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Linguistic Hypersoft Set with Application to Multi-Criteria Decision-Making to Enhance Rural Health Services

Muhammad Saqlain[1], Poom Kumam[1*], Wiyada Kumam[2]

[1] Departments of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand.

[2] Applied Mathematics for Science and Engineering Research Unit (AMSERU), Program in Applied Statistics, Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Pathum Thani 12110, Thailand.

Corresponding author: Poom Kumam **e-mail:** poom.kum@kmutt.ac.th

Abstract: Language, as an abstract system and a creative act, possesses inherent complexity due to its contextual nature and the variability of its meaning. The context of language is shaped by an individual's empirical knowledge, derived from observation and experience. Decision-making challenges related to language encompass both quantitative and qualitative factors, which further contribute to the intricacy of the process. Decision-making challenges may involve both quantitative and qualitative aspects of further subdivided attributes. However, linguistic knowledge cannot be easily quantified by existing methods. Therefore, current methods are ineffective in handling linguistic knowledge. Using mathematical values, such as fuzzy, intuitionistic, and neutrosophic, in decision-making problems without following linguistic knowledge rules can result in vagueness and imprecision. To address these issues, this paper presents a comprehensive generic model. The model introduces the linguistic set structure of the hypersoft set (LHSS) as a solution for decision-making problems. The definition of fundamental operations, including AND, NOT, OR complement, and negation, is proposed alongside illustrative examples and their respective properties. Additionally, operational laws for the linguistic hypersoft set are introduced to effectively address decision-making challenges. By implementing the proposed aggregate operators and operational laws, linguistic quantifiers can be converted into numerical values, thereby enhancing the accuracy and precision of the hypersoft set structure in decision-making scenarios.

Keywords: Linguistic quantifiers; linguistic set; hypersoft set; aggregate operators; multi-criteria decision-making (MCDM).

1. Introduction

These influential 1975 papers, (Zadeh, 1975, 1975a, 1975b) introduces the concept of a linguistic variable and explores its application in approximate reasoning, specifically focusing on their use in decision-making. These work expands on the understanding of linguistic variables' potential in practical scenarios. To implement these concepts in real life problems the scientists explored the areas, and now these concepts are widely used in decision-making process i.e. multi-criteria decision-making (Hwang & Yoon, 1981). By considering multiple criteria, MCDM aims to enhance the decision-making process, improve transparency, and facilitate the selection of robust solutions that align with the desired goals and objectives. To address the challenges associated with decision-making and linguistic preferences (Delgado, Verdegay, & Vila, 1992) presented a paper focuses on linguistic decision-making models. It presents different approaches and techniques for modeling decision making in linguistic contexts, contributing to the understanding of decision-making processes. The method based on linguistic aggregation operators for decision-making with linguistic preference relations was proposed by (Xu, 2004). A semantic model for computing with flexible linguistic expressions was proposed by (Jiang et al., 2021), and (Wu et al., 2023) paper presents a multiple criteria decision-making method that incorporates heterogeneous linguistic expressions. it enhances the understanding of linguistic expressions and their use in decision-making processes. It was difficult to deal with the problems having doubt, uncertainty, vagueness, ambiguity, and indeterminacy. The concept of linguistic to mathematic was unclear, then to overcome the problem some set theories were proposed by the researchers. In next paragraph we present those theories with application to MCDM.

The groundwork for the concept of fuzzy set (membership values) to deal with uncertainty and its use in information and control systems was presented by (Zadeh, 1965). Ambiguity was another problem faced by the decision-makers then (Attanasov, 1986) came up with the concept of intuitionistic set, in which each alternative is assigned a membership and non-membership degree with the condition that their sum is not greater than 1. To deal with the problem having indeterminacy (Smarandache, 1998, 2002, 2002a, 2003, 2005, 2006) came up with the concept of neutrosophic set theory, which has membership, non-membership, and indeterminacy values. Many researchers came up with the concept of extensions and by merging linguistic with fuzzy, intuitionistic, neutrosophic sets, and other hybrid structures. The application in multi-criteria decision-making under fuzzy linguistic sets was proposed by (Joyce, 1976), the study illustrates the use of fuzzy in linguistic environment. The application in multi-criteria decision-making under hesitant fuzzy linguistic term sets was proposed by (Dinesh et al., 2022) The group decision-making process under linguistic intuitionistic fuzzy sets using aggregation operators was presented by (Garg & Kumar, 2018). The application of linguistic sets to group decision making and a method to handle complex decision scenarios was presented by (Wang, Ju, & Liu, 2019) based on q-rung orthopair fuzzy linguistic sets. The novel method for multi-attribute decision-making using interval-valued Pythagorean fuzzy linguistic information was published by (Du et al., 2017). The research addresses the challenges of decision making when dealing with interval-valued linguistic data. The new methods for addressing MCDM problems using linguistic neutrosophic sets in which the interrelationships among individual data are considered was proposed by (Li, Zhang & Wang, 2017).

In decision-making problems decision-makers deal with the alternatives having attributes, the mathematical notation given by (Molodtsov, 1999), the paper presents the foundational concepts of soft set theory. It establishes the groundwork for the study of soft sets and their applications in various domains, including decision-making and data analysis. (Maji & Roy, 2002) the paper applies soft sets to a decision-making problem. The research demonstrates the practical utility of soft sets in real-world decision-making scenarios, showcasing their effectiveness in capturing uncertainty. To further enhance the capabilities in decision-making and data analysis (Ali et al., 2009) came up with some new operations in soft set theory. The concept of the soft set was later extended to fuzzy soft set (Maji, Biswas, & Roy, 2001), intuitionistic soft set (Deli & Çağman, 2013), neutrosophic soft set (Amalini et al., 2020), and other hybrid structures.

To incorporate uncertainty a comprehensive framework for group decision-making was proposed by (Tao et al., 2015). The paper introduces uncertain linguistic fuzzy soft sets and their applications in group decision-making. The research of (Aiwu & Hongjun, 2016) proposes fuzzy-valued linguistic soft set theory and applies it to multi-attribute decision making. This work presents a novel approach to handle linguistic uncertainty and supports decision-making processes. The multi-attribute decision-making method using belief-based probabilistic linguistic term sets was proposed by (Liu, Fei, & Mi, 2023). For the selection of medical waste treatment stations based on linguistic q-rung orthopair fuzzy numbers (Ling, Li, & Lin, 2021) proposes a methodology. To handle linguistic uncertainty in group decision-making processes (Vijayabalaji & Ramesh, 2018) proposed a method to solve these problems.

Novel approaches have been demonstrated by recent studies that have advanced a variety of sectors (Saqlain, 2023). Decision-making utilizing Pythagorean fuzzy Hamacher aggregation operators has been extended by (Paul, Jana, & Pal, 2023), (Du, Wang, & Lu, 2023), maximized wireless power transmission with an improved approach, and (Haq & Saqlain, 2023) used machine learning for attendance tracking in a pandemic (Zulqarnain & Saqlain, 2023) using convolutional neural networks were used to evaluate text readability in higher education, while (Saqlain et al., 2023) presented a multi-polar interval-valued neutrosophic hypersoft set for uncertainty and decision-making. These projects demonstrate (Stević et al., 2023) a dedication to creativity and cross-domain problem-solving (Tešić et al., 2023). The strategic framework for leveraging artificial intelligence in future marketing decision-making has been explored by (Hicham, Nassera, & Karim, 2023). Furthermore, (Saqlain et al., 2023) introduced proportional distribution-based Pythagorean fuzzy fairly aggregation operators in multi-criteria decision-making .

In MCDM, if attributes are further sub-divided, then existing set structures cannot be applied, thus (Smarandache , 2018) proposed the concept of a hypersoft set, which is the generalization of soft set theory. Hypersoft set (HSS) theory tends to consider further divided attributes or attributes bifurcation. The theory of HSS has been applied to solve both, MCDM and MADM problems [23]. Another beauty of HSS, it can be molded as per the DM requirements. The hypersoft set structure have been extended to a fuzzy hypersoft set (Yolcu & Öztürk, 2021; Jafar & Saeed, 2021; Debnath, 2021) , intuitionistic hypersoft set (Yolcu, Smarandache & Öztürk, 2021) and neutrosophic hypersoft set (Smarandache, 2018) and (Saqlain et al., 2020). These papers represent a diverse range of research contributions in the field of linguistic variables, fuzzy sets, soft sets, hypersoft sets and their applications in decision-making and data analysis.

1.1. Novelty

Comprehending language as an abstract system and a creative process poses significant complexity due to its inherent reliance on context. This context is intricately influenced by an individual's empirical knowledge, which is acquired through keen observation and personal experience. When confronted with the need to make decisions involving further subdivided attributes, a combination of quantitative and qualitative factors comes into play. Nevertheless, the absence of a standardized methodology for assigning numerical values to language hinders existing approaches from effectively managing linguistic knowledge operations. The practice of indiscriminately assigning mathematical values (such as fuzzy, intuitionistic, and neutrosophic) to decision-making problems, without taking linguistic rules into account, leads to ambiguity and inaccuracy. Consequently, the primary objective of this paper is to propose an inclusive model that directly addresses these issues. The paper introduces the concept of the linguistic set structure of the hypersoft set (LHSS) as a proficient approach to tackle the challenges encountered in decision-making processes.

1.2. Contribution

This paper makes significant contributions to the field of decision-making by addressing the limitations of existing approaches in dealing with linguistic knowledge. By introducing the LHSS model, this research offers a novel solution to the challenges posed by the abstract and context-dependent nature of language. The definition of basic operations and the proposal of operational laws for the LHSS provide a systematic framework for converting linguistic quantifiers into numerical values. This framework increases the accuracy and precision of decision-making processes, enabling more reliable and effective outcomes. The implementation of the proposed aggregate operators and operational laws offers a practical tool for solving decision-making issues and improving the overall understanding and application of linguistic knowledge. This contribution has the potential to benefit various fields that rely on language-based decision-making, such as natural language processing, sentiment analysis, and artificial intelligence, among others.

1.3. Scientific Validity

The scientific validity of this paper's approach lies in its rigorous and systematic treatment of the challenges associated with language and decision-making. By acknowledging the abstract and context-dependent nature of language, the authors have developed a novel model, the linguistic set structure of the hypersoft set (LHSS), to address these complexities. The paper provides a clear definition of basic operations and operational laws for the LHSS, ensuring the consistency and reproducibility of the proposed framework. Additionally, the authors illustrate the application of the LHSS through examples and properties, further enhancing the scientific validity of their approach. The proposed model offers a systematic and mathematically grounded methodology to convert linguistic quantifiers into numerical values, thereby improving the accuracy and precision of decision-making processes. The scientific validity of this research is further supported by its potential applicability to various domains that rely on linguistic knowledge. Overall, the systematic approach, rigorous analysis, and practical examples presented in this paper contribute to its scientific validity and establish a foundation for further research in the field of decision-making with linguistic elements. operational laws and aggregate operators are indispensable in the development of Mechanics of advanced manufacturing and robotics. They provide the necessary tools and frameworks for decision-making, optimization, and performance evaluation. By leveraging these tools effectively, engineers and researchers can enhance the efficiency, effectiveness, and overall performance of advanced manufacturing processes and robotic systems, leading to advancements in these fields and enabling the realization of advanced technologies and automation.

The power of the proposed method explored in this research lies in its ability to effectively address the challenges posed by the abstract nature of language and its context-dependent meaning in decision-making processes. The method offers a systematic and mathematically grounded framework, the linguistic set structure of the hypersoft set (LHSS), which enables the conversion of linguistic quantifiers into numerical values. One of the key strengths of this method is its ability to handle complicated decision-making scenarios. By incorporating weighted linguistic quantifiers or linguistic variables, the method allows for the consideration of multiple factors and attributes with varying degrees of importance. This extension enhances the versatility of the proposed framework and enables a more comprehensive evaluation of complex decision criteria. Furthermore, the proposed method enhances the accuracy and precision of decision-making processes by providing operational laws and aggregate operators that facilitate the conversion of linguistic knowledge into numerical values. This conversion allows for quantitative analysis and comparison, leading to more reliable and informed decision outcomes.

Moreover, the generic nature of the proposed model makes it applicable to various domains that rely on language-based decision-making. From natural language processing to sentiment analysis and artificial intelligence, the method has the potential to contribute to a wide range of fields. In advanced manufacturing and robotics, operational laws establish the logical rules and principles for manipulating and transforming data, whether it is linguistic or numerical in nature. These laws provide a foundation for modeling and analyzing various aspects of manufacturing processes and robotic systems. By applying operational laws, engineers and researchers can develop algorithms, control strategies, and optimization techniques that ensure the efficient and effective operation of advanced manufacturing systems and robotic devices.

1.4. Layout of proposed research

The following shows that, how the work has been organized: The fundamental ideas of linguistic hypersoft set (LHSS) are broken down in detail in section 2. In section 3, we present a definition, notions, and examples of LHSS with basic properties and operations. Operational laws on LHSS has been proposed in section 4. The aggregate operator Linguistic Hypersoft Ordered Weighted Geometric Averaging Operator (LHSOWGAO) and Linguistic Hypersoft Weighted Geometric Averaging Operator (LHSWGAO) has been presented in section 5. In part 6, an MCDM framework is described for the "LHSS Algorithm to solve MCDM Problem" with a case study to demonstrate the benefits of the proposed algorithm. The findings of the study have been summarized, along with their significance, in section 7, and concluded with future directions. The layout of the paper also presented by Figure 1.

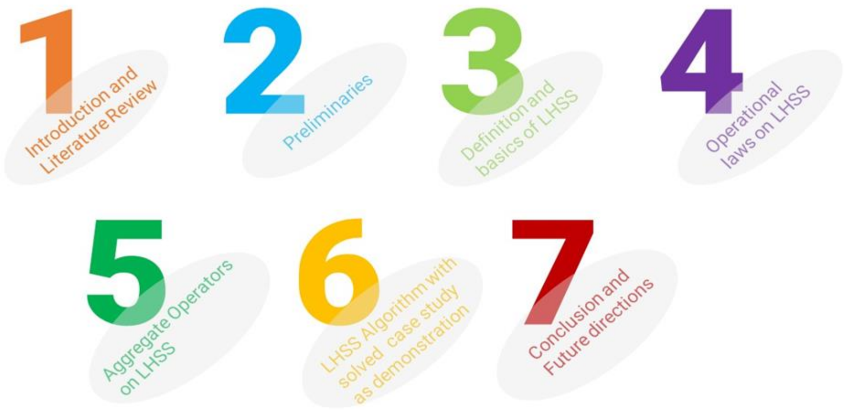

Figure 1. Layout of the paper

2. Preliminary section

In this section, we go through some basic definitions that support the construction of the framework of this paper: linguistic set, linguistic quantifiers, soft set, and hypersoft set (HSS).

Definition 2.1. Linguistic Set

Let $\mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}$ $where\ \ t = 2n+1:\ \ n \geq 1\ \ and\ \ n\ \in \mathbb{R}^+,$ be a finite strictly increasing set. For example, if n = 1 then,

$\mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3\} = \{very\ \ bad, fair, very\ \ good\}$

For Linguistic set, which is under consideration, the relationship to its elements κ^t and the superscript t will be strictly increasing. To define the continuity this set is extended to $\mathrm{K} = \{\kappa^\beta : \beta \in \mathbb{R}\}$ $where\ \ \beta\ \ is\ \ also\ \ stricly\ \ incrasing.$

Definition 2.2. Linguistic Quantifiers

The linguistic quantifiers were introduced by Zadeh [48-51] also known as absolute quantifiers and are represented below in Table 1. Let $\mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}$ $where\ \ t = 2n+1:\ \ n \geq 1\ \ and\ \ n\ \in \mathbb{R}^+,$ be a finite strictly increasing set.

Table 1: linguistic quantifiers

Quantifiers						
Low		Medium			High	
None	Very-Low	Low	Medium	High	Very-High	Perfect

Definition 2.3. Soft Set

A pair $(\mathcal{F}, Å)$ is known as soft set (over $\mho$) iff $\mathcal{F}: Å \longrightarrow P(\mho)$. It means, soft set is the parametrized subset of the universe $\mho$.

Definition 2.4. Hypersoft Set

Let, $a^1, a^2, a^3, \dots, a^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3, \dots, \mathcal{L}^t$ with $\mathcal{L}^i \cap \mathcal{L}^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Then the pair $(\mathcal{F}, \mathbb{L})$ where $\mathbb{L} = \{\mathcal{L}^1 \times \mathcal{L}^2 \times \mathcal{L}^3 \times \dots \times \mathcal{L}^t$: t is finite and real valued} is known as Hypersoft set over $\mho$ with mapping $\mathcal{F}: \mathbb{L} = \mathcal{L}^1 \times \mathcal{L}^2 \times \mathcal{L}^3 \times \dots \times \mathcal{L}^t \longrightarrow P(\mho)$.

3. Linguistic Hypersoft Set (LHSS)

In this section, we propose LHSS with its set structure properties.

Definition 3.1: Linguistic Hypersoft Set (LHSS)

Let, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Then the pair (Γ, Λ) where $\Lambda = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t$: t is finite and real valued} is known as hypersoft set over Ω with mapping $\Gamma: \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \longrightarrow P(\Omega)$.

Then the linguistic hypersoft set will be,

$$\Gamma(\{\mathrm{M}(\Omega)(\imath)\}) : M \subseteq \Lambda \quad \& \quad \imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\} \ where \ \ t = 2n + 1 : \ n \geq 1, \quad n \in \mathbb{R}^+\}$$

Numerical Example 3.1.1:

Let $\Omega = \{\sigma^1, \sigma^2, \sigma^3, \sigma^4\}$ and set $M = \{\sigma^2, \sigma^3\} \subset \Omega$.

Consider the parameters be: $\alpha^1 = nationality, \alpha^2 =$ gender, $\alpha^3 =$ color, and their respective parametric values are:

Nationality = $\Upsilon^1 = \{Pakistani, Chinese, American\}$

Gender = $\Upsilon^2 = \{Male, Female\}$

Color = $\Upsilon^3 = \{Pink, Black, Orange\}$

Then the function $\Gamma: \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \longrightarrow P(\Omega)$ and assume the hypersoft set,

$\Gamma(\{Pakistani, Male, Orange\}) = \{\sigma^2, \sigma^3\} = M$

The linguistic hypersoft set (LHSS), $\Gamma(\{\mathrm{M}(\Omega)(\imath)\}) : M \subseteq \Lambda \quad \& \quad \imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$ $where \ \ t = 2n + 1 : \ n \geq 1, \ n \in \mathbb{R}^+\}$ Can be given as;

$\Gamma(\{Pakistani, Male, Orange\}) = \{\sigma^2, \sigma^3\} = \{\sigma^2(High), \sigma^3(None)\} = L.$

Similarly,

$\Gamma_1(\{Pakistani, Male, Pink\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$

$\Gamma_2(\{Chinese, Female, Pink\}) = \{\sigma^1(high), \sigma^4(low)\} = L_2$

$\Gamma_3(\{American, male, black\}) = \{\sigma^1(medium), \sigma^3(none)\} = L_3$

Definition 3.2: Let $(\Gamma_1, \Lambda_1) = L_1$ be a LHSS, then the subset L_s can be defined as;

$\Gamma_2 : \Lambda_s = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^f \longrightarrow P(\Omega) \ with \ \ s \leq n. Also, \ \Gamma_2(\{L_s(\Omega)(\imath)\}) : \Lambda_s \subseteq \Lambda \quad \& \quad \imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$

$$where \ \ t = 2n + 1 : \ n \geq 1, \quad n \in \mathbb{R}^+\}$$

1. $L_s \subseteq L_1$;
2. $\forall \ell \in L_s, \Gamma_2(\ell) \subseteq \Gamma_1(\ell)$.

This holds only when linguistic variables K^i satisfy the property i.e. each K^i of $(\Gamma_s, \Lambda_s) = K^i$ of (Γ_1, Λ_1).

Example 3.2.1: Recall Example 1. The function $\Gamma_2 : \Lambda_s = \Upsilon^1 \times \Upsilon^2 \longrightarrow P(\Omega)$ and assume the hypersoft set, $\Gamma_2(\{Pakistani, Male\}) = \{\sigma^2(high)\} = L_s$. Where $\Lambda_s \subseteq \Lambda$ and $L_s \subseteq L_1$.

Definition 3.3: Empty linguistic hypersoft set (ELHSS) can be defined as;

$\Gamma_1 : \Lambda_E = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n \longrightarrow P(\Omega)$

$such\ that\ each\ \Upsilon^i\ (i \leq n) is\ empty.$ $\Gamma_1(\{L_E(\Omega)(\imath)\}) : \Lambda_E \subseteq \Lambda$ & $\imath \in K = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}$ $where\ t = 2n + 1 :\ n \geq 1,\ n \in \mathbb{R}^+\}$.

1. $(\Gamma_1, \Lambda_E)^{\phi} = L_E$ **if** $\forall \Gamma_1(\ell) = \phi : \forall \ell \in \Lambda_E$.

Example 3.3.1: Recall Example 1. The function $\Gamma_1 : \Lambda_E = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \longrightarrow P(\Omega)$ and assume the Hypersoft set, $\Gamma_1(\emptyset) = \emptyset = L_E$. Where $\Lambda_E \subseteq \Lambda$.

Definition 3.4: The AND operation on two $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ linguistic hypersoft set LHSS can be defined by;

1. $L_1 \wedge L_2 = (\Gamma_3, \Lambda_3) = L_3$; max of (K^i)
2. $(\ell_i, \ell_j) = \ell_k = L_3$ where $\ell_i \in L_1$ and $\ell_j \in L_2$ with $i \neq j$;
3. $\Gamma_3(\ell_i, \ell_j) = \Gamma_1(\ell_i) \cup \Gamma_2(\ell_j)$

Definition 3.5: The OR operation on two $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ linguistic hypersoft set LHSS can be defined by.

1. $L_1 \vee L_2 = (\Gamma_3, \Lambda_3) = L_3$;
2. $(\ell_i, \ell_j) = \ell_k = L_3$ where $\ell_i \in L_1$ and $\ell_j \in L_2$ with $i \neq j$;
3. $\Gamma_3(\ell_i, \ell_j) = \Gamma_1(\ell_i) \cap \Gamma_2(\ell_j)$

Definition 3.6: The NOT operation on (Γ, Λ) linguistic hypersoft set LHSS can be defined by;

1. $\sim L = \sim (\Gamma, \Lambda) = \sim \Upsilon^1 \times \sim \Upsilon^2 \times \sim \Upsilon^3 \times \ldots \times \sim \Upsilon^n$;
2. $\sim L = \sim \prod \ell_i : i = 1,2,3, \ldots, n$
3. $|\sim L| = n - Tuple$

Definition 3.7: The Complement on $(\Gamma, \Lambda) = L$ linguistic hypersoft set LHSS can be defined by;

1. $(\Gamma, \Lambda)^{\sim} = (\Gamma^{\sim}, \sim L)$; $\Gamma^{\sim} : \sim L \longrightarrow P(\Omega)$.
2. $\Gamma^{\sim}(\sim \ell) = \Omega \backslash \Gamma(\ell)$; $\forall \ell \in L$

Proposition 3.8: Let $(\Gamma, \Lambda) = L$, $(\Gamma_1, \Lambda_1) = L_1$, $(\Gamma_2, \Lambda_2) = L_2$ and $(\Gamma_3, \Lambda_3) = L_3$ be linguistic hypersoft set LHSS then following holds;

1. $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_1, \Lambda_1)$
2. $(\Gamma_1, \Lambda_E)^{\Phi} \subseteq (\Gamma_1, \Lambda_1)$
3. $\sim (\sim L) = L$
4. $\sim (\Gamma_1, \Lambda_E)^{\Phi} = \Omega$
5. If $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_2, \Lambda_2)$ and $(\Gamma_2, \Lambda_2) \subseteq (\Gamma_2, \Lambda_2)$ **then** $(\Gamma_1, \Lambda_1) = (\Gamma_2, \Lambda_2)$ Iff each K^i of $(\Gamma_1, \Lambda_1) = K^i$ of (Γ_2, Λ_2).

This property holds only when linguistic variables satisfy the property i.e. each $K^i$of $(\Gamma_1,\Lambda_1) = K^i$of (Γ_2,Λ_2).

6. If $(\Gamma_1,\Lambda_1) \subseteq (\Gamma_2,\Lambda_2)$ *and* $(\Gamma_2,\Lambda_2) \subseteq (\Gamma_3,\Lambda_3)$ then $(\Gamma_1,\Lambda_1) \subseteq (\Gamma_3,\Lambda_3)$.

This property holds only when linguistic variables satisfy the property i.e. each K^i of $(\Gamma_1,\Lambda_1) = K^i$ of $(\Gamma_2,\Lambda_2) = K^i$of (Γ_3,Λ_3).

Proof: Recall L, L_1, L_2 *and* L_3 from example 3.3.1.

1. $\Gamma_1(\{Pakistani, Male, Pink\}) = \{\sigma^2,\sigma^3\} = \{\sigma^2(Perfect),\sigma^3(low)\} = L_1$ $\because$ $\sigma^2(Perfect) \in L_1$ *also* $\sigma^3(low) \in L_1$

$$\Rightarrow \sigma^2,\sigma^3 \in L_1$$

Thus $(\Gamma_1,\Lambda_1) \subseteq L_1 = (\Gamma_1,\Lambda_1)$.

2. Consider $L_1 = (\Gamma_1,\Lambda_1)$

$\because$ $\phi \in L_1$ $\Rightarrow$ $(\Gamma_1,\Lambda_E)^{\phi} \in L_1$

Thus $(\Gamma_1,\Lambda_E)^{\phi} \subseteq L_1 = (\Gamma_1,\Lambda_1)$ $(\Gamma_1,\Lambda_E)^{\phi} \subseteq (\Gamma_1,\Lambda_1)$.

3. Consider $L = \{\sigma^2(Perfect),\sigma^3(None)\}$, apply definition 6, *we get*, $(\sim L) = \{\sigma^1(none),\sigma^4(perfect)\}$ again apply definition 6, we get;

$\sim(\sim L) = \{\sigma^2(Perfect),\sigma^3(None)\} = L$

4. Consider $(\Gamma_1,\Lambda_E)^{\phi} = \phi \Rightarrow \phi \in L_E$ taking complement, $\sim(L_E) = \Omega\backslash\ \Gamma_1(\ell) = \phi$;

$\Rightarrow \sim(L_E) = \Omega$

hence $\sim(\Gamma_1,\Lambda_E)^{\phi} = \Omega$.

5. Consider, $(\Gamma_1,\Lambda_1) = \{\sigma^1(high),\sigma^3(low)\}$

$$(\Gamma_2,\Lambda_2) = \{\sigma^1(high),\sigma^3(low)\}$$

Each linguistic variable K^i of $(\Gamma_1,\Lambda_1) =$ linguistic variable K^i of (Γ_2,Λ_2) then this implies that $(\Gamma_1,\Lambda_1) \subseteq (\Gamma_2,\Lambda_2)$ *also* $(\Gamma_2,\Lambda_2) \subseteq (\Gamma_1,\Lambda_1)$

thus $(\Gamma_2,\Lambda_2) = (\Gamma_1,\Lambda_1)$.

Counter Example:

Consider,

$(\Gamma_1,\Lambda_1) = \{\sigma^2(high),\sigma^3(very\ \ low)\}$

and

$$(\Gamma_2,\Lambda_2) = \{\sigma^2(perfect),\sigma^3(low)\}$$

Each linguistic variable K^i of $(\Gamma_1,\Lambda_1) <$ linguistic variable K^i of (Γ_2,Λ_2) then this implies that $(\Gamma_1,\Lambda_1) \subseteq (\Gamma_2,\Lambda_2)$ *But* $(\Gamma_2,\Lambda_2) \nsubseteq (\Gamma_1,\Lambda_1)$ since linguistic variable of $(\Gamma_2,\Lambda_2) >$ linguistic variable of (Γ_1,Λ_1).

$$(\Gamma_2,\Lambda_2) \neq (\Gamma_1,\Lambda_1)$$

6. Same as 5.

4. Operational Laws on LHSS

In this section, we discuss the importance of operational laws and theorems and propose for LHSS. Let $(\Gamma_1,\Lambda_1) = L_1$ and $(\Gamma_2,\Lambda_2) = L_2$ be two LHSS and $\mu \geq 0$, where $\Lambda_1 = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n$: n is finite and real valued} over Ω with mapping $\Gamma : \Lambda_1 = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n \longrightarrow P(\Omega)$ and $\Lambda_2 = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^m$: m is finite and real valued} over Ω with mapping $\Gamma_2 : \Lambda_2 = \Upsilon^1 \times \Upsilon^2 \times$

$\Upsilon^3 \times \ldots \times \Upsilon^m \longrightarrow P(\Omega)$. Then the operational laws on LHSS can be defined with some necessary conditions;

Definition 4.1 Union of LHSS

Case 1: $L_1 \cup L_2 = \left\{\prod \alpha^i(K^i) \times \prod \alpha^j(K^j) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\right\}$

Where, $\alpha^i(K^i) \in \prod_{i=1}^{n} \Upsilon^i, and \quad \alpha^j(K^j) \in \prod_{j=1}^{n} \Upsilon^j \, should \;\; be \;\; distinct$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, t\}$.

Case 2: $L_1 \cup L_2 = \left\{\alpha^i(K^i) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\right\}$

$with \;\; i = j \;\; , and \;\; linguistic$ variable $K^i \;\; of \;\; \sigma^i$ $should \;\; be \;\; same.$

Example: Consider,

Case 1;

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$$

$$\Gamma_2(\{American, Female, Pink\}) = \{\sigma^1(high), \sigma^4(low)\} = L_2$$

$\because \Upsilon^i \cap \Upsilon^j = \emptyset$

$L_1 \cup L_2 = \{\sigma^2(Perfect), \sigma^3(low), \sigma^1(high), \sigma^4(low)\}$.

Case 2;

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$$

$$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect), \sigma^3(low)\} = L_2$$

$\because \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $i = j$

$L_1 \cup L_2 = \{\sigma^2(Perfect), \sigma^3(low)\}$.

Case 3; (Counter example) \Restriction

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(high), \sigma^3(verylow)\} = L_1$$

$$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect), \sigma^3(low)\} = L_2$$

$\because \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $i = j$

Each linguistic variable K^i of $L_1 <$ linguistic variable K^i of L_2 then this implies $L_1 \cup L_2$ can be defined with some restriction i.e. consider highest linguistic value K^i of each attribute.

Example: $L_1 = \{\sigma^2(high), \sigma^3(verylow)\}$

$$L_2 = \{\sigma^2(perfect), \sigma^3(low)\}$$

As, $\sigma^2(high) < \sigma^2(perfect), and$

$$\sigma^3(verylow) < \sigma^3(low)$$

Then $L_1 \cup L_2 = \{\sigma^2(perfect), \sigma^3(low)\}$.

Definition 4.2 Intersection of LHSS

Let $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ be two LHSS and $\mu \geq 0$, then the intersection can be defined as;

$$L_1 \cap L_2 = \left\{\prod \alpha^i(K^i) \times \prod \alpha^j(K^j) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\right\} = \emptyset$$

Where, $\alpha^i(K^i) \in \prod_{i=1}^{n} \Upsilon^i$, *and* $\alpha^j(K^j) \in \prod_{j=1}^{n} \Upsilon^j$ *should be distinct* with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i = j$, and $i, j \in \{1, 2, \ldots, t\}$.

Case 2: $L_1 \cap L_2 = \{\alpha^i(K^i) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\}$

with $i = j$ *,and linguistic* variable K^i *of* σ^i Then $L_1 \cap L_2 = L_1$ *or* L_2

Example: Consider,

Case 1;

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$$

$$\Gamma_2(\{American, Female, Pink\}) = \{\sigma^1(high), \sigma^4(low)\} = L_2$$

$$\because \Upsilon^i \cap \Upsilon^j = \emptyset$$

$$L_1 \cap L_2 = \{\emptyset\}$$

Case 2;

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$$

$$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect), \sigma^3(low)\} = L_2$$

$\because \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $i = j$

$L_1 \cap L_2 = \{\sigma^2(Perfect), \sigma^3(low)\}$.

Case 3; (Counter example) \Restriction

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(high), \sigma^3(verylow)\} = L_1$$

$$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect), \sigma^3(low)\} = L_2$$

$\because \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $i = j$

Each linguistic variable K^i of $L_1 <$ linguistic variable K^i of L_2 then this implies $L_1 \cup L_2$ can be defined with some restriction i.e. consider highest linguistic value K^i of each attribute.

Example: $L_1 = \{\sigma^2(high), \sigma^3(verylow)\}$

$$L_2 = \{\sigma^2(perfect), \sigma^3(low)\}$$

As, $\sigma^2(high) < \sigma^2(perfect)$, *and*

$$\sigma^3(verylow) < \sigma^3(low)$$

Then $L_1 \cap L_2 = \emptyset$.

Theorem 4.3: If $\mathbf{L_1}$, $\mathbf{L_2}$ and $\mathbf{L_3}$ be three LHSS then the following holds:

i. $L_1 \cup L_1 = L_1$
ii. $L_1 \cup \emptyset = L_1$
iii. $L_1 \cap L_1 = L_1$
iv. $L_1 \cap \emptyset = \emptyset$
v. $L_1 \cup L_2 = L_2 \cup L_1$
vi. $L_1 \cap L_2 = L_2 \cap L_1$
vii. $L_1 \cup (L_2 \cup L_3) = (L_1 \cup L_2) \cup L_3)$
viii. If $L_1 \subset L_2$ and $L_2 \subset L_1$ the $L_1 = L_2$.
ix. $\mu(L_1) = \mu L_1$; $\mu \geq 0$.
x. $\mu(L_1 \cup L_2) = \mu(L_2 \cup L_1)$

The proofs are straight forward. ∎

Theorem 4.4

If L_1, L_2 be two LHSS then the operations are given as follows:

1. $\mu \times L_1 = L_{\mu\times 1}$; μ (linguistic variable);
2. $L_1 \oplus L_2 = L_{1\oplus 2}$;
3. $L_1 \otimes L_2 = L_{1\otimes 2}$;
4. $(L_1)^{\mu} = L_{1^{\mu}}$.

Proof:

1. Consider, $\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$ and $\mu =$ low,

The proofs are straight forward. ∎

5. Some Aggregation Operators

Aggregate operators play a crucial role in decision-making processes, and their importance cannot be overstated. These operators are responsible for combining and aggregating individual linguistic quantifiers or numerical values to derive a comprehensive assessment of various factors and attributes. By employing aggregate operators, decision-makers can effectively analyze and evaluate complex information, facilitating informed decision-making. One of the key benefits of aggregate operators is their ability to handle multiple criteria simultaneously. Decision-making often involves considering various factors, such as cost, quality, reliability, and customer satisfaction. Aggregate operators enable decision-makers to combine and weigh these criteria appropriately, considering their relative importance. This allows for a comprehensive evaluation and comparison of different options or alternatives.

Aggregate operators also provide a means to summarize and condense large amounts of data into manageable and meaningful information. They enable decision-makers to reduce complex and diverse inputs into a single aggregated value or linguistic quantifier. This simplification aids in understanding and interpreting the information, making it easier to make decisions based on the aggregated results. Moreover, aggregate operators facilitate the integration of subjective or qualitative assessments into the decision-making process. They provide a mechanism to convert linguistic expressions, which often involve subjective opinions or judgments, into numerical values that can be analyzed and compared objectively. This enables decision-makers to incorporate both objective and subjective information, leading to more comprehensive and well-rounded decisions. Additionally, aggregate operators allow for flexibility and adaptability in decision-making. Different aggregate operators, such as weighted averages, minimum or maximum operations, and fuzzy logic operators, offer diverse ways to combine and aggregate data. This flexibility enables decision-makers to tailor the aggregation process to their specific needs and preferences, accommodating different decision contexts and requirements. In conclusion, aggregate operators are vital tools in decision-making processes, enabling the integration, evaluation, and comparison of diverse criteria and information. They enhance the ability to handle multiple factors, summarize complex data, incorporate subjective assessments, and provide flexibility in decision-making. By leveraging aggregate operators effectively, decision-makers can make more informed and well-founded decisions, leading to improved outcomes and increased overall effectiveness in various domains.

Aggregate operators are required by decision-makers (DMs) to rank the given alternatives. The ordered weighted averaging operator (OWAO) proposed by Yager [25] is the most widely used methodology for aggregating decision information. Later, various new OWAO were introduced [26]. The OWAO has been employed in an amazingly wide range of applications [24, 27]. The majority of these operators, on the other hand, can only be employed when the input arguments are exact values, and only a few of them can be used to aggregate linguistic preference data.

Decision-making, on the other hand, is influenced by personal psychological factors such as experience, learning, situation, mood, like-dislike, and so on. It is more appropriate to express their preferences using linguistic parameters rather than numerical variables. Thus, in this section the aggregation operators for hypersoft set have been proposed.

Definition 5.1 LHSWGAO

Consider, $\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \ldots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, t\}$.

Let $\mathfrak{A}: \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^t \longrightarrow P(\Omega) = \{\mathrm{M}(\Omega)(i)\} \subseteq \mathbb{R}^+$ (1)

if $\mathfrak{A}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$

Such that

$$\mathfrak{A}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) = {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \ldots \otimes {\alpha^t}_i^{\omega^t} = \sigma_i$$

Where $\omega = (\omega^1, \omega^2, \omega^3, \ldots, \omega^t)^T$ is the exponential weighting vector of the $\alpha^t(i) \in \{\mathrm{M}(\Omega)(i)\}$ and $\omega^t \in [0, 1]$ with $\sum_{t=1}^{n} \omega^t = 1$, and $i \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}$ Then $\mathfrak{A}$ is called Linguistic Hypersoft Weighted Geometric Averaging Operator (LHSWGAO).

Example: Assume $\omega = (0.4, 0.3, 0.3)^T$ then LHSWGAO $\{\sigma^2(Pakistani, Male, Orange)$, $\sigma^3(Pakistani, Male, Orange)\}$

The linguistic set of definition 2, is labeled as

Table 2. linguistic quantifiers

	Quantifiers					
Low		Medium			High	
None	Very-Low	Low	Medium	High	Very-High	Perfect
0	1	2	3	4	5	6

$= \sigma^2 \begin{pmatrix} Pakistani(low), Male(medium), \\ Orange(high) \end{pmatrix}$

$$\because \mathfrak{A}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$$

$= \alpha 1_i^{\omega^1} \otimes \alpha 2_i^{\omega^2} \otimes \alpha 3_i^{\omega^3} \otimes \dots \otimes \alpha t_i^{\omega^t} = \sigma_i$

= {Pakistani$(low)^{0.4}$, Male$(medium)^{0.3}$,

Orange$(high)^{0.3}$}

$= \sigma^2\{(2)^{0.4} + (3)^{0.3} + (4)^{0.3}\}$

$= \sigma^2(i)^{4.22}$

$= \sigma^2(i)^{4.22} = \sigma^2(High)$ From Table 2, the linguistic value for $i = 4$ is $high$.

Now,

$= \sigma^3 \begin{pmatrix} Pakistani(none), Male(none), \\ Orange(none) \end{pmatrix}$

$$\because \mathfrak{A}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$$

$= \alpha 1_i^{\omega^1} \otimes \alpha 2_i^{\omega^2} \otimes \alpha 3_i^{\omega^3} \otimes \dots \otimes \alpha t_i^{\omega^t} = \sigma_i$

= {Pakistani$(none)^{0.4}$, Male$(none)^{0.3}$,

Orange$(none)^{0.3}$}

$= \sigma^2\{(0)^{0.4} + (0)^{0.3} + (0)^{0.3}\}$

$= \sigma^2(0)$

$= \sigma^2(none)$

From Table 2, the linguistic value for $i = 0$ is $none$.

Definition 5.2 LHSOWGAO

Consider, $\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \ldots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, t\}$.

Let $\mathfrak{D}$: $\Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^t \longrightarrow P(\Omega) = \{\mathrm{M}(\Omega)(i)\} \subseteq \mathbb{R}^+$ (2)

If $\mathfrak{D}^\omega \ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$

Such that $\mathfrak{D}^\omega \ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) =$

$$\alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \ldots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i$$

Subject to the condition, the linguistic values of α_i should be in ascending order. Where $\omega = (\omega^1, \omega^2, \omega^3, \ldots, \omega^t)^T$ is the exponential weighting vector of the $\alpha^t(i) \in \{\mathrm{M}(\Omega)(i)\}$ and $\omega^t \in [0, 1]$ with $\sum_{t=1}^{n} \omega^t = 1$, and $i \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}$ then $\mathfrak{D}$ is called Linguistic Hypersoft Ordered Weighted Geometric Averaging Operator (LHSOWGAO).

Example:

Assume $\omega = (0.4, 0.3, 0.3)^T$ then LHSOWGAO $\{\sigma^2(Pakistani, Male, Orange),$
$\sigma^3(Pakistani, Male, Orange)\}$

The linguistic set of definition 2, is labeled as;

$$= \sigma^2 \begin{pmatrix} Pakistani(low), Male(medium), \\ Orange(high) \end{pmatrix}$$

$$\because \mathfrak{D}^\omega \ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$$

$= \alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \ldots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i$
$= \{\mathrm{Pakistani}(high)^{0.4}, \mathrm{Male}(medium)^{0.3},$
$\mathrm{Orange}(low)^{0.3}\}$

$= \sigma^2\{(4)^{0.4} + (3)^{0.3} + (2)^{0.3}\}$
$= \sigma^2(i)^{4.36}$

$= \sigma^2(i)^{4.36} = \sigma^2(High)$ From Table 2, the linguistic value for $i = 4$ is $high$.

Now,

$$= \sigma^3 \begin{pmatrix} Pakistani(none), Male(none), \\ Orange(none) \end{pmatrix}$$

$$\because \mathfrak{D}^\omega \ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$$

$= \alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \ldots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i$

$= \{\text{Pakistani}(none)^{0.4}, \text{Male}(none)^{0.3},$
$\text{Orange}(none)^{0.3}\}$
$= \sigma^2\{(0)^{0.4} + (0)^{0.3} + (0)^{0.3}\}$
$= \sigma^2(0)$
$= \sigma^2(none)$

From Table 2, the linguistic value for $i = 0$ is $none$.

Theorem 5.1:

1. $\min_i(\alpha^t(i)) \leq \mathfrak{A}^{\omega}(\alpha^1, \alpha^2, .., \alpha^t) \leq \max_i(\alpha^t(i))$

2. $\min_i(\alpha^t(i)) \leq \mathfrak{D}^{\omega}(\alpha^1, \alpha^2, .., \alpha^t) \leq \max_i(\alpha^t(i))$

Proof: The proofs are straight forward. ■

Theorem 5.2:

1. $\mathfrak{D}^{\omega}(\alpha^t(i)) = \mathfrak{D}^{\omega}(\alpha^t(i'))$

Where $(\alpha^t(i'))$ is any permutation of $(\alpha^t(i))$

2. If $\forall(\alpha^t(i)) = (\alpha(i))$ for all t, then $\mathfrak{D}^{\omega}(\alpha^t(i)) = \boldsymbol{\sigma}_i$

3. If $(\alpha^t(i)) \leq (\widehat{\alpha}^t(i))$ for all t, then

$$\mathfrak{D}^{\omega}(\alpha^t(i)) \leq \mathfrak{D}^{\omega}(\widehat{\alpha}^t(i))$$

Proof: The proofs are straight forward. ■

6. Multi-Criteria Decision-Making Method (LHSS Algorithm to solve MCDM Problem)

A decision-making technique based on linguistic hypersoft weighted geometric averaging operator (LHSWGAO) has been used to construct an algorithm known as linguistic hypersoft set based multi-criteria group decision-making method (LHSS algorithm to solve MCGDM problem). The graphical representation of the proposed LHSS algorithm is presented in Figure 2.

Step1: *Consider,* $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ *for* $t \geq 1$ *be* t *distinct parameters, whose corresponding parametric values are respectively the sets* $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ *with* $\Upsilon^i \cap \Upsilon^j = \emptyset$, *for* $i \neq j$, *and* $i, j \in \{1, 2, \dots, t\}$.
Let $\omega = (\omega^1, \omega^2, \omega^3, \dots, \omega^t)^T$ *be the exponential weighting vector. Where* $\omega^t \geq 0$ *, and* $\sum_{t=1}^{n} \omega^t = 1$.
Let $\mathfrak{A}$: $\Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \longrightarrow P(\Omega) = \{M(\Omega)(i)\} \subseteq \mathbb{R}^+$
The decision-makers $\mathcal{D}^m$ *compare the values with the linguistic quantifiers and assign linguistic variable to each alternative as* $H_i = \{(\alpha^t(i) : \ i = 1, 2, \dots, t\}$, *and construct a linguistic preference table for* $(\alpha^t(i))^{(\omega^t)}$.

Step2: *Construct a matrix* $[\sigma_i^j, s_i^j]_{i \times j}$ *for each* $\mathcal{D}^m$ *using linguistic hypersoft weighted geometric averaging operator (LHSWGAO),*

$$\sigma_i^t = {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \dots \otimes {\alpha^t}_i^{\omega^t}$$

Construct a matrix individually for each $\mathcal{D}^m$ *using linguistic hypersoft ordered weighted geometric averaging operator (LHSOWGAO)*

$$\mathfrak{D}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$$

Such that $\mathfrak{D}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) = {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \ldots \otimes {\alpha^t}_i^{\omega^t} = s_i$

Step3: *Construct a matrix using* $[min(\sigma_i^j, s_i^j)]_{\ i\ \times j}$ *for each* $\mathcal{D}^m$.

Step4: List *max value among all the decision-makers.*

$$\max\ [\mathcal{D}^1 \min(\sigma_i^j, s_i^j), \mathcal{D}^2 \min(\sigma_i^j, s_i^j), \ldots, \mathcal{D}^m \min(\sigma_i^j, s_i^j)]_{\ i\ \times j}$$

Step5: *Write value from linguistic table or reference table* known as total score.

Step6: *Finally, list the alternatives with total scores* H_i *and rank highest value.*

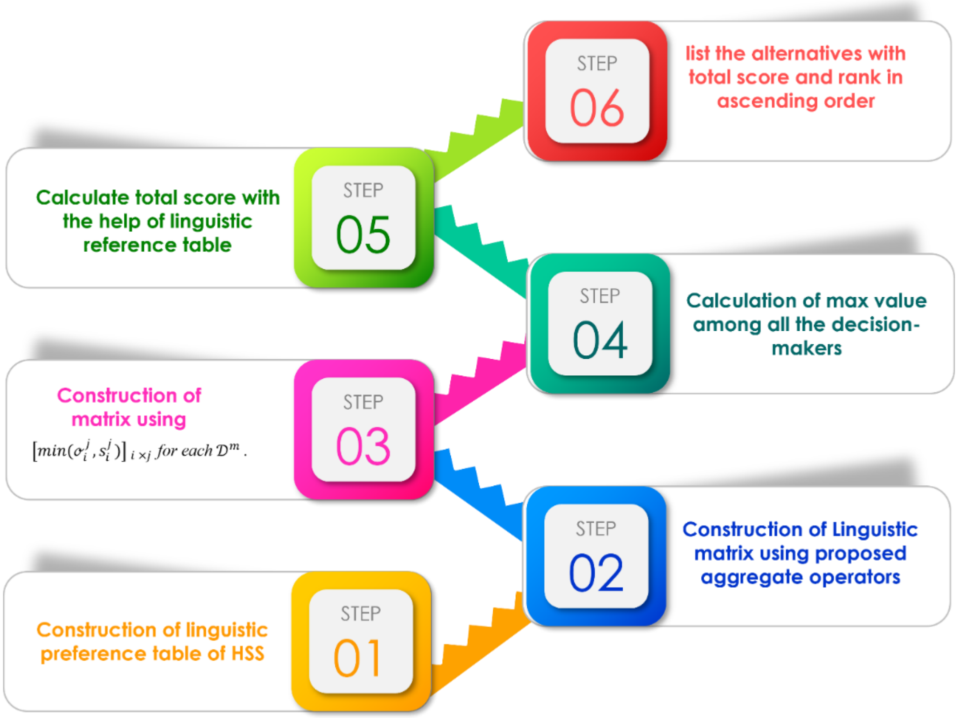

Figure 2. Graphical representation of Proposed LHSS algorithm

6.1 Illustrative example

Problem: Inequitable access to healthcare in rural areas

Rural areas often have limited access to healthcare services due to a shortage of healthcare providers and facilities. This can result in poorer health outcomes for rural residents compared to their urban counterparts. The current policy approach to addressing this problem includes initiatives such as the Sehat Card in Pakistan, which aims to increase access to healthcare services for all Pakistanis. However, the health service has faced challenges related to hospital affordability, accessibility, and political opposition.

1. *Increase funding for rural healthcare services and providers:*

This solution would involve providing more resources for rural healthcare services and providers, such as funding for healthcare facilities, equipment, and staff. It could potentially improve access to healthcare services in rural areas and address the issue of healthcare workforce shortages. However, it may be costly and could face political opposition.

2. *Incentivize healthcare providers to work in rural areas:*

This solution could involve offering financial incentives or loan forgiveness to healthcare providers who work in rural areas. It could help address the issue of healthcare workforce shortages in these areas and improve access to healthcare services. However, it may be difficult to implement and could face resistance from healthcare providers who prefer to work in urban areas.

3. *Invest in infrastructure improvements to support healthcare delivery in rural areas:*

There is no fixed percentage or rule for how much focus should be given to each solution when addressing a public policy problem. The appropriate mix of solutions will depend on various factors, such as the context of the problem, the goals of the policy, the available resources, and the political environment. In general, when developing policy solutions, it is important to consider a range of options and evaluate their feasibility, effectiveness, and potential impacts on equity. This can involve conducting research, consulting with stakeholders, and considering multiple perspectives. The goal should be to identify a set of solutions that are likely to achieve the desired outcomes, while minimizing any unintended negative consequences.

6.2 Demonstration of proposed example

Consider $H = \{H^1, H^2, H^3\}$ *be* three hospitals as alternatives in rural area, and we want to improve the health services. The services of the experts in this domain has been taken and known as decision-makers $\mathcal{D} = \{\mathcal{D}^{\mathrm{m}} \;\; ; m = 1,2,3\}$. The goal should be to identify a set of solutions that are likely to achieve the desired outcomes, while minimizing any unintended negative consequences. Consider the parameters be: $\alpha^1 =$ Increase in funding, $\alpha^2 =$ Incentivize healthcare, $\alpha^3 =$ Invest in infrastructure, and their respective parametric values are:

Increase in funding = $\Upsilon^1 = \{fascilities,\ equipment, staff\}$

Incentivize healthcare = $\Upsilon^2 = \{healthcare\ \ workforce, improve\ \ access\ \}$

Invest in infrastructure = $\Upsilon^3 = \{Transportation, telemedicine\}$

Then the function $\Gamma : \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \longrightarrow P(\Omega)$ and assume the hypersoft set $M = \{H^1, H^2, H^3\} \subset \Omega$ where $\Omega = \{H^1, H^2, H^3\}$ be the universal set.

$\Gamma(\{fascilities, healthcare\ \ workforce, telemedicine\}) = \{H^1, H^2, H^3\} = M$ The linguistic hypersoft set (LHSS), $\Gamma(\{\mathrm{M}(\Omega)(\imath)\}) : M \subseteq \Lambda$ & $\imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}$ *where* $t = 2n + 1 : \ n \geq 1, n \in \mathbb{R}^+\}$ can be given by three decision-makers $\mathcal{D} = \{\mathcal{D}^{\mathrm{m}} \;\; ; m = 1,2,3\}$.

$\mathcal{D}^1 \quad define \quad \Gamma_1(\{fascilities, healthcare\ \ workforce, telemedicine\})$

$= \{H^1, H^2, H^3\}$

$$= \left\{ \begin{matrix} H^1 < fascilities\ \ (low), healthcare\ \ workforce\ \ (high), telemedicine(medium) >, \\ H^2 < fascilities\ \ (v.high), healthcare\ \ workforce\ \ (medium), telemedicine(high) >, \\ H^3 < fascilities\ \ (medium), healthcare\ \ workforce(high), telemedicine(low) > \end{matrix} \right\} = L_1$$

$\mathcal{D}^2 \quad define \quad \Gamma_2(\{equipment, improve\ \ access, transportation\}) = \{H^1, H^2, H^3\}$

$= \{H^1 < equipment(high), improve\ \ access\ \ (v.high), transportation(low) >,$

$H^2 \quad < equipment(low), improve \quad access(v.high), transportation(high) >,$

$H^3 < equipment(medium), improve \quad access(high), transportation(medium) >\}$

$= L_2$

$\mathcal{D}^3 \quad define \quad \Gamma_3(\{staff, healthcare \quad workforce, telemedicine\})$

$= \{H^1 < staff(medium), healthcare \quad workforce(high), telemedicine(v.low) >,$

$H^2 \quad < staff(low), healthcare \quad workforce(low), telemedicine(high) >,$

$H^3 < staff(high), healthcare \quad workforce(high), telemedicine(low) >\}$

$= L_3$

Step1: The decision matrix by decision-makers $\mathcal{D} = \{\mathcal{D}^{\mathrm{m}} \quad : \; m = 1,2,3\}$, presented below.

Refer Table 3.

Step2: Construct a matrix using LHSWGAO, and LHSOWGAO.

$$\begin{matrix} . \\ H^1 \\ H^2 \\ H^3 \end{matrix} = \begin{bmatrix} \mathcal{D}^1 & \mathcal{D}^2 & \mathcal{D}^3 \\ high, medium & medium, high & high, medium \\ v.high, \quad perfect & high, medium & high, v.high \\ high, medium & high, high & high, medium \end{bmatrix}$$

Step3: Find the min of matrix values of step2.

$$\begin{matrix} . \\ H^1 \\ H^2 \\ H^3 \end{matrix} = \begin{bmatrix} \mathcal{D}^1 & \mathcal{D}^2 & \mathcal{D}^3 \\ medium & medium & medium \\ v.high & medium & high \\ medium & high & medium \end{bmatrix}$$

Step4: *Write max value among all the decision-makers.*

$$S = \{H^1 < medium >, H^2 < v.high >, H^3 < high >\}$$

Step5: *Write value from linguistic table.*

$$S = \{H^1 < 4 >, H^2 < 5 >, H^3 < 4 >\}$$

Step6: *Finally, list the alternatives with total scores* $\mathcal{S}_i$ *and rank highest value.*

Alternative	*Score Value*	*Rank*
$\boldsymbol{H^1}$	*3*	*1*
$\boldsymbol{H^2}$	*5*	*3*
$\boldsymbol{H^3}$	*4*	*2*

This solution shows that $H^1 < H^3 < H^2$ involve investing in infrastructure improvements, such as broadband internet access and transportation infrastructure, to support healthcare delivery in rural areas. It could improve access to telemedicine and other remote healthcare services, as well as address transportation barriers to accessing healthcare services. The results are presented in Figure 3.

6.3 Result discussion comparison and future directions

The comparison analysis presented highlights the prioritization of infrastructure improvements for healthcare delivery in rural areas, with the order being $H^1 < H^3 < H^2$. The analysis suggests that investing in infrastructure improvements, such as broadband internet access and transportation infrastructure, can have significant benefits for healthcare accessibility in rural communities.

H^1 represents the highest priority, indicating that addressing healthcare infrastructure deficiencies in rural areas should be the primary focus. This may involve initiatives to improve broadband internet access, which can facilitate telemedicine and remote healthcare services. By enhancing connectivity, individuals in rural areas can access healthcare professionals and receive medical consultations without the need for in-person visits, thereby reducing barriers to healthcare access.

Alternatives	Linguistic Score	Score Value	Ranking
Alternative H^1	$medium$	3	1
Alternative H^2	$v.high$	5	3
Alternative H^3	$high$	4	2

Figure 3. Result and ranking of the alternatives.

H^3 denotes a relatively lower priority compared to H^1but higher than H^2This suggests that while health care infrastructure improvements are crucial, other factors may also need attention. These factors could include policy reforms, financial support, or workforce development to complement the infrastructure enhancements. A comprehensive approach that combines infrastructure improvements with these additional measures can yield a more effective and sustainable healthcare system in rural areas.

H^2 represents the lowest priority, indicating that while still important, addressing transportation barriers to healthcare access may be of lesser immediate significance compared to infrastructure enhancements. Transportation infrastructure improvements could include better roads, public transportation systems, or medical transport services to ensure that individuals can reach healthcare facilities conveniently and efficiently.

The power of the proposed method lies in its ability to overcome the limitations of existing approaches by providing a systematic and effective framework for dealing with linguistic knowledge in decision-making. By offering a mathematically grounded solution, the method enhances the accuracy, precision, and applicability of decision-making processes, contributing to advancements in the field.

To extend the proposed method for complicated cases, additional considerations and modifications can be made to the existing framework. This can be demonstrated through numerical examples that illustrate the application of the extended method. Here is an explanation of how the proposed method can be extended along with numerical examples:

1. Handling Complicated Cases: In complex decision-making scenarios, where multiple factors and attributes need to be considered, the proposed method can be expanded to accommodate these complexities. This can be achieved by incorporating weighted linguistic quantifiers or linguistic variables that represent the relative importance or degree of each factor.
2. Numerical Examples: Let's consider a decision-making problem involving the selection of a new supplier for a company. The decision criteria include factors such as price, quality, delivery time, and customer service. Each factor can be represented by linguistic quantifiers, such as "low," "medium," or "high." To extend the method, weights can be assigned to these linguistic quantifiers based on their relative importance.

For instance, let's assume the weight assigned to price is 0.4, quality is 0.3, delivery time is 0.2, and customer service is 0.1. The linguistic quantifiers for each factor can be mapped to numerical values using the proposed LHSS framework. Suppose "low" corresponds to 1, "medium" corresponds to 3, and "high" corresponds to 5. Now, let's assume we have three potential suppliers: Supplier A, Supplier B, and Supplier C. We can evaluate each supplier's performance on each factor and calculate an overall score based on the assigned weights. The scores can be computed by multiplying the numerical value of each linguistic quantifier by its weight and summing up the results. For example, Supplier A may have a price score of (1 * 0.4), a quality score of (3 * 0.3), a delivery time score of (3 * 0.2), and a customer service score of (5 * 0.1). Summing up these scores, we obtain an overall score for Supplier A. Similarly, we can calculate the overall scores for Supplier B and Supplier C. Based on the calculated overall scores, the company can then make an informed decision on selecting the most suitable supplier. These numerical examples demonstrate the extension of the proposed method to handle complicated decision-making scenarios by incorporating weighted linguistic quantifiers. By assigning weights and mapping linguistic quantifiers to numerical values, the method allows for a more comprehensive and precise evaluation of complex decision criteria.

7. Conclusion

This paper acknowledges the complexity of the relationship between language and meaning, highlighting the challenges in assigning numerical values to linguistic variables. To address this issue, the concept of LHSS (Linguistic Hypersoft Set Structure) along with operational laws, aggregate operators, and MCGDM (Multi-Criteria Group Decision Making) techniques has been proposed. The application of these concepts has been demonstrated through a real-life case study, yielding promising results. The findings indicate that assigning numerical values to linguistic variables enhances accuracy in decision-making. Future directions include extending the framework to complex decision-making scenarios, integrating it with machine learning and AI techniques, exploring hybrid set structures, conducting real-world case studies, focusing on human-computer interaction, addressing ethical and social implications, and advancing user-centric approaches. These future directions aim to enhance the applicability, effectiveness, and ethical use of linguistic set structures in decision-making across various domains.

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