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SV NM/NM/c Queuing Model with Encouraged Arrival and Heterogeneous servers

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Abstract. This article shows the neutrosophic abstraction of M/M/c Queuing model (QM) with Encouraged Arrival (En. A) and heterogeneous servers. Here we derive the system's performance indicators of NM/NM/c QM with En. A and heterogeneous servers. The numerical example for the above model with its respective graphs are also depicted.

Keywords: NM/NM/c queuing model, Encouraged arrival, Heterogeneous service, Performance measures.

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1. **Introduction**

A Mathematical study of queues or waiting in lines was given by Erlang in the year 1909 is defined to be queuing which plays a significant role in almost all-fields. Finding the amount of customers waiting in line and system and the waiting lines of customers in both queue and system are the basic components of queuing theory defined by Shortle and Thompson [23]. Applying fuzzy logic to queuing theory makes solution for imprecise cases or uncertainty in the Fuzzy logic was firstly introduced by Zadeh in 1965 [24]. Rather than crisp queues, fuzzy queues are much more sensible in many real circumstances.

Neutrosophic Philosophy in queuing deals with situations in which the queue parameters are inaccurate. Neutrosophic logic which is a generalisation of fuzzy logic and intuitionistic fuzzy logic [17, 18, 19, 20] was introduced by Florentin Smarandache in the year 1995. This deals with indeterminacy data realistic and thereby gives understandable efficient outcomes [14, 15, 16, 22]. If the parameters of the queuing system are neutrosophic numbers, the system is said to be a neutrosophic queue.

The Neutrosophic set is employed to explain the uncertainty and indeterminacy in any information. This set is characterized by a truth 'T', indeterminacy 'I' and false 'F' membership functions, where $T, I, F \in]^{-}0, 1^{+}[$. There is no restriction on the sum and so $0^{-} \leq T + I + F \leq 3^{+}$. Neutrosophic set needs to be specified from a technical point of view. To this effect, we define certain set-theoretic operators on neutrosophic set, which in turn called as Single valued Neutrosophic set [21]. The membership functions for truth (T), indeterminacy (I), and falseness (F) define a single valued neutrosophic set, where $T, I, F \in [0, 1]$, which fulfills the following requirements: $0 \leq T + I + F \leq 3$. We examine single valued neutrosophic encouraged arrival and heterogeneous service rate. Encouraged arrival defines where the customers are drew towards profitable deals or offers. For instances, people rush towards ticket counters in railway station during vacation or special holidays. Also, Heterogeneous server defines the service occurs in a varied service rate. This in combination with Neutrosophy gives precise output.

Patro with Smarandache discussed more problems and solutions on Neutrosophic Statistical Distribution [13]. Bisher Zeina [11,12] studied Erlang Service Queuing Model and Event-Based Queuing Model on Neutrosophic basis in the year 2020. Deepa and Julia Rose Mary studied Heterogeneous Bulk tandem fluid multiple vacations queuing model for encouraged arrival with catastrophe[3]. Krishnakumar and Maheshwari [4] found the transient solution of M/M/2 queue with heterogeneous servers subject to catastrophes. Maissam Jdid, Smarandache and Said Browmi [5] inspected the assignment form of Product quality control using Neutrosophic logic. Jdid and Smarandache [6] explained the use of Neutrosophic Methods of Operations Research in the Management of Corporate work. Manas et. al. [7] found the solution of transportation issues in a neutrosophic setting.

Bisher Zeina [8,9] studied the M/M/1 Queue's Performance measures on fuzzy environment and Neutrosophic concept of M/M/1, M/M/c, M/M/1/b Queuing system utilizing interval-valued neutrosophic sets in the year 2020. Also in the year 2021, Mohamed Bisher Zeina [10] analysed Single Valued Neutrosophic M/M/1 Queue in Linguistic terms. Some operations of Single Valued Neutrosophic numbers mentioned by Bisher Zeina is utilized here. Bhupender Singh Som and Sunny Seth [1, 2] developed M/M/c queuing system with encouraged arrivals with N number of customers, Impatient customers and Retention of Impatient Customers queuing system in the year 2018.

Here, we consider a case where the customer's arrival and service are neutrosophic. We deduce the steady state equations and thereby deriving the system's performance indicators.

Some Procedures on Single Valued Neutrosophic

Here we provide some basic operations on Neutrosophic sets dealing with addition, subtraction, multiplication, division, power and scalar multiplication. This is one of the necessary factor in performing any operation between two different sets.

In this paper, These Neutrosophic processes are used to determine Neutrosophic queuing models' performance metrics, including as L_s, L_q, W_s and W_q respectively.

Suppose that we have two neutrosophic numbers given by $X = (t_1, i_1, f_1)$,
$Y = (t_2, i_2, f_2)$ where:

$$0 \leq t_1, i_1, f_1, t_2, i_2, f_2$$
$$0 \leq t_1 + i_1 + f_1 \leq 3 \text{ and } 0 \leq t_2 + i_2 + f_2 \leq 3$$

Then:

Neutrosophic Summation

$$X \oplus Y = (t_1 + t_2 - t_1t_2, i_1i_2, f_1f_2)$$

Neutrosophic Multiplication

$$X \otimes Y = (t_1t_2, i_1 + i_2 - i_1i_2, f_1 + f_2 - f_1f_2)$$

Neutrosophic Subtraction

$$X \ominus Y = \left(\frac{t_1-t_2}{1-t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right); t_2 \neq 1, i_2 \neq 0, f_2 \neq 0$$

Neutrosophic Division

$$\frac{X}{Y} = \left(\frac{t_1}{t_2}, \frac{i_1-i_2}{1-i_2}, \frac{f_1-f_2}{1-f_2}\right); t_2 \neq 0, i_2 \neq 1, f_2 \neq 1$$

Neutrosophic Scalar Multiplication

$$\lambda X = \left(1 - (1 - t_1)^{\lambda}, i_1^{\lambda}, f_1^{\lambda}\right); \lambda > 0$$

Neutrosophic Power

$$X^{\lambda} = \left(t_1^{\lambda}, 1 - (1 - i_1)^{\lambda}, 1 - (1 - f_1)^{\lambda}\right)$$

With the aid of the above operations, our model is explained.

2. (NM/NM/c):(FIFO/∞/∞) QM with En. A and Heterogeneous service - Model Description:

Here the customers arrive with a mean arrival rate λ_N and a maximum of c customers may be served simultaneously. The encouraged arrival rate=$\lambda_N(1 + \eta)$. The service rate per busy server is equal to $\sum_{i=1}^{n} \mu_{Ni}$. Also we get $\lambda_{Neff} = \lambda_N$. Neutrosophic Philosophy used here is to present precise information dealing with with uncertainty, untruth, and truth (i.,e) The arrival rate λ_N is assumed to be $\lambda_N = (T_\lambda, I_\lambda, F_\lambda)$ and the service rate $\sum_{i=1}^{n} \mu_N = (T_{\mu_i}, I_{\mu_i}, F_{\mu_i})$. If the customer base in the system , n equals or exceed c, the combined departure rate from the facility is $c\sum_{i=1}^{n}\mu_{Ni}$. Else, if $n < c$, the service rate $n\sum_{i=1}^{n}\mu_{Ni}$. Thus in terms of generalized model, $\lambda_n = \lambda$. Here $\lambda_{N_n} = \lambda_N\Big(1+\eta\Big)$ if $n \geq 0$

$$\sum_{i=1}^{n}\mu_{Nn_i} = \begin{cases} n\sum_{i=1}^{n}\mu_{Ni}; n < c \\ c\sum_{i=1}^{n}\mu_{Ni}; n \geq c \end{cases}$$

Also, the intensity ρ_N is given by,

$$\begin{aligned}\rho_N &= \frac{\lambda_N(1+\eta)}{c\sum_{i=1}^{n}\mu_N} \\ &= \frac{\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)}{c\sum_{i=1}^{n}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)} \\ &= \frac{1}{c}\left(\frac{T_{\lambda(1+\eta)}}{\sum_{i=1}^{n}T_{\mu_i}}, \frac{I_{\lambda(1+\eta)} - \sum_{i=1}^{n}I_{\mu_i}}{1-\sum_{i=1}^{n}I_{\mu_i}}, \frac{F_{\lambda(1+\eta)} - \sum_{i=1}^{n}F_{\mu_i}}{1-\sum_{i=1}^{n}F_{\mu_i}}\right)\end{aligned}$$

where $\rho_N = \frac{\lambda_N}{c\sum_{i=1}^{n}\mu_{Ni}} < 1$.

For the Neutrosophic M/M/c QM with En. A and Heterogeneous server, the steady state equation becomes,

Case I:

$$\frac{dNP_0(t)}{dt} = -\lambda_N\Big(1+\eta\Big)NP_0(t) + \mu_{N_1}NP_1(t); n = 0$$

$$\frac{dNP_0(t)}{dt} = -\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_0(t) + \Big(T_{\mu_1}, I_{\mu_1}, F_{\mu_1}\Big)NP_1(t); n = 0 \tag{1}$$

$$\begin{aligned}\frac{dNP_n(t)}{dt} &= -\left[\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big) + \sum_{i=1}^{n}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)\right]NP_n(t) \\ &+ \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_{n-1}(t) + \sum_{i=1}^{n+1}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{n+1}(t) \quad ; n = 1, 2, ...c-1.\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{dNP_c(t)}{dt} &= -\left[\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big) + \sum_{i=1}^{c}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)\right]NP_c(t) \\ &+ \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_{c-1}(t) + \sum_{i=1}^{c}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{c+1}(t) \quad ; n = c.\end{aligned} \tag{3}$$

$$\frac{dNP_n(t)}{dt} = -\left[\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right) + \sum_{i=1}^{n}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)\right]NP_n(t)$$
$$+\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)NP_{n-1}(t) + \sum_{i=1}^{c}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)NP_{n+1}(t) \quad ; n \geq c+1. \quad (4)$$

In steady state,

$lim_{t\to\infty}\ NP_n(t) = NP_n$

$lim_{t\to\infty}\ \frac{dNP_n(t)}{dt} = 0$

Then,

$$(1) \Rightarrow 0 = -\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)NP_0 + \left(T_{\mu_1}, I_{\mu_1}, F_{\mu_1}\right)NP_1; n = 0 \quad (5)$$

$$(2) \Rightarrow 0 = -\left[\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right) + \sum_{i=1}^{n}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)\right]NP_n$$
$$+\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)NP_{n-1} + \sum_{i=1}^{n+1}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)NP_{n+1} \quad ; n = 1, 2, ...c-1. \quad (6)$$

$$(3) \Rightarrow 0 = -\left[\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right) + \sum_{i=1}^{n}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)\right]NP_c$$
$$+\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)NP_{c-1} + \sum_{i=1}^{c}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)NP_{c+1} \quad ; n = c. \quad (7)$$

$$(4) \Rightarrow 0 = -\left[\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right) + \sum_{i=1}^{n}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)\right]NP_n$$
$$+\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)NP_{n-1} + \sum_{i=1}^{c}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)NP_{n+1} \quad ; n \geq c+1. \quad (8)$$

From (5), we have

$$NP_1 = \frac{\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)}{\left(T_{\mu_1}, I_{\mu_1}, F_{\mu_1}\right)}NP_0$$

$$NP_1 = \frac{\lambda_N\left(1+\eta\right)}{\mu_{N_1}}NP_0$$

From (6), we have

$$\sum_{i=1}^{n+1}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{n+1} = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_n + \sum_{i=1}^{n}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_n - \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_{n-1} \quad (9)$$

Put $n = 1$ in eq (9)

$$\sum_{i=1}^{2}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_2 = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_1 + \Big(T_{\mu_1}, I_{\mu_1}, F_{\mu_1}\Big)NP_1 - \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_0$$

$$NP_2 = \frac{\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)}{\sum_{i=1}^{2}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)}NP_1$$

$$NP_2 = \left(\frac{\lambda_N\Big(1+\eta\Big)}{\sum_{i=1}^{2}\mu_{Ni}}\right)NP_1$$

$$NP_2 = \left(\frac{\lambda_N\Big(1+\eta\Big)^2}{\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0$$

Putting $n = 2$ in (9), we get

$$\sum_{i=1}^{3}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_3 = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_2 + \sum_{i=1}^{2}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_2 - \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_1$$

$$\sum_{i=1}^{3}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_3 = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_2$$

$$NP_3 = \frac{\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)}{\sum_{i=1}^{3}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)} NP_2$$

$$NP_3 = \Big(\frac{\lambda_N\left(1+\eta\right)}{\sum_{i=1}^{3}\mu_{Ni}}\Big) NP_2$$

$$NP_3 = \Big(\frac{\lambda_N\left(1+\eta\right)^3}{\sum_{i=1}^{3}\mu_{Ni}.\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\Big) NP_0$$

Similarly,

$$NP_4 = \Big(\frac{\lambda_N\left(1+\eta\right)^4}{\sum_{i=1}^{4}\mu_{Ni}.\sum_{i=1}^{3}\mu_{Ni}.\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\Big) NP_0$$

$$NP_n = \Big(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\Big) NP_0 \tag{10}$$

If $n = c-1$, we get from eq (9)

$$\sum_{i=1}^{c}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right) NP_c = \left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right) NP_{c-1}$$

$$NP_c = \frac{\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)}{\sum_{i=1}^{c}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)} NP_{c-1}$$

$$NP_c = \Big(\frac{\lambda_N\left(1+\eta\right)}{\sum_{i=1}^{c}\mu_{Ni}}\Big) NP_{c-1}$$

$$NP_{c-1} = \Big(\frac{\lambda_N\left(1+\eta\right)^{c-1}}{\sum_{i=1}^{c-1}\mu_{Ni}.\sum_{i=1}^{c-2}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\Big) NP_0$$

$$NP_c = \Big(\frac{\lambda_N\left(1+\eta\right)^{c}}{\sum_{i=1}^{c}\mu_{Ni}.\sum_{i=1}^{c-1}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\Big) NP_0$$

Case II: When n=c in eq (9),

$$\sum_{i=1}^{c+1}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{c+1} = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_c$$

$$NP_{c+1} = \left(\frac{\lambda_N\Big(1+\eta\Big)}{\sum_{i=1}^{c+1}\mu_{Ni}}\right)NP_c$$

$$NP_{c+1} = \left(\frac{\lambda_N\Big(1+\eta\Big)^{c+1}}{\sum_{i=1}^{c+1}\mu_{Ni}.\sum_{i=1}^{c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0$$

Case III: When n=c+1 in eq(9),

$$\sum_{i=1}^{c+2}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{c+2} = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_{c+1}$$

$$NP_{c+2} = \left(\frac{\lambda_N\Big(1+\eta\Big)^{c+2}}{\sum_{i=1}^{c+2}\mu_{Ni}.\sum_{i=1}^{c+1}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0$$

$$NP_{c+(n-c)} = \left(\frac{\lambda_N\Big(1+\eta\Big)^{n}}{\sum_{i=1}^{c+(n-c)}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0 \tag{11}$$

Now, to find: NP_0

$$\sum_{n=0}^{\infty}NP_n = 1$$

$$\Rightarrow \sum_{n=0}^{c-1}NP_n + NP_c + \sum_{n=c+1}^{\infty}NP_n = 1$$

$$\Rightarrow \sum_{n=0}^{c-1}\left(\frac{\lambda_N\Big(1+\eta\Big)^{n}}{\sum_{i=1}^{n}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0 + \frac{\lambda_N\Big(1+\eta\Big)^{c}}{\sum_{i=1}^{c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}NP_0$$
$$+ \sum_{n=c+1}^{\infty}\left(\frac{\lambda_N\Big(1+\eta\Big)^{n}}{\sum_{i=1}^{n}\mu_{Ni}.....\sum_{i=1}^{c+1}\mu_{Ni}.\sum_{i=1}^{c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0 = 1$$

$$\Rightarrow NP_0[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}$$
$$+\sum_{n=c+1}^{\infty}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\sum_{i=1}^{c+1}\mu_{Ni}.\sum_{i=1}^{c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)]=1$$

$$\Rightarrow NP_0[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\mu_{N_1}}\right)+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}....\mu_{N_1}}$$
$$+\left(\frac{\lambda_N\left(1+\eta\right)^{c+1}}{\sum_{i=1}^{c+1}\mu_{Ni}.....\mu_{N_1}}\right)+\left(\frac{\lambda_N\left(1+\eta\right)^{c+2}}{\sum_{i=1}^{c+2}\mu_{Ni}.....\mu_{N_1}}\right)+........]=1$$

$$\Rightarrow NP_0[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\mu_{N_1}}\right)+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}....\mu_{N_1}}[1+\frac{\lambda_N\left(1+\eta\right)}{\mu_{N_1}}$$
$$+\frac{\lambda_N\left(1+\eta\right)^2}{\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}+........]]=1$$

$$\Rightarrow \quad NP_0 \quad = \quad \left[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\mu_{N_1}}\right) + \frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}....\mu_{N_1}}\left[1 + \sum_{n=1}^{\infty}\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{Nn_i}}\right]\right]^{-1}$$

Sub NP_0 in eq(10) and (11),

$$(10)\Rightarrow NP_n=\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{Ni}....\mu_{N_1}}[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\mu_{N_1}}\right)$$
$$+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}....\mu_{N_1}}[1+\sum_{n=1}^{\infty}\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{Nn_i}}]]^{-1}$$

$$(11)\Rightarrow NP_n=\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{Ni}....\sum_{i=1}^{c+1}\mu_{Ni}....\mu_{N_1}}[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\mu_{N_1}}\right)$$
$$+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}....\mu_{N_1}}[1+\sum_{n=1}^{\infty}\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{Nn_i}}]]]^{-1}$$

for $\frac{\rho_N}{c} < 1$ or $\frac{\lambda_N\left(1+\eta\right)}{c\sum_{i=1}^{n}\mu_{Ni}} < 1$,

$$\begin{aligned}
NL_q &= \sum_{n=c}^{\infty}\left(n-c\right)NP_n \qquad [\text{Take } n-c=k] \\
&= \sum_{k=0}^{\infty} kNP_{k+c} \\
&= \sum_{k=0}^{\infty} k\left(\frac{\lambda_N\left(1+\eta\right)^{k+c}}{\sum_{i=1}^{k+c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0 \\
&= \frac{\lambda_N\left(1+\eta\right)^{c}}{\sum_{i=1}^{c}\mu_{Ni}.....\mu_{N_1}}NP_0\left[\sum_{k=0}^{\infty} k\left(\frac{\lambda_N\left(1+\eta\right)^{k}}{\sum_{i=1}^{k}\mu_{Ni}.....\sum_{i=1}^{c+1}\mu_{Ni}}\right)\right] \\
&= \frac{\lambda_N\left(1+\eta\right)^{c}}{\sum_{i=1}^{c}\mu_{Ni}.....\mu_{N_1}}NP_0\left[\frac{\lambda_N\left(1+\eta\right)}{\mu_{N_1}} + 2\frac{\lambda_N\left(1+\eta\right)^{2}}{\sum_{i=1}^{2}\mu_{Ni}} + ...\right] \\
&= \frac{\lambda_N\left(1+\eta\right)^{c}}{\sum_{i=1}^{c}\mu_{Ni}.....\mu_{N_1}}NP_0\left[\sum_{n=1}^{\infty}\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{N_n}}\right]
\end{aligned}$$

Also, the system's performance indicators for a neutrosophic M/M/c QM with En. A and Heterogeneous server was given by,

As, $NL_q = \rho_N^c \sum_{n=1}^{\infty}\rho_{N_n}.NP_0$

$$\boxed{NL_q = \sum_{n=1}^{\infty}\rho_{N_n}.NP_c} \tag{12}$$

$NL_s = NL_q + \rho_N$

The neutrosophic form of ρ_N is already defined, using that we get

$$\boxed{Nl_s = NL_q + \frac{\lambda_N(1+\eta)}{\sum_{i=1}^{n}\mu_{N_i}}} \tag{13}$$

Similarly, we can also find NW_q and NW_s.

$$\boxed{NW_q = \frac{NL_q}{\lambda_N(1+\eta)}} \tag{14}$$

$$\boxed{NW_s = NW_q + \frac{1}{\sum_{i=1}^{n} \mu_{N_i}}} \tag{15}$$

3. Numerical Example

In this portion, we take some observed neutrosophic values for $\lambda_N(1+\eta)$ and $\sum_{i=1}^{n} \mu_{N_i}$. The values of $T_{\lambda(1+\eta)}$, $I_{\lambda(1+\eta)}$ and $F_{\lambda(1+\eta)}$ denoting the arrival rate $\lambda_N(1+\eta)$, and $\sum_{i=1}^{n}(T_{\mu_{N_i}}, I_{\mu_{N_i}}, F_{\mu_{N_i}})$ representing the service rate. Also, the observed data takes η to be 0.005 and 0.01 and considering the number of servers as c=1, 2 and 3. By letting the heterogeneous service rate $\sum_{i=1}^{n} \mu_{N_i}$ as $(T_{\mu_{N_1}}, I_{\mu_{N_1}}, F_{\mu_{N_1}}) = (0.5, 0.8, 0.7)$, $(T_{\mu_{N_2}}, I_{\mu_{N_2}}, F_{\mu_{N_2}}) = (0.6, 0.7, 0.6)$, $(T_{\mu_{N_3}}, I_{\mu_{N_3}}, F_{\mu_{N_3}}) = (0.7, 0.6, 0.5)$, $(T_{\mu_{N_4}}, I_{\mu_{N_4}}, F_{\mu_{N_4}}) = (0.8, 0.5, 0.4)$.

$(T_{\mu_{N_1}}, I_{\mu_{N_1}}, F_{\mu_{N_1}})$ values can be viewed as first varied service rate of validity, ambiguity, and fake membership values. the system's performance indicators were calculated using (12), (13), (14) and (15) respectively. By considering all the above data and by using equations (12) and (13), we obtain NL_s as $(0.123, 0.8032, 0.6493)$ which means the anticipated size of customers in the system to be truth, indeterminate and false. We apply all the observed values in the concerned system measures of performance and depict a line graph. Here we use the operations of Single valued neutrosophic numbers. The resulted system measures of performance provides the degree to which the system's reported client count, queue length, and wait time are true, unreliable, or false respectively. The evaluated values are tabulated below:

Various Performance measures by varying λ_N with respect to c and $\eta = 0.005$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.0237, 0.9135, 0.8317)$	$(0.123, 0.8032, 0.6493)$	$(0.2358, 0.1393, 0.1623)$	$(1.0093, -0.0281, -0.0149)$
$(0.2, 0.8, 0.9)$	$(0.1101, -0.027, 0.9544)$	$(0.291, -0.0205, 0.8497)$	$(0.548, -4.0916, -0.3065)$	$(1.0055, 0.8261, 0.0281)$
$(0.3, 0.7, 0.6)$	$(0.2847, 3.0318, -6.7481)$	$(0.5028, 1.9343, -3.7904)$	$(0.9452, 7.7457, -18.2979)$	$(1.0007, -1.5639, 1.6779)$
$(0.4, 0.6, 0.7)$	$(0.5923, 2.0323, -0.0168)$	$(0.758, 1.0515, -0.0113)$	$(1.4752, 3.5711, -2.3758)$	$(0.9942, -0.721, 0.2179)$

TABLE 1. When c=1, $\eta = 0.005$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.003, 0.9786, 0.8984)$	$(0.1044, 0.8604, 0.7014)$	$(0.0299, 0.7871, 0.4943)$	$(1.0118, -0.1589, -0.0453)$
$(0.2, 0.8, 0.9)$	$(0.0287, 0.5347, 0.9918)$	$(0.2262, 0.8876, 0.9991)$	$(0.1429, -1.3161, 0.9184)$	$(1.0105, 0.2657, -0.0842)$
$(0.3, 0.7, 0.6)$	$(0.1146, -0.3405, -0.2547)$	$(0.3846, -0.2172 +0.1431)$	$(0.3805, -3.4505, -2.125)$	$(1.0076, 0.6967, 0.1949)$
$(0.4, 0.6, 0.7)$	$(0.3092, 0.3442, 0.4747)$	$(0.5899, 0.1781, 0.3186)$	$(0.7701, -0.6334, -0.744)$	$(1.0028, 0.1279, 0.0682)$

TABLE 2. When c=2, $\eta = 0.005$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.0003, 0.9968, 0.4369)$	$(0.102, 0.8764, 0.3411)$	$(0.003, 0.9682, -1.8029)$	$(1.0122, -0.1955, 0.1653)$
$(0.2, 0.8, 0.9)$	$(0.0062, 0.8595, 0.999)$	$(0.2082, 0.6519, 0.8894)$	$(0.0309, 0.3007, 0.9901)$	$(1.0118, -0.0607, -0.0908)$
$(0.3, 0.7, 0.6)$	$(0.0373, 0.3749, 0.4626)$	$(0.3308, 0.2392, 0.2598)$	$(0.1238, -1.0754, -0.3385)$	$(1.0107, 0.2171, 0.031)$
$(0.4, 0.6, 0.7)$	$(0.1344, -1.7047, 0.8)$	$(0.4862, -0.882, 0.537)$	$(0.3348, -5.7365, 0.336)$	$(1.0081, 1.1582, -0.0308)$

TABLE 3. When c=3, $\eta = 0.005$

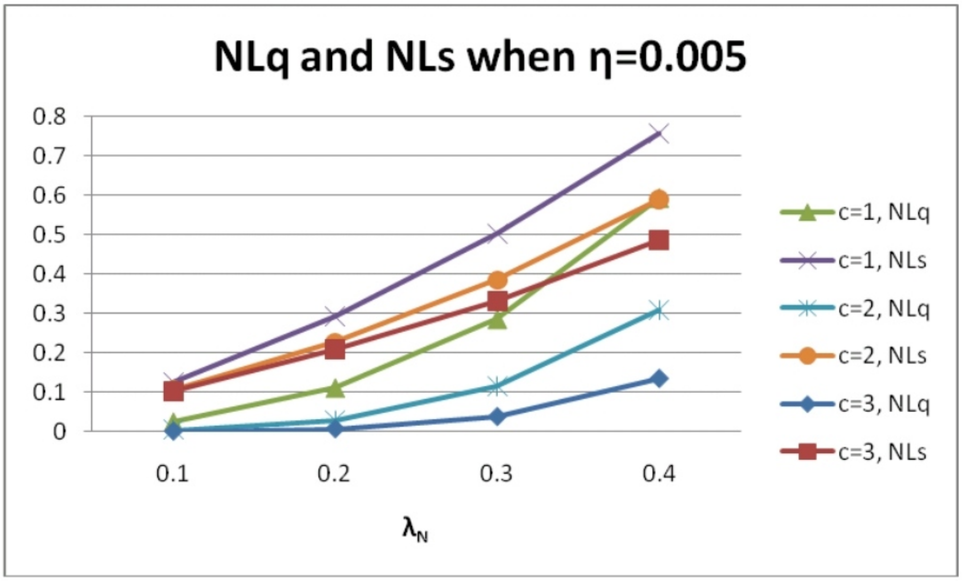

FIGURE 1. NL_q and NL_s when $\eta = 0.005$

In the above line graph, NL_q and NL_s values are plotted with the number of servers as 1, 2 and 3. Here, we consider a parameter η which is considered as encouraged arrival parameter and it takes the value 0.005 and the heterogeneous service rate is taken as mentioned above. At c=1, NL_q and NL_s increases steadily with the increase in λ_N. When c=2, NL_q increases measurably and so NL_s.

Also, when c=3, both NL_q and NL_s increases with λ_N. Now we find that the amount of customers waiting in line and system increases with steady increase of λ_N it demonstrates that it is adequate. for the customers to get served with 2 heterogeneous services. As there is only minute differences between c=2 and c=3, customers can get served with 2 servers as it is unnecessary to increase the server which lead to the loss to the service provider.

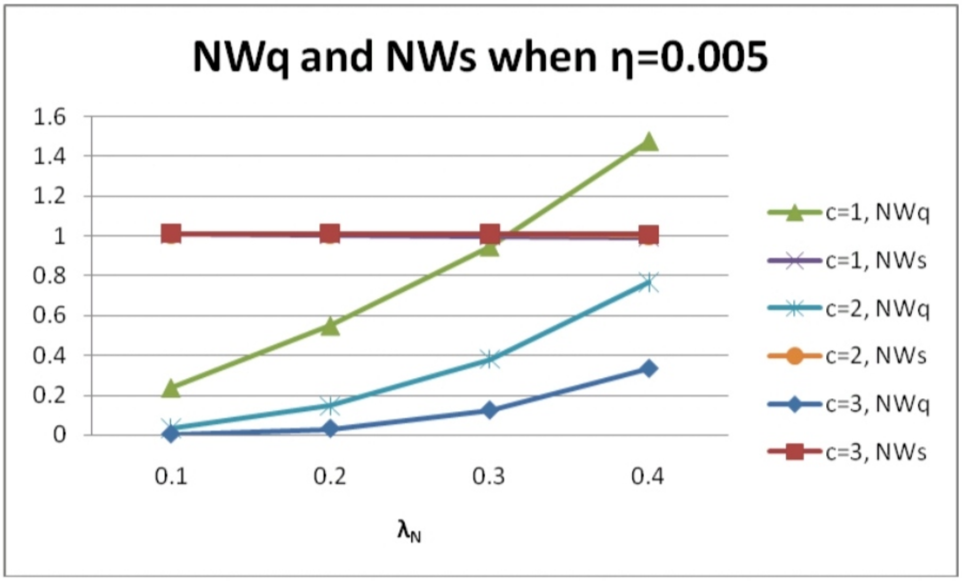

FIGURE 2. NW_q and NW_s when $\eta = 0.005$

In Fig 2, NW_q and NW_s values are plotted with the number of servers to be taken 1, 2 and 3. Here, we consider a parameter η which is considered as encouraged arrival parameter and it takes the value 0.005. At c=1, 2 and 3, the clients' wait times in the system remains almost the same throughout the λ_N. And the waiting time of customers in the queue gradually increasing with increase in λ_N. As the NW_q at c=2 increases step by step, it is sufficient for the customers to get the effective service. Increasing the server is inessential as it leads to loss.

Various Performance measures by varying λ_N with respect to c and $\eta = 0.01$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.0231, 0.9171, 0.7455)$	$(0.1228, 0.8059, 0.5813)$	$(0.2289, 0.1784, -0.2612)$	$(1.0094, -0.036, 0.024)$
$(0.2, 0.8, 0.9)$	$(0.1112, -0.0543, 0.954)$	$(0.2928, -0.0411, 0.849)$	$(0.551, -4.2245, 0.5441)$	$(1.0055, 0.8529, -0.00499)$
$(0.3, 0.7, 0.6)$	$(0.2877, 2.9952, -7.2821)$	$(0.5058, 1.9062, -4.0773)$	$(0.9511, 7.5957, -19.546)$	$(1.0006, -1.5336, 1.7924)$
$(0.4, 0.6, 0.7)$	$(0.581, 2.0263, -0.0356)$	$(0.752, 1.0446, -0.0239)$	$(1.4413, 3.546, -2.4235)$	$(0.9946, -0.7159, 0.2223)$

Table 4. When c=1, $\eta = 0.01$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.0031, 0.9782, 0.8969)$	$(0.1049, 0.8595, 0.6993)$	$(0.0307, 0.784, 0.4891)$	$(1.0118, -0.1583, -0.0449)$
$(0.2, 0.8, 0.9)$	$(0.0292, 0.5246, 0.9917)$	$(0.2275, 0.3974, 0.8825)$	$(0.1447, -1.3558, 0.9177)$	$(1.0104, 0.2737, -0.084)$
$(0.3, 0.7, 0.6)$	$(0.0505, 2.9182, -7.1678)$	$(0.3412, 1.8571, -4.0133)$	$(0.1669, 0.7341, -19.2625)$	$(1.0102, -0.1482, 1.7664)$
$(0.4, 0.6, 0.7)$	$(0.3135, 0.3515, 0.4656)$	$(0.5936, 0.1812, 0.3119)$	$(0.7777, -0.6088, -0.7666)$	$(1.0027, 0.1229, 0.0703)$

Table 5. When c=2, $\eta = 0.01$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.0003, 0.9967, 0.9731)$	$(0.1024, 0.8758, 0.7587)$	$(0.003, 0.9673, -0.1957)$	$(1.0122, -0.1953, 0.018)$
$(0.2, 0.8, 0.9)$	$(0.0063, 0.856, 0.999)$	$(0.2093, 0.6484, 0.889)$	$(0.0312, 0.2864, 0.9901)$	$(1.0118, -0.0578, -0.0908)$
$(0.3, 0.7, 0.6)$	$(0.0165, 1.9074, -3.0607)$	$(0.3177, 1.2139, -1.7137)$	$(0.0546, 3.9997, -9.0737)$	$(1.0115, -0.8075, 0.8321)$
$(0.4, 0.6, 0.7)$	$(0.1369, -0.2329, 0.7905)$	$(0.4891, -0.1201, 0.5295)$	$(0.3396, -2.0586, 0.3074)$	$(1.0081, 0.4156, -0.0282)$

Table 6. When c=3, $\eta = 0.01$

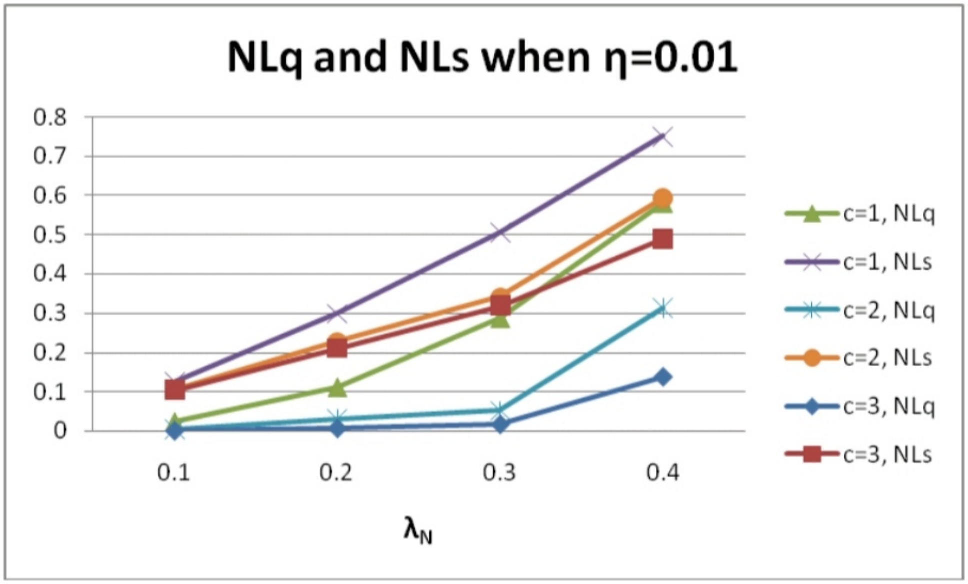

FIGURE 3. NL_q and NL_s when $\eta = 0.01$

In the above line graph, NL_q and NL_s values are plotted with the number of servers to be taken as 1, 2 and 3. Here, we consider a parameter η which is considered as encouraged arrival parameter and it takes the value 0.01 and the heterogeneous service rate is taken as mentioned above. At c=1, NL_q and NL_s gradually increases with increase in λ_N. Also when c=2, NL_q and NL_s increases steadily.

When c=3, both NL_q and NL_s increases slowly with increase in λ_N. To get effective service, a number of 2 servers are enough as the amount of customers waiting in line and system increases with the increasing rate of λ_N with the heterogeneous service rather than increasing the servers as there is slight difference between c=2 and c=3. When the number of servers is increased, it may steer to loss for the service provider.

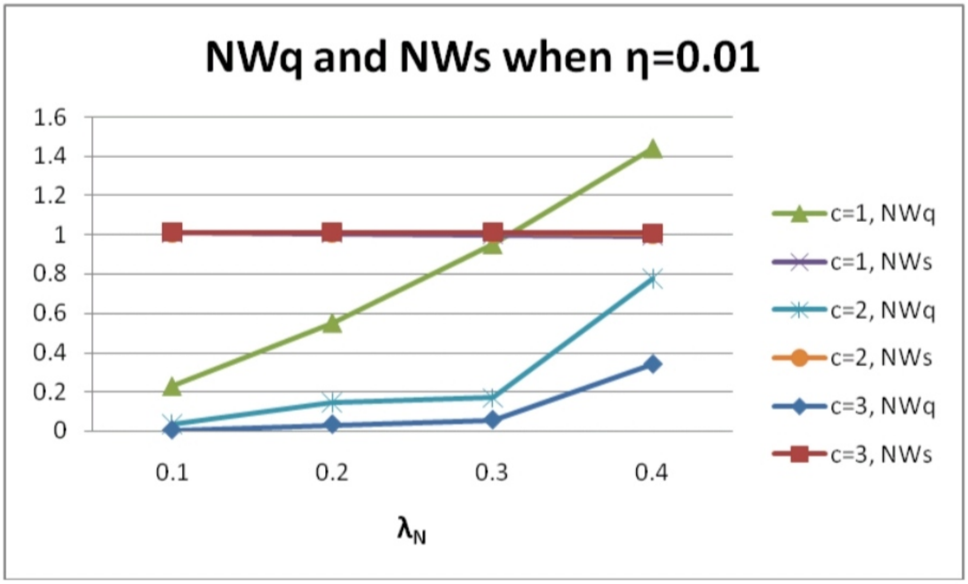

FIGURE 4. NW_q and NW_s when $\eta = 0.01$

In Fig 4, NW_q and NW_s values are plotted with the number of servers are taken as 1, 2 and 3. Here, we consider a parameter η which is considered as encouraged arrival parameter and it takes the value 0.01. At c=1, 2 and 3, the clients' wait times in the system remains unaltered throughout λ_N. And the waiting time of customers in the queue increases steadily with the increase in λ_N. NW_q at c=2, raises step by step which is effectual to get served besides increasing the server. It may give loss for the service provider.

In the tables above, the values of NW_s and NW_q are not negatives, its their membership values, and it is a single valued neutrosophic off numbers. Here the Neutrosophic M/M/c QM with En. A and heterogeneous service are calculated. When $\eta = 0.005$ and c=1, 2 and 3, the number of customers in the system and queue increases gradually with the increasing λ_N. Also the clients' wait times in the system shows some difference when λ_N increases and waiting time of customers in the queue steadily increases. Also, when $\eta = 0.01$ and c=1, 2 and 3, the number of customers in the system and queue increases steadily with the increasing λ_N. Similarly, the clients' wait times in the system shows slight variation when λ_N increases and the customers in line are waiting longer and longer. As a result of the outcome and line graph, it is easy to suggest that it is sufficient to provide 2 servers for the effective service of customers. Increasing the servers may give loss for the service provider when the arrival rate increases.

4. Result

For single valued neutrosophic M/M/c with encouraged arrival and heterogeneous service rate queuing system, the performance measures with respect to varied arrival and service rates are calculated. Numerical examples of the model are tabulated. In the example, we only know their degrees of membership, so that we assume single valued neutrosophic number to calculate it. Here in the neutrosophic model, both the arrival and service rates depends on Truth, Indeterminacy and False membership functions. They are not considered as valued, but its a membership functions. Also, observed numerical values are examined with suitable line graph. With the obtained result and line graph, it is easy to suggest that it is sufficient to provide **2 servers** for the effective service of customers since they provide effective services and thereby yielding maximum profit. Increasing the servers may give loss for the service provider when the arrival rate increases. At heterogeneous services (i.e) in a varied rate, there may occur a sudden destruction and a customer may balk from the system. From this analysis, we find that even in these conditions, the service provider can maintain the same servers, and there is no need to increase the servers as per our model.

5. Conclusion

M/M/c queuing model with Neutrosophic abstraction with encouraged arrivals and heterogeneous service are depicted here. The description studied here shows that the neutrosophic M/M/c QM with En. A and heterogeneous services can be dealt with uncertain and imprecise cases. It can be further developed with other classical queuing models, also it can be extended to the three types of neutrosophic sets which are under, off, and over respectively. Also the system can be analyzed with different other situations.

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