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Gateaux And Frechet Derivative In Neutrosophic Normed Linear Spaces

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Abstract: In this study, we present the neutrosophic derivatives, the neutrosophic Gateaux derivative, and the neutrosophic Frechet derivative, and we examine some of their features. The relationship between the neutrophilic Frechet derivative and the neutrophilic Gateaux derivative is examined.

Keywords: Neutrosophic differentiation; Neutrosophic continuity; Neutrosophic Gateaux derivative; Neutrosophic Frechet derivative.

1. Introduction

The notion of normed linear space is essential to functional analysis. Dimension in normed linear space is catching the attention of researchers more and more. In recent years, many researchers have worked to expand the idea of n-normed linear space. The fuzzy set is a great theory for handling uncertainty that was invented by Zadeh [26]. This idea served as the cornerstone for a broad range of mathematical applications, as well as a large number of situations in everyday life.

In 1986, Atanassov [2] investigated intuitionistic fuzzy sets, which are distinguished by a membership function and non-membership function for each in the universe. Smarandache [23–25] later developed another concept known as neutrosophic set by introducing an intermediate membership function. Katsaras [16] presented the idea of a fuzzy norm in 1984. The fuzzy norm on a linear space was first proposed by Felbin [10] in 1992. Cheng Moderson [5] proposed another fuzzy norm idea for a linear space. After Cheng and Moderson's fuzzy norm formulation was refined by Bag and Samanta [3], they created the concepts of continuity and boundedness of a linear operator with respect to their fuzzy norm. Frechet and Gateaux Bivas Dinda, Samanta, and Bera [4] are the ones who originally introduced derivative in intuitionistic fuzzy normed linear spaces. Neutrosophic norm in a linear space was proposed by Dass Sarath Kumar and Prakasam Muralikrishna [19].

In this article, we define neutrosophic derivative in R, neutrosophic Gateaux derivative, and neutrosophic Frechet derivative on a linear space and examine some of their features. The relationships between the neutrosophic Gateaux derivative and the neutrosophic Frechet derivative are also discussed.

2. Preliminaries

Definition 2.1. [19] A 7-tuple $(\Xi, \mathfrak{A}, \mathfrak{B}, \mathfrak{W}, *, \diamond, \otimes)$ is said to be a Neutrosophic Normed Space [NNS], if Ξ be a linear space over the field $F = (\mathbb{R}$ or $\mathbb{C})$, Let $*$ be a continuous t-norm,$\diamond$, $\otimes$ be a continuous t-conorm and $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{W}$ are functions from $\Xi \times \mathbb{R}^{+} \rightarrow [0,1]$, fulfilling the following conditions for every $\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}} \in \mathbb{R}^{+}$ and $\mathfrak{d}, \tau \in \mathbb{R}$.
(i) $0 \leq \mathfrak{A}(\tilde{\mathfrak{p}}, \tau) \leq 1; 0 \leq \mathfrak{B}(\tilde{\mathfrak{p}}, \tau) \leq 1; 0 \leq \mathfrak{W}(\tilde{\mathfrak{p}}, \tau) \leq 1;$
(ii) $0 \leq \mathfrak{A}(\tilde{\mathfrak{p}}, \tau) + \mathfrak{B}(\tilde{\mathfrak{p}}, \tau) + \mathfrak{W}(\tilde{\mathfrak{p}}, \tau) \leq 3;$
(iii) $\mathfrak{A}(\tilde{\mathfrak{p}}, \tau) > 0;$
(iv) $\mathfrak{A}(\tilde{\mathfrak{p}}, \tau) = 1 \Leftrightarrow \tilde{\mathfrak{p}} = \theta, \theta$ is null vector;
(v) $\mathfrak{A}(c\tilde{\mathfrak{p}}, \tau) = \mathfrak{A}\left(\tilde{\mathfrak{p}}, \frac{\tau}{|c|}\right), \forall c \in F$ and $c \neq 0;$
(vi) $\mathfrak{A}(\tilde{\mathfrak{p}}, \mathfrak{d}) * \mathfrak{A}(\tilde{\mathfrak{q}}, \tau) \leq \mathfrak{A}(\tilde{\mathfrak{p}} + \tilde{\mathfrak{q}}, \mathfrak{d} + \tau);$
(vii) $\mathfrak{A}(\tilde{\mathfrak{p}}, \cdot)$ is non-decreasing function of $\mathbb{R}^{+}$ and $\lim_{\tau \to \infty} \mathfrak{A}(\tilde{\mathfrak{p}}, \tau) = 1;$
(viii) $\mathfrak{B}(\tilde{\mathfrak{p}}, \tau) < 1;$
(ix) $\mathfrak{B}(\tilde{\mathfrak{p}}, \tau) = 0 \Leftrightarrow \tilde{\mathfrak{p}} = \theta;$
(x) $\mathfrak{B}(c\tilde{\mathfrak{p}}, \tau) = \mathfrak{B}\left(\tilde{\mathfrak{p}}, \frac{\tau}{|c|}\right) \forall c \in F$ and $c \neq 0;$
(xi) $\mathfrak{B}(\tilde{\mathfrak{p}}, \mathfrak{d}) \diamond \mathfrak{B}(\tilde{\mathfrak{q}}, \tau) \geq \mathfrak{B}(\tilde{\mathfrak{p}} + \tilde{\mathfrak{q}}, \mathfrak{d} + \tau);$
(xii) $\mathfrak{B}(\tilde{\mathfrak{p}}, \cdot)$ is non-increasing function of $\mathbb{R}^{+}$and $\lim_{\tau \to \infty} \mathfrak{B}(\tilde{\mathfrak{p}}, \tau) = 0.$
(xiii) $\mathfrak{W}(\tilde{\mathfrak{p}}, \tau) < 1;$
(xiv) $\mathfrak{W}(\tilde{\mathfrak{p}}, \tau) = 0 \Leftrightarrow \tilde{\mathfrak{p}} = \theta;$
(xv)$\mathfrak{W}(c\tilde{\mathfrak{p}}, \tau) = \mathfrak{W}\left(\tilde{\mathfrak{p}}, \frac{\tau}{|c|}\right) \forall c \in F$ and $c \neq 0;$
(xvi) $\mathfrak{W}(\tilde{\mathfrak{p}}, \mathfrak{d}) \otimes \mathfrak{W}(\tilde{\mathfrak{q}}, \tau) \geq \mathfrak{W}(\tilde{\mathfrak{p}} + \tilde{\mathfrak{q}}, \mathfrak{d} + \tau);$
(xvii) $\mathfrak{W}(\tilde{\mathfrak{p}}, \cdot)$ is non-increasing function of $\mathbb{R}^{+}$and $\lim_{\tau \to \infty} \mathfrak{W}(\tilde{\mathfrak{p}}, \tau) = 0.$

Definition 2.2. [19] The pair (Ξ, A) is called a Neutrosophic Normed Linear Space [NNLS], If A is a Neutrosophic norm on a linear space Ξ.
For the NNLS (Ξ, A), We also suppose that $\mathfrak{A}, \mathfrak{B}, \mathfrak{W}, *, \diamond, \otimes$ fulfilling the axioms listed below:
(xviii) $\left\{\begin{matrix} \dot{\mathfrak{a}} * \dot{\mathfrak{a}} = \dot{\mathfrak{a}} \\ \dot{\mathfrak{a}} \diamond \dot{\mathfrak{a}} = \dot{\mathfrak{a}} \\ \dot{\mathfrak{a}} \otimes \dot{\mathfrak{a}} = \dot{\mathfrak{a}} \end{matrix}\right\}$, for all $\dot{\mathfrak{a}} \in [0,1]$.
(xix) $\mathfrak{A}(\tilde{\mathfrak{p}}, \tau) > 0$, for every $\tau > 0 \Rightarrow \tilde{\mathfrak{p}} = \theta$.
(xx) $\mathfrak{B}(\tilde{\mathfrak{p}}, \tau) < 1$, for every $\tau > 0 \Rightarrow \tilde{\mathfrak{p}} = \theta$.
(xi) $\mathfrak{W}(\tilde{\mathfrak{p}}, \tau) < 1$, for every $\tau > 0 \Rightarrow \tilde{\mathfrak{p}} = \theta$.
(xii) For $\tilde{\mathfrak{p}} \neq \theta, \mathfrak{A}(\tilde{\mathfrak{p}}, \cdot)$ is strictly increasing on the subset $\{\tau : \mathfrak{A}(\tilde{\mathfrak{p}}, \tau) \in (0,1)\}$ of $\mathbb{R}$ and continuous function of $\mathbb{R}$.
(xiii) For $\tilde{\mathfrak{p}} \neq \theta, \mathfrak{B}(\tilde{\mathfrak{p}}, \cdot)$ is strictly decreasing on the subset $\{\tau : \mathfrak{B}(\tilde{\mathfrak{p}}, \tau) \in (0,1)\}$ of $\mathbb{R}$ and continuous function of $\mathbb{R}$.
(xiv) For $\tilde{\mathfrak{p}} \neq \theta, \mathfrak{W}(\tilde{\mathfrak{p}}, \cdot)$ is strictly decreasing on the subset $\{\tau : \mathfrak{W}(\tilde{\mathfrak{p}}, \tau) \in (0,1)\}$ of $\mathbb{R}$ and continuous function of $\mathbb{R}$.

Definition 2.3. Let $\{\tilde{\mathfrak{p}}_n\}_n$ be a sequence in a NNLS $(\Xi, \mathfrak{J})$, if for given $\dot{r} > 0;\ \tau > 0;\ 0 < \dot{r} < 1$, there exist an integer $n_0 \in \mathbb{N}$ such that $\mathfrak{A}(\tilde{\mathfrak{p}}_n - \tilde{\mathfrak{p}}, \tau) > 1 - \dot{r}$, $\mathfrak{B}(\tilde{\mathfrak{p}}_n - \tilde{\mathfrak{p}}, \tau) < \dot{r}$ and $\mathfrak{W}(\tilde{\mathfrak{p}}_n - \tilde{\mathfrak{p}}, \tau) < \dot{r}$ for all $n \geq n_0$ then the sequence is named to be converge to $\tilde{\mathfrak{p}} \in \Xi$.

Definition 2.4. A mapping $\zeta : (\widetilde{\Theta}, \mathfrak{I}) \rightarrow (\Xi, \mathcal{J})$ is named to be Neutrosophic continuous at $\tilde{\mathfrak{p}}_0 \in \widetilde{\Theta}$, where $(\widetilde{\Theta}, \mathfrak{I})$ and $(\Xi, \mathcal{J})$ are NNLS over the same field F, if for any given $\epsilon > 0$, $\varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho, \epsilon) > 0, \dot{\eta} = \dot{\eta}(\varrho, \epsilon) \in (0,1)$ such that for every $\tilde{\mathfrak{p}} \in \widetilde{\Theta}$,

$$\mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma > 1 - \dot{\eta}) \Rightarrow \mathfrak{A}_{\Xi}(\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0), \epsilon) > 1 - \varrho,$$
$$\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_{\Xi}(\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0), \epsilon) < \varrho,$$
$$\mathfrak{W}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_{\Xi}(\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0), \epsilon) < \varrho.$$

3. Neutrosophic Gateaux Derivative

Definition 3.1. A function $\zeta : (\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes) \rightarrow (\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ is named to be Neutrosophic Differentiable [ND] at $\tilde{\mathfrak{p}} \in \mathbb{R}$, where $(\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes)$ and $(\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ are NNLS over the same field F, if for any given $\epsilon > 0, \varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho, \epsilon) > 0$, $\dot{\eta} = \dot{\eta}(\varrho, \epsilon) \in (0,1)$ such that for every $\tilde{\mathfrak{p}}(\neq \tilde{\mathfrak{p}}_0) \in \mathbb{R}$,

$$\mathfrak{A}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) > 1 - \dot{\eta} \Rightarrow \mathfrak{A}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) > 1 - \varrho,$$
$$\mathfrak{B}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) < \varrho,$$
$$\mathfrak{W}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) < \varrho.$$

We denote ND of ζ at $\tilde{\mathfrak{p}}_0$ by $\zeta'(\tilde{\mathfrak{p}}_0)$.

Alternative definition: A mapping $\zeta : (\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes) \rightarrow (\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ is named to be ND at $\tilde{\mathfrak{p}} \in \mathbb{R}$, where $(\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes)$ and $(\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ are NNLS over the same field F, if for every $\tau > 0$.

$$\lim_{\mathfrak{A}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 1} \mathfrak{A}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 1,$$
$$\lim_{\mathfrak{B}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 0} \mathfrak{B}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 0,$$
$$\lim_{\mathfrak{W}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 0} \mathfrak{W}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 0,$$

$\zeta'(\tilde{\mathfrak{p}}_0)$ is called ND of ζ at $\tilde{\mathfrak{p}}_0$.

Example 3.2. Let $(\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes)$ and $(\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ be two NNLS over the same field $\mathbb{R}$. Let $\mathfrak{A}_1(\tilde{\mathfrak{p}}, \tau) = \mathfrak{A}_2(\tilde{\mathfrak{p}}, \tau) = \frac{\tau}{\tau + |\tilde{\mathfrak{p}}|'}$, $\mathfrak{B}_1(\tilde{\mathfrak{p}}, \tau) = \mathfrak{B}_2(\tilde{\mathfrak{p}}, \tau) = \frac{|\tilde{\mathfrak{p}}|}{\tau + |\tilde{\mathfrak{p}}|}$ and $\mathfrak{W}_1(\tilde{\mathfrak{p}}, \tau) = \mathfrak{W}_2(\tilde{\mathfrak{p}}, \tau) = \frac{|\tilde{\mathfrak{p}}|}{\tau}$. Let $\dot{\mathfrak{a}} * \dot{\mathfrak{b}} = \dot{\mathfrak{a}}\dot{\mathfrak{b}}$ and $\dot{\mathfrak{a}} \diamond \dot{\mathfrak{b}} = \dot{\mathfrak{a}} \otimes \dot{\mathfrak{b}} = \dot{\mathfrak{a}} + \dot{\mathfrak{b}} - \dot{\mathfrak{a}}\dot{\mathfrak{b}}$. A mapping $\zeta : (\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes) \rightarrow (\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ defined by $\zeta(\tilde{\mathfrak{p}}) = \tilde{\mathfrak{p}}^2$. Let $\tilde{\mathfrak{p}}_0 \in \mathbb{R}$ be any point. Clearly, $\lim_{\mathfrak{A}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 1} \mathfrak{A}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 1$,

$$\lim_{\mathfrak{B}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 0} \mathfrak{B}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 0,$$

$\lim_{\mathfrak{W}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 0} \mathfrak{W}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - f(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 0$. Therefore ζ is ND at $\tilde{\mathfrak{p}}_0$.

Theorem 3.3. Let $\zeta: (\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes) \rightarrow (\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ and $g: (\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes) \rightarrow (\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ are two Neutrosophic differentiable functions differentiable at $\tilde{\mathfrak{p}}_0$ and $(\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes)$ and $(\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ fulfilling the condition (xviii). Then for $K \in F, K\zeta + g$ is ND at $\tilde{\mathfrak{p}}_0$ and $(K\zeta + g)'(\tilde{\mathfrak{p}}_0) = K\zeta'(\tilde{\mathfrak{p}}_0) + g'(\tilde{\mathfrak{p}}_0)$.

Proof. Since ζ and g are ND at $\tilde{\mathfrak{p}}_0$. So that, for any given $\epsilon > 0, \varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho, \epsilon) > 0, \dot{\eta} = \dot{\eta}(\varrho, \epsilon) \in (0,1)$ such that for every $\tilde{\mathfrak{p}} \in \mathbb{R}$,

$$\mathfrak{A}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) > 1 - \dot{\eta} \Rightarrow \mathfrak{A}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) > 1 - \varrho$$
$$\mathfrak{B}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) < \varrho,$$
$$\mathfrak{W}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) < \varrho.$$

$$\mathfrak{A}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma) > 1-\dot{\eta} \Rightarrow \mathfrak{A}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - g'(\tilde{\mathfrak{p}}_0),\epsilon\right) > 1-\varrho,$$
$$\mathfrak{B}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - g'(\tilde{\mathfrak{p}}_0),\epsilon\right) < \varrho,$$
$$\mathfrak{W}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - g'(\tilde{\mathfrak{p}}_0),\epsilon\right) < \varrho.$$

Now,

$$\begin{aligned}
&\mathfrak{A}_2\left(\frac{(K\zeta+g)(\tilde{\mathfrak{p}})-(K\zeta+g)(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - \left(K\zeta'(\tilde{\mathfrak{p}}_0)+g'(\tilde{\mathfrak{p}}_0)\right),\epsilon\right)\\
&= \mathfrak{A}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})+g(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - K\zeta'(\tilde{\mathfrak{p}}_0)-g'(\tilde{\mathfrak{p}}_0),\epsilon\right)\\
&\geq \mathfrak{A}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - K\zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right) * \mathfrak{A}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)\\
&= \mathfrak{A}_2\left(\frac{\zeta(\tilde{\mathfrak{p}})-\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2|K|}\right) * \mathfrak{A}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)\\
&> (1-\varrho)*(1-\varrho) = (1-\varrho), \text{ whenever } \mathfrak{A}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma) > 1-\dot{\eta},
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{B}_2\left(\frac{(K\zeta+g)(\tilde{\mathfrak{p}})-(K\zeta+g)(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - \left(K\zeta'(\tilde{\mathfrak{p}}_0)+g'(\tilde{\mathfrak{p}}_0)\right),\epsilon\right)\\
&= \mathfrak{B}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})+g(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - K\zeta'(\tilde{\mathfrak{p}}_0)-g'(\tilde{\mathfrak{p}}_0),\epsilon\right)\\
&\geq \mathfrak{B}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - K\zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right) \diamond \mathfrak{B}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)\\
&= \mathfrak{B}_2\left(\frac{\zeta(\tilde{\mathfrak{p}})-\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2|K|}\right) \diamond \mathfrak{B}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)\\
&< \varrho \diamond \varrho = \varrho, \text{ whenever } \mathfrak{B}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma) < \dot{\eta} \text{ and}
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{W}_2\left(\frac{(K\zeta+g)(\tilde{\mathfrak{p}})-(K\zeta+g)(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - \left(K\zeta'(\tilde{\mathfrak{p}}_0)+g'(\tilde{\mathfrak{p}}_0)\right),\epsilon\right)\\
&= \mathfrak{W}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})+g(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - K\zeta'(\tilde{\mathfrak{p}}_0)-g'(\tilde{\mathfrak{p}}_0),\epsilon\right)\\
&\geq \mathfrak{W}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - K\zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right) \otimes \mathfrak{W}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)\\
&= \mathfrak{W}_2\left(\frac{\zeta(\tilde{\mathfrak{p}})-\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2|K|}\right) \otimes \mathfrak{W}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0} - g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)\\
&< \varrho \otimes \varrho = \varrho, \text{ whenever } \mathfrak{W}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma) < \dot{\eta}.
\end{aligned}$$

So, $K\zeta + g$ is ND at $\tilde{\mathfrak{p}}_0 \in \mathbb{R}$ and $(K\zeta+g)'(\tilde{\mathfrak{p}}_0) = K\zeta'(\tilde{\mathfrak{p}}_0) + g'(\tilde{\mathfrak{p}}_0)$.

Definition 3.4. Let $(\widetilde{\Theta},\mathfrak{I})$ and $(\Xi,\mathcal{J})$ be two NNLS over the same field F. An operator Υ from $(\widetilde{\Theta},\mathfrak{I})$ to $(\Xi,\mathcal{J})$ is named to be Neutrosophic Gateaux differentiable [NGD] at $\tilde{\mathfrak{p}}_0 \in \widetilde{\Theta}$, where, $(\widetilde{\Theta},\mathfrak{I})$ and $(\Xi,\mathcal{J})$ are NNLS over the same field F, if there exists a Neutrosophic continuous linear operator $G: (\widetilde{\Theta},\mathfrak{I}) \to (\Xi,\mathcal{J})$ (generally depends upon $\tilde{\mathfrak{p}}_0$) and for any given $\epsilon > 0, \varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho,\epsilon) > 0$, $\dot{\eta} = \dot{\eta}(\varrho,\epsilon) \in (0,1)$ such that for every $\tilde{\mathfrak{p}} \in \widetilde{\Theta}$ and $s(\neq 0) \in \mathbb{R}$,

$$\mathfrak{A}_{\widetilde{\Theta}}(\mathfrak{d},\sigma) > 1-\dot{\eta} \Rightarrow \mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G(\tilde{\mathfrak{p}}),\epsilon\right) > 1-\epsilon,$$
$$\mathfrak{B}_{\widetilde{\Theta}}(\mathfrak{d},\delta) < \dot{\eta} \Rightarrow \mathfrak{B}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G(\tilde{\mathfrak{p}}),\epsilon\right) < \sigma,$$
$$\mathfrak{W}_{\widetilde{\Theta}}(\mathfrak{d},\delta) < \dot{\eta} \Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G(\tilde{\mathfrak{p}}),\epsilon\right) < \sigma,$$

The operator G becomes known to as NGD of Υ at $\tilde{\mathfrak{p}}_0$. and it is represented by $D_{\zeta(\tilde{\mathfrak{p}}_0)}$.

Alternative definition: An operator Υ from $(\widetilde{\Theta},\mathfrak{I})$ to $(\Xi,\mathcal{J})$ is said to be Neutrosophic Gateaux and Frechet Derivative differentiable at $\tilde{\mathfrak{p}}_0 \in \widetilde{\Theta}$, where, $(\widetilde{\Theta},\mathfrak{I})$ and $(\Xi,\mathcal{J})$ are NNLS over the same field F,if there exists a Neutrosophic continuous linear operator $G: (\widetilde{\Theta},\mathfrak{I}) \to (\Xi,\mathcal{J})$such that for every $\tilde{\mathfrak{p}} \in \widetilde{\Theta}, \tau > 0$ and $\mathfrak{d}(\neq 0) \in \mathbb{R}$

$$\lim_{\mathfrak{A}_{\widetilde{\Theta}}(\mathfrak{d},\tau)\to 1} \mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G(\tilde{\mathfrak{p}}),\tau\right) = 1,$$

$$\lim_{\mathfrak{V}_{\tilde{\Theta}}(\mathfrak{d},\tau)\to 0} \mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G(\tilde{\mathfrak{p}}),\tau\right)=0,$$
$$\lim_{\mathfrak{W}_{\tilde{\Theta}}(\mathfrak{d},\tau)\to 0} \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G(\tilde{\mathfrak{p}}),\tau\right)=0.$$

Here, the operator G is called NGD of Υ at $\tilde{\mathfrak{p}}_0$ and it is denoted by $D_{\zeta(\tilde{\mathfrak{p}}_0)}$.

Example 3.5. Let $\tilde{\Theta}=\Xi=F=\mathbb{R}$ and let $\tilde{\mathfrak{p}}_0\in\mathbb{R}$ be any point. Let $(\mathbb{R},\mathfrak{A}_1,\mathfrak{V}_1,\mathfrak{W}_1 *,\diamond,\otimes)$ and $(\mathbb{R},\mathfrak{A}_2,\mathfrak{V}_2,\mathfrak{W}_2,*,\diamond,\otimes)$ are NNLS over the same field $\mathbb{R}$. Let $\mathfrak{A}_1(\tilde{\mathfrak{p}},\tau)=\mathfrak{A}_2(\tilde{\mathfrak{p}},\tau)=\frac{\tau}{\tau+|\tilde{\mathfrak{p}}|}$, $\mathfrak{V}_1(\tilde{\mathfrak{p}},\tau)=\mathfrak{V}_2(\tilde{\mathfrak{p}},\tau)=\frac{|\tilde{\mathfrak{p}}|}{\tau+|\tilde{\mathfrak{p}}|}$ and $\mathfrak{W}_1(\tilde{\mathfrak{p}},\tau)=\mathfrak{W}_2(\tilde{\mathfrak{p}},\tau)=\frac{|\tilde{\mathfrak{p}}|}{\tau}$. Let $\dot{a}*\dot{b}=\dot{a}\dot{b}$ and $\dot{a}\diamond\dot{b}=\dot{a}\otimes\dot{b}=\dot{a}+\dot{b}-\dot{a}\dot{b}$. An operator $\Upsilon:(\tilde{\Theta},\mathfrak{I})\to(\Xi,\mathcal{J})$ be defined by $\Upsilon(\tilde{\mathfrak{p}})=\tilde{\mathfrak{p}}$. There exist a neutrosophic continuous linear operator $G:(\tilde{\Theta},\mathfrak{I})\to(\Xi,\mathcal{J})$ be defined by $G(\tilde{\mathfrak{p}})=\frac{\tilde{\mathfrak{p}}}{2}$ such that

$$\lim_{\mathfrak{A}_{\tilde{\Theta}}(\mathfrak{d},\tau)\to 1} \mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G(\tilde{\mathfrak{p}}),\tau\right)=1,$$
$$\lim_{\mathfrak{V}_{\tilde{\Theta}}(\mathfrak{d},\tau)\to 0} \mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{s}-G(\tilde{\mathfrak{p}}),\tau\right)=0,$$

$\lim_{\mathfrak{W}_{\tilde{\Theta}}(\mathfrak{d},\tau)\to 0} \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G(\tilde{\mathfrak{p}}),\tau\right)=0$. Hence Υ is Neutrosophic Gateaux and Frechet differentiable at $\tilde{\mathfrak{p}}_0$.

Theorem 3.6. Let $\Upsilon:(\tilde{\Theta},\mathfrak{I})\to(\Xi,\mathcal{J})$ be a linear operator, where $(\tilde{\Theta},\mathfrak{I})$ and $(\Xi,\mathcal{J})$ are two NNLS satisfying (xviii), (xix), (xx) and (xxi). If Υ is NGD at $\tilde{\mathfrak{p}}_0$ then it is unique at $\tilde{\mathfrak{p}}_0$.

Proof. Let G_1,G_2 be two NGD of Υ at $\tilde{\mathfrak{p}}_0$. Then for any given $\epsilon>0,\varrho\in(0,1),\exists\,\sigma=\sigma(\varrho,\epsilon)>0,\dot{\eta}=\dot{\eta}(\varrho,\epsilon)\in(0,1)$ such that for every $\tilde{\mathfrak{p}}\in U$ and
$\mathfrak{d}(\neq 0)\in\mathbb{R}$,

$$\mathfrak{A}_{\tilde{\Theta}}(\mathfrak{d},\sigma)>1-\dot{\eta}\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_1(\tilde{\mathfrak{p}}),\epsilon\right)>1-\varrho,$$
$$\mathfrak{V}_{\tilde{\Theta}}(\mathfrak{d},\sigma)<\dot{\eta}\Rightarrow\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_1(\tilde{\mathfrak{p}}),\epsilon\right)<\varrho,$$
$$\mathfrak{W}_{\tilde{\Theta}}(\mathfrak{d},\sigma)<\dot{\eta}\Rightarrow\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_1(\tilde{\mathfrak{p}}),\epsilon\right)<\varrho \text{ and}$$
$$\mathfrak{A}_{\tilde{\Theta}}(\mathfrak{d},\sigma)>1-\dot{\eta}\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_2(\tilde{\mathfrak{p}}),\epsilon\right)>1-\varrho,$$
$$\mathfrak{V}_{\tilde{\Theta}}(\mathfrak{d},\sigma)<\dot{\eta}\Rightarrow\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_2(\tilde{\mathfrak{p}}),\epsilon\right)<\varrho,$$
$$\mathfrak{W}_{\tilde{\Theta}}(\mathfrak{d},\sigma)<\dot{\eta}\Rightarrow\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_2(\tilde{\mathfrak{p}}),\epsilon\right)<\varrho.$$

$$\mathfrak{A}_{\Xi}(G_1(\tilde{\mathfrak{p}})-G_2(\tilde{\mathfrak{p}}),\tau)=\mathfrak{A}_{\Xi}\left(\left\{\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_1(\tilde{\mathfrak{p}})\right\}-\left\{\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_2(\tilde{\mathfrak{p}})\right\},\tau\right)$$
$$=\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_1(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right)*\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_2(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right)$$
$$>(1-\varrho)*(1-\varrho)=(1-\varrho),\ \forall\,\varrho\in(0,1).$$

Therefore, $\mathfrak{A}_{\Xi}(G_1(\tilde{\mathfrak{p}})-G_2(\tilde{\mathfrak{p}}),\tau)>0,\ \forall\,t>0,$ (3.1)

$$\mathfrak{V}_{\Xi}(G_1(\tilde{\mathfrak{p}})-G_2(\tilde{\mathfrak{p}}),\tau)\le\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_1(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right)\diamond\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_2(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right)$$
$$<\varrho\diamond\varrho=\varrho\ \ \forall\,\varrho\in(0,1).$$

$\mathfrak{V}_{\Xi}(G_1(\tilde{\mathfrak{p}})-G_2(\tilde{\mathfrak{p}}),\tau)<1\ \ \forall\,\tau>0,$ (3.2)

$$\mathfrak{W}_{\Xi}(G_1(\tilde{\mathfrak{p}})-G_2(\tilde{\mathfrak{p}}),\tau)\le\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_1(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right)\otimes\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G_2(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right)$$
$$<\varrho\otimes\varrho=\varrho\ \ \forall\,\varrho\in(0,1).$$

$\mathfrak{W}_{\Xi}(G_1(\tilde{\mathfrak{p}})-G_2(\tilde{\mathfrak{p}}),\tau)<1,\ \forall\,\tau>0.$ (3.3)

From (3.1) , (3.2) and (3.3) we get $G_1(\tilde{\mathfrak{p}}) - G_2(\tilde{\mathfrak{p}}) = \theta$. Thus, $G_1(\tilde{\mathfrak{p}}) - G_2(\tilde{\mathfrak{p}})$.

Theorem 3.7. If Υ_1 and Υ_2 have NGD at $\tilde{\mathfrak{p}}_0$ then $\Upsilon = c\Upsilon_1 + \Upsilon_2$ has NGD at $\tilde{\mathfrak{p}}_0$, where c is a scalar.
Proof. Straight forward.

4. Neutrosophic Frechet Derivative

Definition 4.1. An operator $\Upsilon : (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ is named to be Neutrosophic Frechet Differentiable [NFD] at an interior $\tilde{\mathfrak{p}}_0 \in \widetilde{\Theta}$, where, $(\widetilde{\Theta}, \mathfrak{I})$ and $(\Xi, \mathcal{J})$ be two NNLS over the same field F, if there exists a continuous linear operator $\zeta : (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$(in general depends on $\tilde{\mathfrak{p}}_0$) and if for any given $\epsilon > 0$, $\varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho, \epsilon) > 0, \dot{\eta} = \dot{\eta}(\varrho, \epsilon) \in (0,1)$ such that for all $\tilde{\mathfrak{p}} \in \widetilde{\Theta}$,

$$\mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) > 1 - \dot{\eta} \Rightarrow \mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{1 - \mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \epsilon\right) > 1 - \varrho,$$
$$\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \epsilon\right) < \varrho \text{ and}$$
$$\mathfrak{W}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{W}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \epsilon\right) < \varrho$$

Here, ζ is called NFD of Υ at $\tilde{\mathfrak{p}}_0$ and is represented by $D\Upsilon(\tilde{\mathfrak{p}}_0)$.

Alternative definition: An operator $\Upsilon : (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ is said to be NFD at an interior $\tilde{\mathfrak{p}}_0 \in U$, where, $(\widetilde{\Theta}, \mathfrak{I})$ and $(\Xi, \mathcal{J})$ be two NNLS over the same field F, if there exists a continuous linear operator $\zeta : (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ (in general depends on $\tilde{\mathfrak{p}}_0$) such that for every $\tau > 0$

$$\lim_{\mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 1} \mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{1 - \mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \tau\right) = 1,$$
$$\lim_{\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 0} \mathfrak{B}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \tau\right) = 0 \text{ and}$$
$$\lim_{\mathfrak{W}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 0} \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{W}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \tau\right) = 0.$$

Here, ζ is called NFD of Υ at $\tilde{\mathfrak{p}}_0$ and is represented by $D\Upsilon(\tilde{\mathfrak{p}}_0)$.

Theorem 4.2. Let $\Upsilon : (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ be a linear operator, where $(\widetilde{\Theta}, \mathfrak{I})$ and $(\Xi, \mathcal{J})$ are two NNLS satisfying (xix), (xx) and (xxi). If Υ is NFD at $\tilde{\mathfrak{p}}_0$ then it is unique at $\tilde{\mathfrak{p}}_0$.
Proof. Straight forward.

Example 4.3. Let $\widetilde{\Theta} = \Xi = \mathbb{R}$ and $[a, b]$ be an interval of $\mathbb{R}$ and $\Upsilon : [a, b] \to \mathbb{R}$. For every $\tau > 0$ define $\mathfrak{A}(\tilde{\mathfrak{p}}, \tau) = \frac{\tau}{\tau + |\tilde{\mathfrak{p}}|}$, $\mathfrak{B}(x, \tau) = \frac{|\tilde{\mathfrak{p}}|}{\tau + |\tilde{\mathfrak{p}}|}$ and $\mathfrak{W}(\tilde{\mathfrak{p}}, \tau) = \frac{|\tilde{\mathfrak{p}}|}{\tau}$, then the NFD of Υ at $\tilde{\mathfrak{p}}_0$ is ND.

Proof. If Υ is NFD at $\tilde{\mathfrak{p}}_0$ then for any given $\epsilon > 0$, $\varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho, \epsilon) > 0$,
$\dot{\eta} = \dot{\eta}(\varrho, \epsilon) \in (0,1)$ such that for all $\tilde{\mathfrak{p}} \in \widetilde{\Theta}$,

$$\mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) > 1 - \dot{\eta} \Rightarrow \mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{1 - \mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \epsilon\right) > 1 - \varrho$$
$$\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{|\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0|}, \frac{\epsilon}{\tau + |\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0|}\right) > 1 - \varrho$$
$$\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta, \frac{\epsilon}{\tau + |\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0|}\right) > 1 - \varrho,$$
$$\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \epsilon\right) < \varrho$$
$$\Rightarrow \mathfrak{B}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{|\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0|}, \frac{\epsilon}{\tau + |\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0|}\right) < \varrho$$

$$\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-\zeta,\frac{\epsilon}{\tau+|\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0|}\right)<\varrho \text{ and}$$

$$\mathfrak{W}_{\tilde{\Theta}}(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma)<\dot{\eta}\Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)-(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{W}_{\tilde{\Theta}}(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\tau)},\epsilon\right)<\varrho$$

$$\Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)-(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0)\zeta}{|\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0|},\frac{\epsilon}{\tau+|\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0|}\right)<\varrho$$

$$\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-\zeta,\frac{\epsilon}{\tau+|\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0|}\right)<\varrho$$

Hence, NFD of Υ at $\tilde{\mathfrak{p}}_0$ implies ND Υ at $\tilde{\mathfrak{p}}_0$ and $\Upsilon'^{(\tilde{\mathfrak{p}}_0)}=D\Upsilon(\tilde{\mathfrak{p}}_0)$.

Theorem 4.4. An operator $\Upsilon:(\tilde{\Theta},\mathfrak{I})\to(\Xi,\mathcal{J})$ is NFD at $\tilde{\mathfrak{p}}_0\in U$ then Υ is NGD at $\tilde{\mathfrak{p}}_0$.
Proof. Since Υ is NFD at $\tilde{\mathfrak{p}}_0$, therefore, for $\tau>0$ wehave

$$\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{1-\mathfrak{A}_{\tilde{\Theta}}(h,\tau)},\tau\right)>1-\varrho,$$

$$\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{V}_{\tilde{\Theta}}(h,\tau)},\tau\right)<\varrho \text{ and}$$

$$\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{W}_{\tilde{\Theta}}(h,\tau)},\tau\right)<\varrho.$$

Now, $\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{1-\mathfrak{A}_{\tilde{\Theta}}(h,\tau)},\tau\right)>1-\varrho\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)-\mathfrak{d}D\Upsilon(\tilde{\mathfrak{p}}_0)h}{1-\mathfrak{A}_{\tilde{\Theta}}(\mathfrak{d}h,\tau)},\tau\right)>1-\varrho.$
Putting $h=\mathfrak{d}h,\mathfrak{d}\neq 0$

$$\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\frac{1}{\mathfrak{d}}\left(1-\mathfrak{A}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)\right)},\tau\right)>1-\varrho,$$

$$\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h,\frac{\tau}{|\mathfrak{d}|}\left(1-\mathfrak{A}_{\tilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)\right)\right)>1-\varrho,$$

$\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h,\tau_1\right)>1-\varrho,$ where $\tau_1=\frac{\tau}{|\mathfrak{d}|}\left(1-\mathfrak{A}_{\tilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)\right),$

$\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{V}_{\tilde{\Theta}}(h,\tau)},\tau\right)<\varrho$

$\Rightarrow\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)-\mathfrak{d}D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{V}_{\tilde{\Theta}}(\mathfrak{d}h,\tau)},\tau\right)<\varrho.$

Putting $h=\mathfrak{d}h,\mathfrak{d}\neq 0$.

$$\Rightarrow\mathfrak{V}_{\Xi}\left(\frac{\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\frac{1}{\mathfrak{d}}\mathfrak{V}_{\tilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)},\tau\right)<\varrho,$$

$$\Rightarrow\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h,\quad\frac{\tau}{|\mathfrak{d}|}\mathfrak{V}_{\tilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)\right)<\varrho,$$

$\Rightarrow\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h,\tau_2\right)<\varrho,$ where $\tau_2=\frac{\tau}{|\mathfrak{d}|}\mathfrak{V}_{\tilde{\Theta}}\left(h,\frac{\tau}{\mathfrak{d}}\right)$ and

$\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{W}_{\tilde{\Theta}}(h,\tau)},\tau\right)<\varrho,$

$\Rightarrow\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)-\mathfrak{d}D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{W}_{\tilde{\Theta}}(\mathfrak{d}h,\tau)},\tau\right)<\varrho.$

Putting $h=\mathfrak{d}h,\mathfrak{d}\neq 0$

$$\Rightarrow\mathfrak{W}_{\Xi}\left(\frac{\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\frac{1}{\mathfrak{d}}\mathfrak{W}_{\tilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)},\tau\right)<\varrho,$$

$$\Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h, \quad \frac{\tau}{|\mathfrak{d}|}\mathfrak{W}_{\widetilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)\right)<\varrho,$$

$\Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h,\tau_3\right)<\varrho$, where $\tau_3=\frac{\tau}{|\mathfrak{d}|}\mathfrak{W}_{\widetilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)$.

Hence, Υ is NGD at $\tilde{\mathfrak{p}}_0$ and $D_{\Upsilon(\tilde{\mathfrak{p}}_0)h}=D\Upsilon(\tilde{\mathfrak{p}}_0)h$.

Theorem 4.5. Let $P:\widetilde{\Theta}\subset X\to\Xi\subset Y$ and $Q:\Xi\to Z$ be two linear operator satisfying (xviii). Suppose P is Neutrosophic continuous and has NGD at $\tilde{\mathfrak{p}}_0\in\widetilde{\Theta}$ and Q has NFD at Neutrosophic Gateaux and Frechet Derivative $y_0=P(\tilde{\mathfrak{p}}_0)$. Then $R=QP$ has NGD at $\tilde{\mathfrak{p}}_0$ and $D_{R(\tilde{\mathfrak{p}}_0)}=DQ(\tilde{\mathfrak{q}}_0)D_{P(\tilde{\mathfrak{p}}_0)}$.

Proof. For convenience, we write $G=D_{P(\tilde{\mathfrak{p}}_0)}$ and $\zeta=DQ(\tilde{\mathfrak{q}}_0)$.

Let $\tilde{\mathfrak{p}}\in X$ and we further write $\Delta\tilde{\mathfrak{q}}=P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)$. Then

$\mathfrak{A}\left(\frac{R(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-R(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{A}\left(\frac{QP(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-QP(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{A}\left(\frac{\zeta(\Delta\tilde{\mathfrak{q}})+A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$,

where $A(\Delta\tilde{\mathfrak{q}})=Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})$

$$=\mathfrak{A}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}+\frac{A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$$

$$\geq\mathfrak{A}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\frac{\tau}{2}\right)*\mathfrak{A}\left(\frac{A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{A}(\Delta\tilde{\mathfrak{q}},\tau)}\frac{\mathfrak{A}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)}{\mathfrak{d}},\frac{\tau}{2}\right)$$

$$=\mathfrak{A}\left(\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G,\frac{\tau}{2\mathfrak{A}(\zeta,\tau_2)}\right)*\mathfrak{A}\left(\frac{A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{A}(\Delta\tilde{\mathfrak{q}},\tau_1)}\frac{\tau s}{2\mathfrak{A}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)}\right)$$

$$=\mathfrak{A}\left(\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G,\frac{\tau}{2\mathfrak{A}(\zeta,\tau_2)}\right)*\mathfrak{A}\left(\frac{Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})}{\mathfrak{A}(\Delta\tilde{\mathfrak{q}},\tau_1)},\frac{\tau\mathfrak{d}}{2\mathfrak{A}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)}\right)$$

$$>(1-\varrho)*(1-\varrho)=(1-\varrho).$$

Since P has NGD and Q has NFD.

$$\mathfrak{B}\left(\frac{R(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-R(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{B}\left(\frac{QP(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-QP(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{B}\left(\frac{\zeta(\Delta\tilde{\mathfrak{q}})+A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$$

where $A(\Delta\tilde{\mathfrak{q}})=Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})$

$$=\mathfrak{B}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}+\frac{A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$$

$$\leq\mathfrak{B}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\frac{\tau}{2}\right)\diamond\mathfrak{B}\left(\frac{A(\Delta\tilde{\mathfrak{q}})}{1-\mathfrak{B}(\Delta\tilde{\mathfrak{q}},\tau_1)}\frac{1-\mathfrak{B}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)}{\mathfrak{d}},\frac{\tau}{2}\right)$$

$$=\mathfrak{B}\left(\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G,\frac{\tau}{2\mathfrak{B}(\zeta,\tau_2)}\right)$$

$$\diamond\mathfrak{B}\left(\frac{Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})}{1-\mathfrak{B}(\Delta\tilde{\mathfrak{q}},\tau_1)},\frac{\tau\mathfrak{d}}{2\left(1-\mathfrak{B}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)\right)}\right)$$

$<\varrho\diamond\varrho=\varrho$ and

$\mathfrak{W}\left(\frac{R(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-R(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{W}\left(\frac{QP(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-QP(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{W}\left(\frac{\zeta(\Delta\tilde{\mathfrak{q}})+A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$,

where $A(\Delta\tilde{\mathfrak{q}})=Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})$

$$=\mathfrak{W}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}+\frac{A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$$

$$\leq\mathfrak{W}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\frac{\tau}{2}\right)\otimes\mathfrak{W}\left(\frac{A(\Delta\tilde{\mathfrak{q}})}{1-\mathfrak{W}(\Delta\tilde{\mathfrak{q}},\tau_1)}\frac{1-\mathfrak{W}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)}{\mathfrak{d}},\frac{\tau}{2}\right)$$

$$=\mathfrak{W}\left(\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G,\frac{\tau}{2\mathfrak{W}(\zeta,\tau_2)}\right)$$

$$\otimes\mathfrak{W}\left(\frac{Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})}{1-\mathfrak{W}(\Delta\tilde{\mathfrak{q}},\tau_1)},\frac{\tau\mathfrak{d}}{2\left(1-\mathfrak{W}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)\right)}\right)$$

$<\varrho\otimes\varrho=\varrho$.

Since P has NGD and Q has NFD. Hence $\boldsymbol{R}=\boldsymbol{QP}$ has NGD at $\tilde{\mathfrak{p}}_0$ and $\boldsymbol{D}_{\boldsymbol{R}(\tilde{\mathfrak{p}}_0)}=\boldsymbol{DQ}(\tilde{\mathfrak{q}}_0)\boldsymbol{D}_{\boldsymbol{P}(\tilde{\mathfrak{p}}_0)}$.

Conclusion: In this article we present the idea of Neutrosophic derivative, Neutrosophic Gateaux derivative and Neutrosophic Frechet derivative and we explore some of the properties of this concepts . Moreover, we provide non-trivial examples. We have discussed about the relation between Neutrosophic Gateaux derivative and Neutrosophic Frechet derivative.

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