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On The Orthogonality in Real Symbolic 2-Plithogenic and 3-Plithogenic Vector Spaces

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Abstract:

This paper is dedicated to study the real inner product defined over symbolic 2-plithogenic vector spaces, where we discuss the concept of inner (scalar) products over the symbolic 2-plithogenic vector spaces by using the corresponding Euclidean scalar products to get theorems that describe the conditions of orthogonality in this class of spaces. Also, we give many examples to explain the ortho-normed symbolic 2-plithogenic spaces.

Key words: Symbolic 2-plithogenic vector space, inner product, orthogonal basis, ortho-normed basis.

Introduction

Symbolic 2-plithogenic vector spaces and modules were defined in [1-5] as a new generalization of classical vector spaces, and with a similar algebraic structure of refined neutrosophic vector spaces .

Many algebraic properties of these spaces such as basis, and semi homomorphic images were studied on a wide range.

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Also, we can see symbolic 3-plithogenic vector spaces/modules defined over 3-plithogenic rings. Symbolic plithogenic algebraic structures are generally very rich in their concepts and meta-properties, see [5-8 ,23-28].

In this work, we will study the concept of real inner product over symbolic 2-plithogenic vector spaces, where we use these inner products to study the orthogonality between symbolic 2-plithogenic vectors and ortho-normed basis.

This work is motivated by the previous published works in [9-22] that study neutrosophic vector spaces, matrices and their refined neutrosophic extensions.

For basic definitions about symbolic 2-plithogenic vector spaces, check [1].

Main Discussion

Definition:

Let V be a vector space over the field R.

Let $2-SP_R=\{l_0+l_1P_1+l_2P_2; l_i\in R\}$ be the corresponding symbolic 2-plithogenic field, $2-SP_V=\{q_0+q_1P_1+q_2P_2; q_i\in V\}$ the corresponding symbolic 2-plithogenic vector space, then:

$\varphi: 2-SP_V\times 2-SP_V\to 2-SP_R$ is called a symbolic 2-plithogenic real inner product if and only if:

1). $\varphi(w,n)=\varphi(n,w)$ for all $n,w\in 2-SP_V$.

2). $\varphi(w,w)\geq 0$ for all $w\in 2-SP_V$ (with respect to the corresponding partial order relation defined on $2-SP_R$).

3). $\varphi(u+v,w)=\varphi(u,w)+\varphi(v,w)$.

4). $\varphi(a.w,v)=a\varphi(w,v); w,v\in 2-SP_V, a\in 2-SP_R$.

Theorem.

If there exists a classical inner product $f: V\times V\to R$ such that:

For $w=w_0+w_1P_1+w_2P_2, n=n_0+n_1P_1+n_2P_2\in 2-SP_V$, we define:

$\varphi(w,n)=f(w_0,n_0)+P_1[f(w_0+w_1,n_0+n_1)-f(w_0,n_0)]+P_2[f(w_0+w_1+w_2,n_0+n_1+n_2)-f(w_0+w_1,n_0+n_1)]$.

Then, $\varphi: 2-SP_V \times 2-SP_V \to 2-SP_R$ is a symbolic 2-plithogenic real inner product.

Proof.

Assume that $f: V \times V \to R$ is a real inner product defined on V.

We put $\varphi: 2-SP_V \times 2-SP_V \to 2-SP_R$, where:

$\varphi(w,n) = f(w_0,n_0) + P_1[f(w_0+w_1,n_0+n_1) - f(w_0,n_0)] + P_2[f(w_0+w_1+w_2,n_0+n_1+n_2) - f(w_0+w_1,n_0+n_1)]$.

We must prove that φ is a symbolic 2-plithogenic real inner product on $2-SP_V$.

$\varphi(w,n) = f(w_0,n_0) + P_1[f(w_0+w_1,n_0+n_1) - f(w_0,n_0)] + P_2[f(w_0+w_1+w_2,n_0+n_1+n_2) - f(w_0+w_1,n_0+n_1)] = f(n_0,w_0) + P_1[f(n_0+n_1,w_0+w_1) - f(n_0,w_0)] + P_2[f(n_0+n_1+n_2,w_0+w_1+w_2) - f(n_0+n_1,w_0+w_1)] = \varphi(n,w)$.

$\varphi(w,w) = f(w_0,w_0) + P_1[f(w_0+w_1,w_0+w_1) - f(w_0,w_0)] + P_2[f(w_0+w_1+w_2,w_0+w_1+w_2) - f(w_0+w_1,w_0+w_1)] = \|w_0\|^2 + P_1[\|w_0+w_1\|^2 - \|w_0\|^2] + P_2[\|w_0+w_1+w_2\|^2 - \|w_0+w_1\|^2]$.

And that is because:

$$\begin{cases} \|w_0\|^2 \geq 0 \\ \|w_0\|^2 + [\|w_0+w_1\|^2 - \|w_0\|^2] = \|w_0+w_1\|^2 \geq 0 \\ \|w_0\|^2 + [\|w_0+w_1\|^2 - \|w_0\|^2] + [\|w_0+w_1+w_2\|^2 - \|w_0+w_1\|^2] = \|w_0+w_1+w_2\|^2 \geq 0 \end{cases}$$

Now, we let $u = u_0 + u_1P_1 + u_2P_2 \in 2-SP_V$, we have:

$$\begin{aligned}\varphi(u+w,n) = {} & f(u_0+w_0,n_0) + P_1[f(u_0+u_1+w_0+w_1,n_0+n_1) - f(u_0+w_0,n_0)] \\ & + P_2[f(u_0+u_1+u_2+w_0+w_1+w_2,n_0+n_1+n_2) \\ & - f(u_0+u_1+w_0+w_1,n_0+n_1)] = \varphi(u,n) + \varphi(w,n)\end{aligned}$$

Let $a = a_0 + a_1P_1 + a_2P_2 \in 2-SP_R$, then:

$\varphi(a.w,n) = f(a_0w_0,n_0) + P_1\big[f\big((a_0+a_1+a_2)(u_0+u_1+w_0+w_1),n_0+n_1\big) - f(a_0w_0,n_0)\big] + P_2\big[f\big((a_0+a_1)(u_0+u_1+u_2+w_0+w_1+w_2),n_0+n_1+n_2\big) - f\big((a_0+a_1+a_2)(u_0+u_1+w_0+w_1),n_0+n_1\big)\big] = (a_0+a_1P_1+a_2P_2)\varphi(w,n) = a.\varphi(w,n)$.

So that, φ is a symbolic 2-plithogenic inner product.

Remark:

Suppose that $\varphi: 2-SP_V \times 2-SP_V \to 2-SP_R$ is a symbolic 2-plithogenic inner product.

Put $f: V \times V \to R$, with $f(w_0, n_0) = \varphi(w_0 + 0.P_1 + 0.P_2, n_0 + 0.P_1 + 0.P_2)$, where $w_0, n_0 \in V$.

f is a classical real inner product, that is because:

$f(w_0, n_0) = \varphi(w_0, n_0) = \varphi(n_0, w_0) = f(n_0, w_0)$

$f(w_0, w_0) = \varphi(w_0, w_0) = \|w_0\|^2 \geq 0$

$f(u_0 + w_0, n_0) = \varphi(u_0 + w_0, n_0) = \varphi(u_0, n_0) + \varphi(w_0, n_0) = f(u_0, n_0) + f(w_0, n_0)$

$f(a_0 w_0, n_0) = \varphi(a_0 w_0, n_0) = a_0 \varphi(w_0, n_0) = a_0 f(w_0, n_0); w_0 \in V, a_0 \in R$

Example.

Consider the Euclidean real inner product defined on $V = R^2$ with:

$f(X_1, X_2) = f[(x_1', x_1''), (x_2', x_2'')] = x_1' x_2' + x_1'' x_2''$.

The corresponding symbolic 2-plithogenic inner product defined on $2-SP_V$ is:

$\varphi: 2-SP_V \times 2-SP_V \to 2-SP_R$;

$$\begin{aligned}
&\varphi[(x_0, y_0) + (x_1, y_1)P_1 + (x_2, y_2)P_2, (z_0, t_0) + (z_1, t_1)P_1 + (z_2, t_2)P_2] \\
&\quad = f\big((x_0, y_0), (z_0, t_0)\big) \\
&\quad + P_1\big[f\big((x_0 + x_1, y_0 + y_1), (z_0 + z_1, t_0 + t_1)\big) - f\big((x_0, y_0), (z_0, t_0)\big)\big] \\
&\quad + P_2\big[f\big((x_0 + x_1 + x_2, y_0 + y_1 + y_2), (z_0 + z_1 + z_2, t_0 + t_1 + t_2)\big) \\
&\quad - f\big((x_0 + x_1, y_0 + y_1), (z_0 + z_1, t_0 + t_1)\big)\big] \\
&\quad = x_0 z_0 + y_0 t_0 \\
&\quad + P_1[(x_0 + x_1)(z_0 + z_1) + (y_0 + y_1)(t_0 + t_1) - x_0 z_0 - y_0 t_0] \\
&\quad + P_2[(x_0 + x_1 + x_2)(z_0 + z_1 + z_2) + (y_0 + y_1 + y_2)(t_0 + t_1 + t_2) \\
&\quad - (x_0 + x_1)(z_0 + z_1) - (y_0 + y_1)(t_0 + t_1)]
\end{aligned}$$

For example:

Consider $X = (1,1) + (1,2)P_1 + (0,1)P_2, Y = (1,0) + (-1,0)P_1 + (1,1)P_2$, we have:

$$\begin{cases} (x_0, y_0) = (1,1), (z_0, t_0) = (1,0) \\ (x_0 + x_1, y_0 + y_1) = (2,3), (z_0 + z_1, t_0 + t_1) = (0,0) \\ (x_0 + x_1 + x_2, y_0 + y_1 + y_2) = (2,4), (z_0 + z_1 + z_2, t_0 + t_1 + t_2) = (1,1) \end{cases}$$

And:

$$\begin{cases} f\big((x_0,y_0),(z_0,t_0)\big)=1 \\ f\big((x_0+x_1,y_0+y_1),(z_0+z_1,t_0+t_1)\big)=0 \\ f\big((x_0+x_1+x_2,y_0+y_1+y_2),(z_0+z_1+z_2,t_0+t_1+t_2)\big)=6 \end{cases}$$

This implies that:

$\varphi(X,Y)=1+P_1[0-1]+P_2[6-0]=1-P_1+6P_2$

Remark.

The norm of $w=w_0+w_1P_1+w_2P_2$ is defined with respect to φ as follows:

$\|w\|=\sqrt{\varphi(w,w)}=\|w_0\|+P_1[\|w_0+w_1\|-\|w_0\|]+P_2[\|w_0+w_1+w_2\|-\|w_0+w_1\|]$.

In the previous example, we can see:

$\|X\|=\|(1,1)\|+P_1[\|(2,3)\|-\|(1,1)\|]+P_2[\|(2,4)\|-\|(2,3)\|]=\sqrt{2}+P_1\big[\sqrt{13}-\sqrt{2}\big]+P_2\big[\sqrt{20}-\sqrt{13}\big]$.

Theorem.

Let φ be a symbolic 2-plithogenic inner product on $2-SP_V$, then:

1). $\|w\|\geq 0; w\in 2-SP_V$

2). $\|a.w\|=|a|.\|w\|; a\in 2-SP_R$

3). $\|w+n\|\leq\|w\|+\|n\|; n\in 2-SP_V$

4). $|\varphi(w,n)|\leq\|w\|.\|n\|$

5). $w\perp n$ if and only if $w_0\perp n_0, w_0+w_1\perp n_0+n_1, w_0+w_1+w_2\perp n_0+n_1+n_2$

6). If $w\perp n$, then $\|w+n\|^2=\|w\|^2+\|n\|^2$

Proof.

1). It holds directly from the definition of norm, and from the partial order relation defined on $2-SP_R$.

2). Let $a=a_0+a_1P_1+a_2P_2\in 2-SP_R, w=w_0+w_1P_1+w_2P_2\in 2-SP_V$, we have:

$\|a.w\|^2=\varphi(a.w,a.w)=a^2\varphi(w,w)$, thus $\|a.w\|=|a|.\|w\|$.

3). $\|w+n\|=\|w_0+n_0\|+P_1[\|w_0+w_1+n_0+n_1\|-\|w_0+n_0\|]+P_2[\|w_0+w_1+w_2+n_0+n_1+n_2\|-\|w_0+w_1+n_0+n_1\|]$, we have:

$$\begin{cases} \|w_0+n_0\|\leq\|w_0\|+\|n_0\| \\ \|w_0+w_1+n_0+n_1\|\leq\|w_0+w_1\|+\|n_0+n_1\| \\ \|w_0+w_1+w_2+n_0+n_1+n_2\|\leq\|w_0+w_1+w_2\|+\|n_0+n_1+n_2\| \end{cases}$$

So that:

$\|w+n\| \leq \|w_0\| + \|n_0\| + P_1[\|w_0+w_1\| + \|n_0+n_1\| - \|w_0\| - \|n_0\|] + P_2[\|w_0 + w_1 + w_2\| + \|n_0+n_1+n_2\| - \|w_0+w_1\| - \|n_0+n_1\|] \leq \|w\| + \|n\|$.

4). We have:

$|\varphi(w,n)| = |\varphi(w_0,n_0)| + P_1[|\varphi(w_0+w_1,n_0+n_1)| - |\varphi(w_0,n_0)|] + P_2[|\varphi(w_0+w_1+w_2,n_0+n_1+n_2)| - |\varphi(w_0+w_1,n_0+n_1)|]$.

According to Cauchy-Shwartz inequality, we can write:

$$\begin{cases} |f(w_0,n_0)| \leq \|w_0\| + \|n_0\| \\ |f(w_0+w_1,n_0+n_1)| \leq \|w_0+w_1\| + \|n_0+n_1\| \\ |f(w_0+w_1+w_2,n_0+n_1+n_2)| \leq \|w_0+w_1+w_2\| + \|n_0+n_1+n_2\| \end{cases}$$

So that,

$$\begin{aligned} |\varphi(w,n)| \leq{} & \|w_0\|.\|n_0\| + P_1[\|w_0+w_1\|.\|n_0+n_1\| - \|w_0\|.\|n_0\|] \\ & + P_2[\|w_0+w_1+w_2\|.\|n_0+n_1+n_2\| - \|w_0+w_1\|.\|n_0+n_1\|] \\ & = \|w\|.\|n\| \end{aligned}$$

5). $w \perp n$ if and only if $\varphi(w,n) = 0$, which is equivalent to:

$$\begin{cases} f(w_0,n_0) = 0 \Longrightarrow w_0 \perp n_0 \\ f(w_0+w_1,n_0+n_1) \Longrightarrow w_0+w_1 \perp n_0+n_1 \\ f(w_0+w_1+w_2,n_0+n_1+n_2) \Longrightarrow w_0+w_1+w_2 \perp n_0+n_1+n_2 \end{cases}$$

6). Assume that $w \perp n$, then $\|w+n\|^2 = \varphi(w+n,w+n) = \varphi(w,w) + \varphi(n,n) + 2\varphi(w,n) = \|w\|^2 + \|n\|^2$.

Example.

Consider the Euclidean real inner symbolic 2-plithogenic product defined previously on $2-SP_R$, we have:

$w = (1,0) + (0,1)P_1, u = w = (2,2) + (1,3)P_1 - (1,1)P_2, s = (0,1) + (1,-2)P_1$, we can see:

$\varphi(w,s) = 0, \|w\| = 1 + (\sqrt{2}-1)P_1, \|s\| = 1 + (\sqrt{2}-1)P_1, w+s = (1,1) + (1,-1)P_1$

$\|w+s\|^2 = 2 + 2P_1 = \|w\|^2 + \|s\|^2$

$\varphi(w,u) = 2 + (8-2)P_1 + (6-8)P_2 = 2 + 6P_1 - 2P_2$

$|\varphi(w,u)| = |2| + [|8| - |2|]P_1 + [|6| - |8|]P_2 = 2 + 6P_1 - 2P_2$

$\|u\| = \sqrt{8} + (\sqrt{34} - \sqrt{8})P_1 + (\sqrt{20} - \sqrt{34})P_2$

$\|w\|.\|u\| = 2\sqrt{8} + (2\sqrt{34} - 2\sqrt{8} + 4\sqrt{8} + 2\sqrt{34} - 2\sqrt{8})P_1$
$+ (2\sqrt{20} - 2\sqrt{34} + 2\sqrt{20} - 2\sqrt{34})P_2$
$= 2\sqrt{8} + 4\sqrt{34}P_1 + (4\sqrt{20} - 4\sqrt{34})P_2$

We have $2 \le 2\sqrt{8}, 2+6 = 8 \le 2\sqrt{8} + 4\sqrt{34}, 2+6-2 = 8 \le 2\sqrt{8} + 4\sqrt{34} + 4\sqrt{20} - 4\sqrt{34} = 2\sqrt{8} + 4\sqrt{20}$

Hence $|\varphi(w,u)| \le \|w\|.\|u\|$.

Symbolic 2-plithogenic orthogonal basis.

Let $N = \{V_1, \dots, V_n\}$ be a basis of symbolic 2-plithogenic vector space $2 - SP_V$, where n is the number of elements in the basis of V.

We say that N is the orthogonal if and only if:

$\varphi(V_i, V_j) = 0; i \ne j, 1 \le i, j \le 3n$

It is called ortho-normed if and only if:

$\begin{cases} \varphi(V_i, V_j) = 0 \\ \varphi(V_i, V_i) = 1 \end{cases}; i \ne j, 1 \le i, j \le 3n$

We will answer the following question:

How can we build a symbolic 2-plithogenic ortho-normed basis of $2 - SP_V$?

Theorem.

Let $\Delta = \{q_1, q_2, \dots, q_n\}$ be an ortho-normed basis of V with respect to the inner product $f: V \times V \to R$.

The set $\Delta_P = \{m_i + (n_j - m_i)P_1 + (s_k - n_j)P_2; m_i, n_j, s_k \in \Delta, 1 \le i, j, k \le n\}$ is an ortho-normed basis of $2 - SP_V$.

Proof.

According to [4], the set Δ_P is a basis of $2 - SP_V$.

Let

$M_1 = m_i + (n_j - m_i)P_1 + (s_k - n_j)P_2 \in \Delta, M_2 = \acute{m}_i + (\acute{n}_j - \acute{m}_i)P_1 + (\acute{s}_k - \acute{n}_j)P_2 \in \Delta$,

we have:

$\varphi(M_1, M_2) = f(m_i, \acute{m}_i) + P_1[f(n_j, \acute{n}_j) - f(m_i, \acute{m}_i)] + P_2[f(s_k, \acute{s}_k) - f(n_j, \acute{n}_j)] = 0$,

thus $M_1 \perp M_2$.

On the other hand,

$\|M_1\| = \|m_i\| + P_1\big[\|n_j\| - \|m_i\|\big] + P_2\big[\|s_k\| - \|n_j\|\big] = 1 + (1-1)P_1 + (1-1)P_2 = 1$,

so that Δ_P is ortho-normed basis.

Example.

Let $V = R^3$ be the Euclidean with three dimensions.

Consider $2 - SP_V = \{(x_0, y_0, z_0) + (x_1, y_1, z_1)P_1 + (x_2, y_2, z_2)P_2; x_i, y_i, z_i \in R\}$ be the corresponding symbolic 2-plithogenic vector space.

It is known that $\Delta = \{q_1 = (1,0,0), q_2 = (0,1,0), q_3 = (0,0,1)\}$ is an ortho-normed of $V = R^3$.

The corresponding ortho-normed basis of $2 - SP_V$ is:

$M_1 = q_1 = (1,0,0)$

$M_2 = q_1 + (q_1 - q_1)P_1 + (q_2 - q_1)P_2 = (1,0,0) + (-1,1,0)P_2$

$M_3 = q_1 + (q_2 - q_1)P_1 + (q_1 - q_2)P_2 = (1,0,0) + (-1,1,0)P_1 + (1,1,0)P_2$

$M_4 = q_1 + (q_2 - q_1)P_1 + (q_2 - q_2)P_2 = (1,0,0) + (-1,1,0)P_1$

$M_5 = q_1 + (q_3 - q_1)P_1 + (q_3 - q_3)P_2 = (-1,0,0) + (-1,0,1)P_1$

$M_6 = q_1 + (q_3 - q_1)P_1 + (q_1 - q_3)P_2 = (1,0,0) + (-1,0,1)P_1 + (1,0,-1)P_2$

$M_7 = q_1 + (q_3 - q_1)P_1 + (q_2 - q_3)P_2 = (1,0,0) + (-1,0,1)P_1 + (0,1,-1)P_2$

$M_8 = q_1 + (q_1 - q_1)P_1 + (q_3 - q_1)P_2 = (1,0,0) + (-1,0,1)P_2$

$M_9 = q_1 + (q_2 - q_1)P_1 + (q_3 - q_2)P_2 = (1,0,0) + (-1,1,0)P_1 + (0,-1,1)P_2$

$M_{10} = q_2 = (0,1,0)$

$M_{11} = q_2 + (q_1 - q_2)P_1 + (q_1 - q_1)P_2 = (0,1,0) + (1,-1,0)P_1$

$M_{12} = q_2 + (q_1 - q_2)P_1 + (q_2 - q_1)P_2 = (0,1,0) + (1,-1,0)P_1 + (-1,1,0)P_2$

$M_{13} = q_2 + (q_1 - q_2)P_1 + (q_3 - q_1)P_2 = (0,1,0) + (1,-1,0)P_1 + (-1,0,1)P_2$

$M_{14} = q_2 + (q_2 - q_2)P_1 + (q_1 - q_2)P_2 = (0,1,0)P_1 + (1,-1,0)P_2$

$M_{15} = q_2 + (q_2 - q_2)P_1 + (q_1 - q_2)P_2 = (0,1,0) + (0,-1,1)P_2$

$M_{16} = q_2 + (q_3 - q_2)P_1 + (q_1 - q_3)P_2 = (0,1,0) + (0,-1,1)P_1 + (1,0,-1)P_2$

$M_{17} = q_2 + (q_3 - q_2)P_1 + (q_2 - q_3)P_2 = (0,1,0) + (0,-1,1)P_1 + (0,1,-1)P_2$

$M_{18} = q_2 + (q_3 - q_2)P_1 + (q_3 - q_3)P_2 = (0,1,0) + (0,-1,1)P_1$

$M_{19} = q_3 = (0,0,1)$

$M_{20} = q_3 + (q_1 - q_3)P_1 + (q_1 - q_1)P_2 = (0,0,1) + (1,0,-1)P_1$

$M_{21} = q_3 + (q_1 - q_3)P_1 + (q_2 - q_1)P_2 = (0,0,1) + (1,0,-1)P_1 + (-1,1,0)P_2$

$M_{22} = q_3 + (q_1 - q_3)P_1 + (q_3 - q_1)P_2 = (0,0,1) + (1,0,-1)P_1 + (-1,0,1)P_2$

$M_{23} = q_3 + (q_2 - q_3)P_1 + (q_1 - q_2)P_2 = (0,0,1) + (0,1,-1)P_1 + (1,-1,0)P_2$

$M_{24} = q_3 + (q_2 - q_3)P_1 + (q_2 - q_2)P_2 = (0,0,1) + (0,1,-1)P_1$

$M_{25} = q_3 + (q_2 - q_3)P_1 + (q_3 - q_2)P_2 = (0,0,1) + (0,1,-1)P_1 + (0,-1,1)P_2$

$M_{26} = q_3 + (q_3 - q_3)P_1 + (q_1 - q_3)P_2 = (0,0,1) + (1,0,-1)P_2$

$M_{27} = q_3 + (q_3 - q_3)P_1 + (q_2 - q_3)P_2 = (0,0,1) + (0,1,-1)P_2$

Example.

Consider the ortho-normed basis of $V = R^2, \Delta = \{q_1 = (1,0), q_2 = (0,1)\}$.

We find the corresponding ortho-normed symbolic 2-plithogenic basis of $2 - SP_V$.

$M_1 = q_1 + (q_1 - q_1)P_1 + (q_1 - q_1)P_2 = (1,0)$

$M_2 = q_1 + (q_1 - q_1)P_1 + (q_2 - q_1)P_2 = (1,0) + (-1,1)P_2$

$M_3 = q_1 + (q_2 - q_1)P_1 + (q_2 - q_2)P_2 = (1,0) + (-1,1)P_1$

$M_4 = q_1 + (q_2 - q_1)P_1 + (q_1 - q_2)P_2 = (1,0) + (-1,1)P_1 + (1,-1)P_2$

$M_5 = q_2 + (q_2 - q_2)P_1 + (q_2 - q_2)P_2 = (0,1)$

$M_6 = q_2 + (q_2 - q_2)P_1 + (q_1 - q_2)P_2 = (0,1) + (1,-1)P_2$

$M_7 = q_2 + (q_1 - q_2)P_1 + (q_1 - q_1)P_2 = (0,1) + (1,-1)P_1$

$M_8 = q_2 + (q_1 - q_2)P_1 + (q_2 - q_1)P_2 = (0,1) + (1,-1)P_1 + (-1,1)P_2$

Remark.

Let $S = \{V_1, \dots, V_{3n}\}$ be the ortho-normed basis of $2 - SP_V$, let $X = X_0 + X_1P_1 + X_2P_2 \in 2 - SP_V$, then:

$X = A_1V_1 + A_2V_2 + \cdots + A_{3n}V_{3n}$, we can write:

$\varphi(X, V_1) = A_1\varphi(V_1, V_1) + A_2\varphi(V_2, V_2) + \cdots + A_{3n}\varphi(V_{3n}, V_{3n}) = A_i\|V_i\|^2$, thus:

$A_1 = \frac{\varphi(X,V_1)}{\|V_1\|^2} = \varphi(X, V_1), A_2 = \varphi(X, V_2), \dots$.

And so on:

So that, we get the following result:

$X = \varphi(X, V_1)V_1 + \varphi(X, V_2)V_2 + \cdots + \varphi(X, V_{3n})V_{3n} = \sum_{i=1}^{3n} \varphi(X, V_i)V_i$.

Symbolic 3-plithogenic inner product

Definition:

Let V be a vector space over the field R.

Let $3-SP_R=\{l_0+l_1P_1+l_2P_2+l_3P_3; l_i\in R\}$ be the corresponding symbolic 3-plithogenic field, $3-SP_V=\{q_0+q_1P_1+q_2P_2+q_3P_3; q_i\in V\}$ the corresponding symbolic 3-plithogenic vector space, then:

$\varphi: 3-SP_V\times 3-SP_V\to 3-SP_R$ is called a symbolic 3-plithogenic real inner product if and only if:

1). $\varphi(w,n)=\varphi(n,w)$ for all $n,w\in 3-SP_V$.

2). $\varphi(w,w)\geq 0$ for all $w\in 3-SP_V$ (with respect to the corresponding partial order relation defined on $3-SP_R$).

3). $\varphi(u+v,w)=\varphi(u,w)+\varphi(v,w)$.

4). $\varphi(a.w,v)=a\varphi(w,v); w,v\in 2-SP_V, a\in 3-SP_R$.

Theorem.

If there exists a classical inner product $f:V\times V\to R$ such that:

For $w=w_0+w_1P_1+w_2P_2+w_3P_3, n=n_0+n_1P_1+n_2P_2+n_3P_3\in 3-SP_V$, we define:

$\varphi(w,n)=f(w_0,n_0)+P_1[f(w_0+w_1,n_0+n_1)-f(w_0,n_0)]+P_2[f(w_0+w_1+w_2,n_0+n_1+n_2)-f(w_0+w_1,n_0+n_1)]+P_3[f(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)-f(w_0+w_1+w_2,n_0+n_1+n_2)]$.

Then, $\varphi: 3-SP_V\times 3-SP_V\to 3-SP_R$ is a symbolic 3-plithogenic real inner product.

Proof.

Assume that $f:V\times V\to R$ is a real inner product defined on V.

We put $\varphi: 3-SP_V\times 3-SP_V\to 3-SP_R$, where:

$\varphi(w,n)=f(w_0,n_0)+P_1[f(w_0+w_1,n_0+n_1)-f(w_0,n_0)]+P_2[f(w_0+w_1+w_2,n_0+n_1+n_2)-f(w_0+w_1,n_0+n_1)]+P_3[f(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)-f(w_0+w_1+w_2,n_0+n_1+n_2)]$.

We must prove that φ is a symbolic 2-plithogenic real inner product on $3-SP_V$.

$\varphi(w,n)=f(w_0,n_0)+P_1[f(w_0+w_1,n_0+n_1)-f(w_0,n_0)]+P_2[f(w_0+w_1+w_2,n_0+n_1+n_2)-f(w_0+w_1,n_0+n_1)]+P_3[f(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)-f(w_0+w_1+w_2,n_0+n_1+n_2)]=f(n_0,w_0)+P_1[f(n_0+n_1,w_0+w_1)-f(n_0,w_0)]+$

$P_2[f(n_0+n_1+n_2, w_0+w_1+w_2) - f(n_0+n_1, w_0+w_1)] + P_3[f(n_0+n_1+n_2+n_3, w_0+w_1+w_2+w_3) - f(n_0+n_1+n_2, w_0+w_1+w_2)] = \varphi(n, w)$.

$\varphi(w, w) = f(w_0, w_0) + P_1[f(w_0+w_1, w_0+w_1) - f(w_0, w_0)] + P_2[f(w_0+w_1+w_2, w_0+w_1+w_2) - f(w_0+w_1, w_0+w_1)] + P_3[f(w_0+w_1+w_2+w_3, w_0+w_1+w_2+w_3) - f(w_0+w_1+w_2, w_0+w_1+w_2)] = \|w_0\|^2 + P_1[\|w_0+w_1\|^2 - \|w_0\|^2] + P_2[\|w_0+w_1+w_2\|^2 - \|w_0+w_1\|^2] + P_3[\|w_0+w_1+w_2+w_3\|^2 - \|w_0+w_1+w_2\|^2]$.

And that is because:

$$\begin{cases} \|w_0\|^2 \geq 0 \\ \|w_0\|^2 + [\|w_0+w_1\|^2 - \|w_0\|^2] = \|w_0+w_1\|^2 \geq 0 \\ \|w_0\|^2 + [\|w_0+w_1\|^2 - \|w_0\|^2] + [\|w_0+w_1+w_2\|^2 - \|w_0+w_1\|^2] = \|w_0+w_1+w_2\|^2 \geq 0 \\ \|w_0+w_1+w_2+w_3\|^2 \geq 0 \end{cases}$$

Now, we let $u = u_0 + u_1P_1 + u_2P_2 + u_3P_3 \in 3 - SP_V$, we have:

$\varphi(u+w, n) = \varphi(u, n) + \varphi(w, n)$

Let $a = a_0 + a_1P_1 + a_2P_2 + a_3P_3 \in 3 - SP_R$, then:

$\varphi(a.w, n) = a.\varphi(w, n)$.

So that, φ is a symbolic 3-plithogenic inner product.

Remark.

The norm of $w = w_0 + w_1P_1 + w_2P_2 + w_3P_3$ is defined with respect to φ as follows:

$\|w\| = \sqrt{\varphi(w, w)} = \|w_0\| + P_1[\|w_0+w_1\| - \|w_0\|] + P_2[\|w_0+w_1+w_2\| - \|w_0+w_1\|] + P_3[\|w_0+w_1+w_2+w_3\| - \|w_0+w_1+w_2\|]$.

Theorem.

Let φ be a symbolic 3-plithogenic inner product on $3 - SP_V$, then:

1). $\|w\| \geq 0; w \in 3 - SP_V$

2). $\|a.w\| = |a|.\|w\|; a \in 3 - SP_R$

3). $\|w+n\| \leq \|w\| + \|n\|; n \in 3 - SP_V$

4). $|\varphi(w, n)| \leq \|w\|.\|n\|$

5). $w \perp n$ if and only if $w_0 \perp n_0, w_0+w_1 \perp n_0+n_1, w_0+w_1+w_2 \perp n_0+n_1+n_2, w_0+w_1+w_2+w_3 \perp n_0+n_1+n_2+n_3$

6). If $w \perp n$, then $\|w+n\|^2 = \|w\|^2 + \|n\|^2$

Proof.

1). It holds directly from the definition of norm, and from the partial order relation defined on $3-SP_R$.

2). Let $a=a_0+a_1P_1+a_2P_2+a_3P_3 \in 3-SP_R, w=w_0+w_1P_1+w_2P_2+w_3P_3 \in 3-SP_V$, we have:

$\|a.w\|^2=\varphi(a.w,a.w)=a^2\varphi(w,w)$, thus $\|a.w\|=|a|.\|w\|$.

3). $\|w+n\|=\|w_0+n_0\|+P_1[\|w_0+w_1+n_0+n_1\|-\|w_0+n_0\|]+P_2[\|w_0+w_1+w_2+n_0+n_1+n_2\|-\|w_0+w_1+n_0+n_1\|]+P_3[\|w_0+w_1+w_2+w_3+n_0+n_1+n_2+n_3\|-\|w_0+w_1+w_2+n_0+n_1+n_2\|]$, we have:

$$\begin{cases} \|w_0+n_0\| \le \|w_0\|+\|n_0\| \\ \|w_0+w_1+n_0+n_1\| \le \|w_0+w_1\|+\|n_0+n_1\| \\ \|w_0+w_1+w_2+n_0+n_1+n_2\| \le \|w_0+w_1+w_2\|+\|n_0+n_1+n_2\| \\ \|w_0+w_1+w_2+w_3+n_0+n_1+n_2+n_3\| \le \|w_0+w_1+w_2+w_3\|+\|n_0+n_1+n_2+n_3\| \end{cases}$$

So that:

$\|w+n\| \le \|w\|+\|n\|$.

4). We have:

$|\varphi(w,n)|=|\varphi(w_0,n_0)|+P_1[|\varphi(w_0+w_1,n_0+n_1)|-|\varphi(w_0,n_0)|]+P_2[|\varphi(w_0+w_1+w_2,n_0+n_1+n_2)|-|\varphi(w_0+w_1,n_0+n_1)|]+P_3[|\varphi(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)|-|\varphi(w_0+w_1+w_2,n_0+n_1+n_2)|]$.

According to Cauchy-Shwartz inequality, we can write:

$$\begin{cases} |f(w_0,n_0)| \le \|w_0\|+\|n_0\| \\ |f(w_0+w_1,n_0+n_1)| \le \|w_0+w_1\|+\|n_0+n_1\| \\ |f(w_0+w_1+w_2,n_0+n_1+n_2)| \le \|w_0+w_1+w_2\|+\|n_0+n_1+n_2\| \\ |f(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)| \le \|w_0+w_1+w_2+w_3\|+\|n_0+n_1+n_2+n_3\| \end{cases}$$

So that,

$|\varphi(w,n)| \le \|w\|.\|n\|$

5). $w \perp n$ if and only if $\varphi(w,n)=0$, which is equivalent to:

$$\begin{cases} f(w_0,n_0)=0 \Longrightarrow w_0 \perp n_0 \\ f(w_0+w_1,n_0+n_1) \Longrightarrow w_0+w_1 \perp n_0+n_1 \\ f(w_0+w_1+w_2,n_0+n_1+n_2) \Longrightarrow w_0+w_1+w_2 \perp n_0+n_1+n_2 \\ f(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3) \Longrightarrow w_0+w_1+w_2+w_3 \perp n_0+n_1+n_2+n_3 \end{cases}$$

6). Assume that $w \perp n$, then $\|w+n\|^2=\varphi(w+n,w+n)=\varphi(w,w)+\varphi(n,n)+2\varphi(w,n)=\|w\|^2+\|n\|^2$.

Conclusion

In This paper we have studied the real inner product defined over symbolic 2-plithogenic vector spaces, where we discussed the concept of inner (scalar) products over the symbolic 2-plithogenic vector spaces by using the corresponding Euclidean scalar products to get theorems that describe the conditions of orthogonality in this class of spaces. Also, we illustrated many examples to explain the ortho-normed symbolic 2-plithogenic spaces.

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