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The Neutrosophic Regular and Most Important Properties that Bind Neutrosophic Ring Elements

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Abstract: This research has broadened the definition of the neutrosophic regular in neutrosophic rings, similar to what is known in classical rings. We have studied the properties of neutrosophic regular elements and the most important properties that link them to the neutrosophic (zero divisor, idempotent, unit and nilpotent) elements in neutrosophic rings, and we reached several important results linking these elements to each other, as some of them are different from what is known in classical rings. The most important of which are:
If $R(\mathrm{I})$ is a neutrosophic right (left) strongly regular neutrosophic ring, then $NSNil_{R(\mathrm{I})} = \{0\}$. In any neutrosophic field $R(\mathrm{I})$ is achieved: $R(\mathrm{I}) = NReg_{R(I)}$, although there are some elements that are not neutrosophic unit, $NU_{R(I)} \cap NZ_{R(I)} = \{bI\ , b \in U_R\}$, and $NU_{R(I)} \cap NId_{R(I)} = \{1, I\}$.

Keywords: Neutrosophic ring, Neutrosophic field, Neutrosophic regular, strongly regular, unit, simple nilpotent, zero divisor.

1. Introduction

The concept of the regular element in the rings appeared in the hands of researcher J.Von Neumann [1]. Regular rings and their properties have been extensively examined by many authors and researchers [2-3-4-5]. Neutosophy is a comprehensive perspective on intuitionistic fuzzy logic that represents a fresh expansion of fuzzy ideas. This method has an intriguing influence on applied science [6-7-8-9]. More neutrosophical applications in many areas may be found in [10-11-12-13-14].

Pure mathematics has various applications, including neutrosophic groups [15], metric spaces [16], and rings [17-18-19-20]. In 1980, Smarandache first introduced the neutrosophic theory.
This idea has created a new notion in algebraic structures, known as neutrosophic structures.

Kandasamy and her colleague, Smarandache, introduced the idea of neutrosophic algebraic structures in [21]. Vasantha Kandasamy and Smarandache introduced the notion of neutrosophic zero divisors, idempotent, and unit elements in neutrosophic rings and fields [22].
Agboola, Akinola, and Oyebolain conducted further research on neutrosophic rings [23-24].
Chalapathi and Kiran examined the enumeration of neutrosophic units in neutrosophic rings and fields [25]. A novel multiplication operation based on neutrosophic theory has been developed to enhance the algebraic structures of classical rings and enable easier derivation of elementary structural theorems for indeterminate situations. Therefore, when a real-world problem involves indeterminacy, the use of neutrosophic algebraic theory is necessary.

This paper explores the concepts of neutrosophic regularity in neutrosophic rings using relevant examples of key points. In addition, we analyzed the properties of specific elements of the neutrosophic rings to determine the properties that bind these elements together.

2. Definitions and notations

Given that researchers interested in the subject are well aware of classical rings and other fields, in this section, we provide various definitions and key findings of neutrosophic rings. For those interested in delving deeper into the topic of neutrosophic rings, we recommend referring to references.

Definition.2.1 [22] Assume that we have a ring denoted by R. The set $\langle \mathrm{R} \cup I \rangle = \{a + bI\,; a, b \in R \text{ and } I^2 = I\}$ is called the neutrosophic ring. $\langle \mathrm{R} \cup I \rangle$ is referred to as a neutrosophic field when R is a field.

Properties.2.2 [19-22]

1. If R is a unity ring, then $\langle R \cup I \rangle$ is a unity neutrosophic ring with neutrosophic unity I.
2. $I^n = I$ for each $n \in \mathbb{Z}^+$
3. $aI = Ia\ \forall a \in R$.
4. $0I = 0\ ,\ I + I + \cdots + I = nI$

Definition.2.3 [22] If $\langle \mathrm{R} \cup I \rangle$ is a neutrosophic ring, then $x \in \langle \mathrm{R} \cup I \rangle$, where $x \neq 0$ is considered a neutrosophic zero divisor if found $y \neq 0$ of $\langle \mathrm{R} \cup I \rangle$, such that $xy = yx = 0$.

Definition.2.4 [22-23-25] Assume that $\langle \mathrm{R} \cup I \rangle$ is a neutrosophic ring, then

1. If $e \in \langle \mathrm{R} \cup I \rangle$ satisfies $e^2 = e$, it is considered to be a neutrosophic idempotent.
2. Any element x $\in \langle \mathrm{R} \cup I \rangle$ is considered neutrosophic nilpotent if it satisfies the condition $x^n = 0$, where $n \in \mathbb{Z}^+$.
3. Any element x $\in \langle \mathrm{R} \cup I \rangle$ is considered a unit if there is y in $\langle \mathrm{R} \cup I \rangle$, where, $xy = yx = 1$.
4. Any element x $\in \langle \mathrm{R} \cup I \rangle$ is considered a neutrosophic unit if there is y in $\langle \mathrm{R} \cup I \rangle$, where $xy = yx =$I.

Theorem.2.5 [19] If $\langle K \cup\ I \rangle$ is a neutrosophic field, then each element in the form of $a + bI$ is unit $\Leftrightarrow a \neq 0\ and\ a \neq -b$.

To represent the neutrosophic (field) ring, we use the symbol $R(I)$ instead of $\langle R \cup\ I \rangle$.

3. Results

We present the idea of regularity and its effects on the components of neutrosophic rings in this section, and we explain the most important properties that link the elements of the neutrosophic ring to each other.

In a neutrosophic ring $R(I)$, we indicate by $NZ_{R(I)}$ the collection of neutrosophic zero divisor elements, $NId_{R(I)}$ the collection of neutrosophic idempotent elements, $U_{R(I)} = \{x \in R(I);\ \exists y \in R(I); xy = yx = 1\}$ the collection of unit elements, $NU_{R(I)} = \{x \in R(I);\ \exists y \in R(I); xy = yx = I\}$ the collection of neutrosophic unit elements, $NNil_{R(I)}$ the collection of neutrosophic nilpotent elements, $NReg_{R(I)}$ the collection of neutrosophic regular elements. In addition, in classical ring R we are going indicate by Z_R the set of zero divisors, Id_R is the collection of idempotent, U_R is the collection of unit, Nil_R is the collection of nilpotent, Reg_R is the collection of regular. We will indicate by $\mathbb{R}$ to the collection of real numbers, $\mathbb{Z}$ is the collection of integers.

Definition.3.1 Assume that $R(\mathrm{I})$ is a neutrosophic ring and let x be its element. We can say that in $R(\mathrm{I})$, if there is an element y where $x = xyx$, then x is a neutrosophic regular element. We call $R(\mathrm{I})$ a regular neutrosophic ring if $NReg_{R(\mathrm{I})} = R(\mathrm{I})$.

Example.3.2 The element $3 + 4I \in NReg_{\mathbb{Z}_7(I)}$ because it achieves $3 + 4I = (3 + 4I)(5 + 6I)(3 + 4I)$.

Definition.3.3 Assume that $R(\mathrm{I})$ is a neutrosophic ring. $x \in R(\mathrm{I})$ is a neutrosophic right (left) neutrosophic strongly regular if found y in $R(\mathrm{I})$ where $x = yxx$ $(x = xxy)$. If each element in R(I) is a right (left) neutrosophic regular element, we call $R(\mathrm{I})$ a right (left) neutrosophic strongly regular. If $R(\mathrm{I})$ is a right and left neutrosophic strongly regular, we call it a neutrosophic strongly regular. It is clear that $R(\mathrm{I})$ is a neutrosophic strongly regular, when $R(\mathrm{I})$ is a commutative neutrosophic regular.

Definition.3.4 In $R(\mathrm{I})$, we define the set of neutrosophic simple nilpotent elements, which we denote by $NSNil_{R(\mathrm{I})}$ as follows $NSNil_{R(\mathrm{I})} = \{x \in R(\mathrm{I}); x^2 = 0\}$. clear that $NSNil_{R(\mathrm{I})} \subseteq NNil_{R(\mathrm{I})}$

Example.3.5 In $\mathbb{Z}_4(\mathrm{I})$, we have $(2 + 2I)^2 = 0 \Rightarrow 2 + 2I \in NSNil_{\mathbb{Z}_4(\mathrm{I})}$.

Corollary.3.6 Assume that $R(\mathrm{I})$ is a neutrosophic ring.

1. If $x \in NReg_{R(\mathrm{I})}$, then there is z of $R(\mathrm{I})$ where $x = xzx$ and there is $y = zxz$ of $R(\mathrm{I})$ where $x = xyx\ and\ y = yxy$.
2. If $x \in NReg_{R(\mathrm{I})}$, then there is $y \in R(\mathrm{I})$ where $x = xyx$. Now if we put $f = xy$ and $g = yx$, then f and g are neutrosophic idempotent. (It can be easily verified that $f^2 = f$ and $g^2 = g$).

Example.3.7 In $\mathbb{Z}_{11}(\mathrm{I})$, we have $3 + 8I \in NReg_{\mathbb{Z}_{11}(I)}$; $3 + 8I = (3 + 8I)4(3 + 8I)$. If we put $f = (3 + 8I)4 = 1 + 10I$ and $g = 4(3 + 8I) = 1 + 10I$, then we note that $f^2 = g^2 = (1 + 10I)^2 = 1 + 10I = f = g$.

Example.3.8 In $\mathbb{Z}_7(\mathrm{I})$, we have $5 + 6I \in NReg_{\mathbb{Z}_7(I)}$; $5 + 6I = (5 + 6I)(3 + 6I)(5 + 6I)$.
If we put $f = (5 + 6I)(3 + 6I) = 1$ and $g = (3 + 6I)(5 + 6I) = 1$, we note that $f^2 = g^2 = 1 = f = g$.

Theorem.3.9 If $R(\mathrm{I})$ is an infinite (finite) neutrosophic field, then every element of the form aI ; $a \neq 0$ it has an infinite (finite) number of neutrosophic inverses of the shape $b + cI \in R(\mathrm{I})$ where $b \neq -c$.

Proof.

We have $aI(b + cI) = I \Rightarrow (ab + ac)I = I \Rightarrow a(b + c) = 1$. Since $a \neq 0$ so $b + c = a^{-1}$. Therefore $\forall b \in R$, $c = a^{-1} - b \in R$.

Corollary.3.10 In any neutrosophic field is achieved. Every element of the form $a + bI$, where, $a \neq$

$-b$ it has a neutrosophic inverse dI such that $c=\frac{1}{a+b}$. Since $a+bI \in U_{R(I)}$ according to the theorem.2.5, therefore $U_{R(I)} \subset NU_{R(I)}$.

Example.3.11

1. In the neutrosophic field $\mathbb{R}(I)$, the neutrosophic inverse of $3+5I$ is $\frac{1}{8}I$.

2. In the neutrosophic field $\mathbb{R}(I)$, the neutrosophic inverse of $aI=3I$ is $b+cI$; $\forall b \in \mathbb{R}$, $c=a^{-1}-b$.

$$\text{Suppose that,} \quad b=3 \Rightarrow c=\frac{-8}{3} \quad ; \quad 3I\left(3-\frac{8}{3}I\right)=I$$

$$\text{Suppose that,} \quad b=\sqrt{2} \Rightarrow c=\frac{1}{3}-\sqrt{2}=\frac{1-3\sqrt{2}}{3} \quad ; \quad 3I\left(\sqrt{2}+(\frac{1-3\sqrt{2}}{3})I\right)=I$$

3. In the neutrosophic field $\mathbb{Z}_3(I)$, the element $2I$ has a finite number of neutrosophic inverses of the shape $b+cI$; $\forall b \in \mathbb{Z}_3, c=2^{-1}-b$.

If $b=0$ *then* $c=2^{-1}-0=2,$ *thus* $b+cI=2I$

If $b=1$ *then* $c=2^{-1}-1=1,$ *thus* $b+cI=1+I$

If $b=2$ *then* $c=2^{-1}-2=2+1=0,$ *thus* $b+cI=2$

Theorem.3.12 Let $R(\mathrm{I})$ be unity. If $x \neq 0$ has a right inverse (right neutrosophic inverse) and let it be y and has a left inverse (left neutrosophic inverse) and let it be z then we can distinguish the following cases:

$$\text{If} \quad x.y=1 \ \textit{and} \ z.x=1 \ \textit{then} \ y=z$$
$$\text{If} \quad x.y=I \ \textit{and} \ z.x=1 \ \textit{then} \ y=zI$$
$$\text{If} \quad x.y=1 \ \textit{and} \ z.x=I \ \textit{then} \ z=yI$$
$$\text{If} \quad x.y=I \ \textit{and} \ z.x=I \ \textit{then} \ yI=zI$$

Proof.

In the first case, it is clear.

In the rest of the cases

$$\text{If} \quad x.y=I \ \textit{and} \ z.x=1, \text{then we note} \quad y=1.y=(zx).y=z.(xy)=zI$$
$$\text{If} \quad x.y=1 \ \textit{and} \ z.x=I, \text{then we note} \quad z=z.1=z.(xy)=(zx).y=Iy=yI$$
$$\text{If} \quad x.y=I \ \textit{and} \ z.x=I, \text{then we note} \quad zI=z.(xy)=(zx).y=Iy=yI$$

Example.3.13

1. In $\mathbb{Z}_8(\mathrm{I})$, the element $4+I$ is a neutrosophic unit and achieves $(4+I)(4+I)=I$ and $5I(4+I)=I$. And we note $(5I)I=(4+I)I=5I$

2. In the neutrosophic ring $\mathbb{R}\,(I)$ we have $(3+5I)\left(\frac{1}{3}-\frac{5}{24}I\right)=1$ and also $(3+5I)\left(\frac{1}{8}I\right)=I$. And we note $\left(\frac{1}{3}-\frac{5}{24}I\right)I=\frac{1}{8}I$.

3. In the neutrosophic ring $\mathbb{Z}_8(I)$, the element $4+3I$ is a neutrosophic unit and achieves $(4+3I)7I=I$ and also $(4+3I)(4+3I)=I$ And we note $(4+3I)I=(7I)I=7I$.

Theorem.3.14 In any neutrosophic field $R(\mathrm{I})$ is achieved $NU_{R(I)} \cap NZ_{R(I)}=\{bI\,, b \in U_R\}$.

Proof. We have a first $b \in U_R \Rightarrow bI \in NU_{R(I)}$

and also $bI(b^{-1}-b^{-1}I)=(b^{-1}-b^{-1})bI=0 \Rightarrow bI \in NZ_{R(I)}$, therefore $bI \in NU_{R(I)} \cap NZ_{R(I)} \neq \emptyset$.

On the other hand, $\forall x = a + bI \in NU_{R(I)} \cap NZ_{R(I)}$ where $a \neq 0$ *or* $b \neq 0$ thus $a + bI \in NU_{R(I)}$ *and* $a + bI \in NZ_{R(I)}$. Since $a + bI \in NU_{R(I)}$ so $a \neq -b$ and since $a + bI \in NZ_{R(I)}$, there is $c + dI \in R(I)$ where $c \neq 0$ *or* $d \neq 0$ such that $(a + bI)(c + dI) = 0$. In fact $a = 0$ *and* $b \neq 0$ because if we suppose $a \neq 0$, then we distinguish two cases, *if* $a \neq 0$ and $b = 0$ then $a(c + dI) = 0$ thus $ac + adI = 0$, since $a \neq 0$ hence $c = d = 0$ and this is contradictory to that $x = a + bI \in NZ_{R(I)}$.

Now if $a \neq 0$ and $b \neq 0$ then $(a + bI)(c + dI) = 0 \Rightarrow ac + (ad + bc + bd)I = 0 \Rightarrow ac = 0$ and $ad + bc + bd = 0 \underset{a\neq 0}{\Longrightarrow} c = 0$ and $(a + b)d = 0$. Since $a \neq -b$ so $d = 0$. This is contradictory to that $x \in NZ_{R(I)}$. Therefore, $x = bI$; $b \neq 0$.

Corollary.3.15 In general, it is not necessarily only that $NU_{R(I)} \cap NZ_{R(I)} = \{bI , b \in U_R\}$, when R(I) is a unity neutrosophic ring.

Example.3.16 In $\mathbb{Z}_8$(I), the element $(4 + I) \in NU_{\mathbb{Z}_8(\mathrm{I})} \cap NZ_{\mathbb{Z}_8(\mathrm{I})}$, where it achieves $5I(4 + I) = I$ *and* $(4 + I)(4 + 4I) = 0$.

Theorem.3.17 In any neutrosophic field R(I) is achieved $\mathrm{NU}_{R(I)} \cap \mathrm{NId}_{R(I)} = \{1, \mathrm{I}\}$.

Proof.

We note $1, I \in NU_{R(I)}$ and $1, I \in NId_{R(I)}$, thus $1, I \in NU_{R(I)} \cap NId_{R(I)}$.

At other hand, $\forall x = a + bI \in NU_{R(I)} \cap NId_{R(I)}$ where $(a \neq 0$ *or* $b \neq 0)$ and $a \neq -b$.

$$\Rightarrow (\exists x^{-1} \in R(I); x^{-1}\mathrm{x} = \mathrm{x}x^{-1} = 1 \text{ or } \mathrm{I}) \text{ and } x^2 = \mathrm{x}$$

Now if $x^{-1}\mathrm{x} = 1$ and $x^2 = x$, then $x^{-1}\mathrm{x} = x^{-1}x^2 = 1$. Subsequently $x = 1$.

If $x^{-1}\mathrm{x} = \mathrm{I}, x^2 = x$, then $x^{-1}x^2 = x^{-1}\mathrm{x} \Rightarrow (x^{-1}\mathrm{x})\mathrm{x} = \mathrm{I} \Rightarrow Ix = I$

Since $I(a + bI) = I$, so $a + b = 1$

We have $x^2 = \mathrm{x} \Rightarrow (a + bI)^2 = a + bI \Rightarrow a^2 + (2ab + b^2)I = a + bI \Rightarrow a^2 = a$ and $2ab + b^2 = b$.

Now we have $a^2 = a$ and $2ab + b^2 = b$ and $a + b = 1$.

If $a \neq 0$, we have $a^2 = a$ so $a(a - 1) = 0$ *thus* $a - 1 = 0$. Therefore, $a = 1$. And since $a + b = 1$ *thus* $b = 0$. Therefore $x = 1$.

Now if $b \neq 0$, we have $2ab + b^2 = b$. Since $a = 1 - b$ so $2(1 - b)b + b^2 = b$

$\Rightarrow 2b - 2b^2 - b = 0 \Rightarrow b(1 - b) = 0 \Rightarrow 1 - b = 0 \Rightarrow b = 1$. Since $a + b = 1$, so $a = 0$. Therefore, $x = I$. So $\mathrm{NU}_{R(I)} \cap \mathrm{NId}_{R(I)} = \{1, \mathrm{I}\}$

Example.3.18 In the neutrosophic field $\mathbb{Z}_3(I)$, we have $\mathrm{NId}_{\mathbb{Z}_3(I)} = \{0,1, I, 1 + 2I\}$, $\mathrm{NU}_{\mathbb{Z}_3(I)} = \{1,2, I, 2I, 1 + I\}$. Clear that $\mathrm{NId}_{\mathbb{Z}_3(I)} \cap \mathrm{NU}_{\mathbb{Z}_3(I)} = \{1, \mathrm{I}\}$

Corollary.3.19 In general, it is not necessarily only that $\mathrm{NU}_{R(I)} \cap \mathrm{NId}_{R(I)} = \{1, \mathrm{I}\}$, when R(I) is unity.

Example.3.20 In the neutrosophic ring $\mathbb{Z}_6$(I), the element $(3 + 4I) \in NU_{\mathbb{Z}_6(\mathrm{I})} \cap NId_{\mathbb{Z}_6(\mathrm{I})}$, where $I(3 + 4I) = I$ and $(3 + 4I)^2 = 3 + 4I$

Theorem.3.21 Assume that R(I) is unity. Therefore, every unit element is a neutrosophic regular.

Proof. $\forall x \in U_{R(I)} \Rightarrow \exists x^{-1} \in R(I);\ xx^{-1} = 1 \Rightarrow xx^{-1}x = x \in NReg_{R(I)}$.

Theorem.3.22 Assume that R(I) is unity. Then, for every neutrosophic unit element of shape bI , $b \neq 0$ is a neutrosophic regular.

Proof.

Since $bI \in NU_{R(I)}$ thus $\exists x \in R(I)$ such that $(bI)x = x(bI) = I$. So $(bI)x(bI) = bI \in NReg_{R(I)}$.

Theorem.3.23 In any neutrosophic field $R(\mathrm{I})$, every neutrosophic unit element of the shape $a + bI$; $a \neq 0$ and $a \neq -b$ is a neutrosophic regular.

Proof. Using theorem.2.5. We have $a + bI$ is a unit. Therefore, it is a neutrosophic regular according to the theorem.3.21.

Corollary.3.24 In general, in a unity neutrosophic ring, every neutrosophic unit element of the shape $a + bI$; $a \neq 0 \neq b$ is not necessarily a neutrosophic regular.

Example.3.25 In the neutrosophic ring $\mathbb{Z}_8(\mathrm{I})$, the element $4 + I$ is a neutrosophic unit and achieves $(4 + I)(4 + I) = I$. We note $4 + I \notin NReg_{\mathbb{Z}_8(\mathrm{I})}$.

Theorem.3.26 Assume that $R(\mathrm{I})$ is unity. If $x \in NNil_{R(I)}$, then $1 - x$, $I(1 - x) = I - Ix$, $1 + x$, $I(1 + x) = I + Ix \in NReg_{R(I)}$.

Proof. We have $x \in NNil_{R(\mathrm{I})} \Rightarrow \exists n \in \mathbb{Z}^{+};\ x^n = 0$. On the other hand, we note $(I - Ix)(I + x + x^2 + \dots . + x^{n-1}) = I + Ix + Ix^2 + \dots . + Ix^{n-1} - Ix - Ix^2 - \dots . - Ix^{n-1} - Ix^n = 1 - Ix^n = I$. Therefore, $I - Ix = I(1 - x) \in NU_{R(\mathrm{I})}$. Using theorem.3.22, $I - Ix \in NReg_{R(I)}$.

Finally, $(I + Ix)(I - x + x^2 - x^3 + \dots . + (-1)^{n-1}x^{n-1}) = I - Ix + Ix^2 - I\,x^3 + \dots\dots + I(-1)^{n-1}x^{n-1} + Ix - Ix^2 + Ix^3 - \dots . + I(-1)^{n-2}x^{n-1} + I(-1)^{n-1}x^n = I + I(-1)^{n-1}x^n = I \Rightarrow I + Ix = (1 + x)I \in NU_{R(\mathrm{I})}$. Using theorem.3.22, $I + Ix \in NReg_{R(I)}$.

Similarly, we prove that if $x \in NNil_{R(\mathrm{I})}$, then $1 - x, x - 1$, $x + 1$, $Ix - I \in NReg_{R(\mathrm{I})}$.

Example.3.27 In the neutrosophic ring $\mathbb{Z}_9(\mathrm{I})$, the element $(3 + 3I)$ is a neutrosophic simple nilpotent,,and we have $1 - (3 + 3I) = 1 + 6 + 6I = 7 + 6I$. We note

$$(7 + 6I)(4 + 3I)(7 + 6I) = 7 + 6I.\ \text{Therefore, } 7 + 6I \in NReg_{\mathbb{Z}_9(\mathrm{I})}.$$

$$I - I(3 + 3I) = I + 3I = 4I.\ \text{We note } (4I)(7)(4I) = 4I \in NReg_{\mathbb{Z}_9(\mathrm{I})}.$$

$$3 + 3I - 1 = 2 + 3\mathrm{I}.\ \text{We have } (2 + 3\mathrm{I})(5 + 6\mathrm{I})(2 + 3\mathrm{I}) = 2 + 3I \in NReg_{\mathbb{Z}_9(\mathrm{I})}.$$

$$I(3 + 3I) - I = 5I;\ (5I)(2)(5I) = 5I \in NReg_{\mathbb{Z}_9(\mathrm{I})}.$$

On the other hand, $3 + 3I + 1 = 4 + 3I$, where $(4 + 3I)(7 + 6I)(4 + 3I) = 4 + 3I \in NReg_{\mathbb{Z}_9(\mathrm{I})}$. $I(3 + 3I) + I = 7I; (7I)4(7I) = 7I \in NReg_{\mathbb{Z}_9(\mathrm{I})}$.

Corollary.3.28 Assume that $R(\mathrm{I})$ is unity. Now if $x = bI \in NNil_{R(I)}$, then $I - x$ and $I + x \in NReg_{R(I)}$.

Proof. We have $x = bI \in NNil_{R(I)} \Rightarrow \exists n \in \mathbb{Z}^{+};\ x^n = (bI)^n = b^nI^n = 0 \underset{I^n \neq 0}{\Longrightarrow} b^n = 0 \Rightarrow\ b \in Nil_{R(I)}$. On the other hand, $I - x = I - bI = I(1 - b)$. Using theorem.3.26, we have $I(1 - b) \in NReg_{R(I)}$.

Similarly, we prove that $I + x \in NReg_{R(I)}$.

Corollary.3.29 In general, if $R(\mathrm{I})$ is a unity neutrosophic ring and $x = a + bI \in NNil_{R(I)}$, then it is not necessarily that $I - x$, $I + x \in NReg_{R(I)}$.

Example.3.30 In the neutrosophic ring $\mathbb{Z}_8(\mathrm{I})$, the element $(4 + 4I)$ is a neutrosophic simple nilpotent, while $I - (4 + 4I) = I + 4 + 4I = 4 + 5I \notin NReg_{\mathbb{Z}_8(\mathrm{I})}$, because if $a + bI \in \mathbb{Z}_8(\mathrm{I})$; $a \neq 0$ *or* $b \neq 0$. We have $(4 + 5I)(a + bI)(4 + 5I) = (4 + 5I)(4 + 5I)(a + bI) = I(a + bI) = (a + b)I \neq 4 + 5I\ \ \forall a, b \in \mathbb{Z}_8$.

Corollary.3.31 Assume that $R(\mathrm{I})$ is unity. Now, if $x_1 = a + bI \in NU_{R(I)}$ *and* $x_2 = c + dI \in NNil_{R(I)}$, then it is not necessarily that $x_1 - x_2$, $x_1 + x_2 \in NReg_{R(I)}$.

Example.3.32 In the neutrosophic ring $\mathbb{Z}_9(I)$, the element $(3 + 3I) \in NSNil_{\mathbb{Z}_9(\mathrm{I})}$,and $8I \in NU_{R(I)}$, we have $8I - (3 + 3I) = 8I + 6 + 6I = 6 + 5I \notin NReg_{\mathbb{Z}_9(\mathrm{I})}$, because if $a + bI \in \mathbb{Z}_9(\mathrm{I})$; $a \neq 0$ *or* $b \neq 0$. We have $(6+5I)(a+bI)(6+5I)=(6+5I)(6+5I)(a+bI)=4I(a+bI)=(4a+4b)I\neq 6+5I\ \forall a,b\in\mathbb{Z}_9$.

Theorem.3.33 In any neutrosophic field, R(I) is achieved $R(\mathrm{I})=NReg_{R(I)}$.

Proof.

$\forall x=a+bI\in R(I)$. If $x=0$, then $x\in NReg_{R(I)}$, because $\forall y\in R(I)$ so $0=0.y.0$

If $x\neq 0$, then $a\neq 0\ or\ b\neq 0$.

Now if $a\neq 0\ and\ b=0\Rightarrow x=a\in U_{R(I)}$. Using theorem. 3.21, $x\in NReg_{R(I)}$

If $a=0\ and\ b\neq 0\Rightarrow x=bI\in NU_{R(I)}$. Using theorem. 3.22, $x\in NReg_{R(I)}$

If $a\neq 0\ and\ b\neq 0\ and\ a\neq -b$. Using theorem. 2.6, $x\in U_{R(I)}$. Using theorem. 3.21, $x\in NReg_{R(I)}$

If $a\neq 0\ and\ b\neq 0\ and\ a=-b\Rightarrow x=a-aI$; $(a-aI)a^{-1}(a-aI)=(1-I)(a-aI)=a-aI=x$

$$\Rightarrow x\in NReg_{R(I)}$$

Another way to prove (in case $a\neq 0\ and\ b\neq 0$).

If $a\neq 0\ and\ b\neq 0\Rightarrow\exists\ x+yI\in R(\mathrm{I})$; $x\neq 0\ or\ y\neq 0\ and\ x,y\in R$.

And it is achieved $a+bI=(a+bI)(x+yI)(a+bI)=[a^2+(2ab+b^2)I](x+yI)$

Suppose that $c=a^2$, $d=2ab+b^2\Rightarrow a+bI=(c+dI)(x+yI)$

It's clear $c\neq 0$ in R(I) and that $d=0\ or\ d\neq 0$.

If $d=0$ then $a+bI=c(x+yI)=cx+cyI\Rightarrow a=cx,b=cy\Rightarrow x=c^{-1}a, y=c^{-1}b\in R(\mathrm{I})$

If $d\neq 0\Rightarrow a+bI=(c+dI)(x+yI)=cx+(cy+dx+dy)I\Rightarrow a=cx\ ,b=cy+dx+dy$

$$\Rightarrow x=c^{-1}a\in R(\mathrm{I})\Rightarrow b=cy+dc^{-1}a+dy\Rightarrow b=(c+d)y+dc^{-1}a$$

If $c+d=0\Rightarrow b=0y+dc^{-1}a$, $\forall y\in R(\mathrm{I})$, in this case we will consider $y=0$ for ease.

If, $c+d\neq 0\Rightarrow b-dc^{-1}a=(c+d)y\Rightarrow y=(c+d)^{-1}(b-dc^{-1}a)\in R(\mathrm{I})$.

Corollary.3.34 Let R(I) be a neutrophilic field, then every element $a+bI$; $a\neq 0, a=-b$ is neutrosophic regular, so there is $a^{-1}+cI$; $\forall c\in R(I)$ where $(a+bI)(a^{-1}+cI)(a+bI)=a+bI$.

Proof. Since $a\neq 0\ and\ a=-b$, so $\forall c\in R(I)$. We note that

$$(a-aI)(a^{-1}+cI)(a-aI)=(1+acI-I-acI)(a-aI)=(1-I)(a-aI)=a-aI.$$

Example.3.35 In the neutrosophic field $\mathbb{Z}_7(I)$, the element $5+6I$ is a neutrosophic regular, where $a=5$, $b=6$. Using theorem.3.33, we can find the element $x+yI$ that achieves neutrosophic regularity, $c=\overline{a^2}=4$, $d=\overline{2ab+b^2}=\overline{60+36}=5$; $c+d=4+5=2\Rightarrow x=c^{-1}a=2(5)=3$

$$\Rightarrow y=(c+d)^{-1}(b-dc^{-1}a)=(2)^{-1}[6-(5.2.5)]=4[6-(1)]=4[6+6]=4(5)=6$$

Now easily it can be verified that $5+6I=(5+6I)(3+6I)(5+6I)$

Example.3.36 In the neutrosophic field $\mathbb{Z}_7(I)$, the element $3+4I$ is a neutrosophic regular element where $a=3$, $b=4$. Using theorem.3.33, we can find the element $x+yI$ that achieves neutrosophic regularity

$$c=\overline{a^2}=2\ ,d=\overline{2ab+b^2}=\overline{24+16}=5\ ;c+d=2+5=0\Rightarrow y\in\mathbb{Z}_7$$
$$\Rightarrow x=c^{-1}a=4(3)=5$$

Now easily it can be verified that $\forall y\in\mathbb{Z}_7$; $3+4I=(3+4I)(5+yI)(3+4I)$

Example.3.37 In the neutrosophic field $\mathbb{Z}_{11}(I)$, the element $3+8I$ is a neutrosophic regular element where $a=3$, $b=8$. Using theorem.3.33, we can find the element $x+yI$ that achieves neutrosophic regularity

$$c=\overline{a^2}=9\ ,d=\overline{2ab+b^2}=\overline{48+64}=2\ ;c+d=9+2=0\Rightarrow y\in\mathbb{Z}_{11}\Rightarrow x=c^{-1}a=5(3)=4$$

Now easily it can be verified that

$3+8I=(3+8I)(4+yI)(3+8I)\ \forall y\in\mathbb{Z}_{11}$. Suppose that $y=0\ \Rightarrow 3+8I=(3+8I)4(3+8I)$

Corollary.3.38 Assume that $R(\mathrm{I})$ is a unity and $a+bI\in R(I)$ where $a,b\in Reg_{R(I)}$, then it is not necessarily that $a+bI\in NReg_{R(I)}$, and also if $a+bI\in NReg_{R(I)}$, then it is not necessarily $a,b\in Reg_R$.

Example.3.39 In the neutrosophic ring $\mathbb{Z}_4(I)$, the element $3+3I$ is neutrosophic irregular, although $a=b=3=3.3.3\in Reg_{\mathbb{Z}_4}$, because if we assume that

$3+3I=(3+3I)(x+yI)(3+3I)=(3+3I)(3+3I)(x+yI)=(1+3I)(x+yI)$

$=x+yI+3xI+3yI=x+3xI\ ;x,y\in\mathbb{Z}_4$

$\Rightarrow 3+3I=x+3xI\Rightarrow x=3$ and $3x=3$ $thus\ x=3$ and $x=1$, but this is a contradiction. Therefore, $3+3I$ is a neutrosophic irregular.

Example.3.40 In the neutrosophic ring $\mathbb{Z}_8(I)$, the element $x=3+2I$ is a neutrosophic regular although $2\notin Reg_{\mathbb{Z}_8}$, where $x=xxx$.

Theorem.3.41 If $R(I)$ is a unity neutrosophic regular neutrosophic ring, then $R(I)=NU_{R(I)}\cup NZ_{R(I)}$.

Proof. Always be an investigator $NU_{R(I)}\cup NZ_{R(I)}\subseteq R(I)$.

On other hand, $\forall\ x\in R(I)$ where $x\notin NZ_{R(I)}$. Now since $x\in R(I)$ so there is y belonging to $R(I)$ that achieves

$x=xyx$ so $x-xyx=0$ thus $x(1-yx)=0\xRightarrow[x\notin NZ_{R(I)}]{}1-yx=0$ thus $yx=1$

$x=xyx\Rightarrow\ x-xyx=0\Rightarrow\ (1-xy)x=0\xRightarrow[x\notin NZ_{R(I)}]{}1-xy=0$ thus $xy=1$

Since $x\in NU_{R(I)}$ so $x\in NU_{R(I)}\cup NZ_{R(I)}$. Therefore, $R(I)\subseteq NU_{R(I)}\cup NZ_{R(I)}$.

Example.3.42 We have $\mathbb{Z}_3(I)$ is a neutrosophic regular, which $NZ_{\mathbb{Z}_3(I)}=\{0,I,2I,1+2I,2+I\}$, $NU_{\mathbb{Z}_3(I)}=\{1,2,I,2I,1+I,2+2I\}$, and we note $\mathbb{Z}_3(I)=NU_{\mathbb{Z}_3(I)}\cup NZ_{\mathbb{Z}_3(I)}$.

Note.3.43 The condition of neutrosophic regularity in the unity neutrosophic ring is necessary for it to satisfy $R(I)=NU_{R(I)}\cup NZ_{R(I)}$.

Example.3.44 We have $\mathbb{Z}(\mathrm{I})$ is a unity neutrosophic irregular neutrosophic ring and is not achieved $\mathbb{Z}(\mathrm{I})=NU_{\mathbb{Z}(\mathrm{I})}\cup NZ_{\mathbb{Z}(\mathrm{I})}$.

Theorem.3.45 If $R(\mathrm{I})$ is a neutrosophic right (left) strongly regular neutrosophic ring, then $NSNil_{R(\mathrm{I})}=\{0\}$.

Proof. $\forall\ x\in NSNil_{R(\mathrm{I})}\Rightarrow x^2=xx=0$. Since $x\in R(\mathrm{I})$, there is y belongs to $R(I)$ that achieves $x=yxx$, therefore $x=yxx=y0=0\Rightarrow NSN_{R(\mathrm{I})}=\{0\}$.

Similarly, we prove that $NSNil_{R(\mathrm{I})}=\{0\}$ in the case of $R(\mathrm{I})$ is a neutrosophic left strongly regular.

Table 1. key distinctions between the classical and neutrosophic rings.

	unity classical ring R	unity neutrosophic ring R(I)
1	$U_R\cap Z_R=\emptyset$	$NU_{R(I)}\cap NZ_{R(I)}\neq\emptyset$

2	$\mathrm{U}_R \cap \mathrm{Id}_R = \{1\}$	If $R(\mathrm{I})$ be a neutrosophic field, then $\mathrm{NU}_{R(I)} \cap \mathrm{NId}_{R(I)} = \{1, \mathrm{I}\}$
3	Suppose R is a field then then every element $x \neq 0$ has inverse.	If $R(\mathrm{I})$ be an infinite (finite) neutrosophic field, then there are elements that have an infinite (finite) number of neutrosophic inverse.
		Suppose $R(\mathrm{I})$ is a field then there are elements that non unit.
4	If $u \in U_R$ *and* $a \in Nil_R$, then $u - a,\ u + a \in\ Reg_R$	If $x_1 \in NU_{R(I)}$ *and* $x_2 \in NNil_{R(I)}$ then it is not necessarily that $x_1 - x_2,\ x_1 + x_2 \in NReg_{R(I)}$
5	Every unit element is regular.	Every neutrosophic unit element is not necessarily a neutrosophic regular.

4. Conclusion and Future Works

This study broadened the idea of neutrosophic regularity in neutrosophic rings. We investigated the qualities of neutrosophic regular elements and the most significant properties that connect them to neutrosophic elements in neutrosophic rings. We discovered numerous key findings that connect these components, some of which differ from what is known about classic rings. Furthermore, various examples were constructed to demonstrate the reliability of the study.

We intend to investigate the characteristics of ideals in regular neutrosophic rings in the future.

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The author declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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