NSS

University of New Mexico

Bipolar Neutrosophic Projection Based Models for Solving Multi-attribute Decision Making Problems

Surapati Pramanik[1], Partha Pratim Dey[2], Bibhas C. Giri[3], and Florentin Smarandache[4]

[1] Department of Mathematics, Nandalal Ghosh B.T. College, Panpur, P.O.-Narayanpur, District - North 24 Parganas, Pin Code-743126, West Bengal, India. E-mail: sura_pati@yahoo.co.in

[2] Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India. E-mail: parsur.fuzz@gmail.com

[3] Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India. E-mail: bcgiri.ju.math@gmail.com

[4]University of New Mexico. Mathematics & Science Department, 705 Gurley Ave., Gallup, NM 87301, USA. Email: fsmarandache@gmail.com

Abstract. Bipolar neutrosophic sets are the extension of neutrosophic sets and are based on the idea of positive and negative preferences of information. Projection measure is a useful apparatus for modelling real life decision making problems. In the paper, we define projection, bidirectional projection and hybrid projection measures between bipolar neutrosophic sets. Three new methods based on the proposed projection measures are developed for solving multi-attribute decision making problems. In the solution process, the ratings of performance values of the alternatives with respect to the attributes are expressed in terms of bipolar neutrosophic values. We calculate projection, bidirectional projection, and hybrid projection measures between each alternative and ideal alternative with bipolar neutrosophic information. All the alternatives are ranked to identify the best alternative. Finally, a numerical example is provided to demonstrate the applicability and effectiveness of the developed methods. Comparison analysis with the existing methods in the literature in bipolar neutrosophic environment is also performed.

Keywords: Bipolar neutrosophic sets; projection measure; bidirectional projection measure; hybrid projection measure; multi-attribute decision making.

1 Introduction

For describing and managing indeterminate and inconsistent information, Smarandache [1] introduced neutrosophic set which has three independent components namely truth membership degree (*T*), indeterminacy membership degree (*I*) and falsity membership degree (*F*) where *T*, *I*, and *F* lie in$]^-0, 1^+[$. Later, Wang et al. [2] proposed single valued neutrosophic set (SVNS) to deal real decision making problems where *T*, *I*, and *F* lie in [0, 1].

Zhang [3] grounded the notion of bipolar fuzzy sets by extending the concept of fuzzy sets [4]. The value of membership degree of an element of bipolar fuzzy set belongs to [-1, 1]. With reference to a bipolar fuzzy set, the membership degree zero of an element reflects that the element is irrelevant to the corresponding property, the membership degree belongs to (0, 1] of an element reflects that the element somewhat satisfies the property, and the membership degree belongs to $[-1,0)$ of an element reflects that the element somewhat satisfies the implicit counter-property.

Deli et al. [5] extended the concept of bipolar fuzzy set to bipolar neutrosophic set (BNS). With reference to a bipolar neutrosophic set *Q*, the positive membership degrees $T_Q^+(x)$, $I_Q^+(x)$, and $F_Q^+(x)$ represent respectively the truth membership, indeterminate membership and falsity membership of an element $x \in X$ corresponding to the bipolar neutrosophic set *Q* and the negative membership degrees $T_Q^-(x)$, $I_Q^-(x)$, and $F_Q^-(x)$ denote respectively the truth membership, indeterminate membership and false membership degree of an element $x \in X$ to some implicit counter-property corresponding to the bipolar neutrosophic set *Q*.

Projection measure is a useful decision making device as it takes into account the distance as well as the included angle for measuring the closeness degree between two objects [6, 7]. Yue [6] and Zhang et al. [7] studied projection based multi-attribute decision making (MADM) in crisp environment i.e. projections are defined by ordinary numbers or crisp numbers. Yue [8] further investigated a new multi-attribute group decision making (MAGDM) method based on determining the weights of the decision makers by employing projection technique with interval data. Yue and Jia [9] established a methodology for MAGDM based on a new normalized projection measure, in which the attribute values are provided by decision makers in hybrid form with crisp values and interval data.

Xu and Da [10] and Xu [11] studied projection method for decision making in uncertain environment with preference information. Wei [12] discussed a MADM method based on the projection technique, in which the attribute values are presented in terms of intuitionistic fuzzy numbers. Zhang et al. [13] proposed a grey relational projection method for MADM based on intuitionistic trapezoidal fuzzy number. Zeng et al. [14] investigated projections on interval valued intuitionistic fuzzy numbers and developed algorithm to the MAGDM problems with interval-valued intuitionistic fuzzy information. Xu and Hu [15] developed two projection based models for MADM in intuitionistic fuzzy environment and interval valued intuitionistic fuzzy environment. Sun [16] presented a group decision making method based on projection method and score function under interval valued intuitionistic fuzzy environment. Tsao and Chen [17] developed a novel projection based compromising method for multi-criteria decision making (MCDM) method in interval valued intuitionistic fuzzy environment.

In neutrosophic environment, Chen and Ye [18] developed projection based model of neutrosophic numbers and presented MADM method to select clay-bricks in construction field. Bidirectional projection measure [19, 20] considers the distance and included angle between two vectors *x, y*. Ye [19] defined bidirectional projection measure as an improvement of the general projection measure of SVNSs to overcome the drawback of the general projection measure. In the same study, Ye [19] developed MADM method for selecting problems of mechanical design schemes under a single-valued neutrosophic environment. Ye [20] also presented bidirectional projection method for MAGDM with neutrosophic numbers.

Ye [21] defined credibility – induced interval neutrosophic weighted arithmetic averaging operator and credibility – induced interval neutrosophic weighted geometric averaging operator and developed the projection measure based ranking method for MADM problems with interval neutrosophic information and credibility information. Dey et al. [22] proposed a new approach to neutrosophic soft MADM using grey relational projection method. Dey et al. [23] defined weighted projection measure with interval neutrosophic assessments and applied the proposed concept to solve MADM problems with interval valued neutrosophic information. Pramanik et al. [24] defined projection and bidirectional projection measures between rough neutrosophic sets and proposed two new multi-criteria decision making (MCDM) methods based on projection and bidirectional projection measures in rough neutrosophic set environment.

In the field of bipolar neutrosophic environment, Deli et al. [5] defined score, accuracy, and certainty functions in order to compare BNSs and developed bipolar neutrosophic weighted average (BNWA) and bipolar neutrosophic weighted geometric (BNWG) operators to obtain collective bipolar neutrosophic information. In the same study, Deli et al. [5] also proposed a MCDM approach on the basis of score, accuracy, and certainty functions and BNWA, BNWG operators. Deli and Subas [25] presented a single valued bipolar neutrosophic MCDM through correlation coefficient similarity measure. Şahin et al. [26] provided a MCDM method based on Jaccard similarity measure of BNS. Uluçay et al. [27] defined Dice similarity, weighted Dice similarity, hybrid vector similarity, weighted hybrid vector similarity measures under BNSs and developed MCDM methods based on the proposed similarity measures. Dey et al. [28] defined Hamming and Euclidean distance measures to compute the distance between BNSs and investigated a TOPSIS approach to derive the most desirable alternative.

In this study, we define projection, bidirectional projection and hybrid projection measures under bipolar neutrosophic information. Then, we develop three methods for solving MADM problems with bipolar neutrosophic assessments. We organize the rest of the paper in the following way. In Section 2, we recall several useful definitions concerning SVNSs and BNSs. Section 3 defines projection, bidirectional projection and hybrid projection measures between BNSs. Section 4 is devoted to present three models for solving MADM under bipolar neutrosophic environment. In Section 5, we solve a decision making problem with bipolar neutrosophic information on the basis of the proposed measures. Comparison analysis is provided to demonstrate the feasibility and flexibility of the proposed methods in Section 6. Finally, Section 7 provides conclusions and future scope of research.

2 Basic Concepts Regarding SVNSs and BNSs

In this Section, we provide some basic definitions regarding SVNSs, BNSs which are useful for the construction of the paper.

2.1 Single valued neutrosophic sets [2]

Let *X* be a universal space of points with a generic element of *X* denoted by *x*, then a SVNS *P* is characterized by a truth membership function $T_P(x)$, an indeterminate membership function $I_P(x)$ and a falsity membership function $F_P(x)$. A SVNS *P* is expressed in the following way.

$$P = \{x, \langle T_P(x), I_P(x), F_P(x) \rangle \mid x \in X\}$$

where, $T_P(x)$, $I_P(x)$, $F_P(x) : X \to [0, 1]$ and $0 \le T_P(x) + I_P(x) + F_P(x) \le 3$ for each point $x \in X$.

2.2 Bipolar neutrosophic set [5]

Consider *X* be a universal space of objects, then a BNS *Q* in *X* is presented as follows:

$$Q = \{x, \langle T_Q^+(x), I_Q^+(x), F_Q^+(x), T_Q^-(x), I_Q^-(x), F_Q^-(x) \rangle \mid x \in X\},$$

where $T_Q^+(x)$, $I_Q^+(x)$, $F_Q^+(x)$: $X \to [0, 1]$ and $T_Q^-(x)$, $I_Q^-(x)$, $F_Q^-(x)$: $X \to [-1, 0]$. The positive membership degrees $T_Q^+(x)$, $I_Q^+(x)$, $F_Q^+(x)$ denote the truth membership, indeterminate membership, and falsity membership functions of an element $x \in X$ corresponding to a BNS Q and the negative membership degrees $T_Q^-(x)$, $I_Q^-(x)$, $F_Q^-(x)$ denote the truth membership, indeterminate membership, and falsity membership of an element $x \in X$ to several implicit counter property associated with a BNS Q. For convenience, a bipolar neutrosophic value (BNV) is presented as $\tilde{q} = \langle T_Q^+, I_Q^+, F_Q^+, T_Q^-, I_Q^-, F_Q^- \rangle$.

Definition 1 [5]
Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be any two BNSs. Then $Q_1 \subseteq Q_2$ if and only if $T_{Q_1}^+(x) \le T_{Q_2}^+(x)$, $I_{Q_1}^+(x) \le I_{Q_2}^+(x)$, $F_{Q_1}^+(x) \ge F_{Q_2}^+(x)$; $T_{Q_1}^-(x) \ge T_{Q_2}^-(x)$, $I_{Q_1}^-(x) \ge I_{Q_2}^-(x)$, $F_{Q_1}^-(x) \le F_{Q_2}^-(x)$ for all $x \in X$.

Definition 2 [5]
Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be any two BNSs. Then $Q_1 = Q_2$ if and only if $T_{Q_1}^+(x) = T_{Q_2}^+(x)$, $I_{Q_1}^+(x) = I_{Q_2}^+(x)$, $F_{Q_1}^+(x) = F_{Q_2}^+(x)$; $T_{Q_1}^-(x) = T_{Q_2}^-(x)$, $I_{Q_1}^-(x) = I_{Q_2}^-(x)$, $F_{Q_1}^-(x) = F_{Q_2}^-(x)$ for all $x \in X$.

Definition 3 [5]
Let, $Q = \{x, \langle T_Q^+(x), I_Q^+(x), F_Q^+(x), T_Q^-(x), I_Q^-(x), F_Q^-(x) \rangle \mid x \in X\}$ be a BNS. The complement of Q is represented by Q^c and is defined as follows:

$T_{Q^c}^+(x) = \{1^+\} - T_Q^+(x)$, $I_{Q^c}^+(x) = \{1^+\} - I_Q^+(x)$, $F_{Q^c}^+(x) = \{1^+\} - F_Q^+(x)$;

$T_{Q^c}^-(x) = \{1^-\} - T_Q^-(x)$, $I_{Q^c}^-(x) = \{1^-\} - I_Q^-(x)$, $F_{Q^c}^-(x) = \{1^-\} - F_Q^-(x)$.

Definition 4
Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be any two BNSs. Their union $Q_1 \cup Q_2$ is defined as follows:

$Q_1 \cup Q_2$ = {Max ($T_{Q_1}^+(x)$, $T_{Q_2}^+(x)$), Min ($I_{Q_1}^+(x)$, $I_{Q_2}^+(x)$), Min ($F_{Q_1}^+(x)$, $F_{Q_2}^+(x)$), Min ($T_{Q_1}^-(x)$, $T_{Q_2}^-(x)$), Max ($I_{Q_1}^-(x)$, $I_{Q_2}^-(x)$), Max ($F_{Q_1}^-(x)$, $F_{Q_2}^-(x)$)}, $\forall$ $x \in X$.

Their intersection $Q_1 \cap Q_2$ is defined as follows:

$Q_1 \cap Q_2$ = {Min ($T_{Q_1}^+(x)$, $T_{Q_2}^+(x)$), Max ($I_{Q_1}^+(x)$, $I_{Q_2}^+(x)$), Max ($F_{Q_1}^+(x)$, $F_{Q_2}^+(x)$), Max ($T_{Q_1}^-(x)$, $T_{Q_2}^-(x)$), Min ($I_{Q_1}^-(x)$, $I_{Q_2}^-(x)$), Min ($F_{Q_1}^-(x)$, $F_{Q_2}^-(x)$)}, $\forall$ $x \in X$.

Definition 5 [5]
Let $\tilde{q}_1 = \langle T_{Q_1}^+, I_{Q_1}^+, F_{Q_1}^+, T_{Q_1}^-, I_{Q_1}^-, F_{Q_1}^- \rangle$ and $\tilde{q}_2 = \langle T_{Q_2}^+, I_{Q_2}^+, F_{Q_2}^+, T_{Q_2}^-, I_{Q_2}^-, F_{Q_2}^- \rangle$ be any two BNVs, then

i. $\beta . \tilde{q}_1 = \langle 1 - (1 - T_{Q_1}^+)^\beta, (I_{Q_1}^+)^\beta, (F_{Q_1}^+)^\beta, -(-T_{Q_1}^-)^\beta, -(-I_{Q_1}^-)^\beta, -(1 - (1 - (-F_{Q_1}^-))^\beta) \rangle$;

ii. $(\tilde{q}_1)^\beta = \langle (T_{Q_1}^+)^\beta, 1 - (1 - I_{Q_1}^+)^\beta, 1 - (1 - F_{Q_1}^+)^\beta, -(1 - (1 - (-T_{Q_1}^-))^\beta), -(-I_{Q_1}^-)^\beta, (-F_{Q_1}^-)^\beta \rangle$;

iii. $\tilde{q}_1 + \tilde{q}_2 = \langle T_{Q_1}^+ + T_{Q_2}^+ - T_{Q_1}^+ . T_{Q_2}^+, I_{Q_1}^+ . I_{Q_2}^+, F_{Q_1}^+ . F_{Q_2}^+, -T_{Q_1}^- . T_{Q_2}^-, -(-I_{Q_1}^- - I_{Q_2}^- - I_{Q_1}^- . I_{Q_2}^-), -(-F_{Q_1}^- - F_{Q_2}^- - F_{Q_1}^- . F_{Q_2}^-) \rangle$;

iv. $\tilde{q}_1 . \tilde{q}_2 = \langle T_{Q_1}^+ . T_{Q_2}^+, I_{Q_1}^+ + I_{Q_2}^+ - I_{Q_1}^+ . I_{Q_2}^+, F_{Q_1}^+ + F_{Q_2}^+ - F_{Q_1}^+ . F_{Q_2}^+, -(-T_{Q_1}^- - T_{Q_2}^- - T_{Q_1}^- . T_{Q_2}^-), -I_{Q_1}^- . I_{Q_2}^-, -F_{Q_1}^- . F_{Q_2}^- \rangle$ where $\beta > 0$.

3 Projection, bidirectional projection and hybrid projection measures of BNSs

This Section proposes a general projection, a bidirectional projection and a hybrid projection measures for BNSs.

Definition 6
Assume that $X = (x_1, x_2, \ldots, x_m)$ be a finite universe of discourse and Q be a BNS in X, then modulus of Q is defined as follows:

$$\| Q \| = \sqrt{\sum_{j=1}^{m} \alpha_j^2} = \sqrt{\sum_{j=1}^{m} [(T_{Q_j}^+)^2 + (I_{Q_j}^+)^2 + (F_{Q_j}^+)^2 + (T_{Q_j}^-)^2 + (I_{Q_j}^-)^2 + (F_{Q_j}^-)^2]} \quad (1)$$

where $\alpha_j = \langle T_{Q_j}^+(x), I_{Q_j}^+(x), F_{Q_j}^+(x), T_{Q_j}^-(x), I_{Q_j}^-(x), F_{Q_j}^-(x) \rangle$, j = 1, 2, ..., m.

Definition 7 [10, 29]
Assume that $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$ be two vectors, then the projection of vector u onto vector v can be defined as follows:

$$Proj\,(u)_v = \| u \| \operatorname{Cos}(u, v) = \sqrt{\sum_{j=1}^{m} u_j^2} \times \frac{\sum_{j=1}^{m}(u_j v_j)}{\sqrt{\sum_{j=1}^{m} u_j^2} \times \sqrt{\sum_{j=1}^{m} v_j^2}} = \frac{\sum_{j=1}^{m}(u_j v_j)}{\sqrt{\sum_{j=1}^{m} v_j^2}} \quad (2)$$

where, $Proj\,(u)_v$ represents that the closeness of u and v in magnitude.

Definition 8
Assume that $X = (x_1, x_2, \ldots, x_m)$ be a finite universe of discourse and R, S be any two BNSs in X, then

$$Proj\,(R)_S = \| R \| \operatorname{Cos}(R, S) = \frac{1}{\| S \|}(R.S) \quad (3)$$

is called the projection of R on S, where

$\|R\| = \sqrt{\sum_{i=1}^{m}[(T_R^+)^2(x_i) + (I_R^+)^2(x_i) + (F_R^+)^2(x_i) + (T_R^-)^2(x_i) + (I_R^-)^2(x_i) + (F_R^-)^2(x_i)]}$,

$\|S\| = \sqrt{\sum_{i=1}^{m}[(T_S^+)^2(x_i) + (I_S^+)^2(x_i) + (F_S^+)^2(x_i) + (T_S^-)^2(x_i) + (I_S^-)^2(x_i) + (F_S^-)^2(x_i)]}$,

and $R.S = \sum_{i=1}^{m}[T_R^+(x_i)T_S^+(x) + I_R^+(x_i)I_S^+(x_i) + F_R^+(x_i)F_S^+(x_i) + T_R^-(x_i)T_S^-(x_i) + I_R^-(x_i)I_S^-(x_i) + F_R^-(x_i)F_S^-(x_i)]$.

Example 1. Suppose that R = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, S = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be the two BNSs in X, then the projection of R on S is obtained as follows:

$$Proj\,(R)_S = \frac{1}{\| S \|}(R.S) = \frac{(0.5)(0.7) + (0.3)(0.3) + (0.2)(0.1) + (-0.2)(-0.4) + (-0.1)(-0.2) + (-0.05)(-0.3)}{\sqrt{(0.7)^2 + (0.3)^2 + (0.1)^2 + (-0.4)^2 + (-0.2)^2 + (-0.3)^2}}$$

= 0.612952

The bigger value of $Proj\,(R)_S$ reflects that R and S are closer to each other.

However, in single valued neutrosophic environment, Ye [20] observed that the general projection measure cannot describe accurately the degree of α close to β. We also notice that the general projection incorporated by Xu [11] is not reasonable in several cases under bipolar neutrosophic setting, for example let, $\alpha = \beta$ = < a, a, a, -a, -a, -a > and γ = < 2a, 2a, 2a, -2a, -2a, -2a >, then $Proj\,(\alpha)_\beta$ = 2.44949 $\|a\|$ and $Proj\,(\gamma)_\beta$ = 4.898979 $\|a\|$. This shows that β is much closer to γ than α which is not true because $\alpha = \beta$. Ye [20] opined that α is equal to β whenever $Proj\,(\alpha)_\beta$ and $Proj\,(\beta)_\alpha$ should be equal to 1. Therefore, Ye [20] proposed an alternative method called bidirectional projection measure to overcome the limitation of general projection measure as given below.

Definition 9 [20]
Consider x and y be any two vectors, then the bidirectional projection between x and y is defined as follows:

$$B\text{-}proj\,(x, y) = \frac{1}{1 + \left| \frac{x.y}{\| x \|} - \frac{x.y}{\| y \|} \right|} = \frac{\| x \| \, \| y \|}{\| x \| \, \| y \| + | \, \| x \| - \| y \| \, | \, x.y} \quad (4)$$

where $\|x\|$, $\|y\|$ denote the moduli of x and y respectively, and $x.\,y$ is the inner product between x and y.
Here, $B\text{-}Proj\,(x, y) = 1$ if and only if $x = y$ and $0 \le B\text{-}Proj\,(x, y) \le 1$, i.e. bidirectional projection is a normalized measure.

Definition 10
Consider $R = \left\langle T_R^+(x_i), I_R^+(x_i), F_R^+(x_i), T_R^-(x_i), I_R^-(x_i), F_R^-(x_i) \right\rangle$ and $S = \left\langle T_S^+(x_i), I_S^+(x_i), F_S^+(x_i), T_S^-(x_i), I_S^-(x_i), F_S^-(x_i) \right\rangle$ be any two BNSs in $X = (x_1, x_2, \ldots, x_m)$, then the bidirectional projection measure between R and S is defined as follows:

$$B\text{-}Proj\,(R, S) = \frac{1}{1 + \left| \frac{R.S}{\| R \|} - \frac{R.S}{\| S \|} \right|} = \frac{\| R \| \, \| S \|}{\| R \| \, \| S \| + | \, \| R \| - \| S \| \, | \, R.S} \quad (5)$$

where

$\|R\| = \sqrt{\sum_{i=1}^{m}[(T_R^+)^2(x_i) + (I_R^+)^2(x_i) + (F_R^+)^2(x_i) + (T_R^-)^2(x_i) + (I_R^-)^2(x_i) + (F_R^-)^2(x_i)]}$,

$\|S\| = \sqrt{\sum_{i=1}^{m}[(T_S^+)^2(x_i) + (I_S^+)^2(x_i) + (F_S^+)^2(x_i) + (T_S^-)^2(x_i) + (I_S^-)^2(x_i) + (F_S^-)^2(x_i)]}$

and $R.S = \sum_{i=1}^{m}[T_R^+(x_i)T_S^+(x) + I_R^+(x_i)I_S^+(x_i) + F_R^+(x_i)F_S^+(x_i) + T_R^-(x_i)T_S^-(x_i) + I_R^-(x_i)I_S^-(x_i) + F_R^-(x_i)F_S^-(x_i)]$.

Proposition 1. Let $B\text{-}Proj\,(R)_S$ be a bidirectional projection measure between any two BNSs R and S, then

1. $0 \le B\text{-}Proj\,(R, S) \le 1$;
2. $B\text{-}Proj\,(R, S) = B\text{-}Proj\,(S, R)$;
3. $B\text{-}Proj\,(R, S) = 1$ for $R = S$.

Proof.
1. For any two non-zero vectors R and S,

$$\frac{1}{1 + \left| \frac{R.S}{\| R \|} - \frac{R.S}{\| S \|} \right|} > 0, \quad \because \frac{1}{1+x} > 0, \; when \;\; x > 0$$

$\therefore$ *B*-Proj (*R*, *S*) >0, for any two non-zero vectors *R* and *S*.
B-Proj (*R*, *S*) = 0 if and only if either $\| R \| = 0$ or $\| S \| = 0$ i.e. when either *R* = (0, 0, 0, 0, 0, 0) or *S* = (0, 0, 0, 0, 0, 0) which is trivial case.

$\therefore$ *B*-Proj (*R*, *S*) ≥ 0.
For two non-zero vectors *R* and *S*,
$\| R \| \, \| S \| + | \, \| R \| - \| S \| \, | \, R.S \geq \| R \| \, \| S \|$
$\therefore \| R \| \, \| S \| \leq \| R \| \, \| S \| + | \, \| R \| - \| S \| \, | \, R.S$

$$\therefore \frac{\| R \| \, \| S \|}{\| R \| \, \| S \| + | \, \| R \| - \| S \| \, | \, R.S} \leq 1$$

$\therefore$ *B-Proj* (*R*, *S*) ≤ 1.
$\therefore 0 \leq$ *B-Proj* (*R*, *S*) ≤ 1;

2. From definition, *R.S* = *S.R*, therefore,

$$B\text{-}Proj\ (R,\ S) = \frac{\| R \| \| S \|}{\| R \| \| S \| + | \| R \| - \| S \| | \, R.S} = \frac{\| S \| \| R \|}{\| S \| \, \| R \| + | \, \| S \| - \| R \| \, | \, S.R} = B\text{-}Proj\ (S,\ R).$$

Obviously, *B-Proj* (*R*, *S*) = 1, only when $\| R \| = \| S \|$ i. e. when $T_R^+(x_i) = T_S^+(x_i)$, $I_R^+(x_i) = I_S^+(x_i)$, $F_R^+(x_i) = F_S^+(x_i)$, $T_R^-(x_i) = T_S^-(x_i)$, $I_R^-(x_i) = I_S^-(x_i)$, $F_R^-(x_i) = F_S^-.(x_i)$

This completes the proof.

Example 2. Assume that *R* = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, *S* = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be the BNSs in *X*, then the bidirectional projection measure between *R* on *S* is computed as given below.
B-Proj (*R*, *S*) =

$$\frac{(0.6576473).(0.9380832)}{(0.6576473).(0.9380832) + | \, 0.9380832 - 06576473 \, | \, (0.575)}$$
= 0.7927845

Definition 11
Let *R* = $\left\langle T_R^+(x_i), I_R^+(x_i), F_R^+(x_i), T_R^-(x_i),\ I_R^-(x_i), F_R^-(x_i) \right\rangle$ and *S* = $\left\langle T_S^+(x_i), I_S^+(x_i), F_S^+(x_i), T_S^-(x_i), I_S^-(x_i), F_S^-(x_i) \right\rangle$ be any two BNSs in *X* = (x_1, x_2, …, x_m), then hybrid projection measure is defined as the combination of projection measure and bidirectional projection measure. The hybrid projection measure between *R* and *S* is represented as follows:

Hyb-Proj (*R*, *S*) = ρ *Proj* $(R)_S$ + (1 - ρ) *B-Proj* (*R*, *S*)

$$= \rho \, \frac{R.S}{\| S \|} + (1 - \rho) \frac{\| R \| \, \| S \|}{\| R \| \, \| S \| + | \, \| R \| - \| S \| \, | \, R.S} \qquad (6)$$

where

$\|R\|$ =

$$\sqrt{\sum_{i=1}^{m} [(T_R^+)^2(x_i) + (I_R^+)^2(x_i) + (F_R^+)^2(x_i) + (T_R^-)^2(x_i) + (I_R^-)^2(x_i) + (F_R^-)^2(x_i)]},$$

$\|S\|$ =

$$\sqrt{\sum_{i=1}^{m} [(T_S^+)^2(x_i) + (I_S^+)^2(x_i) + (F_S^+)^2(x_i) + (T_S^-)^2(x_i) + (I_S^-)^2(x_i) + (F_S^-)^2(x_i)]},$$

and
R.S =

$$\sum_{i=1}^{m} [T_R^+(x_i)T_S^+(x) + I_R^+(x_i)I_S^+(x_i) + F_R^+(x_i)F_S^+(x_i) + T_R^-(x_i)T_S^-(x_i) + I_R^-(x_i)I_S^-(x_i) + F_R^-(x_i)\, F_S^-(x_i)]$$

where $0 \leq \rho \leq 1$.

Proposition 2
Let *Hyb-Proj* (*R*, *S*) be a hybrid projection measure between any two BNSs *R* and *S*, then
1. $0 \leq$ *Hyb-Proj* (*R*, *S*) ≤ 1;
2. *Hyb-Proj* (*R*, *S*) = *B-Proj* (*S*, *R*);
3. *Hyb-Proj* (*R*, *S*) = 1 for *R* = *S*.

Proof. The proofs of the properties under Proposition 2 are similar as Proposition 1.
Example 3. Assume that *R* = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, *S* = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be the two BNSs, then the hybrid projection measure between *R* on *S* with $\rho = 0.7$ is calculated as given below.
Hyb-Proj (*R*, *S*) = (0.7). (0.612952) + (1 - 0.7). (0.7927845) = 0.6669018.

4 Projection, bidirectional projection and hybrid projection based decision making methods for MADM problems with bipolar neutrosophic information

In this section, we develop projection based decision making models to MADM problems with bipolar neutrosophic assessments. Consider *E* = {E_1, E_2, …, E_m}, ($m \geq 2$) be a discrete set of *m* feasible alternatives, *F* = {F_1, F_2, …, F_n}, ($n \geq 2$) be a set of attributes under consideration and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the attributes such that $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$. Now, we present three algorithms for MADM problems involving bipolar neutrosophic information.

4.1. Method 1

Step 1. The rating of evaluation value of alternative E_i (i = 1, 2, …, *m*) for the predefined attribute F_j (j = 1, 2, …, *n*) is presented by the decision maker in terms of bipolar neutrosophic values and the bipolar neutrosophic decision matrix is constructed as given below.

$$\left\langle q_{ij} \right\rangle_{m \times n} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ q_{m1} & q_{m2} & \cdots & q_{mn} \end{bmatrix}$$

where q_{ij} = < ($T^{+}_{ij}, I^{+}_{ij}, F^{+}_{ij}, T^{-}_{ij}, I^{-}_{ij}, F^{-}_{ij}$) > with $T^{+}_{ij}, I^{+}_{ij}, F^{+}_{ij}$, $-T^{-}_{ij}$, $-I^{-}_{ij}$, $-F^{-}_{ij} \in [0, 1]$ and $0 \le T^{+}_{ij} + I^{+}_{ij} + F^{+}_{ij} - T^{-}_{ij} - I^{-}_{ij} - F^{-}_{ij} \le 6$ for i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. We formulate the bipolar weighted decision matrix by multiplying weights w_j of the attributes as follows:

$$w_j \otimes \langle q_{ij} \rangle_{m\times n} = \langle z_{ij} \rangle_{m\times n} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix}$$

where $z_{ij} = w_j . q_{ij}$ = < 1 – (1 - $T^{+}_{ij})^{w_j}$, $(I^{+}_{ij})^{w_j}$, $(F^{+}_{ij})^{w_j}$, - (- $T^{-}_{ij})^{w_j}$, - (- $I^{-}_{ij})^{w_j}$, - (1 – (1 – (- F^{-}_{ij}))^{w_j}) > = < $\mu^{+}_{ij}, \nu^{+}_{ij}, \omega^{+}_{ij}, \mu^{-}_{ij}, \nu^{-}_{ij}, \omega^{-}_{ij}$ > with $\mu^{+}_{ij}, \nu^{+}_{ij}, \omega^{+}_{ij}, -\mu^{-}_{ij}, -\nu^{-}_{ij}, -\omega^{-}_{ij} \in [0, 1]$ and $0 \le \mu^{+}_{ij} + \nu^{+}_{ij} + \omega^{+}_{ij} - \mu^{-}_{ij} - \nu^{-}_{ij} - \omega^{-}_{ij} \le 6$ for i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. We identify the bipolar neutrosophic positive ideal solution (BNPIS) [27, 28] as follows:

$z^{PIS} = \langle e^{+}_{j}, f^{+}_{j}, g^{+}_{j}, e^{-}_{j}, f^{-}_{j}, g^{-}_{j} \rangle$ = < [{ $\underset{i}{\text{Max}}\,(\mu^{+}_{ij})$ | j ∈ σ }; { $\underset{i}{\text{Min}}\,(\mu^{+}_{ij})$ | j ∈ ς }], [{ $\underset{i}{\text{Min}}\,(\nu^{+}_{ij})$ | j ∈ σ }; { $\underset{i}{\text{Max}}\,(\nu^{+}_{ij})$ | j ∈ ς }], [{ $\underset{i}{\text{Min}}\,(\omega^{+}_{ij})$ | j ∈ σ }; { $\underset{i}{\text{Max}}\,(\omega^{-}_{ij})$ | j ∈ ς }], [{ $\underset{i}{\text{Min}}\,(\mu^{-}_{ij})$ | j ∈ σ }; { $\underset{i}{\text{Max}}\,(\mu^{-}_{ij})$ | j ∈ ς }], [{ $\underset{i}{\text{Max}}\,(\nu^{-}_{ij})$ | j ∈ σ }; { $\underset{i}{\text{Min}}\,(\nu^{-}_{ij})$ | j ∈ ς }], [{ $\underset{i}{\text{Max}}\,(\omega^{-}_{ij})$ | j ∈ σ }; { $\underset{i}{\text{Min}}\,(\omega^{-}_{ij})$ | j ∈ ς }] >, j = 1, 2, …, *n*, where σ and ς are benefit and cost type attributes respectively.

Step 4. Determine the projection measure between z^{PIS} and $Z^{i} = \langle z_{ij} \rangle_{m\times n}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* by using the following Eq.

$$Proj\ (Z^{i})_{z^{PIS}} = \frac{\sum_{j=1}^{n}[\mu^{+}_{ij}e^{+}_{j} + \nu^{+}_{ij}f^{+}_{j} + \omega^{+}_{ij}g^{+}_{j} + \mu^{-}_{ij}e^{-}_{j} + \nu^{-}_{ij}f^{-}_{j} + \omega^{-}_{ij}g^{-}_{j}]}{\sqrt{\sum_{j=1}^{n}[(e^{+}_{j})^{2} + (f^{+}_{j})^{2} + (g^{+}_{j})^{2} + (e^{-}_{j})^{2} + (f^{-}_{j})^{2} + (g^{-}_{j})^{2}]}} \qquad (7)$$

Step 5. Rank the alternatives in a descending order based on the projection measure $Proj\ (Z^{i})_{z^{PIS}}$ for i = 1, 2, …, *m* and bigger value of $Proj\ (Z^{i})_{z^{PIS}}$ determines the best alternative.

4.2. Method 2

Step 1. Give the bipolar neutrosophic decision matrix $\langle q_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. Construct weighted bipolar neutrosophic decision matrix $\langle z_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. Determine $z^{PIS} = \langle e^{+}_{j}, f^{+}_{j}, g^{+}_{j}, e^{-}_{j}, f^{-}_{j}, g^{-}_{j} \rangle$; j = 1, 2, …, *n*.

Step 4. Compute the bidirectional projection measure between z^{PIS} and $Z^{i} = \langle z_{ij} \rangle_{m\times n}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* using the Eq. as given below.

$$B\text{-}Proj\ (Z^{i}, z^{PIS}) = \frac{\| Z^{i} \| \| z^{PIS} \|}{\| Z^{i} \| \| z^{PIS} \| + | \| Z^{i} \| - \| z^{PIS} \| | Z^{i}.z^{PIS}} \qquad (8)$$

where $\| Z^{i} \| = \sqrt{\sum_{j=1}^{n}[(\mu^{+}_{ij})^{2} + (\nu^{+}_{ij})^{2} + (\omega^{+}_{ij})^{2} + (\mu^{-}_{ij})^{2} + (\nu^{-}_{ij})^{2} + (\omega^{-}_{ij})^{2}]}$, i = 1, 2, ..., *m*.

$\| z^{PIS} \| = \sqrt{\sum_{j=1}^{n}[(e^{+}_{j})^{2} + (f^{+}_{j})^{2} + (g^{+}_{j})^{2} + (e^{-}_{j})^{2} + (f^{-}_{j})^{2} + (g^{-}_{j})^{2}]}$ and

$Z^{i}.z^{PIS} = \sum_{j=1}^{n}[\mu^{+}_{ij}e^{+}_{j} + \nu^{+}_{ij}f^{+}_{j} + \omega^{+}_{ij}g^{+}_{j} + \mu^{-}_{ij}e^{-}_{j} + \nu^{-}_{ij}f^{-}_{j} + \omega^{-}_{ij}g^{-}_{j}]$, i = 1, 2, ..., *m*.

Step 5. According to the bidirectional projection measure *B-Proj* (Z^{i}, z^{PIS}) for i = 1, 2, …, *m* the alternatives are ranked and highest value of *B-Proj* (Z^{i}, z^{PIS}) reflects the best option.

4.3. Method 3

Step 1. Construct the bipolar neutrosophic decision matrix $\langle q_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. Formulate the weighted bipolar neutrosophic decision matrix $\langle z_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. Identify $z^{PIS} = \langle e^{+}_{j}, f^{+}_{j}, g^{+}_{j}, e^{-}_{j}, f^{-}_{j}, g^{-}_{j} \rangle$, j = 1, 2, …, *n*.

Step 4. By combining projection measure $Proj\ (Z^{i})_{z^{PIS}}$ and bidirectional projection measure *B-Proj* (Z^{i}, z^{PIS}), we calculate the hybrid projection measure between z^{PIS} and $Z^{i} = \langle z_{ij} \rangle_{m\times n}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* as follows.

$$Hyb\text{-}Proj\,(Z^{i},\ z^{PIS}) = \rho\ Proj\ (Z^{i})_{z^{PIS}} + (1-\rho)\ B\text{-}Proj\,(Z^{i},\ z^{PIS}) =$$

$$\rho\ \frac{Z^{i}.z^{PIS}}{\| z^{PIS} \|} + (1-\rho)\ \frac{\| Z^{i} \| \| z^{PIS} \|}{\| Z^{i} \| \| z^{PIS} \| + | \| Z^{i} \| - \| z^{PIS} \| | Z^{i}.z^{PIS}} \tag{9}$$

where $\| Z^{i} \| = \sqrt{\sum_{j=1}^{n}[(\mu_{ij}^{+})^{2}+(\nu_{ij}^{+})^{2}+(\omega_{ij}^{+})^{2}+(\mu_{ij}^{-})^{2}+(\nu_{ij}^{-})^{2}+(\omega_{ij}^{-})^{2}]}$, i = 1, 2, …, m,

$\| z^{PIS} \| = \sqrt{\sum_{j=1}^{n}[(e_{j}^{+})^{2}+(f_{j}^{+})^{2}+(g_{j}^{+})^{2}+(e_{j}^{-})^{2}+(f_{j}^{-})^{2}+(g_{j}^{-})^{2}]}$,

$Z^{i}.z^{PIS} = \sum_{j=1}^{n}[\mu_{ij}^{+}e_{j}^{+}+\nu_{ij}^{+}f_{j}^{+}+\omega_{ij}^{+}g_{j}^{+}+\mu_{ij}^{-}e_{j}^{-}+\nu_{ij}^{-}f_{j}^{-}+\omega_{ij}^{-}g_{j}^{-}]$, i = 1, 2, …, m, with $0 \le \rho \le 1$.

Step 5. We rank all the alternatives in accordance with the hybrid projection measure *Hyb-Proj* (Z^{i}, z^{PIS}) and greater value of *Hyb-Proj* (Z^{i}, z^{PIS}) indicates the better alternative.

5 A numerical example

We solve the MADM studied in [5, 28] where a customer desires to purchase a car. Suppose four types of car (alternatives) Ei, (i = 1, 2, 3, 4) are taken into consideration in the decision making situation. Four attributes namely Fuel economy (F_1), Aerod (F_2), Comfort (F_3) and Safety (F_4) are considered to evaluate the alternatives. Assume the weight vector [5] of the attribute is given by $w = (w_1, w_2, w_3, w_4) = (0.5, 0.25, 0.125, 0.125)$.

Method 1: The proposed projection measure based decision making with bipolar neutrosophic information for car selection is presented in the following steps:

Step 1: Construct the bipolar neutrosophic decision matrix

The bipolar neutrosophic decision matrix $\langle q_{ij} \rangle_{m\times n}$ presented by the decision maker as given below (see Table 1)

Table 1. The bipolar neutrosophic decision matrix

	F_1	F_2	F_3	F_4
E_1	<0.5, 0.7, 0.2, -0.7, -0.3, -0.6>	<0.4, 0.5, 0.4, -0.7, -0.8, -0.4>	<0.7, 0.7, 0.5, -0.8, -0.7, -0.6>	<0.1, 0.5, 0.7, -0.5, -0.2, -0.8>
E_2	<0.9, 0.7, 0.5, -0.7, -0.7, -0.1>	<0.7, 0.6, 0.8, -0.7, -0.5, -0.1>	<0.9, 0.4, 0.6, -0.1, -0.7, -0.5>	<0.5, 0.2, 0.7, -0.5, -0.1, -0.9>
E_3	<0.3, 0.4, 0.2, -0.6, -0.3, -0.7>	<0.2, 0.2, 0.2, -0.4, -0.7, -0.4>	<0.9, 0.5, 0.5, -0.6, -0.5, -0.2>	<0.7, 0.5, 0.3, -0.4, -0.2, -0.2>
E_4	<0.9, 0.7, 0.2, -0.8, -0.6, -0.1>	<0.3, 0.5, 0.2, -0.5, -0.5, -0.2>	<0.5, 0.4, 0.5, -0.1, -0.7, -0.2>	<0.2, 0.4, 0.8, -0.5, -0.5, -0.6>

Step 2. Construction of weighted bipolar neutrosophic decision matrix

The weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ is obtained by multiplying weights of the attributes to the bipolar neutrosophic decision matrix as follows (see Table 2).

Table 2. The weighted bipolar neutrosophic decision matrix

	F_1	F_2	F_3	F_4
E_1	<0.293, 0.837, 0.447,-0.837, -0.818, -0.182 >	<0.120, 0.795, 0.841, 0.915, -0.946, -0.120>	<0.140, 0.956, 0.917, 0.972, -0.956, -0.108>	<0.013, 0.917, 0.956, -0.917, -0.818, -0.182>
E_2	<0.684, 0.837, 0.707, -0.837, -0.837, -0.051>	<0.260, 0.880, 0.946, -0.915, -0.841, -0.026>	<0.250, 0.892, 0.938, -0.750, -0.956, -0.083>	<.083, 0.818, 0.956, 0.917, -0.750, -0.250>
E_3	<0.163, 0.632, 0.447, -0.774, -0.548, -0.452>	<0.054, 0.669, 0.669, - 0.795, -0.915, -0.120>	<0.250, 0.917, 0.917, -0.938, -0.917, -0.028>	<.140, 0.917, 0.860, -0.892, -0.818, -0.028>
E_4	<0.648, 0.837, 0.447, , -0.894,--0.774, -0.051>	<0.085, 0.841, 0.669, -0.841, -0.841, -0.054>	<0.083, 0.892, 0.917, -0.750, -0.956, -0.028>	<0.062, 0.818, 0.972, -0.917, -0.917, -0.108>

Step 3. Selection of BNPIS

The BNRPIS (z^{PIS}) = $\langle e_{j}^{+}, f_{j}^{+}, g_{j}^{+}, e_{j}^{-}, f_{j}^{-}, g_{j}^{-} \rangle$, (j = 1, 2, 3, 4) is computed from the weighted decision matrix as follows:

$\langle e_{1}^{+}, f_{1}^{+}, g_{1}^{+}, e_{1}^{-}, f_{1}^{-}, g_{1}^{-} \rangle$ = < 0.684, 0.632, 0.447, -0.894, -0.548, -0.051 >;

$\langle e_{2}^{+}, f_{2}^{+}, g_{2}^{+}, e_{2}^{-}, f_{2}^{-}, g_{2}^{-} \rangle$ = < 0.26, 0.669, 0.669, -0.915, -0.841, -0.026 >;

$\langle e_{3}^{+}, f_{3}^{+}, g_{3}^{+}, e_{3}^{-}, f_{3}^{-}, g_{3}^{-} \rangle$ = < 0.25, 0.892, 0.917, -0.972, -0.917, -0.028 >;

$\langle e_{4}^{+}, f_{4}^{+}, g_{4}^{+}, e_{4}^{-}, f_{4}^{-}, g_{4}^{-} \rangle$ = < 0.14, 0.818, 0.86, -0.917, -0.75, -0.028 >.

Step 4. Determination of weighted projection measure
The projection measure between positive ideal bipolar neutrosophic solution z^{PIS} and each weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ can be obtained as follows:

$Proj\ (Z^1)_{z^{PIS}}$ = 3.4214, $Proj\ (Z^2)_{z^{PIS}}$ = 3.4972, $Proj\ (Z^3)_{z^{PIS}}$ = 3.1821, $Proj\ (Z^4)_{z^{PIS}}$ = 3.3904.

Step 5. Rank the alternatives
We observe that $Proj\ (Z^2)_{z^{PIS}} > Proj\ (Z^1)_{z^{PIS}} > Proj\ (Z^4)_{z^{PIS}} > Proj\ (Z^3)_{z^{PIS}}$. Therefore, the ranking order of the cars is $E_2 \succ E_1 \succ E_4 \succ E_3$. Hence, E_2 is the best alternative for the customer.

Method 2: The proposed bidirectional projection measure based decision making for car selection is presented as follows:
Step 1. Same as Method 1
Step 2. Same as Method 1
Step 3. Same as Method 1
Step 4. Calculation of bidirectional projection measure
The bidirectional projection measure between positive ideal bipolar neutrosophic solution z^{PIS} and each weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ can be determined as given below.
B-Proj (Z^1, z^{PIS}) = 0.8556, *B-Proj* (Z^2, z^{PIS}) = 0.8101, *B-Proj* (Z^3, z^{PIS}) = 0.9503, *B-Proj* (Z^4, z^{PIS}) = 0.8969.
Step 5. Ranking the alternatives
Here, we notice that *B-Proj* (Z^3, z^{PIS}) > *B-Proj* (Z^4, z^{PIS}) > *B-Proj* (Z^1, z^{PIS}) > *B-Proj* (Z^2, z^{PIS}) and therefore, the ranking order of the alternatives is obtained as $E_3 \succ E_4 \succ E_1 \succ E_2$. Hence, E_3 is the best choice among the alternatives.

Method 3: The proposed hybrid projection measure based MADM with bipolar neutrosophic information is provided as follows:
Step 1. Same as Method 1
Step 2. Same as Method 1
Step 3. Same as Method 1
Step 4. Computation of hybrid projection measure
The hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are shown in the Table 3.

Table 3. Results of hybrid projection measure for different valus of ρ

Similarity measure	ρ	Measure values	Ranking order
Hyb-Proj (Z^i, z^{PIS})	0.25	*Hyb-Proj* (Z^1, z^{PIS}) = 1.4573 *Hyb-Proj* (Z^2, z^{PIS}) = 1.4551 *Hyb-Proj* (Z^3, z^{PIS}) = 1.5297 *Hyb-Proj* (Z^4, z^{PIS}) = 1.5622	$E_4 > E_3 > E_1 > E_2$
Hyb-Proj (Z^i, z^{PIS})	0.50	*Hyb-Proj* (Z^1, z^{PIS}) = 2.1034 *Hyb-Proj* (Z^2, z^{PIS}) = 2.0991 *Hyb-Proj* (Z^3, z^{PIS}) = 2.0740 *Hyb-Proj* (Z^4, z^{PIS}) = 2.1270	$E_4 > E_1 > E_2 > E_3$
Hyb-Proj (Z^i, z^{PIS})	0.75	*Hyb-Proj* (Z^1, z^{PIS}) = 2.4940 *Hyb-Proj* (Z^2, z^{PIS}) = 2.7432 *Hyb-Proj* (Z^3, z^{PIS}) = 2.6182 *Hyb-Proj* (Z^4, z^{PIS}) = 2.6919	$E_2 > E_4 > E_3 > E_1$
Hyb-Proj (Z^i, z^{PIS})	0.90	*Hyb-Proj* (Z^1, z^{PIS}) = 3.1370 *Hyb-Proj* (Z^2, z^{PIS}) = 3.1296 *Hyb-Proj* (Z^3, z^{PIS}) = 2.9448 *Hyb-Proj* (Z^4, z^{PIS}) = 3.0308	$E_1 > E_2 > E_4 > E_3$

6 Comparative analysis

In the Section, we compare the results obtained from the proposed methods with the results derived from other existing methods under bipolar neutrosophic environment to show the effectiveness of the developed methods.

Dey et al. [28] assume that the weights of the attributes are not identical and weights are fully unknown to the decision maker. Dey et al. [28] formulated maximizing deviation model under bipolar neutrosophic assessment to compute unknown weights of the attributes as *w* = (0.2585, 0.2552, 0.2278, 0.2585). By considering *w* = (0.2585, 0.2552, 0.2278, 0.2585), the proposed projection measures are shown as follows:

$Proj\ (Z^1)_{z^{PIS}}$ = 3.3954, $Proj\ (Z^2)_{z^{PIS}}$ = 3.3872, $Proj\ (Z^3)_{z^{PIS}}$ = 3.1625, $Proj\ (Z^4)_{z^{PIS}}$ = 3.2567.

Since, $Proj\ (Z^1)_{z^{PIS}} > Proj\ (Z^2)_{z^{PIS}} > Proj\ (Z^4)_{z^{PIS}} > Proj\ (Z^3)_{z^{PIS}}$, therefore the ranking order of the four alternatives is given by $E_1 \succ E_2 \succ E_4 \succ E_3$. Thus, E_1 is the best choice for the customer.

Now, by taking *w* = (0.2585, 0.2552, 0.2278, 0.2585), the bidirectional projection measures are calculated as given below.
B-Proj (Z^1, z^{PIS}) = 0.8113, *B-Proj* (Z^2, z^{PIS}) = 0.8111, *B-Proj* (Z^3, z^{PIS}) = 0.9854, *B-Proj* (Z^4, z^{PIS}) = 0.9974.

Since, *B-Proj* (Z^4, z^{PIS}) > *B-Proj* (Z^3, z^{PIS}) > *B-Proj* (Z^1, z^{PIS}) > *B-Proj* (Z^2, z^{PIS}), consequently the ranking

order of the four alternatives is given by $E_4 \succ E_3 \succ E_1 \succ E_2$. Hence, E_4 is the best option for the customer.
Also, by taking w = (0.2585, 0.2552, 0.2278, 0.2585), the proposed hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are revealed in the Table 4.

Deli et al. [5] assume the weight vector of the attributes as w = (0.5, 0.25, 0.125, 0.125) and the ranking order based on score values is presented as follows:

$$E_3 \succ E_4 \succ E_2 \succ E_1$$

Thus, E_3 was the most desirable alternative.

Dey et al. [28] employed maximizing deviation method to find unknown attribute weights as w = (0.2585, 0.2552, 0.2278, 0.2585). The ranking order of the alternatives is presented based on the relative closeness coefficient as given below.

$$E_3 \succ E_2 \succ E_4 \succ E_{1.}$$

Obviously, E_3 is the most suitable option for the customer.

Dey et al. [28] also consider the weight vector of the attributes as w = (0.5, 0.25, 0.125, 0.125), then using TOPSIS method, the ranking order of the cars is represented as follows:

$$E_4 \succ E_2 \succ E_3 \succ E_{1.}$$

So, E_4 is the most preferable alternative for the buyer. We observe that different projection measure provides different ranking order and the projection measure is weight sensitive. Therefore, decision maker should choose the projection measure and weights of the attributes in the decision making context according to his/her needs, desires and practical situation.

Conclusion

In this paper, we have defined projection, bidirectional projection measures between bipolar neutrosophic sets. Further, we have defined a hybrid projection measure by combining projection and bidirectional projection measures. Through these projection measures we have developed three methods for multi-attribute decision making models under bipolar neutrosophic environment. Finally, a car selection problem has been solved to show the flexibility and applicability of the proposed methods. Furthermore, comparison analysis of the proposed methods with the other existing methods has also been demonstrated.

The proposed methods can be extended to interval bipolar neutrosophic set environment. In future, we shall apply projection, bidirectional projection, and hybrid projection measures of interval bipolar neutrosophic sets for group decision making, medical diagnosis, weaver selection, pattern recognition problems, etc.

Table 4. Results of hybrid projection measure for different values of ρ

Similarity measure	ρ	Measure values	Ranking order
Hyb-Proj $(Z^i, \boldsymbol{Z}^{\mathrm{PIS}})$	0.25	*Hyb-Proj* $(Z^1, \boldsymbol{Z}^{\mathrm{PIS}})$ = 1.4970 *Hyb-Proj* $(Z^2, \boldsymbol{Z}^{\mathrm{PIS}})$ = 1.4819 *Hyb-Proj* $(Z^3, \boldsymbol{Z}^{\mathrm{PIS}})$ = 1.5082 *Hyb-Proj* $(Z^4, \boldsymbol{Z}^{\mathrm{PIS}})$ = 1.5203	$E_4 > E_3 > E_1 > E_2$
Hyb-Proj $(Z^i, \boldsymbol{Z}^{\mathrm{PIS}})$	0.50	*Hyb-Proj* $(Z^1, \boldsymbol{Z}^{\mathrm{PIS}})$ = 2.1385 *Hyb-Proj* $(Z^2, \boldsymbol{Z}^{\mathrm{PIS}})$ = 2.1536 *Hyb-Proj* $(Z^3, \boldsymbol{Z}^{\mathrm{PIS}})$ = 2.0662 *Hyb-Proj* $(Z^4, \boldsymbol{Z}^{\mathrm{PIS}})$ = 2.1436	$E_4 > E_1 > E_2 > E_3$
Hyb-Proj $(Z^i, \boldsymbol{Z}^{\mathrm{PIS}})$	0.75	*Hyb-Proj* $(Z^1, \boldsymbol{Z}^{\mathrm{PIS}})$ = 2.7800 *Hyb-Proj* $(Z^2, \boldsymbol{Z}^{\mathrm{PIS}})$ = 2.8254 *Hyb-Proj* $(Z^3, \boldsymbol{Z}^{\mathrm{PIS}})$ = 2.6241 *Hyb-Proj* $(Z^4, \boldsymbol{Z}^{\mathrm{PIS}})$ = 2.7670	$E_2 > E_4 > E_3 > E_1$
Hyb-Proj $(Z^i, \boldsymbol{Z}^{\mathrm{PIS}})$	0.90	*Hyb-Proj* $(Z^1, \boldsymbol{Z}^{\mathrm{PIS}})$ = 3.1648 *Hyb-Proj* $(Z^2, \boldsymbol{Z}^{\mathrm{PIS}})$ = 3.2285 *Hyb-Proj* $(Z^3, \boldsymbol{Z}^{\mathrm{PIS}})$ = 2.9589 *Hyb-Proj* $(Z^4, \boldsymbol{Z}^{\mathrm{PIS}})$ = 3.1410	$E_2 > E_1 > E_4 > E_3$

References

[1] F. Smarandache. A unifying field of logics. Neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, 1998.

[2] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman. Single valued neutrosophic sets. Multi-space and Multi-Structure, 4 (2010), 410-413.

[3] W. R. Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. In: Proc. of IEEE Conf., 1994, 305-309. DOI: 10.1109/IJCF.1994.375115.

[4] L. A. Zadeh. Fuzzy sets. Information and Control 8 (3) (1965), 338-353.

[5] I. Deli, M. Ali, and F. Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. In: International conference on advanced mechatronic systems (ICAMechS), IEEE, 2015, 249-254.

[6] Z. L. Yue. Approach to group decision making based on determining the weights of experts by using projection method. Applied Mathematical Modelling, 36(7) (2012), 2900-2910.

[7] G. Zhang, Y. Jing, H. Huang, and Y. Gao. Application of improved grey relational projection method to evaluate sustainable building envelope performance. Applied Energy, 87(2) (2010), 710-720.

[8] Z. Yue. Application of the projection method to determine weights of decision makers for group decision making. Scientia Iranica, 19(3) (2012), 872-878.

[9] Z. Yue, and Y. Jia. A direct projection-based group decision-making methodology with crisp values and interval data. Soft Computing (2015). DOI 10.1007/s00500-015-1953-5.

[10] Z. S. Xu, and, Q. Da. Projection method for uncertain multi-attribute decision making with preference information on alternatives. International Journal of Information & Decision Making, 3 (2004), 429-434.

[11] Z. Xu. On method for uncertain multiple attribute decision making problems with uncertain multiplicative preference information on alternatives. Fuzzy optimization and Decision Making, 4 (2005), 131-139.

[12] G. W. Wei. Decision-making based on projection for intuitionistic fuzzy multiple attributes. Chinese Journal of Management, 6(9) (2009), 1154-1156.

[13] X. Zhang, F. Jin, and P. Liu. A grey relational projection method for multi-attribute decision making based on intuitionistic trapezoidal fuzzy number. Applied Mathematical Modelling, 37(5) (2013), 3467-3477.

[14] S. Zeng, T. Baležentis, J. Chen, and G. Luo. A projection method for multiple attribute group decision making with intuitionistic fuzzy information. Informatica, 24(3) (2013), 485-503.

[15] Z. Xu, and H. Hu. Projection models for intuitionistic fuzzy multiple attribute decision making. International Journal of Technology & Decision Making, 9(2) (2010), 267–280.

[16] G. Sun. A group decision making method based on projection method and score function under IVIFS environment. British Journal of Mathematics & Computer Science, 9(1) (2015), 62-72.
DOI: 10.9734/BJMCS/2015/9549.

[17] C. Y. Tsao, and T. Y. Chen. A projection-based compromising method for multiple criteria decision analysis with interval-valued intuitionistic fuzzy information. Applied Soft Computing, 45 (2016), 207-223.

[18] J. Chen, and J. Ye. A projection model of neutrosophic numbers for multiple attribute decision making of claybrick selection. Neutrosophic Sets and Systems, 12 (2016), 139-142.

[19] J. Ye. Projection and bidirectional projection measures of single valued neutrosophic sets and their decision – making method for mechanical design scheme. Journal of Experimental & Theoretical Artificial Intelligence, (2016). DOI:10.1080/0952813X.2016.1259263.

[20] J. Ye. Bidirectional projection method for multiple attribute group decision making with neutrosophic numbers. Neural Computing and Applications, (2015). DOI: 10.1007/s00521-015-2123-5.

[21] J. Ye. Interval neutrosophic multiple attribute decision-making method with credibility information. International Journal of Fuzzy Systems, (2016). DOI: 10.1007/s40815-015-0122-4.

[22] P. P. Dey, S. Pramanik, and B. C. Giri. Neutrosophic soft multi-attribute decision making based on grey relational projection method. Neutrosophic Sets and Systems, 11 (2016), 98-106.

[23] P. P. Dey, S. Pramanik, and B. C. Giri. Extended projection based models for solving multiple attribute decision making problems with interval valued neutrosophic information. In: New Trends in Neutrosophic Theory and Applications, eds. F. Smarandache and S. Pramanik, Pons Editions, Brussels, 2016, 127-140.

[24] S. Pramanik, R. Roy, and T. K. Roy, Multi criteria decision making based on projection and bidirectional projection measures of rough neutrosophic sets, New Trends in Neutrosophic Theory and Applications-Vol-II. Pons Editions, Brussels, 2017. In Press.

[25] I. Deli, and Y. A. Subas. Multiple criteria decision making method on single valued bipolar neutrosophic set based on correlation coefficient similarity measure. In: International Conference on mathematics and mathematics education (ICMME-2016), Frat University, Elazg, Turkey, 2016.

[26] M. Şahin, I. Deli, and V. Uluçay. Jaccard vector similarity measure of bipolar neutrosophic set based on multi-criteria decision making. In: International Conference on Natural Science and Engineering (ICNASE'16), Killis, 2016.

[27] V. Uluçay, I. Deli, and M. Şahin. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. Neural Computing and Applications, (2016). DOI: 10.1007/s00521-016-2479-1.

[28] P. P. Dey, S. Pramanik, and B. C. Giri. TOPSIS for solving multi-attribute decision making problems under bipolar neutrosophic environment. In: New Trends in Neutrosophic Theory and Applications, eds. F. Smarandache and S. Pramanik, Pons Editions, Brussels, 2016, 65-77.

[29] Z. S. Xu. Theory method and applications for multiple attribute decision-making with uncertainty, Tsinghua University Press, Beijing, 2004.

Received: February 3, 2017. Accepted: February 21, 2017.