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Neutro Open Set-Based Strong Neutro Metric Space

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Abstract. Memet et al. interposed the concept of neutro metric spaces. They established this concept depend on the neutro axioms from the neutro function one of the ideas of Smarandache. In the concept of neutro metric spaces used the non-negative and negative real numbers. We introduced this notion depend on the non-negative real numbers and redefined the neutro metric space depend on the strong neutro metrics. We investigate the properties of strong neutro metric spaces and present the notion of neutro-open sets and neutro-closed sets. We distinguished between the operations on neutro-open sets in strong neutro metric space and open sets in metric spaces.

Keywords: Strong neutro metric(space), neutro open set, neutro closed set, unified set.

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1. Introduction

Florentin Smarandache [8], considering the facts in the real world, he introduced the theory of neutro algebra. According to him, a system in which everything is right or everything is wrong either does not exist, or if it exists, it is not real. In the theory of neutro algebra, he deals with the issue that some principles may be true and some principles may not be true, and this is closer to the problems in the real world. In 1906 M. Frechet introduced metric spaces [6] as a mathematical tool in the rela world and other researchers continued it in some branches. Recently, Memet et al. introduced the notion of neutro metric spaces [7]. Some researchers have investigated the neutro structures such as [1–6, 9]

We introduce a drawing out of metric spaces, whatever is a distribution of topologic spaces. Our intention in headlining this matter is to design the principles of the matter in kind to contest the matter of metric space theory. We exercise the axioms of metric spaces and illustrate the notion of strong neutro metric spaces and investigate their properties. This paper introduces the notion of open balls in strong metric space with the same as the notion of open balls in metric space and depend on this notion, we present the notion of neutro open balls. We show that the union of any family of unified neutro open sets is a neutro open set and the intersection of any family of chain neutro open sets is a neutro open set, while the intersection of a finite set of an open set is an open set in metric space. Indeed, we distinguished the fundamental structures of metric spaces and fundamental structures of neutro metric spaces.

2. Preliminaries

We need materials that have been reviewed before and are effective in our article, so we will address them in this section.

Definition 2.1. [8] For any non-avoided set X, (X, σ) is a neutro algebra, if σ is a neutro operation.

Definition 2.2. [7] Allow $X \neq \emptyset$ and $\sigma : X^2 \to \mathbb{R}$. Then, (X, σ) is titled a *neutro metric space (neutro M. S)* if, there is

$(NM\text{-}1)$ $\big(x, y \in X$ intent, $x\sigma y \geq 0$, $\big(r, s \in X$ intent, $x\sigma y < 0$ or inconclusive $\big)$;

$(NM\text{-}2)$ $\big(x \in X$ intent, $x\sigma x = 0$, $\big(y \in X$ intent, $x\sigma x \neq 0$ or inconclusive $\big)$;

$(NM\text{-}3)$ $\big(x, y, z, w \in X$, intent, $x\sigma z \leq x\sigma y + y\sigma z$, $\big(r, s, t, v \in X$, intent, $x\sigma z > x\sigma y + y\sigma z$ or inconclusive $\big)$;

$(NM\text{-}4)$ $\big(x, y, z, w \in X$, intent, $x\sigma y = y\sigma x$, $\big(r, s \in X$, intent, $r\sigma s \neq s\sigma r$ or inconclusive $\big)$.

3. Strong neutro open sets

Definition 3.1. Allow $X \neq \emptyset$ and $\sigma : X^2 \to \mathbb{R}^{\geq 0}$. Then, (X, σ) is titled a *strong neutro M. S* if,

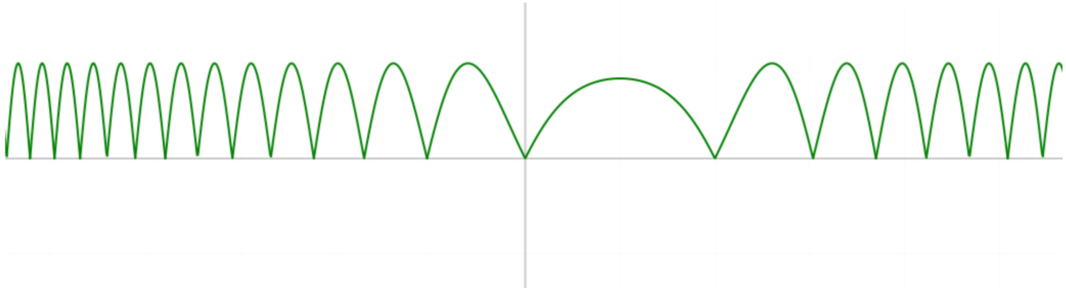
$(NM\text{-}1)$ $\big(x \in X$ intent, $x\sigma x = 0$ and $\big(y \in X$ intent, $y\sigma y \neq 0$ or inconclusive $\big)$;

$(NM\text{-}2)$ $\big(x, y, z, w \in X$, intent, $x\sigma z \leq x\sigma y + y\sigma z$ and $\big(r, s, t, v \in X$, intent, $r\sigma t > r\sigma s + s\sigma t$ or inconclusive $\big)$;

$(NM\text{-}3)$ $\big(x, y, z, w \in X$, intent, $x\sigma y = y\sigma x$ and $\big(r, s \in X$, intent, $r\sigma s \neq s\sigma r$ or inconclusive $\big)$.

Example 3.2. Illustrate $\sigma : \mathbb{R}^2 \to \mathbb{R}^{>0}$ by $x\sigma y = |Sin(xy - x)|$, where $x, y \in \mathbb{R}$. By Figure 1, $x\sigma x = |Sin(x^2 - x)|$ and easy to see that there exists $y \in \mathbb{R}$ in kind $y\sigma y = 0$ and there exists $z \in \mathbb{R}$ in kind $z\sigma z \neq 0$. Accordingly the item $(NM) - 1$ is valid. Inaddition, $x\sigma y = y\sigma x$, infers $|Sin(xy - x)| = |Sin(xy - y)|$. If $x = y$, then $x\sigma y = y\sigma x$ and for $x = 0$ and $y = \frac{\pi}{2}$, we have $x\sigma y \neq y\sigma x$. Accordingly the item $(NM) - 2$ is valid. If $yz = y$ and $y = z$, then $x\sigma z \leq x\sigma y + y\sigma z$ and for $x = \frac{\pi}{2}, y = 1$ and $z = 0$, we get that $x\sigma z > x\sigma y + y\sigma z$ and so the item $(NM) - 3$ is valid.

FIGURE 1. $|Sin(x^2 - x)|$



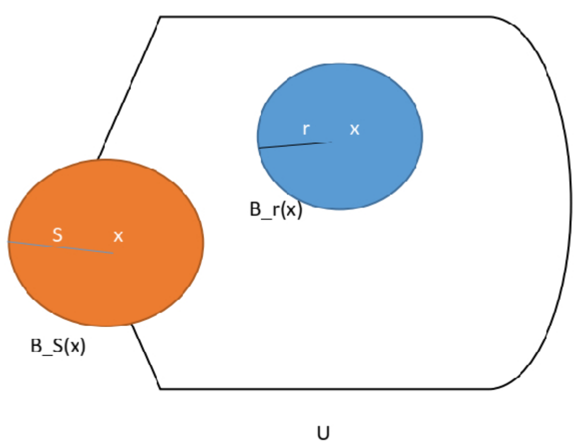
Allow (X, σ) be a strog neutro M. S and $r \in \mathbb{R}^{>0}$. Then $B_r(x) = \{y \in X \mid x\sigma y < r\}$, is the open ball of radius r and center $x \in X$.

Example 3.3. Consider the strog neutro M. S $(\mathbb{R}, \sigma)$ by $x\sigma y = |Sin(xy - x)|$. Computation show that $B_1(x) = \mathbb{R}$ and for all $x \neq 0, B_1(x) = \mathbb{R} \setminus \{\frac{x \pm \frac{\pi}{2}}{x}, \frac{x \pm \frac{3\pi}{2}}{x}, \frac{x \pm \frac{5\pi}{2}}{x}, \ldots\}$. Also for any $r > 1$, we get that $B_r(x) = \mathbb{R}$.

Definition 3.4. Allow (X, σ) be a strog neutro M. S and $U \subseteq X$. Then U is a neutro open set(N. O. S), if $x \in U, r \in \mathbb{R}^{>0}$ in kind $B_r(x) \subseteq U$ and $y \in X, s \in \mathbb{R}^{>0}$ in kind $B_s(x) \nsubseteq U$. From now on, will denote the set of all strog N. O. S of X by $\mathcal{NO}(X)$.

Example 3.5. (i) Allow X be a nonavoid set and $U \subseteq X$. One can see a sample unified set X in Figure 2.

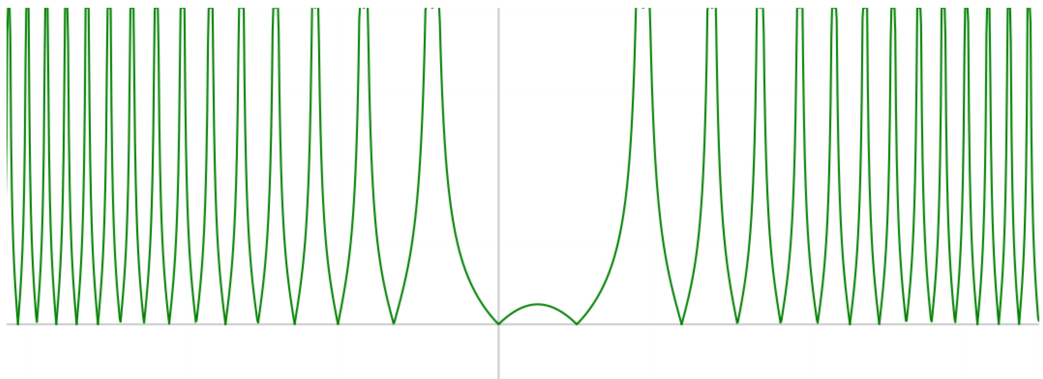
FIGURE 2. Strog N. O. S U



(ii) Illustrate $\sigma : \mathbb{R}^2 \to \mathbb{R}^{>0}$ by $x\sigma y = |tan(xy - x)|$, where $x, y \in \mathbb{R}$. By Figure 3, $x\sigma x = |tan(x^2 - x)|$ and easy to see that there exists $y \in \mathbb{R}$ in kind $y\sigma y = 0$ and there exists $z \in \mathbb{R}$

in kind $z\sigma z \neq 0$. Accordingly the item $(NM)-1$ is valid. As well, $x\sigma y = y\sigma x$, infers $|tan(xy-x)| = |tan(xy-y)|$. If $x = y$, then $x\sigma y = y\sigma x$ and for $x = 0$ and $y = \frac{\pi}{4}$, we have $x\sigma y \neq y\sigma x$. Accordingly the item $(NM)-2$ is valid. If $yz = y$ and $y = z$, then $x\sigma z \leq x\sigma y + y\sigma z$ and for $x = \frac{\pi}{4}, y = 1$ and $z = 0$, we get that $x\sigma z > x\sigma y + y\sigma z$ and so the item $(NM)-3$ is valid.

FIGURE 3. $|tan(x^2-x)|$



Computations show that

$$B_1(\frac{\pi}{4}) = \{y \in \mathbb{R} \mid y\sigma\frac{\pi}{4} < 1\}$$
$$= \{y \in \mathbb{R} \mid |tan(\frac{\pi}{4}y - \frac{\pi}{4})| < 1\}$$
$$= \big((-\infty,2)\cap(-\infty,6)\cap(-\infty,10)\cap\ldots\big)\cup(4,\infty)\cap(8,\infty)\cap(12,\infty)\cap\ldots$$
$$= \big(\bigcap_{k\in\mathbb{O}}(-\infty,2k)\big)\cap\big(\bigcap_{k\in\mathbb{N}}(2k,\infty)\big)$$
$$= (-\infty,2)\cap\big(\bigcap_{k\in\mathbb{N}}(2k,\infty)\big)$$
$$= \emptyset.$$

Consider $0 \in U = (-1,\infty)$. Then

$B_1(0) = \{y \in \mathbb{R} \mid y\sigma 0 < 1\} = \{y \in \mathbb{R} \mid |tan(0)| < 1\} = \mathbb{R}$, which $B_1(0) \not\subseteq U$.

Consider $-\frac{\pi}{4} \in U = (-1,\infty)$. Then

$$B_1(-\frac{\pi}{4}) = \{y \in \mathbb{R} \mid y\sigma - \frac{\pi}{4} < 1\}$$
$$= \{y \in \mathbb{R} \mid |tan(-\frac{\pi}{4}y + \frac{\pi}{4})| < 1\}$$
$$= \big((0,\infty)\cap(-4,\infty)\cap(-8,\infty)\cap\ldots\big)\cap\big((-\infty,-2)\cap(-\infty,-6)\cap(-\infty,-10)\cap\ldots\big)$$
$$= \bigcap_{k\in\mathbb{W}}(-4k,\infty)\cap\bigcap_{k\in\mathbb{O}}(\infty,-2k) = \emptyset,$$

which $B_1(-\frac{\pi}{4}) \subseteq U$. Thence, $(-1, \infty) \in \mathcal{NO}(\mathbb{R})$.

Theorem 3.6. *Allow (X, σ) be a strog neutro M. S. Then $Card(X)geq2$.*

Proof. Allow $Card(X) = 1$ and $x = \{x\}$. By the axiom $NM - 1$, if $x\sigma x = 0$, then must exis $x \neq y \in X$ in kind $y\sigma y \neq 0$, which is a contradication. Accordingly $Card(X) \geq 2$. □

Theorem 3.7. *Any M. S, can be a strong neutro M. S.*

Proof. Allow (X, d) be a M. S and $\lambda \notin X$. Then $(X \cup \{\lambda\}, \sigma)$ is a strong neutro M. S, whichever for $x, y \in X$

$$x\sigma y = \begin{cases} d(x, y) & \text{if } x, y \in X \\ \lambda & \text{if } x = y = \lambda, \end{cases}$$

$\lambda\sigma x = x_0, x\sigma\lambda = y_0$ and $\lambda > \lambda\sigma x + x\sigma\lambda$, which $x_0 \neq y_0, x_0, y_0 \in X$. One can see that the neutro axioms $NM - 1, NM - 2$ and $NM - 3$ are valid. □

Theorem 3.8. *Allow (X, σ) be a strog neutro M. S and $Card(X) = 2$. Then $Card(Range(\sigma)) = 4$.*

Proof. Allow $X = \{a, b\}$. Then illustrate the map $\sigma : X^2 \to \mathbb{R}$ go after:

σ	a	b
a	0	s
b	s'	r

We claim that $0 \neq r \neq s \neq s'$. Since $a\sigma a = 0$, $r \in \mathbb{R}$ which that $b\sigma b = r \neq 0$. If for $s \in \mathbb{R}$ consider $a\sigma b = s$, then $s \neq s' \in \mathbb{R}$ in kind $b\sigma a = s'$. Now, we investigate the coming cases:

case 1: if $a\sigma a > a\sigma b + b\sigma b$, then $0 > s + s'$, which is a contradication. Thence, $a\sigma a \leq a\sigma b + b\sigma b$ or $0 \leq s + s'$.

case 2: if $a\sigma b > a\sigma a + a\sigma b$, then $s > s + s'$, which is a contradication. Thence, $a\sigma b \leq a\sigma a + a\sigma b$.

case 3: if $b\sigma a > b\sigma b + b\sigma a$, then $s' > s + s'$, which is a contradication. Thence, $b\sigma a \leq b\sigma b + b\sigma a$.

Since (X, σ) is a strog neutro M. S, we get that $b\sigma b \leq b\sigma a + a\sigma b$. Accordingly $r \leq s + s'$ and so $Card(Range(\sigma)) = 4$. □

Theorem 3.9. *Allow (X, σ) be a strog neutro M. S. Then*

(i) $X \notin \mathcal{NO}(X)$.

(ii) $\emptyset \in \mathcal{NO}(X)$.

Proof. (i) Allow $y \in X$ in kind for all $r \in \mathcal{R}^{>0}$, $B_r(y) \nsubseteq X$. If $y\sigma y = 0$, then $\emptyset \neq B_r(y) \subseteq X$, which makes a contradication. If $y\sigma y < 0$, then $\emptyset \neq B_r(y) \subseteq X$, which makes a contradication. If $y\sigma y > 0$, then $\emptyset = B_r(y) \subseteq X$, which makes a contradication. They follow that there for all $y \in X$, $r \in \mathcal{R}^{>0}$, in kind $B_r(y) \subseteq X$ and so $X \notin \mathcal{NO}(X)$.

(i) Because there is no point in the empty set, we get that $\emptyset \in \mathcal{NO}(X)$. □

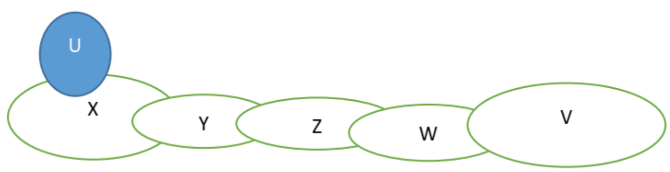
Theorem 3.10. *Allow (X, σ) be a strog neutro M. S, $r \in \mathbb{R}^{>0}$ and $x \in X$. Then $B_r(x)$ is not an O. S.*

Proof. Allow $x \in X$ be an arbitrary and for $r \in \mathbb{R}^{>0}$, $B_r(x)$ be an O. S. Thence, $X = \bigcup_{x \in X} B_r(x)$ is an O. S, which it is a contradication by Theorem 3.18. □

Allow X, Y, U be nonavoid sets. Accordingly, X, Y are unified, if $U \nsubseteq X$, then $U \nsubseteq X \cup Y$.

Example 3.11. Allow X, Y, Z, W, U, V be nonavoid sets. Then one can can see the unified sets X, Y, Z, W, U, V in Figure 4.

FIGURE 4. Unified set X

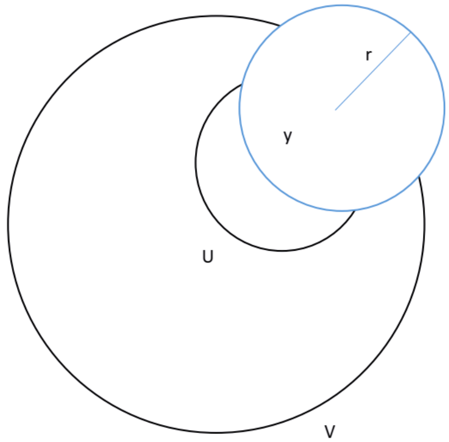


Theorem 3.12. *Allow (X, σ) be a strog neutro M. S and $\{U_i\}_{i \in I} \subseteq \mathcal{NO}(X)$ be unified. Then $\bigcup_{i \in I} U_i \in \mathcal{NO}(X)$.*

Proof. Since for all $i \in I, U_i \in \mathcal{NO}(X)$, we get $x \in U_i$ and $r \in \mathcal{R}^{>0}$ in kind $B_r(x) \subseteq U_i \subseteq \bigcup_{i \in I} U_i$. As well, $y \in U_i$ and $s \in \mathcal{R}^{>0}$ in kind $B_r(y) \nsubseteq U_i \subseteq$. Since for all $i \in I, U_i$ are unified, we get that $B_r(y) \nsubseteq \bigcup_{i \in I} U_i$. They conclude that $\bigcup_{i \in I} U_i \in \mathcal{NO}(X)$. □

Theorem 3.13. *Allow (X, σ) be a strog neutro M. S and $\{U_i\}_{i \in I} \subseteq \mathcal{NO}(X)$. If $\{U_i\}_{i \in I}$ is a chain, then $\bigcap_{i \in I} U_i \in \mathcal{NO}(X)$.*

FIGURE 5. Unified sets U, V



Proof. If $\bigcap_{i \in I} U_i = \emptyset$, then by Theorem 3.18, $\bigcap_{i \in I} U_i \in \mathcal{NO}(X)$. Allow $\bigcap_{i \in I} U_i \neq \emptyset$. Since $\{U_i\}_{i \in I}$ is a chain, $j \in I$ in kind $\bigcap_{i \in I} U_i = U_j$ and so $\bigcap_{i \in I} U_i \in \mathcal{NO}(X)$. □

Theorem 3.14. *Allow (X, σ) be a strog neutro M. S, $x \in X$ and $r \in \mathcal{R}^{>0}$. Then $B_r(x) \in \mathcal{NO}(X)$.*

Proof. We claim that $x \in X$ in kind $x\sigma x = 0$. If for all $x \in X, x\sigma x \neq 0$, since (X, σ) is a strog neutro M. S, $y \in X$ in kind $x\sigma y > x\sigma x + x\sigma y$, which is contradiaction. Because $x \in X$ in kind $x\sigma x = 0$, we get that $x \in B_r(x)$. Thence, $B_r(x) \neq \emptyset$ and $B_r(x) \subseteq B_r(x)$. As well, we claim that $x \in X$ in kind $x\sigma x \neq 0$. If for all $x \in X, x\sigma x = 0$, since (X, σ) is a strog neutro M. S, $y \in X$ in kind $y\sigma y \leq y\sigma x + x\sigma y$, which is a contradiaction. Allow $x\sigma x = r$. Since (X, σ) is a strog neutro M. S, $z \in X$ in kind $x\sigma z > x\sigma x + x\sigma z > r + x\sigma z$. Thence, $z \notin B_r(x)$ and so $B_r(z) \nsubseteq B_r(x)$. Therefore, Then $B_r(x) \in \mathcal{NO}(X)$. □

Theorem 3.15. *Allow (X, σ) be a strog neutro M. S, $U, V \subseteq X$ and U be unified. If $U \in \mathcal{NO}(X)$ and $U \subseteq V$, then $V \in \mathcal{NO}(X)$.*

Proof. Since $U \in \mathcal{NO}(X)$, we get that there exists $x \in U$ and $s \in \mathbb{R}$, in kind $B_s(x) \subseteq U$. Thence, there exists $x \in V$ and $s \in \mathbb{R}$, in kind $B_s(x) \subseteq V$, because of $U \subseteq V$. As well, there exists $y \in U$ and $r \in \mathbb{R}$, in kind $B_r(y) \nsubseteq U$ as shown in Figure 5, since U, V are unified sets. □

Theorem 3.16. *Allow (X, σ) be a strog neutro M. S. Then*

(i) if a set contains a union of unified open balls, then it is strog N. O. S.
(ii) if a set is strog N. O. S, then it contains a union of unified open balls.

(iii) if a set is strog N. O. S, then it may dose't contain a union of unified open balls.

Proof. (i) Allow $U \subseteq X$ and $U \supseteq \bigcup_{x\in X} B_r(x)$. Then by Theorems 3.12 and 3.14, $\bigcup_{x\in X} B_r(x)$ is a strog N. O. S in X. Now, using Theorem 3.16, U is a N. O. S.

(ii, iii) Suppose that U is a strog N. O. S. Thence, $x \in U$ and $r \in \mathbb{R}^{>0}$ in kind $B_r(x) \subseteq U$ and $y \in U$ and $s \in \mathbb{R}^{>0}$ in kind $B_s(y) \nsubseteq U$. Since $\{x\} \subseteq B_r(x) \subseteq U$, we get that $\bigcup_{x\in U} \{x\} \subseteq \bigcup_{\exists\ x\in U} B_r(x) \subseteq U$. Moreover, $\{y\} \subseteq B_s(y) \nsubseteq U$, infers $\bigcup_{\exists\ y\in U} \{y\} \subseteq \bigcup_{\exists\ y\in U} B_s(y) \nsubseteq U$. □

3.1. Strong neutro closed sets

Definition 3.17. Allow (X, σ) be a strog neutro M. S and $F \subseteq X$. Then F is a neutro closed set, if $F^c = \{x \in X \mid x \notin F\}$ is a N. O. S From now on, will denote the set of all strog neutro closed set of X by $\mathcal{NC}(X)$.

Theorem 3.18. *Allow(X, σ) be a strog neutro M. S. Then*

(i) $X \in \mathcal{NC}(X)$.

(ii) $\emptyset \notin \mathcal{NC}(X)$.

Proof. (i) Since $X^c = \emptyset$, by Theorem 3.18, $\emptyset \in \mathcal{NO}(X)$ and so $X \in \mathcal{NC}(X)$.

(ii) Since $\emptyset^c = X$, by Theorem 3.18, $X \notin \mathcal{NO}(X)$ and so $\emptyset \notin \mathcal{NC}(X)$. □

Theorem 3.19. *Allow (X, σ) be a strog neutro M. S and $\{F_i\}_{i\in I} \subseteq \mathcal{NC}(X)$ be unified. Then* $\bigcap_{i\in I} F_i \in \mathcal{NC}(X)$.

Proof. Since for all $i \in I, F_i \in \mathcal{NC}(X)$, by Theorem 3.12, we get $X \setminus \bigcap_{i\in I} F_i = \bigcup_{i\in I}(X \setminus F_i) \in \mathcal{NO}(X)$. Thence, $\bigcap_{i\in I} F_i \in \mathcal{NC}(X)$. □

Theorem 3.20. *Allow (X, σ) be a strog neutro M. S and $\{F_i\}_{i\in I} \subseteq \mathcal{NC}(X)$. If $\{F_i\}_{i\in I}$ is a chain, then* $\bigcup_{i\in I} F_i \in \mathcal{NC}(X)$.

Proof. Allow $\{F_i\}_{i\in I}$ and $F_i \subseteq F_j \subseteq F_k \subseteq \ldots$ be a chain. Then $F_i^c \supseteq F_j^c \supseteq F_k^c \supseteq \ldots$ and so $\{F_i^c\}_{i\in I}$ is a chain. Since for all $i \in I, F_i \in \mathcal{NC}(X)$, by Theorem 3.13, we get $X \setminus \bigcup_{i\in I} F_i = \bigcap_{i\in I}(X \setminus F_i) \in \mathcal{NO}(X)$. Thence, $\bigcup_{i\in I} F_i \in \mathcal{NC}(X)$. □

4. Conclusions

In this research work, we have been able to deal the connection of M. S and neutro M. S. Indeed, we investigate the diets in M. Ss and add some conditions on the axioms of M. Ss as the neutro metric axioms and so introduce the concepts of open balls, N. O. S, and neutro closed sets. The notion of unified sets recreates an essential role in the basic concepts of strong neutro spaces. We hope to extend these concepts in the real analysis in the next works.

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