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On The Computing of Symbolic 2-Plithogenic And 3-Plithogenic Complex Roots of Unity

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Abstract:

The concept of unity roots plays a central role in the theory of field extensions and polynomials roots' computing.

The objective of this paper is to find the algebraic formula for computing the symbolic 2-plithogenic and 3-plithogenic complex roots of unity, where a general formula will be provided with many related examples up to the exponent 3.

Keywords: symbolic 2-plithogenic complex number, symbolic 3-plithogenic complex number, symbolic n-plithogenic roots of unity.

Introduction and preliminaries.

The symbolic n-plithogenic set was supposed by Smarandache in [1-3]. Symbolic n-plithogenic sets were very helpful in algebra, where this concept has helped with developing algebraic structures, where we can see easily that for any value of n, we get a bigger structure.

Symbolic 2-plithogenic structures and 3-plithogenic were defined and handled by many authors around the globe.

For example, by now we have symbolic 2-plithogenic spaces, modules, matrix [4-7], and same thing for 3-plithogenic structures, see [8-11].

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In this paper, we are trying to close an important research gap by answering the following question.

How can we find all the roots of unity in the symbolic 2-plithogenic complex ring, and in the 3-plithogenic complex ring?

The symbolic 2-plithogenic or 3-plithogenic root of unity is a symbolic plithogenic number $\boldsymbol{x}$ with the following algebraic property $\boldsymbol{x^n = 1}$.

Definition.

Let C be the complex field, we have:

1). $2 - SP_C = \{v_0 + v_1P_1 + v_2P_2;\ v_i \in C\}$ is called the symbolic 2-plithogenic complex ring.

2). $3 - SP_C = \{v_0 + v_1P_1 + v_2P_2 + v_3P_3;\ v_i \in C\}$ is called the symbolic 3-plithogenic complex ring.

Algebraic operations on $2 - SP_C, 3 - SP_C$ are defined as follows:

$(+)$: $2 - SP_C * 2 - SP_C \to 2 - SP_C$ such that:

$(v_0 + v_1P_1 + v_2P_2) + (u_0 + u_1P_1 + u_2P_2) = (v_0 + u_0) + (v_1 + u_1)P_1 + (v_2 + u_2)P_2.$

$(+)$: $3 - SP_C * 3 - SP_C \to 3 - SP_C$ such that:

$(v_0 + v_1P_1 + v_2P_2 + v_3P_3) + (u_0 + u_1P_1 + u_2P_2 + u_3P_3) = (v_0 + u_0) + (v_1 + u_1)P_1 + (v_2 + u_2)P_2 + (v_3 + u_3)P_3.$

$(.)$: $2 - SP_C * 2 - SP_C \to 2 - SP_C$ such that:

$(v_0 + v_1P_1 + v_2P_2)(u_0 + u_1P_1 + u_2P_2) = v_0u_0 + (v_0u_1 + v_1u_0 + v_1u_1)P_1 + (v_0u_2 + v_2u_1 + v_2u_2 + v_2u_0 + v_1u_2)P_2.$

$(.)$: $3 - SP_C * 3 - SP_C \to 3 - SP_C$ such that:

$(v_0 + v_1P_1 + v_2P_2 + v_3P_3).(u_0 + u_1P_1 + u_2P_2 + u_3P_3) = v_0u_0 + (v_0u_1 + v_1u_0 + v_1u_1)P_1 + (v_0u_2 + v_2u_1 + v_2u_2 + v_2y_0 + v_1u_2)P_2 + (v_0u_3 + v_1u_3 + v_2u_3 + v_3u_3 + v_3u_0 + v_3u_1 + v_3u_2)P_3.$

Multiplication is defined with the following property:

$P_i \times P_j = P_{\max(i,j)}, P_i \times P_i = P_i; 1 \leq i \leq 3, 1 \leq j \leq 3$

Main discussion.

Definition.

Let $2-SP_C$ be the symbolic 2-plithogenic complex ring,

then $v=v_0+v_1P_1+v_2P_2\in 2-SP_C$ is called n-th root of unity if and only if $v^n=1$.

Definition.

Let $3-SP_C$ be the symbolic 2-plithogenic complex ring,

then $v=v_0+v_1P_1+v_2P_2+v_3P_3\in 3-SP_C$ is called n-th root of unity if and only if $v^n=1$.

Definition.

Let $v=v_0+v_1P_1+v_2P_2\in 2-SP_C, u=u_0+u_1P_1+u_2P_2+u_3P_3\in 3-SP_C$, then we define:

$\bar{v}=\overline{v_0}+\overline{v_1}P_1+\overline{v_2}P_2, \bar{u}=\overline{u_0}+\overline{u_1}P_1+\overline{u_2}P_2+\overline{u_3}P_3$.

Remark.

For $v=v_0+v_1P_1+v_2P_2\in 2-SP_C$, we have:

$v^n={v_0}^n+[(v_0+v_1)^n-{v_0}^n]P_1+[(v_0+v_1+v_2)^n-(v_0+v_1)^n]P_2; n\in N$.

For $v=v_0+v_1P_1+v_2P_2+v_3P_3\in 3-SP_C$, we have:

$v^n={v_0}^n+[(v_0+v_1)^n-{v_0}^n]P_1+[(v_0+v_1+v_2)^n-(v_0+v_1)^n]P_2+[(v_0+v_1+v_2+v_3)^n-(v_0+v_1+v_2)^n]P_3; n\in N$.

Theorem.

Let $v=v_0+v_1P_1+v_2P_2\in 2-SP_C, u=u_0+u_1P_1+u_2P_2+u_3P_3\in 3-SP_C$, then:

1. $\|v\|=|v_0|+[|(v_0+v_1)|-|v_0|]P_1+[|(v_0+v_1+v_2)|-|(v_0+v_1)|]P_2$
2. $\|u\|=|u_0|+[|(u_0+u_1)|-|u_0|]P_1+[|(u_0+u_1+u_2)|-|(u_0+u_1)|]P_2+[|(u_0+u_1+u_2+u_3)|-|(u_0+u_1+u_2)|]P_3$

Proof.

1. $\|v\|^2=v.\bar{v}=(v_0+v_1P_1+v_2P_2)(\overline{v_0}+\overline{v_1}P_1+\overline{v_2}P_2)=v_0\overline{v_0}+(v_0\overline{v_1}+v_1\overline{v_0}+v_1\overline{v_1})P_1+(v_0\overline{v_2}+v_2\overline{v_1}+v_2\overline{v_2}+v_2\overline{v_0}+v_1\overline{v_2})P_2=|v_0|^2+\big((v_0+v_1)(\overline{v_0}+\overline{v_1})-v_0\overline{v_0}\big)P_1+\big((v_0+v_1+v_2)(\overline{v_0}+\overline{v_1}+\overline{v_2})-(v_0+v_1)(\overline{v_0}+\overline{v_1})\big)P_2=|v_0|^2+[|(v_0+v_1)|^2-|v_0|^2]P_1+[|(v_0+v_1+v_2)|^2-|(v_0+v_1)|^2]P_2$

Now, we put

$R = |v_0| + [|(v_0 + v_1)| - |v_0|]P_1 + [|(v_0 + v_1 + v_2)| - |(v_0 + v_1)|]P_2.$

We get

$$R^2 = |v_0|^2 + [|(v_0 + v_1)|^2 - |v_0|^2]P_1 + [|(v_0 + v_1 + v_2)|^2 - |(v_0 + v_1)|^2]P_2 + 2|v_0|P_1[|(v_0 + v_1)| - |v_0|] + 2|v_0|[|(v_0 + v_1 + v_2)| - |(v_0 + v_1)|]P_2 + 2(|(v_0 + v_1)| - |v_0|)[|(v_0 + v_1 + v_2)| - |(v_0 + v_1)|]P_1P_2$$

$$= |v_0|^2 + [|(v_0 + v_1)|^2 - |v_0|^2 - 2|v_0||v_0 + v_1| + 2|v_0||v_0 + v_1| - 2|v_0|^2]P_1 + [|(v_0 + v_1 + v_2)|^2 - |(v_0 + v_1)|^2 + 2|v_0 + v_1||(v_0 + v_1 + v_2)| + 2|v_0||(v_0 + v_1 + v_2)| - 2|v_0||v_0 + v_1| + 2|v_0 + v_1||(v_0 + v_1 + v_2)| - |v_0 + v_1|^2 - 2|v_0||(v_0 + v_1 + v_2)| + 2|v_0||v_0 + v_1|]P_2$$

$$= |v_0|^2 + [|(v_0 + v_1)|^2 - |v_0|^2]P_1 + [|(v_0 + v_1 + v_2)|^2 - |(v_0 + v_1)|^2]P_2$$

This implies that

$\|v\| = R = |v_0| + [|(v_0 + v_1)| - |v_0|]P_1 + [|(v_0 + v_1 + v_2)| - |(v_0 + v_1)|]P_2.$

2. $\|u\|^2 = u.\bar{u} = (u_0 + u_1P_1 + u_2P_2 + u_3P_3)(\overline{u_0} + \overline{u_1}P_1 + \overline{u_2}P_2 + \overline{u_3}P_3) = u_0\overline{u_0} + (u_0\overline{u_1} + u_1\overline{u_0} + u_1\overline{u_1})P_1 + (u_0\overline{u_2} + u_2\overline{u_1} + u_2\overline{u_2} + u_2\overline{u_0} + u_1\overline{u_2})P_2 + (u_0\overline{u_3} + u_1\overline{u_3} + u_2\overline{u_3} + u_3\overline{u_0} + u_3\overline{u_1} + u_3\overline{u_2} + u_3\overline{u_3})P_3 = |u_0|^2 + \big((u_0 + u_1)(\overline{u_0} + \overline{u_1}) - u_0\overline{u_0}\big)P_1 + \big((u_0 + u_1 + u_2)(\overline{u_0} + \overline{u_1} + \overline{u_2}) - (u_0 + u_1)(\overline{u_0} + \overline{u_1})\big)P_2 + \big((u_0 + u_1 + u_2 + u_3)(\overline{u_0} + \overline{u_1} + \overline{u_2} + \overline{u_3}) - (u_0 + u_1 + u_2)(\overline{u_0} + \overline{u_1} + \overline{u_2})\big)P_3 = |u_0|^2 + [|(u_0 + u_1)|^2 - |u_0|^2]P_1 + [|(u_0 + u_1 + u_2)|^2 - |(u_0 + u_1)|^2]P_2 + [|(u_0 + u_1 + u_2 + u_3)|^2 - |(u_0 + u_1 + u_2)|^2]P_3$

We put

$R = |u_0| + [|(u_0 + u_1)| - |u_0|]P_1 + [|(u_0 + u_1 + u_2)| - |(u_0 + u_1)|]P_2 + [|(u_0 + u_1 + u_2 + u_3)| - |(u_0 + u_1 + u_2)|]P_3$

by an easy computing, we get $R^2 = |u|^2$, thus $|u| = R$.

Example.

Take $v = (2 + i) + (1 - i)P_1 + 2iP_2 \in 2 - SP_C$, we have:

$v_0 = 2 + i, v_1 = 1 - i, v_2 = 2i.$

$\overline{v_0} = 2 - i, \overline{v_1} = 1 + i, \overline{v_2} = -2i$

$\bar{v} = (2 - i) + (1 + i)P_1 - 2iP_2$

Also, $\begin{cases} |v_0| = \sqrt{5}, |v_0 + v_1| = |3| = 3, |v_0 + v_1 + v_2| = |3 + 2i| = \sqrt{13} \\ \|v\| = \sqrt{5} + (3 - \sqrt{5})P_1 + (\sqrt{13} - 3)P_2 \end{cases}$

Example.

Take $v=(1+i)+(1-3i)P_1+(5+i)P_2+(4+3i)P_3$, we have:

$v_0=1+i, v_1=1-3i, v_2=5+i, v_3=4+3i.$

$\overline{v_0}=1-i, \overline{v_1}=1+3i, \overline{v_2}=5-i, \overline{v_3}=4-3i$

$\bar{v}=1-i+(1+3i)P_1+(5-i)P_2+(4-3i)P_3$

Also,

$$\begin{cases} |v_0|=\sqrt{2}, |v_0+v_1|=|2-2i|=\sqrt{8}, |v_0+v_1+v_2|=|7-i|=\sqrt{50}, |v_0+v_1+v_2+v_3|=|11+2i| \\ \|v\|=\sqrt{2}+(\sqrt{8}-\sqrt{2})P_1+(\sqrt{50}-\sqrt{8})P_2+(\sqrt{125}-\sqrt{50})P_3 \end{cases}$$

Theorem.

Let $v=v_0+v_1P_1+v_2P_2 \in 2-SP_C$, then v is a symbolic 2-plithogenic n-th root of unity if and only if $v_0, v_0+v_1, v_0+v_1+v_2$ are classical n-th roots of unity in the field C.

Proof.

It is know $v^n=1$, which is equivalent to:

$${v_0}^n+[(v_0+v_1)^n-{v_0}^n]P_1+[(v_0+v_1+v_2)^n-(v_0+v_1)^n]P_2=1$$

$$\begin{cases} {v_0}^n=1 \\ (v_0+v_1)^n-{v_0}^n=0 \Rightarrow (v_0+v_1)^n={v_0}^n=1 \\ (v_0+v_1+v_2)^n-(v_0+v_1)^n=0 \Rightarrow (v_0+v_1+v_2)^n=(v_0+v_1)^n=1 \end{cases}$$

So that, $v_0, v_0+v_1, v_0+v_1+v_2$ are n-th roots of unity.

Example.

Let us find all a symbolic 2-plithogenic roots of unity order 2.

The classical set of the roots of unity of order 2 is $E_1=\{-1,1\}$.

The corresponding 2-symbolic plithogenic roots of unity of order 2 are:

1). $v_0=v_0+v_1=v_0+v_1+v_2=1 \Rightarrow R_1=1$

2). $v_0=v_0+v_1=1, v_0+v_1+v_2=-1 \Rightarrow R_2=1-2P_2$

3). $v_0=v_0+v_1+v_2=1, v_0+v_1=-1 \Rightarrow R_3=1-2P_1+2P_2$

4). $v_0=1, v_0+v_1=v_0+v_1+v_2=-1 \Rightarrow R_4=1-2P_1$

5). $v_0=v_0+v_1+v_2=1=v_0+v_1=-1 \Rightarrow R_5=-1$

6). $v_0=v_0+v_1=-1, v_0+v_1+v_2s=1 \Rightarrow R_6=-1+2P_2$

7). $v_0=v_0+v_1+v_2=-1, v_0+v_1=-1 \Rightarrow R_7=-1+2P_1-2P_2$

8). $v_0 = -1, v_0 + v_1 + v_2 = v_0 + v_1 = 1 \Longrightarrow R_8 = -1 + 2P_1$

Example.

Let us find all a symbolic 2-plithogenic roots of unity order 3.

The classical set of the roots of unity of order 3 is $E_2 = \left\{-1, e^{2\frac{\pi}{3}i}, e^{4\frac{\pi}{3}i}\right\}$.

The corresponding 2-symbolic plithogenic roots of unity of order 3 are:

1). $v_0 = v_0 + v_1 = v_0 + v_1 + v_2 = 1 \Longrightarrow R_1 = 1$

2). $v_0 = v_0 + v_1 = 1, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i} \Longrightarrow R_2 = 1 + \left(e^{2\frac{\pi}{3}i} - 1\right) P_2$

3). $v_0 = v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_3 = 1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_2$

4). $v_0 = v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_4 = 1 + \left(e^{2\frac{\pi}{3}i} - 1\right) P_1 + \left(e^{2\frac{\pi}{3}i} - 1\right) P_2$

5). $v_0 = v_0 + v_1 + v_2 = 1 = v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_5 = 1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_1 + \left(1 - e^{4\frac{\pi}{3}i}\right) P_2$

6). $v_0 = 1, v_0 + v_1 = v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i} \Longrightarrow R_6 = 1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_1$

7). $v_0 = 1, v_0 + v_1 + v_2 = v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_7 = 1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_2$

8). $v_0 = 1, v_0 + v_1 + v_2 = e^{4\frac{\pi}{3}i}, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_8 = 1 + \left(e^{2\frac{\pi}{3}i} - 1\right) P_1 +$
$\left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right) P_2$

9). $v_0 = 1, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_9 = 1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_1 +$
$\left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right) P_2$

10). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = 1 \Longrightarrow R_{10} = e^{2\frac{\pi}{3}i} + \left(1 - e^{2\frac{\pi}{3}i}\right) P_1$

11). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{11} = e^{2\frac{\pi}{3}i}$

12). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{12} = e^{2\frac{\pi}{3}i} + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right) P_1$

13). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = 1 \Longrightarrow R_{13} = e^{2\frac{\pi}{3}i} + \left(1 - e^{2\frac{\pi}{3}i}\right) P_1 +$
$\left(e^{2\frac{\pi}{3}i} - 1\right) P_2$

14). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{4\frac{\pi}{3}i}, v_0 + v_1 = 1 \Longrightarrow R_{14} = e^{2\frac{\pi}{3}i} + \left(1 - e^{2\frac{\pi}{3}i}\right) P_1 +$
$\left(e^{4\frac{\pi}{3}i} - 1\right) P_2$

15). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{15} = e^{2\frac{\pi}{3}i} + \left(1 - e^{2\frac{\pi}{3}i}\right) P_2$

16). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{16} = e^{2\frac{\pi}{3}i} + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right)P_1 + \left(1 - e^{4\frac{\pi}{3}i}\right)P_2$

17). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{4\frac{\pi}{3}i}, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{17} = e^{2\frac{\pi}{3}i} + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right)P_2$

18). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{18} = e^{2\frac{\pi}{3}i} + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right)P_1 + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right)P_2$

19). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = 1 \Longrightarrow R_{19} = e^{4\frac{\pi}{3}i} + \left(1 - e^{4\frac{\pi}{3}i}\right)P_1$

20). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{20} = e^{4\frac{\pi}{3}i} + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right)P_1$

21). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{21} = e^{4\frac{\pi}{3}i}$

22). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = 1 \Longrightarrow R_{22} = e^{4\frac{\pi}{3}i} + \left(1 - e^{4\frac{\pi}{3}i}\right)P_1 + \left(e^{2\frac{\pi}{3}i} - 1\right)P_2$

23). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{4\frac{\pi}{3}i}, v_0 + v_1 = 1 \Longrightarrow R_{23} = e^{4\frac{\pi}{3}i} + \left(1 - e^{4\frac{\pi}{3}i}\right)P_1 + \left(e^{4\frac{\pi}{3}i} - 1\right)P_2$

24). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{24} = e^{4\frac{\pi}{3}i} + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right)P_1 + \left(1 - e^{2\frac{\pi}{3}i}\right)P_2$

25). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{25} = e^{4\frac{\pi}{3}i} + \left(1 - e^{4\frac{\pi}{3}i}\right)P_2$

26). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{26} = e^{4\frac{\pi}{3}i} + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right)P_1 + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right)P_2$

27). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{27} = e^{4\frac{\pi}{3}i} + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right)P_2$

Theorem.

Let $v = v_0 + v_1P_1 + v_2P_2 + v_3P_3 \in 3 - SP_C$, then v is an n-th root of unity if and only if $v_0, v_0 + v_1, v_0 + v_1 + v_2$ are n-th root of unity in C.

Proof.

It is know that $v^n = 1$, which is equivalent to:

$$v_0{}^n + [(v_0 + v_1)^n - v_0{}^n]P_1 + [(v_0 + v_1 + v_2)^n - (v_0 + v_1)^n]P_2 + [(v_0 + v_1 + v_2 + v_3)^n - (v_0 + v_1 + v_2)^n]P_3 = 1$$

$$\begin{cases} {v_0}^n = 1 \\ (v_0 + v_1)^n - {v_0}^n = 0 \Rightarrow (v_0 + v_1)^n = {v_0}^n = 1 \\ (v_0 + v_1 + v_2)^n - (v_0 + v_1)^n = 0 \Rightarrow (v_0 + v_1 + v_2)^n = (v_0 + v_1)^n = 1 \\ (v_0 + v_1 + v_2 + v_3)^n - (v_0 + v_1 + v_2)^n = 0 \Rightarrow (v_0 + v_1 + v_2 + v_3)^n = (v_0 + v_1 + v_2)^n = 1 \end{cases}$$

Thus the proof holds.

Example.

Let us find all 0f 2-nd roots of unity in $3 - SP_C$.

1). $v_0 = v_0 + v_1 = v_0 + v_1 + v_2 = v_0 + v_1 + v_2 + v_3 = 1 \Rightarrow R_1 = 1$

2). $v_0 = v_0 + v_1 = v_0 + v_1 + v_2 = 1, v_0 + v_1 + v_2 + v_3 = -1 \Rightarrow R_2 = 1 - 2P_3$

3). $v_0 = v_0 + v_1 = v_0 + v_1 + v_2 + v_3 = 1, v_0 + v_1 + v_2 = -1 \Rightarrow R_3 = 1 - 2P_2 + 2P_3$

4). $v_0 = v_0 + v_1 + v_2 = v_0 + v_1 + v_2 + v_3 = 1, v_0 + v_1 = -1 \Rightarrow R_4 = 1 - 2P_1 + 2P_2$

5). $v_0 = v_0 + v_1 = 1, v_0 + v_1 + v_2 = v_0 + v_1 + v_2 + v_3 = -1 \Rightarrow R_5 = 1 - 2P_2$

6). $v_0 = v_0 + v_1 + v_2 = 1, v_0 + v_1 = v_0 + v_1 + v_2 + v_3 = -1 \Rightarrow R_6 = 1 + 2P_2 + 2P_3$

7). $v_0 = v_0 + v_1 + v_2 + v_3 = 1, v_0 + v_1 = v_0 + v_1 + v_2 = -1 \Rightarrow R_7 = 1 - 2P_1 + 2P_3$

8). $v_0 = 1, v_0 + v_1 + v_2 + v_3 == v_0 + v_1 = v_0 + v_1 + v_2 = -1 \Rightarrow R_8 = 1 - 2P_1$

9). $R_9 = -1 = -R_1$

10). $R_{10} = -R_2 = -1 + 2P_3$

11). $R_{11} = -R_3 = -1 + 2P_2 - 2P_3$

12). $R_{12} = -R_4 = -1 + 2P_1 - 2P_2$

13). $R_{13} = -R_5 = -1 + 2P_2$

14). $R_{14} = -R_6 = -1 - 2P_2 - 2P_3$

15). $R_{15} = -R_7 = -1 + 2P_1 - 2P_3$

16). $R_{16} = -R_8 = -1 + 2P_1$.

The group of unity roots classification

It is known that the set of all n-th roots of unity forms a subgroup of C^* denoted by U_C with respect to the multiplication operation and this group is isomorphic to the additive group Z_n (integers modulo n).

By a similar approach, we can see easily that the set of all symbolic 2-plithogenic complex n-th roots of unity forms a group with respect to multiplication operation, and the set of all symbolic 3-plithogenic complex n-th roots of unity forms a group with respect to multiplication operation.

The following theorem classifies the symbolic 2-plithogenic and 3-plithogenic groups of n-th roots of unity.

Theorem:

Let U_{2-SPC} be the group of n-th unity roots of symbolic 2-plithogenic complex numbers, and U_{3-SPC} be the group of n-th unity roots of symbolic 3-plithogenic complex numbers, then:

1-) $U_{2-SPC} \cong Z_n \times Z_n \times Z_n$.

2-) $U_{3-SPC} \cong Z_n \times Z_n \times Z_n \times Z_n$.

Proof:

1-) Define the mapping $f: U_{2-SPC} \to U_C \times U_C \times U_C$ such that:

$f(e_0 + e_1 P_1 + e_2 P_2) = (e_0, e_0 + e_1, e_0 + e_1 + e_2)$.

The mapping f is well defined:

For $M = m_0 + m_1 P_1 + m_2 P_2 = N = n_0 + n_1 P_1 + n_2 P_2$, we get:

$m_0 = n_0, m_0 + m_1 = n_0 + n_1, m_0 + m_1 + m_2 = n_0 + n_1 + n_2$,

Thus $f(M) = f(N)$.

The mapping f preserves multiplication:

For $M = m_0 + m_1 P_1 + m_2 P_2, N = n_0 + n_1 P_1 + n_2 P_2$, we get:

$f(MN) = f(m_0 n_0 + [m_0 n_1 + m_1 n_0 + m_1 n_1] P_1 + [m_0 n_2 + m_1 n_2 + m_2 n_0 + m_2 n_1 + m_2 n_2] P_2) = (m_0 n_0, m_0 n_0 + m_0 n_1 + m_1 n_0 + m_1 n_1, m_0 n_0 + m_0 n_1 + m_1 n_0 + m_1 n_1 + m_0 n_2 + m_1 n_2 + m_2 n_0 + m_2 n_1 + m_2 n_2) = (m_0, m_0 + m_1, m_0 + m_1 + m_2).(n_0, n_0 + n_1, n_0 + n_1 + n_2) = f(M) f(N)$.

The mapping f is injective:

$Ker(f) = \{M = m_0 + m_1 P_1 + m_2 P_2; f(M) = (1,1,1)\}$,

So that, $m_0 = 1, m_1 = m_2 = 0, thus\ Ker(f) = \{1\}$.

The mapping f is surjective:

$Im(f) = U_C \times U_C \times U_C$.

Thus, the mapping f is a group isomorphism, which means that $U_{2-SPC} \cong U_C \times U_C \times U_C$. Since $U_C \cong Z_n$, we get $U_{2-SPC} \cong Z_n \times Z_n \times Z_n$.

2-) Define the mapping $f: U_{3-SPC} \to U_C \times U_C \times U_C \times U_C$ such that:

$f(e_0 + e_1 P_1 + e_2 P_2) = (e_0, e_0 + e_1, e_0 + e_1 + e_2, e_0 + e_1 + e_2 + e_3)$.

The mapping f is well defined:

For $M = m_0 + m_1 P_1 + m_2 P_2 + m_3 P_3 = N = n_0 + n_1 P_1 + n_2 P_2 + n_3 P_3$, we get:

$m_0 = n_0, m_0 + m_1 = n_0 + n_1, m_0 + m_1 + m_2 = n_0 + n_1 + n_2, m_0 + m_1 + m_2 + m_3 = n_0 + n_1 + n_2 + n_3$,

Thus $f(M) = f(N)$.

The mapping f preserves multiplication:

For $M = m_0 + m_1 P_1 + m_2 P_2 + m_3 P_3, N = n_0 + n_1 P_1 + n_2 P_2 + n_3 P_3$, we get:

$f(MN) = f(m_0 n_0 + [m_0 n_1 + m_1 n_0 + m_1 n_1] P_1 + [m_0 n_2 + m_1 n_2 + m_2 n_0 + m_2 n_1 + m_2 n_2] P_2 + [m_0 n_3 + m_1 n_3 + m_2 n_3 + m_3 n_0 + m_3 n_1 + m_3 n_2 + m_3 n_3] P_3) = (m_0 n_0, m_0 n_0 + m_0 n_1 + m_1 n_0 + m_1 n_1, m_0 n_0 + m_0 n_1 + m_1 n_0 + m_1 n_1 + m_0 n_2 + m_1 n_2 + m_2 n_0 + m_2 n_1 + m_2 n_2, m_0 n_0 + m_0 n_1 + m_1 n_0 + m_1 n_1 + m_0 n_2 + m_1 n_2 + m_2 n_0 + m_2 n_1 + m_2 n_2 + m_0 n_3 + m_1 n_3 + m_2 n_3 + m_3 n_0 + m_3 n_1 + m_3 n_2 + m_3 n_3) = (m_0, m_0 + m_1, m_0 + m_1 + m_2, m_0 + m_1 + m_2 + m_3).(n_0, n_0 + n_1, n_0 + n_1 + n_2, n_0 + n_1 + n_2 + n_3) = f(M) f(N)$.

The mapping f is injective:

$Ker(f) = \{M = m_0 + m_1 P_1 + m_2 P_2 + m_3 P_3; f(M) = (1,1,1)\}$,

So that, $m_0 = 1, m_1 = m_2 = m_3 = 0, thus\ Ker(f) = \{1\}$.

The mapping f is surjective:

$Im(f) = U_C \times U_C \times U_C \times U_C$.

Thus, the mapping f is a group isomorphism, which means that $U_{3-SPC} \cong U_C \times U_C \times U_C \times U_C$. Since $U_C \cong Z_n$, we get $U_{3-SPC} \cong Z_n \times Z_n \times Z_n \times Z_n$.

Conclusion.

In this paper, we presented an algebraic algorithm to compute n-th roots of unity in symbolic 2-plithogenic/3-plithogenic complex ring respectively.

Also, we have illustrated some examples to clarify the flow of our algorithm.

In the future, we aim to find n-th roots of unity in symbolic m-plithogenic complex ring for any value of m.

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