs) with the concept of normal approximation.

(ii) Proposed a novel and unique solution technique for non-linear neutrosophic linear programming problem (NLN-LPP) using a modified possibility score function.

(iii) Elaborated different properties of non-linear neutrosophic numbers (NLNNs) in the corrected form in a systematic manner for future researchers.

In nutshell, this paper elaborates on the concepts of non-linear neutrosophic number (NLNN) sets, different forms of NLNNs, α, β, γ cuts on non-linear neutrosophic numbers (NLNNs), possibility mean, possibility standard deviation, and possibility variance of NLNNs. In this paper, we propose the solution technique for non-linear neutrosophic linear programming problems (NLN-LPPs) in which all coefficients/parameters are NLNNs. In this continuation, we suggest a new modified possibility score function for non-linear NNs in terms of possibility means and possibility standard deviations of NLNNs for better use of all parts of information. This modified score function is used to convert NLNN coefficients/parameters of NLN-LPP into equivalent crisp values. Thereafter, the equivalent crisp problem is solved with the usual method to obtain the optimal solution of NLN-LPP. The proposed solution algorithm is unique and new for solving non-linear neutrosophic linear programming problems. A numerical example is solved with the proposed algorithm to legitimate the research output. A case study is also discussed to show its applicability in solving real-life problems.

This paper is organized in a section-wise format: This first section of the paper gives a systematic introduction of the current research problem and focused literature review from the beginning. In sub-section 1.1, literature review on fuzzy and neutrosophic programming problems are presented. Sub-section 1.2 discloses the causes of motivation for proposing this research work, novelty of proposed work and major contributions. Some preliminaries on the neutrosophic set (NN set) are presented in next section 2 and its subsections. α, β, γ cut sets of NLNNs are defined and derived in subsection 2.1. Possibility mean, possibility variance, and possibility standard deviations are defined and derived in subsection 2.2. The modified possibility score function for NL-NNs is proposed in section 3. The formulation of non-linear neutrosophic linear programming problem (NLN-LPP) and suggested solution technique for NLN-LPPs are described and explained in section 4. To better understand the proposed algorithm, one numerical example and a case study of an industrial decision-making problem based on NLN-LPP are illustrated in section 5. Conclusions and research directions for future researchers are proposed in the last section.

2. Neutrosophic Set: Preliminaries

In this section, we shall discuss some generic preliminaries on neutrosophic set (NN set) related to the research area under study. As we know that neutrosophic set (introduced by Smarandche [15]) is a set of objects with membership function values of truthiness, indeterminacy, and falsity of information of objects of concern set. Later, Wang et al. [27] gave the concept of single-valued NN which is NN set with values of membership functions lying within the interval [0, 1]. In continuation of this context, a single-valued neutrosophic set is mathematically defined by the following generic definition:

Definition 1. (Wang et al. [27]): A neutrosophic set A in X is characterized as $A=\left\{x:\mu_A^T(x), \sigma_A^T(x), \upsilon_A^T(x), x\in X\right\}$ where $T_A(x), I_A(x), F_A(x)\in[0,1]$ represents degree of membership for truthiness, indeterminacy and falsity parts of information respectively along with condition $0\le T_A(x)+I_A(x)+F_A(x)\le 3$. If all membership functions of defined SVNNs are linear, then it is called as linear SVNNs. It is being reiterated that NNs are further classified as linear NNs and Non-linear NNs (NLNNs) on the linearity of all membership functions and non-linearity of at least one membership function of NN. Chakrabort et al. [25] presented the definition of non-linear trapezoidal type NN as:

Definition 2. (Chakraborty et al. [25]): A single valued non-linear trapezoidal NN is defined as:

$$A=\left\{x:(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4;c_1,c_2,c_3,c_4;p_1,p_2;q_1,q_2;r_1,r_2;\omega,\rho,\lambda);T_A(x), I_A(x), F_A(x), x\in X\right\} \quad (1)$$

where $T_A(x), I_A(x), F_A(x)\in[0,1]$ represents membership for truthiness, indeterminacy and falsity of information respectively are given as:

$$T_A(x)=\begin{cases}\omega\left(\dfrac{x-a_1}{a_2-a_1}\right)^{p_1}, & a_1\le x<a_2,\\ \omega, & a_2<x\le a_3,\\ \omega\left(\dfrac{a_4-x}{a_4-a_3}\right)^{p_2}, & a_3<x\le a_4,\\ 0, & \text{Otherwise},\end{cases} \tag{2}$$

$$I_A(x)=\begin{cases}\rho\left(\dfrac{b_2-x}{b_2-b_1}\right)^{q_1}, & b_1\le x<b_2,\\ 0, & b_2<x\le b_3,\\ \rho\left(\dfrac{x-b_3}{b_4-b_3}\right)^{p_2}, & b_3<x\le b_4,\\ \rho, & \text{Otherwise},\end{cases} \tag{3}$$

$$F_A(x)=\begin{cases}\lambda\left(\dfrac{c_2-x}{c_2-c_1}\right)^{q_1}, & c_1\le x<c_2,\\ 0, & c_2<x\le c_3,\\ \lambda\left(\dfrac{x-c_3}{c_4-c_3}\right)^{p_2}, & c_3<x\le c_4,\\ \lambda, & \text{Otherwise},\end{cases} \tag{4}$$

along with conditions $0\le T_A(x)+I_A(x)+F_A(x)\le 3$ and $p_1,p_2;q_1,q_2;r_1,r\ne 1$.

Thereafter, Javier and Francisco [26] proposed an alternate definition of non-linear NN for triangular values in view of mapping of parameter values ρ and λ with their minimum and maximum values. According to Javier and Francisco [26], NLNNs are defined as follows:

Definition 3. (Javier and Francisco [26]):

$$A_{(m,n)}=\left\{x:(\underline{a},a,\overline{a};\ w,y,u);T(x),I(x),F(x);\ x\in X\right\} \tag{5}$$

is a single valued non-linear NN (SVNN) whose respective membership function $T(x),I(x)$ and $F(x)$ are defined as:

$$T_A(x)=\begin{cases}\left[1-\left(\dfrac{a-x}{a-\underline{a}}\right)^{m_T}\right]w, & \underline{a}\le x<a,\\ w, & x=a,\\ \left[1-\left(\dfrac{a-x}{a-\overline{a}}\right)^{n_T}\right]w, & a\le x<\overline{a},\\ 0, & \text{Otherwise},\end{cases} \tag{6}$$

$$I(x)=\begin{cases}\left[y+(1-y)\left(\dfrac{a-x}{\underline{a}}\right)^{m_I}\right], & \underline{a}\le x<a,\\ y, & x=a,\\ \left[y+(1-y)\left(\dfrac{a-x}{a-\overline{a}}\right)^{n_I}\right], & a<x\le\overline{a},\\ 1, & \text{Otherwise},\end{cases} \tag{7}$$

$$F(x)=\begin{cases}\left[u+(1-u)\left(\dfrac{a-x}{\underline{a}}\right)^{m_F}\right], & \underline{a}\le x<a,\\ u, & x=a,\\ \left[u+(1-u)\left(\dfrac{a-x}{a-\overline{a}}\right)^{n_F}\right], & a<x\le\overline{a},\\ 1, & \text{Otherwise},\end{cases} \quad (8)$$

Where parameters $m=(m_T,m_I,m_F)\in[1,\infty]$; $n=(n_T,n_I,n_F)\in[1,\infty]$. It can be observed from definition (6)-(8), when $m=(1,1,1)$; $n=(1,1,1)$, then NLNN reduces to a triangular linear NN.

2.1. α,β,γ cut- sets of non-linear neutrosophic numbers (NLNNs)

Definition 4. : The α,β,γ cut sets of NLNN $A_{(m,n)}=\{x:(\underline{a},a,\overline{a};\ w,y,u);T(x),I(x),F(x);\ x\in X\}$ are defined as: $$A_{(\alpha,\beta,\gamma)}=\{\langle x,T(x)\ge\alpha,I(x)\le\beta,F(x)\le\gamma\rangle:x\in X\} \quad (9)$$

With the conditions $0\le\alpha\le w;\ y\le\beta\le1;\ u\le\gamma\le1$ and $\alpha+\beta+\gamma\le3$. Using the definition (5) - (8) of NLNN, we can obtain α,β,γ cut sets as:

For α cut set $T(x)\ge\alpha\Rightarrow\left[1-\left(\dfrac{a-x}{a-\underline{a}}\right)^{m_T}\right]w\ge\alpha\Rightarrow x\ge a-\left(1-\dfrac{\alpha}{w}\right)^{1/m_T}(a-\underline{a})$

And $T(x)\ge\alpha$ gives $x\ge a-\left(1-\dfrac{\alpha}{w}\right)^{1/m_T}(a-\underline{a})$

Thus α cut set of NLNN $A_{(m,n;\alpha,\beta,\gamma)}$ is a closed interval described as:

$$A_{(m_T,n_T;\alpha)}=\left[T^L(\alpha,m_T),T^U(\alpha,n_T)\right] \quad (10)$$

Where $$T^L(\alpha,m_T)=a-\left(1-\frac{\alpha}{w}\right)^{1/m_T}(a-\underline{a}) \quad (11)$$

And $$T^U(\alpha,n_T)=a-\left(1-\frac{\alpha}{w}\right)^{1/n_T}(a-\overline{a}) \quad (12)$$

In similar manner, β and cut set of NLNN $A_{(m,n;\alpha,\beta,\gamma)}$ are closed interval sets described as:

$$A_{(m_I,n_I;\beta)}=\left[I^L(\beta,m_T),\ I^U(\beta,n_T)\right] \quad (13)$$

Where $I^L(\beta,m_I)=a-\left(\dfrac{\beta-y}{1-y}\right)^{1/m_I}(a-\underline{a})$; $\quad I^U(\beta,n_T)=a-\left(\dfrac{\beta-y}{1-y}\right)^{1/n_I}(a-\overline{a})$

$$A_{(m_F,n_F;\gamma)}=\left[F^L(\gamma,m_F),\ F^U(\gamma,n_F)\right] \quad (14)$$

Where $F^L(\gamma,m_F)=a-\left(\dfrac{\gamma-u}{1-u}\right)^{1/m_F}(a-\underline{a})$; $\quad F^U(\gamma,n_F)=a-\left(\dfrac{\gamma-u}{1-u}\right)^{1/n_F}(a-\overline{a})$

2.2 Possibility mean, possibility variance and possibility standard deviation of non-linear neutrosophic numbers (NLNNs)

Definition 5. (Possibility mean of a NLNN): (Javier and Francisco [26]): For a NLNN as defined in (5) – (8), $A_{(m,n)} = \left\{ x : (\underline{a}, a, \overline{a};\ w, y, u);\ x \in X \right\}$ and its α cut set i.e. $A_{(m_T, n_T; \alpha)} = \left[T^L(\alpha, m_T), T^U(\alpha, n_T) \right]$, then *f*-weighted possibility mean of truth membership function is defined as:

$$M_T(A_{(m,n)}) = \frac{w}{2}\left(\frac{2m_T^2 \underline{a} + 3m_T a + a}{(1+2m_T)(1+m_T)} + \frac{2n_T^2 \overline{a} + 3n_T a + a}{(1+2n_T)(1+n_T)} \right) \tag{15}$$

Where *f*-weight is considered as $f = {}^{2\alpha}\!/_{w}$ as suggested by Chakraborty et al. [25]. Similarly, *g*-weighted ($g = {}^{2(1-\beta)}\!/_{(1-y)}$) possibility mean of indeterminacy membership function is defined as:

$$M_I(A_{(m,n)}) = \frac{(1-y)}{2}\left(\frac{2m_I^2 \underline{a} + 3m_I a + a}{(1+2m_I)(1+m_I)} + \frac{2n_I^2 \overline{a} + 3n_I a + a}{(1+2n_I)(1+n_I)} \right) \tag{16}$$

Also, *h*- weighted ($h = {}^{2(1-\gamma)}\!/_{(1-u)}$) possibility mean of indeterminacy membership function is defined as:

$$M_F(A_{(m,n)}) = \frac{(1-u)}{2}\left(\frac{2m_F^2 \underline{a} + 3m_F a + a}{(1+2m_F)(1+m_F)} + \frac{2n_F^2 \overline{a} + 3n_F a + a}{(1+2n_F)(1+n_F)} \right) \tag{17}$$

Definition 6. (Possibility variance of a NLNN): (Javier and Francisco [26]): For a NLNN as defined in (5)–(8), $A_{(m,n)} = \left\{ x : (\underline{a}, a, \overline{a};\ w, y, u);\ x \in X \right\}$ and its α cut set i.e. $A_{(m_T, n_T; \alpha)} = \left[T^L(\alpha, m_T), T^U(\alpha, n_T) \right]$, then *f*- weighted possibility variance of truth membership function is defined as:

$$V_T(A_{(m,n)}) = w\left[\frac{m_T^2 (a-\underline{a})^2}{4(1+m_T)(2+m_T)} + \frac{n_T^2 (\overline{a}-a)^2}{4(1+n_T)(2+n_T)} - \frac{n_T^2 m_T^2 (a^2 - \underline{a}a - a\overline{a} + \underline{a}\overline{a})}{(m_T + n_T + 2m_T n_T)(m_T + n_T + m_T n_T)} \right] \tag{18}$$

Where *f*-weight is considered as $f = {}^{2\alpha}\!/_{w}$ as suggested by Chakraborty et al. [25]. Similarly, *g*-weighted ($g = {}^{2(1-\beta)}\!/_{(1-y)}$) possibility variance of indeterminacy membership function is defined as:

$$V_I(A_{(m,n)}) = (1-y)\left[\frac{m_I^2 (a-\underline{a})^2}{4(1+m_I)(2+m_I)} + \frac{n_I^2 (\overline{a}-a)^2}{4(1+n_I)(2+n_I)} - \frac{n_I^2 m_I^2 (a^2 - \underline{a}a - a\overline{a} + \underline{a}\overline{a})}{(m_I + n_I + 2m_I n_I)(m_I + n_I + m_I n_I)} \right] \tag{19}$$

Also, *h*- weighted ($h = {}^{2(1-\gamma)}\!/_{(1-u)}$) possibility mean of indeterminacy membership function is defined as:

$$V_F(A_{(m,n)}) = (1-u)\left[\frac{m_F^2 (a-\underline{a})^2}{4(1+m_F)(2+m_F)} + \frac{n_F^2 (\overline{a}-a)^2}{4(1+n_F)(2+n_F)} - \frac{n_F^2 m_F^2 (a^2 - \underline{a}a - a\overline{a} + \underline{a}\overline{a})}{(m_F + n_F + 2m_F n_F)(m_F + n_F + m_F n_F)} \right] \tag{20}$$

Definition 7. (Possibility standard deviation of a NLNN): (Javier and Francisco [26]): For a NLNN $A_{(m,n)} = \left\{ x : (\underline{a}, a, \overline{a};\ w, y, u);\ x \in X \right\}$ and its α cut set i.e. $A_{(m_T, n_T; \alpha)} = \left[T^L(\alpha, m_T), T^U(\alpha, n_T) \right]$, then possibility standard deviation of its membership functions are defined as:

Possibility S.D = $\sqrt{\text{Possibility variance}}$

$\Rightarrow$ Possibility standard deviation of truth membership $D_T(A_{(m,n)}) = \sqrt{V_T(A_{(m,n)})}$

$$i.e.\quad D_T(A_{(m,n)}) = \left[w\left[\frac{m_T^2(a-\underline{a})^2}{4(1+m_T)(2+m_T)} + \frac{n_T^2(\overline{a}-a)^2}{4(1+n_T)(2+n_T)} - \frac{n_T^2 m_T^2(a^2-\underline{a}a-a\overline{a}+\underline{a}\overline{a})}{(m_T+n_T+2m_Tn_T)(m_T+n_T+m_Tn_T)} \right]\right]^{1/2} \quad (21)$$

Possibility standard deviation of indeterminacy membership $D_I(A_{(m,n)}) = \sqrt{V_I(A_{(m,n)})}$

$$i.e\quad D_I(A_{(m,n)}) = \left[(1-y)\left[\frac{m_I^2(a-\underline{a})^2}{4(1+m_I)(2+m_I)} + \frac{n_I^2(\overline{a}-a)^2}{4(1+n_I)(2+n_I)} - \frac{n_I^2 m_I^2(a^2-\underline{a}a-a\overline{a}+\underline{a}\overline{a})}{(m_I+n_I+2m_In_I)(m_I+n_I+m_In_I)} \right]\right]^{1/2} \quad (22)$$

and possibility standard deviation of falsity membership function $D_F(A_{(m,n)}) = \sqrt{V_F(A_{(m,n)})}$

$$i.e\quad D_F(A_{(m,n)}) = \left[(1-u)\left[\frac{m_F^2(a-\underline{a})^2}{4(1+m_F)(2+m_F)} + \frac{n_F^2(\overline{a}-a)^2}{4(1+n_F)(2+n_F)} - \frac{n_F^2 m_F^2(a^2-\underline{a}a-a\overline{a}+\underline{a}\overline{a})}{(m_F+n_F+2m_Fn_F)(m_F+n_F+m_Fn_F)} \right]\right]^{1/2} \quad (23)$$

Remark 1. It is to be noted that there were some typo errors in definitions of possibility mean, possibility variance and possibility standard deviations of NLNNs given by Javier and Francisco [26] which are respectively corrected here and presented definitions (5) – (8) are in corrected form.

Remark 2. It can also be observed from definition (5) – (8), when $m=(1,1,1)$; $n=(1,1,1)$, then NLNN reduces to a single valued triangular NN (SVTNN) and accordingly their characteristics as: by definitions (15) – (23),

Possibility means $M_T(A_{(1,1)}) = \dfrac{(\underline{a}+4a+\overline{a})w}{6}$ $M_I(A_{(1,1)}) = \dfrac{(\underline{a}+4a+\overline{a})(1-y)}{6}$; $M_F(A_{(1,1)}) = \dfrac{(\underline{a}+4a+\overline{a})(1-u)}{6}$

Possibility variance $V_T(A_{(1,1)}) = \dfrac{(\overline{a}-\underline{a})^2 w}{24}$; $V_I(A_{(1,1)}) = \dfrac{(\overline{a}-\underline{a})^2(1-y)}{24}$; $V_F(A_{(1,1)}) = \dfrac{(\overline{a}-\underline{a})^2(1-u)}{24}$

Possibility SD $D_T(A_{(1,1)}) = (\overline{a}-\underline{a})\sqrt{\dfrac{w}{24}}$; $D_I(A_{(1,1)}) = (\overline{a}-\underline{a})\sqrt{\dfrac{(1-y)}{24}}$; $D_F(A_{(1,1)}) = (\overline{a}-\underline{a})\sqrt{\dfrac{(1-u)}{24}}$

3. Proposed modified possibility score function for non-linear neutrosophic numbers (NLNNs)

Possibility score functions are used for ranking purposes and conversion of NNs into their equivalent crisp values. Javier and Francisco [26] proposed a possibility score function for NLNNs as a simple addition of the average of possibility means and possibility standard deviations related to truthiness. Indeterminacy and falsity membership values of NLNNs. Here we argue that this score function is a limitation to express all x values of the domain set in decision-making context. Therefore, to better characterize the role of all range values x in expressing the possibility score function, we propose a modified form of possibility score function for NLNNs:

$$PS(A_{(m,n)}) = \frac{PS_T(A_{(m,n)}) + PS_I(A_{(m,n)}) + PS_F(A_{(m,n)})}{3} \quad (24)$$

Where $PS_T(A_{(m,n)}), PS_I(A_{(m,n)}), PS_F(A_{(m,n)})$ are respective possibility score functions for truth, indeterminacy and falsity membership functions which are defined as:

$$PS_T(A_{(m,n)}) = M_T(A_{(m,n)}) + 2.58 D_T(A_{(m,n)}) \quad (25)$$

$$PS_I(A_{(m,n)}) = M_I(A_{(m,n)}) + 2.58 D_I(A_{(m,n)}) \quad (26)$$

$$PS_F(A_{(m,n)}) = M_F(A_{(m,n)}) + 2.58 D_F(A_{(m,n)}) \quad (27)$$

As the normal curve is the best fitted curve for all membership values in general conditions and this curve covers values with 99% confidence in interval $Mean \pm 2.58\, S.D.$ This is the main reason of proposing the possibility score function ((23) – (26)) in the modified form so that membership functions *T(x), I(x)* and *F(x)* graphs can be well approximated to normal curve with statistical parameters - possibility means and possibility standard deviation. The proposed modified possibility score function displays the contribution of all x values as in the normal curve.

4. Formulation of Non-linear Neutrosophic Linear Programming Problem (NLN-LPP) and Proposed Solution Technique

During the literature review, it has already been disclosed in particular that non-linear neutrosophic linear programming problems have not been discussed so far due to the non-linear complexities of functions. So, now we propose non-linear neutrosophic linear programming problems (NLN –LPP) as linear programming problems with non-linear neutrosophic numbers (NLNNs) as parameters/ coefficients of LPPs. In mathematical format, a single objective NLN-LPP with N decision variables can be described as:

$$Maximize / Minimize\ Z = c_{1,(m,n)}x_1 + c_{2,(m,n)}x_2 + ... + c_{N,(m,n)}x_N \quad \text{(Objective function)}$$

Subject to the set of constraints,

$$a_{11,(m,n)}x_1 + a_{12,(m,n)}x_2 + ... + a_{1N,(m,n)}x_N\ (\leq = \geq)\ b_{1,(m,n)}$$

$$a_{21,(m,n)}x_1 + a_{22,(m,n)}x_2 + ... + a_{2N,(m,n)}x_N\ (\leq = \geq)\ b_{2,(m,n)}$$

$$.$$

$$.$$

$$a_{M1,(m,n)}x_1 + a_{M2,(m,n)}x_2 + ... + a_{MN,(m,n)}x_N\ (\leq = \geq)\ b_{M,(m,n)}$$

And Non-negativity restrictions $x_1, x_2, ..., x_N \geq 0$ (28)

Where superscript $\sim$ on coefficients indicates that concern coefficients are single valued NLNNs with the set of values $A_{(m,n)} = \left\{ x : (\underline{a}, a, \overline{a};\ w, y, u); (m,n) = (m_T, m_I, m_F; n_T, n_I, n_F);\ x \in X \right\}$. The other notations have usual meaning in respect of LPPs. Such problems (27) have incomplete, vague and uncertain information on coefficients in terms of NLNNs are defined as NLN-LPPs. In the real world, such decision-making problems are expected to have a crisp optimal solution. Thus, we here propose a solution methodology for NLN-LPPs in which firstly all NLNNs are converted into equivalent crisp values using respective modified possibility score functions. Mathematically, converted equivalent crisp LPP with modified possibility score functions can be described as:

$$Maximize / Minimize\ Z = PS(c_{1,(m,n)})x_1 + PS(c_{2,(m,n)})x_2 + ... + PS(c_{N,(m,n)})x_N \quad \text{(Objective function)}$$

Subject to,

$$PS(a_{11,(m,n)})x_1 + PS(a_{12,(m,n)})x_2 + ... + PS(a_{1N,(m,n)})x_N\ (\leq = \geq)\ PS(b_{1,(m,n)})$$

$$PS(a_{21,(m,n)})x_1 + PS(a_{22,(m,n)})x_2 + ... + PS(a_{2N,(m,n)})x_N\ (\leq = \geq)\ PS(b_{2,(m,n)})$$

$$.$$

$$.$$

$$PS(a_{M1,(m,n)})x_1 + PS(a_{M2,(m,n)})x_2 + ... + PS(a_{MN,(m,n)})x_N\ (\leq = \geq)\ PS(b_{M,(m,n)})$$

And $x_1, x_2, ..., x_N \geq 0$ (29)

Where $PS(A_{ij,(m,n)})$ indicates the corresponding possibility score function values as defined in (24) – (27). The satisfactory solution to original NLN-LPP is the optimal solution of equivalent crisp LPP (29).

5. Numerical illustration and case study

To describe the proposed algorithm, we shall consider the following numerical example and a case study of industrial problem based on NLN-LPP as:

Numerical Example

$$Maximize\ Z = c_{1,(m,n)}x_1 + c_{2,(m,n)}x_2 + c_{3,(m,n)}x_3$$

Subject to, $a_{11,(m,n)}x_1 + a_{12,(m,n)}x_2 + a_{13,(m,n)}x_3 \le b_{1,(m,n)}$

$$a_{21,(m,n)}x_1 + a_{22,(m,n)}x_2 + a_{23,(m,n)}x_3 \le b_{2,(m,n)}$$

and non-negativity restrictions $x_1, x_2, ..., x_N \ge 0$

where neutrosophic coefficients are given as:

$c_1 = ((2,3,4); 0.5, 0.25, 0.25, (2,2,2); (2,2,2))$; $c_2 = ((3,4,5); 0.5, 0.25, 0.25, (2,2,2); (2,2,2))$

$c_3 = ((4,5,6); 0.5, 0.25, 0.25, (2,2,2); (2,2,2))$; $a_{11} = ((3,4,5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

$a_{12} = ((4,5,6); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $a_{13} = ((4,5,6); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

$b_1 = ((6,7,8); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $a_{21} = ((1,2,3); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

$a_{22} = ((3,4,5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $a_{23} = ((2.5,3.5,4.5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

$b_2 = ((5,6,7); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

For the proposed solution technique, we calculate possibility means (by equations (15) – (17)), possibility SD (by equations (21) – (23)) and modified possibility score function (by equations (24) – (27)) corresponding to each NN coefficient of the problem. These values are described in tabular format (Table 1, Appendix A) in correspondence to the neutrosophic coefficients of the problem. Using these values, the given NLN-LPP is converted into equivalent crisp problem as:

$$Maximize\ Z = 3.211x_1 + 3.8192x_2 + 4.2109x_3$$

Subject to, $3.5229x_1 + 4.1896x_2 + 4.1896x_3 \le 5.2313$

$$2.2729x_1 + 3.5229x_2 + 3.2313x_3 \le 4.8559$$

and non-negativity restrictions $x_1, x_2, x_3 \ge 0$

Solving with the Simplex method, the optimal solution obtained is as: $x_1 = 0,\ x_2 = 0,\ x_3 = 1.2486,$ $Z = 5.2578$ which is also the solution to original NLN-LPP. If we use possibility score function as $PS^*(A_{(m,n)}) = M(A_{(m,n)}) + D(A_{(m,n)})$ (as suggested by Javier and Francisco [26]) to convert NLNNs into corresponding equivalent crisp values and solve the converted crisp LPP, we obtain the optimal solution of the problem as: $x_1 = 0,\ x_2 = 0,\ x_3 = 1.2885,\ Z = 4.9001$. On comparison, it is clear that the modified possibility score function gives better values of objective function.

Case Study

Let us consider a case study of 'XYZ' company manufacturing certain fashion items in different production slots. The production variables of these items as well as demands are decided with the help of information gathered via social media networks, reviews, customer comments, etc. It is known to decision-makers of production units that this information is not fully true and reliable. Decision makers assume that related information on social media is in NNs format *i.e.* truthiness, falsity, and indeterminacy also their degree of memberships varies mostly in a non-linear way. For sake of simplicity in this case study, it is assumed that production and demand coefficients are in SVTNN. The profit maximization LP problem of this company with NLNNs can be presented as:

$$Maximize\ Z = c_{1,(m,n)}x_1 + c_{2,(m,n)}x_2 \quad \text{(Profit)}$$

Subject to, $a_{11,(m,n)}x_1 + a_{12,(m,n)}x_2 \le b_{1,(m,n)}$

$$a_{21,(m,n)}x_1 + a_{22,(m,n)}x_2 \le b_{2,(m,n)}$$

$$a_{31,(m,n)}x_1 + a_{32,(m,n)}x_2 \le b_{3,(m,n)}$$

$$a_{41,(m,n)}x_1 + a_{42,(m,n)}x_2 \le b_{4,(m,n)}$$

and non-negativity restrictions $x_1, x_2 \ge 0$

where neutrosophic coefficients are given as:

$c_1 = ((4,5,6);0.5,0.25,0.25,(1,1,1);(1,1,1))$; $c_2 = ((6,7,8);0.5,0.25,0.25,(1,1,1);(1,1,1))$;

$a_{11} = ((4,5,6);0.5,0.25,0.25,(1,1,1);(1,1,1))$; $a_{12} = ((1,2,3);0.5,0.25,0.25,(1,1,1);(1,1,1))$;

$a_{21} = ((2.5,3.5,4.5);0.5,0.25,0.25,(1,1,1);(1,1,1))$; $a_{22} = ((3.5,4.5,5.5);0.5,0.25,0.25,(1,1,1);(1,1,1))$;

$a_{31} = ((5,6,7);0.5,0.25,0.25,(1,1,1);(1,1,1))$; $a_{32} = ((5,6,7);0.5,0.25,0.25,(1,1,1);(1,1,1))$;

$a_{41} = ((5.5,6.5,7.5);0.5,0.25,0.25,(1,1,1);(1,1,1))$; $a_{42} = ((3,4,5);0.5,0.25,0.25,(1,1,1);(1,1,1))$;

$b_1 = ((1,2,3);0.5,0.25,0.25,(1,1,1);(1,1,1))$; $b_2 = ((8,9,10);0.5,0.25,0.25,(1,1,1);(1,1,1))$;

$b_3 = ((5,6,7);0.5,0.25,0.25,(1,1,1);(1,1,1))$; $b_4 = ((1.5,2.5,3.5);0.5,0.25,0.25,(1,1,1);(1,1,1))$

with the proposed solution technique and tabulated values (Table 2, Appendix A), this problem is converted into equivalent crisp problem as:

$$Maximize\ Z = 4.1758x_1 + 5.4305x_2 \qquad \text{(profit)}$$

Subject to,
$$4.2457x_1 + 2.9679x_2 \le 3.3989$$
$$3.0282x_1 + 3.9957x_2 \le 5.1853$$
$$4.6322x_1 + 5.4120x_2 \le 5.1853$$
$$5.7870x_1 + 3.9120x_2 \le 2.1478$$

and non-negativity restrictions $x_1, x_2 \ge 0$

With the help of simplex method, the optimal solution to this crisp problem is obtained as: $x_1 = 0$, $x_2 = 0.5490$, $x_3 = 0$, $Z = 2.9815$ which is also the solution to original NLN-LPP. This is too better solution to the problem than solution by technique based on possibility score function by Javier and Francisco [26] which is $x_1 = 0$, $x_2 = 0.5414$, $x_3 = 0$, $Z = 2.7063$.

6. Conclusions and future research directions

Non-linear neutrosophic numbers (NLNNs) are different kinds of neutrosophic numbers (NNs) with at least one non-linear membership function (either of truthiness, falsity or indeterminacy part) of the information. Furthermore, a non-linear neutrosophic linear programming problem (NLN-LPP) is a special type of linear programming problem in which coefficients/parameters are non-linear neutrosophic numbers. This paper presents comprehensive research on non-linear neutrosophic numbers (NLNNs) and non-linear neutrosophic linear programming problems (NLN-LPPs). Here, the author proposed a novel solution technique for NLN-LPPs based on the proposed modified possibility score function. This proposed modified possibility score function covers the almost entire range of values of NNs. Besides this, this paper elaborates on the concepts of non-linear neutrosophic (NLNN) sets, different forms of NLNNs, α, β , γ - cuts on NLNNs, possibility mean, possibility standard deviation, and possibility variance of NLNNs in corrected forms for clear understanding to future researchers. As future research, this work can be extended to solve non-linear neutrosophic non-linear programming problems (NLN-NLPPs), non-linear neutrosophic multiobjective programming problems (NLN-MOPPs), non-linear neutrosophic bi-level and multi-level programming problems (NLN-BL/MLPPs), etc. There is a scope of research investigations on basic operations on NLNNs – addition, substation, multiplication, and division of two or more NLNNs.

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Appendix A.

Table 1. Possibility score function values for NLNN coefficients of numerical example

NLNN Coefficient	$M_T(A_{(m,n)})$	$M_F(A_{(m,n)})$	$M_I(A_{(m,n)})$	$D_T(A_{(m,n)})$	$D_F(A_{(m,n)})$	$D_I(A_{(m,n)})$	$PS_T(A)$	$PS_T(A)^*$	$PS_F(A)$	$PS_F(A)*$	$PS_I(A)$	$PS_I(A)*$	$PS(A)$	$PS(A)*$
c_1	1.5	2.25	2.25	0.4081	0.5	0.5	2.553	1.9081	3.54	2.75	3.54	2.75	3.211	2.4693
c_2	2	3	2.825	0.4081	0.5	0.5	3.053	2.4081	4.2897	3.5	4.115	3.325	3.8192333	3.0777
c_3	2.5	3.75	3.75	0.4081	0.5	0.5	3.553	2.9081	4.03974	4.25	5.04	4.25	4.2109133	3.8027
a_{11}	2	3	3.75	0.2886	0.3535	0.3535	2.7447	2.2886	3.9121	3.3535	3.9121	4.1035	3.5229667	3.2485
a_{12}	2.5	3.75	3.75	0.2886	0.3535	0.3535	3.2447	2.7886	4.6621	4.1035	4.6621	4.1035	4.1896333	3.6652
a_{13}	2.3333	3.75	3.75	0.2886	0.3535	0.3535	3.2447	2.6219	4.6621	4.1035	4.6621	4.1035	4.1896333	3.6096
a_{21}	0.8333	1.5	1.5	0.2886	0.3535	0.3535	1.7447	1.1219	2.6621	1.8535	2.4121	1.8535	2.2729667	1.6096
a_{22}	1.8333	3	3	0.2886	0.3535	0.3535	2.7447	2.1219	3.9121	3.3535	3.9121	3.3535	3.5229667	2.9429
a_{23}	1.5833	2.75	2.625	0.2886	0.3535	0.3535	2.4947	1.8719	3.6621	3.1035	3.5371	2.9785	3.2313	2.6513
b_1	3.3333	4.375	5.25	0.2886	0.3535	0.3535	4.2447	3.6219	5.2871	4.7285	6.1621	5.6035	5.2313	4.6513
b_2	2.8333	4.5	4.5	0.2886	0.3535	0.3535	3.7437	3.1219	5.4121	4.8535	5.4121	4.8535	4.8559667	4.2763

*Possibility score function $PS^*(A_{(m,n)}) = M(A_{(m,n)}) + D(A_{(m,n)})$ suggested by Javier and Francisco [26]

Table 2. Possibility score function values for NLNN coefficients of case study

NLNN Coefficient	$M_T(A_{(m,n)})$	$M_F(A_{(m,n)})$	$M_I(A_{(m,n)})$	$D_T(A_{(m,n)})$	$D_F(A_{(m,n)})$	$D_I(A_{(m,n)})$	$PS_T(A)$	$PS_T(A)^*$	$PS_F(A)$	$PS_F(A)$ *	$PS_I(A)$	$PS_I(A)$ *	$PS(A)$	$PS(A)$ *
c_1	2.5	3.75	3.75	0.2886	0.3535	0.3535	3.2034	2.7886	4.66203	4.1035	4.66203	4.1035	4.17582	3.6652
c_2	3.5	5.25	5.25	0.2886	0.3535	0.3535	3.9676	3.7886	6.16203	5.6035	6.16203	5.6035	5.430553	4.9985333
a_{11}	2.5	3.75	3.75	0.2886	0.3535	0.3535	3.4132	2.7886	4.66203	4.1035	4.66203	4.1035	4.245753	3.6652
a_{12}	1	1.5	1.5	0.2886	0.3535	0.3535	4.0799	1.2886	2.41203	1.8535	2.41203	1.8535	2.967987	1.6652
a_{21}	1.75	2.625	2.625	0.2886	0.3535	0.3535	2.0108	2.0386	3.53703	2.9785	3.53703	2.9785	3.028287	2.6652
a_{22}	2.25	3.375	3.375	0.2886	0.3535	0.3535	3.4132	2.5386	4.28703	3.7285	4.28703	3.7285	3.995753	3.3318667
a_{31}	3	4.5	4.5	0.2886	0.3535	0.3535	3.07278	3.2886	5.41203	4.8535	5.41203	4.8535	4.63228	4.3318667
a_{32}	3	4.5	4.5	0.2886	0.3535	0.3535	3.7445	3.2886	5.41203	4.8535	5.41203	4.8535	5.41203	4.3318667
a_{41}	3.25	4.875	4.875	0.2886	0.3535	0.3535	3.9945	3.5386	5.78703	5.2285	5.78703	5.2285	5.78703	4.6652
a_{42}	2	3	3	0.2886	0.3535	0.3535	2.7445	2.2886	3.91203	3.3535	3.91203	3.3535	3.91203	2.9985333
b_1	1	1.5	1.5	0.2886	0.3535	0.3535	5.3729	1.2886	2.41203	1.8535	2.41203	1.8535	3.398987	1.6652
b_2	3	4.5	4.5	0.2886	0.3535	0.3535	4.732	3.2886	5.41203	4.8535	5.41203	4.8535	5.185353	4.3318667
b_3	3	4.5	4.5	0.2886	0.3535	0.3535	4.732	3.2886	5.41203	4.8535	5.41203	4.8535	5.185353	4.3318667
b_4	0.125	1.875	1.875	0.2886	0.3535	0.3535	0.8695	0.4136	2.7870	2.2285	2.7870	2.2285	2.1478	1.6235

*Possibility score function $PS^*(A_{(m,n)}) = M(A_{(m,n)}) + D(A_{(m,n)})$ suggested by Javier and Francisco [26]

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