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On neutrosophic $n-$normed linear spaces

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Abstract. In this current paper, our objective is to establish neutrosophic $n-$normed linear space(briefly called $N-n-NLS$) and introduce the convergence structure within these spaces. We also define Cauchy sequences, completeness in $N-n-NLS$ and obtained the relations between these notions.

Keywords: Neutrosophic normed linear spaces, $n-$normed space, convergence and Cauchy sequences.

1. Introduction

Nearly about 60 years ago, Zadeh [42] introduced fuzzy sets as a generalization of crisp set to address those problems which can't be modeled in the framework of crisp sets. These sets have vast applicability in many areas of science and engineering especially, in control engineering, decision making theory [40], fuzzy physics [28], artificial intelligence and robotics [41]. Katsaras [24] first observed that there are some situations in which the precise value of the norm of a vector can't be determined and therefore the concept of a fuzzy norm seems more appropriate as compared to the crisp norm. In view of this, he introduced the concept of fuzzy normed linear spaces and proved that any two fuzzy norms are equivalent. For further developments on these spaces, we recommend to the reader [24], [27-28],[30], [32], etc. Sadati and Park [34] generalized fuzzy normed linear space, called intutionistic fuzzy normed space while studying fuzzy topological spaces. Later, Karakus et al. [20] studied generalized convergence called statistical convergence in intutionistic fuzzy normed space. However, Srinivasan Vijayabalaji et al. [40] defined the intutionistic fuzzy n-normed linear space (IF-n-NLS) and introduced the notions of convergence and Cauchy sequence. Subsequently, these spaces have been developed in [4, 15, 18, 20] etc.

Recently, Kirişci and Şimşek [22] introduce a more generalized form of fuzzy normed space called neutrosophic normed linear space and studied statistical convergence in these spaces. After their pioneer work, many papers have been appeared on $NNLS$ and linked with summability theory. For an extensive view, we refer [21,23, 29-32, 37, 38] etc. In present work, we are motivated by the works of [40] and [22] to define a $N-n-NLS$ and develop some fundamental notions of convergence and Cauchy sequences in these spaces.

2. Preliminaries and background

This section starts with some basic definitions, results and terminology on neutrosophic normed linear spaces and $n-$norm [22].

Definition 2.1 "A binary operation $\circ : \Im \times \Im \rightarrow \Im$, where $\Im = [0,\ 1]$ is continuous $t-$norm, for each τ_1, τ_2, τ_3 and $\tau_4 \in [0,1]$ if the below conditions are satisfied:
(i) $\circ$ is continuous, commutative and associative;
(ii) $\tau_1 \circ 1 = \tau_1$
(iii) $\tau_1 \circ \tau_2 \leq \tau_3 \circ \tau_4$ whenever $\tau_1 \leq \tau_3$ and $\tau_2 \leq \tau_4$."

Definition 2.2 "A binary operation $\diamond : \Im \times \Im \rightarrow \Im$, where $\Im = [0,\ 1]$ is continuous $t-$conorm, for each τ_1, τ_2, τ_3 and $\tau_4 \in [0,1]$ if the below conditions are satisfied:
(i) $\diamond$ is continuous, commutative and associative;
(ii) $\tau_1 \diamond 0 = \tau_1$
(iii) $\tau_1 \diamond \tau_2 \leq \tau_3 \diamond \tau_4$ whenever $\tau_1 \leq \tau_3$ and $\tau_2 \leq \tau_4$.

Kirişci and Şimşek [22] used Definition 2.1 and Definition 2.2 to define neutrosophic normed linear space as follows."

Definition 2.3 [22]"Let U be a linear space over F and $\circ$, $\diamond$ respectively denotes t-norm and t-conorm, let G, B, Y are function from $(\varrho, \lambda_1) \in U \times (0, \infty)$ to $[0,1]$. A six tuple $(U, G, B, Y, \circ, \diamond)$ is called a neutrosophic normed linear space, if the below properties are satisfied:

For every $\varrho, v \in U$, $\lambda_1, \lambda_2 > 0$ and scaler $\alpha \neq 0$, we have

(i) $0 \leq G(\varrho,\ \lambda_1) \leq 1, 0 \leq\ B(\varrho,\ \lambda_1) \leq 1, 0 \leq Y(\varrho,\ \lambda_1) \leq 1$;

(ii) $G(\varrho,\ \lambda_1) +\ B(\varrho,\ \lambda_1) + Y(\varrho,\ \lambda_1) \leq 3$;

(iii) $G(\varrho,\ \lambda_1) = 1,\ B(\varrho,\ \lambda_1) = 0$ and $Y(\varrho,\ \lambda_1) = 0$ if and only if $\varrho = 0$;

(iv) $G(\alpha\varrho,\ \lambda_1) = G\left(\varrho,\ \frac{\lambda_1}{|\alpha|}\right),\ B(\alpha\varrho,\ \lambda_1) = B\left(\varrho,\ \frac{\lambda_1}{|\alpha|}\right)$ and $Y(\alpha\varrho,\ \lambda_1) = Y\left(\varrho,\ \frac{\lambda_1}{|\alpha|}\right)$;

(v) $G(\varrho,\ \lambda_1) \circ G(v,\ \lambda_2) \leq G(\varrho + v,\ \lambda_1 + \lambda_2),\ B(\varrho,\ \lambda_1) \diamond B(v,\ \lambda_2) \geq B(\varrho + v,\ \lambda_1 + \lambda_2)$ and $Y(\varrho,\ \lambda_1) \diamond Y(v,\ \lambda_2) \geq Y(\varrho + v,\ \lambda_1 + \lambda_2)$;

(vi) $G(\varrho,\ .)$ is a non-decreasing continuous function;

(vii) $B(\varrho,\ .)$ and $Y(\varrho,\ .)$ are non-increasing continuous function;

(viii) $\lim_{\lambda_1\to\infty} G(\varrho,\ \lambda_1) = 1$, $\lim_{\lambda_1\to\infty} B(\varrho,\ \lambda_1) = 0$ and $\lim_{\lambda_1\to\infty} Y(\varrho,\ \lambda_1) = 0$;

(ix) If $\lambda_1 \leq 0$, then $G(\varrho,\ \lambda_1) = 0, \quad B(\varrho,\ \lambda_1) = 1$ and $Y(\varrho,\ \lambda_1) = 1$.

We call $N(G, B, Y)$ as the neutrosophic norm and $(U, G, B, Y, \circ, \diamond)$ the neutrosophic normed linear space. For some examples on these spaces, we refer [22]."

Finally we recall the concept of $n-$norm as given in [16].

Definition 2.4 [16] "Let U be a real space of dimension $m \geq n$ (m is finite or infinite, $n \in \mathfrak{n}$) the real valued function $||.||_{\mathfrak{n}}$ on $U \times U \times ... \times U = U^{\mathfrak{n}}$ is called $n-$norm on U if and only if it satisfying the below axioms:

(i): $||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} = 0$ iff $\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}} \in U$ are linearly dependent;

(ii): $||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}$ remains invariant for $1 \leq i \leq n$;

(iii): $||\varrho_1, \varrho_2, \ldots, \alpha\varrho_{\mathfrak{n}}||_{\mathfrak{n}} = |\alpha|||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}$ for any $\alpha \in \mathbb{R}$;

(iv): $||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, v + w||_{\mathfrak{n}} \leq ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, v||_{\mathfrak{n}} + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, w||_{\mathfrak{n}}$.

The pair $(U, ||.||_{\mathfrak{n}})$ is known as $n-$normed linear space."

3. N-n-NLS

We, now turn towards our main results. We start with the following definition of neutrosophic $-n-$ normed space.

Definition 3.1 Let U be a linear space over F and $\circ$, $\diamond$ respectively denotes t-norm and t-conorm, let G_0, B_0, Y_0 are function from $U^{\mathfrak{n}} \times (0, \infty)$ to $[0, 1]$. A six tuple $(U, G_0, B_0, Y_0, \circ, \diamond)$ is called a neutrosophic $n-$normed linear space, $(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \in U^{\mathfrak{n}} \times (0, \infty) \to [0, 1]$, if the below properties are satisfied:

(i)$G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) + B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) + Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \leq 3$;

(ii) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0$;

(iii) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$, iff ϱ_i are dependent for $1 \leq i \leq \mathfrak{n}$;

(iv) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ remains invariant, ϱ_i for $1 \leq i \leq \mathfrak{n}$;

(v) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}, \lambda_1) = G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \frac{\lambda_1}{|\alpha|})$ for $\alpha \neq 0,\ \alpha \in F$;

(vi) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \circ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho'_{\mathfrak{n}}, \lambda_2)$
$\geq G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho'_{\mathfrak{n}}, \lambda_1 + \lambda_2)$;

(vii) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-decreasing continuous in λ_1

(viii) $\lim\limits_{\lambda_1\to\infty} G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$ and $\lim\limits_{\lambda_1\to 0} G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 0$;

(ix) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0$;

(x) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$ iff ϱ_i are dependent for $1 \leq i \leq \mathfrak{n}$

(xi) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ remains invariant, ϱ_i for $1 \leq i \leq \mathfrak{n}$;

(xii) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}, \lambda_1) = B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \frac{\lambda_1}{|\alpha|})$ for $\alpha \neq 0,\ \alpha \in F$;

(xiii) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \diamond B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho'_{\mathfrak{n}}, \lambda_2)$
$\geq B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho'_{\mathfrak{n}}, \lambda_1 + \lambda_2)$;
(xiv) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-increasing continuous in λ_1;
(xv) $\lim_{\lambda_1 \to \infty} B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 0$ and $\lim_{\lambda_1 \to 0} B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$;
(xvi) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0$;
(xvii) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$ iff ϱ_i are dependent for $1 \leq i \leq \mathfrak{n}$
(xviii) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ remains invariant, ϱ_i for $1 \leq i \leq \mathfrak{n}$;
(xix) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}, \lambda_1) = Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \frac{\lambda_1}{|\alpha|})$ for $\alpha \neq 0,\ \alpha \in F$;
(xx) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \diamond Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho'_{\mathfrak{n}}, \lambda_2)$
$\geq Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho'_{\mathfrak{n}}, \lambda_1 + \lambda_2)$;
(xxi) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-increasing continuous in λ_1;
(xxii) $\lim_{\lambda_1 \to \infty} Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 0$ and $\lim_{\lambda_1 \to 0} Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$.
For simplicity, we shall denote the neutrosophic $n-$norm by $N_{\mathfrak{n}}(G_0, B_0, Y_0)$.

Example 3.1 Let $(U, \|.\|_{\mathfrak{n}})$ be an $n-$normed space. For $\tau_1, \tau_2 \in [0,1]$, define, $t-$norm, $t-$conorm by $\tau_1 \circ \tau_2 = \min\{\tau_1, \tau_2\}$ and $\tau_1 \diamond \tau_2 = \max\{\tau_1, \tau_2\}$ and fuzzy sets G_0, B_0, Y_0 on $U^{\mathfrak{n}} \times (0, \infty)$ by

$$G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{\lambda_1}{\lambda_1 + \|\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}\|_{\mathfrak{n}}} \text{ and}$$
$$B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{\|\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}\|_{\mathfrak{n}}}{\lambda_1 + \|\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}\|_{\mathfrak{n}}},$$
$$Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{\|\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}\|_{\mathfrak{n}}}{\lambda_1},$$

then $(U, G_0, B_0, Y_0, \circ, \diamond)$ is a $N - n - NLS$.

Proof (i) and (ii) directly follows from definition of G_0, B_0, Y_0, i.e.,

$$(i)\ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) + B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) + Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \leq 3;$$
$$(ii)\ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0;$$
$$(iii)\ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1 \Leftrightarrow \frac{\lambda_1}{\lambda_1 + \|\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}\|_{\mathfrak{n}}} = 1$$
$$\Leftrightarrow ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} = 0$$
$$\Leftrightarrow \varrho_i \text{ are linearly dependent for } 1 \leq i \leq \mathfrak{n}.$$
$$(iv)\ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}$$
$$= \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}-1}||_{\mathfrak{n}}} = \cdots$$

$$(v)\ G_0\left(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n, \frac{\lambda_1}{|\alpha|}\right) = \frac{\frac{\lambda_1}{|\alpha|}}{\frac{\lambda_1}{|\alpha|} + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n} = \frac{\lambda_1}{\lambda_1 + |\alpha|||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n}$$
$$= \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \alpha\varrho_n||_n} = G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \alpha\varrho_n, \lambda_1).$$

(vi) In general, let's suppose that,

$$\begin{aligned}
&G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}, \lambda_1) \le G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n, \lambda_2)\\
&\Rightarrow \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n} \le \frac{\lambda_2}{\lambda_2 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n}\\
&\Rightarrow \lambda_1(\lambda_2 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n) \le \lambda_2(\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n)\\
&\Rightarrow \lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n \le \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n\\
&\Rightarrow ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n \le \left(\frac{\lambda_2}{\lambda_1}\right)||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n.\\
&\therefore\quad ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n\\
&\qquad \le \left(\frac{\lambda_2}{\lambda_1}\right)||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n\\
&\qquad \le \left(\frac{\lambda_2}{\lambda_1} + 1\right)||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n = \left(\frac{\lambda_2 + \lambda_1}{\lambda_1}\right)||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n.
\end{aligned}$$

Now,

$$\begin{aligned}
&||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}||_n \le ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n\\
&\qquad \le \left(\frac{\lambda_2 + \lambda_1}{\lambda_1}\right)||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n\\
&\Rightarrow \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}||_n}{\lambda_2 + \lambda_1} \le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n}{\lambda_1}\\
&\Rightarrow 1 + \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}||_n}{\lambda_2 + \lambda_1} \le 1 + \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n}{\lambda_1}\\
&\Rightarrow \frac{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}||_n}{\lambda_2 + \lambda_1} \le \frac{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n}{\lambda_1}\\
&\Rightarrow \frac{\lambda_2 + \lambda_1}{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}||_n} \ge \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n}\\
&\Rightarrow G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}, \lambda_2 + \lambda_1)\\
&\qquad \ge \min\{G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n, \lambda_2), G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}, \lambda_1)\}\\
&\qquad = G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n, \lambda_2) \circ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}, \lambda_1).
\end{aligned}$$

(vii) Clearly $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-decreasing continuous in λ_1.

(viii) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0$.

$$(ix)\ B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 0 \Rightarrow \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = 0$$
$$\Rightarrow ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} = 0$$

$\Rightarrow \varrho_i$ are linearly dependent for $1 \le i \le \mathfrak{n}$.

$$(x)\ B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = \cdots$$

$$(xi)\ B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = \frac{|\alpha|||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1 + |\alpha|||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}$$
$$= \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\frac{\lambda_1}{|\alpha|} + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = B_0\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \frac{\lambda_1}{|\alpha|}\right).$$

(xii) In general, let's suppose that,

$$B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2) \le B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_1)$$
$$\Rightarrow \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_2 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} \le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}$$
$$\Rightarrow ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}(\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}})$$
$$\le ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}(\lambda_2 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}})$$
$$\Rightarrow \lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} \le \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}.$$
$$\Rightarrow \lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} - \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}} \le 0.$$

Now,

$$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}} - \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}$$
$$\le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}} - \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}$$
$$= \frac{\lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} - \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{(\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}})(\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}})} \le 0.$$

This implies that, $$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}} \le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}.$$

Similarly, $$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}} \le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}.$$

This implies that,

$$B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}, \lambda_2 + \lambda_1)$$
$$\le \max\{B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2), B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_1)\}.$$
$$B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2) \diamond B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_1).$$

(xiii) Clearly $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-increasing continuous in λ_1.

(xiv) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0$.

$$(xv)\ Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 0 \Rightarrow \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1} = 0$$
$$\Rightarrow ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} = 0$$
$$\Rightarrow \varrho_i \text{ are linearly dependent for } 1 \le i \le \mathfrak{n}.$$

$$(xvi)\ Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1} = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}-1}||_{\mathfrak{n}}}{\lambda_1}$$
$$= Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}-1}, \lambda_1) = \cdots.$$

$$(xvii)\ Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1} = \frac{|\alpha|||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1}$$
$$= \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\frac{\lambda_1}{|\alpha|}} = Y_0\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \frac{\lambda_1}{|\alpha|}\right).$$

$(xviii)$In general, let's suppose that,

$$Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2) \le Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}', \lambda_1)$$
$$\Rightarrow \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_2} \le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}'||_{\mathfrak{n}}}{\lambda_1}$$
$$\Rightarrow \lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} \le \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}'||_{\mathfrak{n}}$$
$$\Rightarrow \lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} - \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}'||_{\mathfrak{n}} \le 0.$$

Now,

$$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}'||_{\mathfrak{n}}}{\lambda_2 + \lambda_1} - \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}'||_{\mathfrak{n}}}{\lambda_1}$$
$$\le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}'||_{\mathfrak{n}}}{\lambda_2 + \lambda_1} - \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}'||_{\mathfrak{n}}}{\lambda_1}$$
$$= \frac{\lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} - \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}'||_{\mathfrak{n}}}{\lambda_1(\lambda_2 + \lambda_1)} \le 0.$$

This implies that,
$$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}'||_{\mathfrak{n}}}{\lambda_2 + \lambda_1} \le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}'||_{\mathfrak{n}}}{\lambda_1}.$$

Similarly,
$$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}'||_{\mathfrak{n}}}{\lambda_2 + \lambda_1} \le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_2}$$

Therefore,
$$Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}', \lambda_2 + \lambda_1)$$
$$\le \max\{Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2), Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}', \lambda_1)\}.$$
$$Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2) \diamond Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}', \lambda_1).$$

(xix) Clearly $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-increasing continuous in λ_1.

Thus, $N_{\mathfrak{n}}(G_0, B_0, Y_0)$ satisfy all conditions of a neutrosophic $n-$norm and the space $(U, G_0, B_0, Y_0, \circ, \diamond)$ is a $N-n-NLS$ becomes a neutrosophic $n-$normed linear space □

Definition 3.2 A sequence $v = (v_k)$ in a $N-n-NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ is said to be convergent to v_0 w.r.t. the norm $N_{\mathfrak{n}}(G_0, B_0, Y_0)$ if for $\epsilon > 0$, $\lambda_1 > 0$ and $\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1} \in U$, $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) > 1 - \epsilon$ and $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $\forall\ k \geq \mathfrak{n}_0$. In this case, we write $N_{\mathfrak{n}}(G_0, B_0, Y_0) - \lim_{k\to\infty} v_k$. □

Example 3.2 Consider the neutrosophic $n-$normed linear space as given in example 3.1. Define a sequence $v = (v_k)$ by $v_k = \frac{1}{k}$, then for each $\epsilon > 0$, $\lambda_1 > 0$ and $\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1} \in U$ we have

$$
\begin{aligned}
G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k, \lambda_1) &= G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}, \lambda_1) \\
&= \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}||_{\mathfrak{n}}} \to 1 \text{ as } k \to \infty \text{ and} \\
B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k, \lambda_1) &= B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}, \lambda_1) \\
&= \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}||_{\mathfrak{n}}} \to 0 \text{ as } k \to \infty, \\
Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k, \lambda_1) &= Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}, \lambda_1) \\
&= \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}||_{\mathfrak{n}}}{\lambda_1} \to 0 \text{ as } k \to \infty,
\end{aligned}
$$

this shows that the sequence $v_k = (\frac{1}{k})$ is convergent to θ where θ denotes the zero element in U w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$.

Theorem 3.1 For any sequence $v = (v_k)$, in a $N-n-NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ with $N_{\mathfrak{n}}(G, B, Y) - \lim_k v_k = v_0$, v_0 is unique.

Proof Let, if possible $N_{\mathfrak{n}}(G_0, B_0, Y_0) - \lim_k v_k = v_0$ and $N_{\mathfrak{n}}(G_0, B_0, Y_0) - \lim_k v_k = w_0$. Let $\epsilon > 0$ and $\lambda_1 > 0$ be given. Choose $\epsilon_1 > 0$ s.t.

$$(1 - \epsilon_1) \circ (1 - \epsilon_1) > 1 - \epsilon \text{ and } \epsilon_1 \diamond \epsilon_1 < \epsilon. \tag{1}$$

Since $N_{\mathfrak{n}}(G_0, B_0, Y_0) - \lim_k v_k = v_0$ so there exists $\mathfrak{n}_1 \in \mathbb{N}$ s.t.$\forall\ k \geq \mathfrak{n}_1$

$$G\left(v_k - v_0, \frac{\lambda_1}{2}\right) > 1 - \epsilon_1 \text{ and } B\left(v_k - v_0, \frac{\lambda_1}{2}\right) < \epsilon_1, Y\left(v_k - v_0, \frac{\lambda_1}{2}\right) < \epsilon_1.$$

Further, $N_{\mathfrak{n}}(G_0, B_0, Y_0) - \lim_k v_k = w_0$, will give another $\mathfrak{n}_2 \in \mathbb{N}$ s.t. $\forall k \geq \mathfrak{n}_2$

$$G\left(v_k - w_0, \frac{\lambda_1}{2}\right) > 1 - \epsilon_1 \text{ and } B\left(v_k - w_0, \frac{\lambda_1}{2}\right) < \epsilon_1, Y\left(v_k - w_0, \frac{\lambda_1}{2}\right) < \epsilon_1.$$

Let, $\mathfrak{n}_0 = \max\{\mathfrak{n}_1, \mathfrak{n}_2\}$, then for all $k \geq \mathfrak{n}_0$

$$G(v_0 - w_0, \lambda_1) = G\left(v_0 - v_k + v_k - w_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\right)$$
$$\geq G\left(v_k - v_0, \frac{\lambda_1}{2}\right) \circ G\left(v_k - w_0, \frac{\lambda_1}{2}\right)$$
$$> (1 - \epsilon_1) \circ (1 - \epsilon_1) > 1 - \epsilon.$$

Since $\epsilon > 0$ is arbitrary so $G(v_0 - w_0, \lambda_1) = 1$ and therefore $v_0 - w_0 = 0$ i.e., $v_0 = w_0$.
Now,

$$B(v_0 - w_0, \lambda_1) = B\left(v_0 - v_k + v_k - w_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\right)$$
$$\leq B\left(v_k - v_0, \frac{\lambda_1}{2}\right) \diamond B\left(v_k - w_0, \frac{\lambda_1}{2}\right)$$
$$< \epsilon_1 \diamond \epsilon_1 < \epsilon,$$
$$Y(v_0 - w_0, \lambda_1) = Y\left(v_0 - v_k + v_k - w_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\right)$$
$$\leq Y\left(v_k - v_0, \frac{\lambda_1}{2}\right) \diamond Y\left(v_k - w_0, \frac{\lambda_1}{2}\right)$$
$$< \epsilon_1 \diamond \epsilon_1 < \epsilon.$$

Since $\epsilon > 0$ is arbitrary so we have $B(v_0 - w_0, \lambda_1) = Y(v_0 - w_0, \lambda_1) = 0$ which gives $v_0 - w_0 = 0$ i.e., $v_0 = w_0$. Hence, in all cases v_0 is uniquely determined.□

Theorem 3.2 If $v = (v_k)$ and $w = (w_k)$ be any two sequences in a $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ s.t. $\lim v_k = v_0$ and $\lim w_k = w_0$ then,

(i) $\lim_k \alpha v_k = \alpha v_0,\ \alpha \neq 0$ for any scalar.

(ii) $\lim_k (v_k + w_k) = v_0 + w_0$.

Proof. The proof of the theorem follow parallel lines of the proof of theorem 3.1, so omitted.
□

Theorem 3.3 A sequence $v = (v_k)$ in a $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ is convergent to v_0 w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$, if and only if, $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 1$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$ as $\mathfrak{n} \to \infty$, where $\lambda_1 > 0$.

Proof Let $v = (v_k)$ converges to v_0 w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$. So, for $0 < \epsilon < 1$ and $\lambda_1 > 0$, $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) > 1 - \epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $\forall\ k \geq \mathfrak{n}_0$. This implies, for $\forall\ k \geq \mathfrak{n}_0$. $1 - G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, which shows $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 1$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$ as $k \to \infty$.

Conversely, suppose that for $\lambda_1 > 0$, $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 1$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$ as $k \to \infty$. For $0 < \epsilon <$

$1, \exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $1 - G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, which gives $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) > 1 - \epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$. Hence, (v_k) converges to v_0 w.r.t. $N_{\mathfrak{n}}(G, B, Y)$. $\square$

Definition 3.3 A sequence $v = (v_k)$ in a $N-n-NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ is said to be Cauchy w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$ if for $\epsilon > 0$ and $\lambda_1 > 0$, $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) > 1-\epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) < \epsilon$, $\forall\ k, p \geq \mathfrak{n}_0$. $\square$

Example 3.3 Let us consider the space $U = (0, 1]$. If we define G_0, B_0 and Y_0 by $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k; \lambda_1) = \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k||_{\mathfrak{n}}}$, $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k; \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k||_{\mathfrak{n}}}$, $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k; \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k||_{\mathfrak{n}}}{\lambda_1}$ and the $t-$norm, $t-$conorm respectively as $\circ = \min$ and $\diamond = \max$, then $(U, G_0, B_0, Y_0, \circ, \diamond)$ becomes a $N-n-NLS$.

Defined a sequence $v = (v_k)$ where $v_k = \frac{1}{k}$ as in example 3.2 then for $\epsilon > 0$, $\lambda_1 > 0$ and $\varrho_1, \varrho_2, \ldots, \varrho_{n-1} \in U$ we have

$G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) = G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}, \lambda_1) = \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}||_{\mathfrak{n}}} \to 1$ as $k, p \to \infty$,

$B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) = B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}||_{\mathfrak{n}}} \to 0$,

$Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) = Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}||_{\mathfrak{n}}}{\lambda_1} \to 0$, as $k, p \to \infty$.

This shows that the sequence $v = (v_k)$ is Cauchy w.r.t. the neutrosophic $n-$norm $N_{\mathfrak{n}}(G_0, B_0, Y_0)$.

Theorem 3.4 Every convergent sequence in a $N-n-NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ is a Cauchy w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$.

Proof Let $v = (v_k)$ converges to v_0 w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$. For $0 < \epsilon < 1$ and $\lambda > 0$, choose $\epsilon_1 \in (0, 1)$ such that (1) is satisfied. Since $v_k \to v_0$ w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$, so $\exists$ $\mathfrak{n}_0 \in \mathbb{N}$ s.t. $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}) > 1 - \epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}) < \epsilon$, $\forall\ k, p \geq \mathfrak{n}_0$.

Now,

$$G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) = G\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0 + v_0 - v_p, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\bigg)$$
$$\geq G\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}\bigg) \circ G\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_p - v_0, \frac{\lambda_1}{2}\bigg)$$
$$> (1 - \epsilon_1) \circ (1 - \epsilon_1)$$

$> 1-\epsilon$ and

$$B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) = B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0 + v_0 - v_p, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\right)$$
$$\leq B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}\right) \diamond B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_p - v_0, \frac{\lambda_1}{2}\right)$$
$$< \epsilon_1 \diamond \epsilon_1$$
$$< \epsilon,$$

$$Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) = Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0 + v_0 - v_p, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\right)$$
$$\leq Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}\right) \diamond Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_p - v_0, \frac{\lambda_1}{2}\right)$$
$$< \epsilon_1 \diamond \epsilon_1$$
$$< \epsilon.$$

This shows that (v_k) is a Cauchy sequence w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$. □

Theorem 3.5 Consider the neutrosophic $n-$norm linear space as defined in Example 3.1. Let $v = (v_k)$ be any sequence in U, then

(i) (v_k) is Cauchy in $(U, ||.||_{\mathfrak{n}})$ iff $(v_{\mathfrak{n}})$ is Cauchy in $(U, G_0, B_0, Y_0, \circ, \diamond)$.

(ii) (v_k) is a convergent in $(U, ||.||_{\mathfrak{n}})$ iff (v_k) is convergent in $(U, G_0, B_0, Y_0, \circ, \diamond)$.

Proof (i) Let $v = (v_k)$ be a Cauchy in $(U, ||.||_{\mathfrak{n}})$, then

$$\lim_{k,p\to\infty} ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p||_{\mathfrak{n}} = 0.$$

Now, for $\lambda_1 > 0$

$$\lim_{k,p\to\infty} G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1)$$
$$= \lim_{k,p\to\infty} \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p||_{\mathfrak{n}}} = 1 \text{ and;}$$
$$\lim_{k,p\to\infty} B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1)$$
$$= \lim_{k,p\to\infty} \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p||_{\mathfrak{n}}} = 0,$$
$$\lim_{k,p\to\infty} Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1)$$
$$= \lim_{k,p\to\infty} \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p||_{\mathfrak{n}}}{\lambda_1} = 0.$$

This shows that, $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) \to 1$, $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) \to 0$ and $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) \to 0$, as $k, p \to \infty$. So for $0 < \epsilon < 1$ and $\lambda_1 > 0$, $\exists$ $\mathfrak{n}_0 \in \mathbb{N}$ s.t. $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) > 1 - \epsilon$ and $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) < \epsilon$, $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) < \epsilon$, and therefore (v_k) is Cauchy sequence in $(U, G_0, B_0, Y_0, \circ, \diamond)$.

Conversely, if (v_k) is Cauchy in $(U, G_0, B_0, Y_0, \circ, \diamond)$ then it is clearly (v_k) is Cauchy in $(U, ||.||_{\mathfrak{n}})$.

(ii) Let, $v = (v_k)$ be a convergent in $(U, ||.||_{\mathfrak{n}})$ and converges to v_0. Then $\lim_{k\to\infty} ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0|| = 0$.
Now,

$$\lim_{k\to\infty} G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = \lim_{k\to\infty} \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0||_{\mathfrak{n}}} = 1 \text{ and;}$$

$$\lim_{k\to\infty} B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = \lim_{k\to\infty} \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0||_{\mathfrak{n}}} = 0,$$

$$\lim_{k\to\infty} Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = \lim_{k\to\infty} \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0||_{\mathfrak{n}}}{\lambda_1} = 0.$$

This shows that, $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 1$, $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$ and $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$, as $k \to \infty$. So for $\epsilon > 0$ and $\lambda_1 > 0$, $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) > 1 - \epsilon$ and $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, and therefore (v_k) is convergent sequence in $(U, G_0, B_0, Y_0, \circ, \diamond)$.

Converse part of the theorem can be obtained similarly and therefore omitted. □

Definition 3.4 A $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ is said to be complete, iff, every Cauchy sequence is convergent in $(U, G_0, B_0, Y_0, \circ, \diamond)$.

Example 3.5 The sequence $v = (v_k) = \frac{1}{k}$ as in Example 3.4 is a Cauchy sequence that converges to 0 w.r.t $N_{\mathfrak{n}}(G_0, B_0, Y_0)$. But, $0 \notin (0, 1] = U$ and therefore the $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ where $U = (0, 1]$ is not complete.

Theorem 3.6 If every Cauchy sequence in a $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ has a convergent subsequence, then it is complete.

Proof Let $v = (v_k)$ be a Cauchy in a $N - n - NLS$ U and (v_{k_p}) be a subsequence of (v_k) that converges to v_0. We shell show that (v_k) converges to v_0. Let $\lambda_1 > 0$ and $\epsilon \in (0, 1)$. Choose $\epsilon_1 \in (0, 1)$ s.t. (1) is satisfied.

Since (v_k) is a Cauchy sequence, so $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $\forall\ k, p \geq \mathfrak{n}_0$
$G\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \frac{\lambda_1}{2}\right) > 1 - \epsilon_1$ and
$B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \frac{\lambda_1}{2}\right) < \epsilon_1, Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \frac{\lambda_1}{2}\right) < \epsilon_1.$

Since (v_{k_p}) converges to v_0, so $\exists\ i_p \in \mathfrak{n}$ with $i_p > \mathfrak{n}_0$ s.t.

$G\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_p} - v_0, \frac{\lambda_1}{2}\right) > 1 - \epsilon_1$ and
$B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_p} - v_0, \frac{\lambda_1}{2}\right) < \epsilon_1, Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_p} - v_0, \frac{\lambda_1}{2}\right) < \epsilon_1.$
Now,

$$G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = G\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p} + v_{i_p} - v_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\right)$$
$$\geq G\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p}, \frac{\lambda_1}{2}\right) \circ G\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_0} - v_p, \frac{\lambda_1}{2}\right)$$
$$> (1 - \epsilon_1) \circ (1 - \epsilon_1) > 1 - \epsilon \text{ and;}$$
$$B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p} + v_{i_p} - v_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\right)$$
$$\leq B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p}, \frac{\lambda_1}{2}\right) \diamond B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_p} - v_0, \frac{\lambda_1}{2}\right) < \epsilon_1 \diamond \epsilon_1 < \epsilon;$$
$$Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p} + v_{i_p} - v_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\right)$$
$$\leq Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p}, \frac{\lambda_1}{2}\right) \diamond Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_p} - v_0, \frac{\lambda_1}{2}\right) < \epsilon_1 \diamond \epsilon_1 < \epsilon.$$

This shows that $v_k \to v_0$ in U w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$ and therefore U is complete. □

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