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Solving Neutrosophic Zero-Sum Two-Person Matrix Game using Mellin's Transform

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Abstract. This paper addresses the research gap in neutrosophic game theory, specifically the resolution of zero-sum two-person matrix games characterized by single-valued neutrosophic triangular numbers. We introduce a novel de-neutrosophication method leveraging Mellin's transform to obtain crisp value indices, thereby translating neutrosophic linear programming problems into their crisp counterparts. The effectiveness and precision of our approach are demonstrated through a real-world telecom sector case study, showcasing its potential for yielding more accurate and dependable solutions.

Keywords: Neutrosophic Triangular Numbers, Neutrosophic Triangular Matrix Games, Neutrosophic Linear Programming Problem, Mellin's transform.

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1. **Introduction**

Taking the right decision in today's competitive and conflicting world is an arduous affair. Game theory has played a pivotal role in decision making to take right decision and achieve desired goals. In today's real world conflicting scenario, where it is challenging to collect the accurate data for players, game theory provides a strategic mathematical procedure that help players to take the precise and perfect decision even with half-baked, imprecise and vague data. This is the reason why researchers all over the globe are attracted to develop new techniques and horizons in the theory of games.

The notion of theory of games was first introduced by Neumann and Morgenstern [1] by their work that published in 1944. During the classical game theory, the data available used to be accurate and crisp, so the classical set theory served the purpose, where the membership is binary {0, 1}. But classical game theory no longer serves the purpose when the data available is inadequate, imprecise, and vague. To overcome this problem of handling the imprecise, inadequate, and vague data, Prof. Zadeh [2] in 1965, pioneered the trailblazing concept of Fuzzy sets. Since then, numerous extensions and horizons like fuzzy triangular numbers (TFNs), fuzzy trapezoidal numbers (TrFNs), fuzzy pentagonal numbers (PFNs) and many more has been added to the fuzzy set theory by various researchers from all over the world. Li [26–28], Seikh et al. [29] have studied TFNs for their work in developing the theory of games. Jana et al. [30] , Chandra et al. [31], Kumar et al. [32], Bandhopadhyay et al. [33], and Dutta et al. [34] have explored TrFNs for their study of matrix games. Chakraborty et al. [35], Nasir et al. [36], Gajalakshmi et al. [42], and Umamageshwari et al. [43] investigated various properties of PFNs and applied it to various competitive game scenarios and achieved wonderful results of economic and social use.

Pawlak Z. [45, 46] in 1982 presented a novel mathematical instrument called ‘Rough set theory’ to deal with vague and uncertain information. He, in rough set theory, made use of two sets –lower and upper approximation intervals denoted as LAI and UAI respectively to handle vague and uncertain data. Later various researchers like Jangid et al. [47], Brikaa et al. [48], and Seikh et al. [49] dig deep to combine fuzziness and roughness to get fuzzy rough sets and used it in many types of MGs.

Atanasov [3,4] introduced Intuitionistic fuzzy sets (IFS) by adding non-membership function to the already existing fuzzy sets, to handle the uncertainty present in the available data, in a better way. The concept of the IFS has been used by various researchers [5–14] to investigate uncertainty in game theory using LPP approach.

Fuzzy sets and its generalisations have served well to handle imprecise and incomplete information in game theory, however, they are no longer suitable to handle inconsistent and indeterminate information that exists quiet often in real life situations. To get over this issue, Smarandache [15] invented a very prudent comprehensive framework of neutral logics called ‘Neutrosophy’, now recognised as a new arm of mathematics. The core theme of Neutrosophy states that beside some degree of truth, each concept possesses some degree of indeterminacy and falsity. Smarandache [16] defined neutrosophic set as a generalisation of IFSs. Smarandache explained indeterminacy in logic of Neutrosophy and clarified that truth, indeterminacy and falsity membership functions are independent of each other. Fuzzy sets and neutrosophic sets are clearly different in their application domain. Fuzzy sets deal with uncertain information i.e., incomplete and imprecise, whereas neutrosophic sets deal with inconsistent and indeterminate information. Neutrosophic sets are viewed from the vision of philosophy and one may find it difficult to apply it in mathematical and scientific problems. To get over this problem Wang et al. [17] defined single-valued neutrosophic sets and gave its various mathematical properties. At the moment, applying neutrosophic sets in game theory is a new thing and is in its initial stage. Now a days it is a very attractive research area for researchers all over the globe. Not much work is available at the moment in this field. However, some researchers like Das et al. [18], Hussain et al. [19], Tamilarasi et al. [20], Das S. K. [21], Seikh et al. [50], Das et al. [51], Bhaumik et al. [52], and Chakraborty et al. [44], have investigated neutrosophic sets in LPP models, integer programming models and MGs.

De-neutrosophication and ranking technique is utmost important while investigating and solving NLPP models. Jangid et al. [22] used a ranking technique by evaluating ambiguity and value of truth, indeterminacy, and falsity membership degree functions using (α, β, γ)-cut of SVNTNs involved in pay-off matrix of a NMG. Mahapatra et al. [23] used de-neutrosophication technique to convert NLPP to crisp LPP using the centroid method. Abdel et al. [24] suggested a novel ranking map to solve fully NLPP with trapezoidal neutrosophic numbers. A new ranking methodology was introduced by Das & Dash [18] to solve NLPP model with mixed parametric constraints. Darehmiraki [25] introduced a parametric de-neutrosophication function for ranking and then solving NLPPs. We in this paper have used a novel ranking technique for SVNTNs using Mellin's transform. [37] introduced a graphical method for solving Neutrosophical nonlinear programming with linear constraints, applicable to various model complexities. [38] reformulated the general model for the optimal distribution of agricultural lands using the concepts of neutrosophic science. In the book [39] discussed industrial engineering and computational intelligence foster intelligent machines for multi-criteria decision-making in smart environments. [40] evaluated the sustainable flue gas treatments in Egypt's steel sector using a new hybrid spherical fuzzy multicriteria decision-making approach. [41] developed a multi-criteria tool to evaluate sustainable battery recycling plant locations, prioritizing environmental factors in Egypt.

This research bridges the gap in neutrosophic game theory by converting neutrosophic matrix games to crisp linear programming, enhancing solution accuracy and reliability. The proposed methodology not only streamlines the process but also enhances the accuracy and reliability of the game's outcomes for both players. We demonstrate the impact of our approach through a real-world case study in the telecommunications sector, showing its potential to yield practical strategies in industry-specific scenarios. However, we must also note that the transition from neutrosophic to crisp values may involve certain trade-offs in terms of capturing the full spectrum of uncertainty inherent in real-world situations. Despite this, the practical benefits of our approach in terms of actionable insights and decision support in complex scenarios hold significant promise.

The structure of the paper has been developed as shown by the following figure -1:

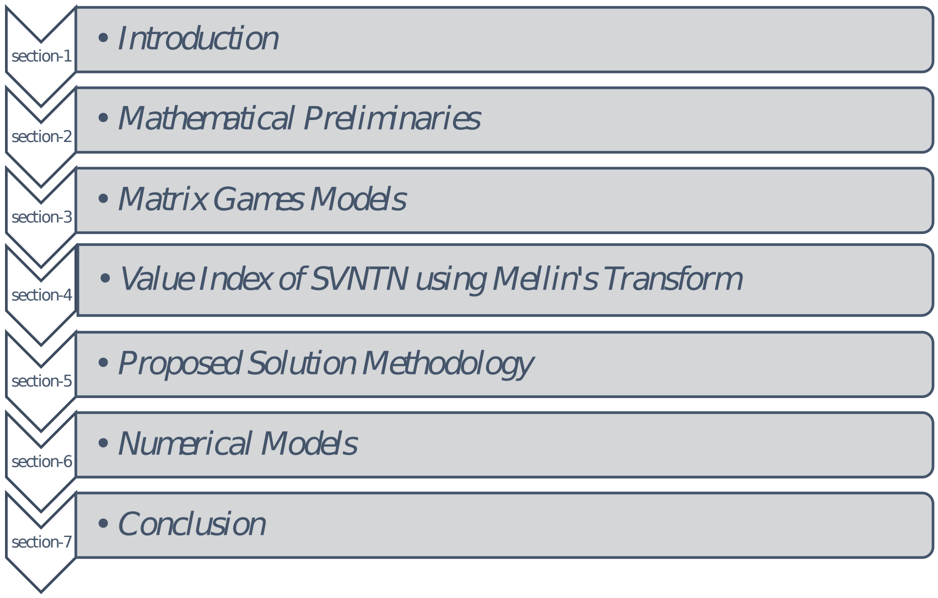


FIGURE 1. Structure of the Paper

2. Mathematical Preliminaries

In the present section, we give some fundamental definitions and symbols that are requisite and will be used throughout this article.

Definition 2.1. (Hussain et al. [19]) Let $X = \{\omega_1, \omega_2, ...\omega_n\}$ be a universe of discourse. A **Neutrosophic Set** ã in X is defined as ã$=\{\langle \omega_i, t_{\tilde{a}}(\omega_i), i_{\tilde{a}}(\omega_i), f_{\tilde{a}}(\omega_i)\rangle : \omega_i \in X\}$ where $t_{\tilde{a}}(\omega_i), i_{\tilde{a}}(\omega_i), f_{\tilde{a}}(\omega_i)$ are truth membership, indeterminacy membership and falsify membership degree mappings respectively with domain X and co-domain $[0, 1]$.

Definition 2.2. (Tamilarasi et al. [20]) A **single valued neutrosophic triangular number** (SVNTN) on $\Re$ (set of reals) is a neutrosophic set, denoted by $\tilde{a}_{SVNTN} = \{(\zeta, \eta, \theta); \rho, \sigma, \tau\}$, whose truth, indeterminacy and falsify functions are respectively written as follows:

$$t_{\tilde{a}_{SVNTN}}(\omega) = \begin{cases} \left(\frac{\omega-\zeta}{\eta-\zeta}\right)\rho & \text{if } \zeta \leq \omega \leq \eta \\ \rho & \text{if } \omega = \eta \\ \left(\frac{\theta-\omega}{\theta-\eta}\right)\rho & \text{if } \eta \leq \omega \leq \theta \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$i_{\tilde{a}_{SVNTN}}(\omega) = \begin{cases} \frac{(\eta-\omega)+\sigma(\omega-\zeta)}{\eta-\zeta} & \text{if } \zeta \leq \omega \leq \eta \\ \rho & \text{if } \omega = \eta \\ \frac{(\omega-\eta)+\sigma(\theta-\omega)}{\theta-\eta} & \text{if } \eta \leq \omega \leq \theta \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$f_{\tilde{a}_{SVNTN}}(\omega) = \begin{cases} \frac{(\eta-\omega)+\tau(\omega-\zeta)}{\eta-\zeta} & \text{if } \zeta \leq \omega \leq \eta \\ \tau & \text{if } \omega = \eta \\ \frac{(\omega-\eta)+\tau(\theta-\omega)}{\theta-\eta} & \text{if } \eta \leq \omega \leq \theta \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Where $0 \leq \rho \leq 1$, $0 \leq \sigma \leq 1$, $0 \leq \tau \leq 1$ such that $0 \leq \rho + \sigma + \tau \leq 3$. Here σ, ρ, τ respectively represents maximum truth membership degree, minimum indeterminacy membership degree, minimum falsify membership degree.

Definition 2.3. (Hussain et al. [19]): Let $\tilde{a}^1_{SVNTN} = \{(\zeta^1, \eta^1, \theta^1); \rho^1, \sigma^1, \tau^1\}$ and $\tilde{a}^2_{SVNTN} = \{(\zeta^2, \eta^2, \theta^2); \rho^2, \sigma^2, \tau^2\}$ be two single valued neutrosophic triangular numbers and $\lambda \in \Re$ then some algebraic operations are as follows:

(1) Addition:

$$\tilde{a}^1_{SVNTN} \oplus \tilde{a}^2_{SVNTN} = \{(\zeta^1 + \zeta^2, \eta^1 + \eta^2, \theta^1 + \theta^2); min(\rho^1, \rho^2), max(\sigma^1, \sigma^2), max(\tau^1, \tau^2)\}$$

(2) Negative Image:
$-\tilde{a}^1_{SVNTN} = \{(-\theta^1, -\eta^1, -\zeta^1); \rho^1, \sigma^1, \tau^1\}$

(3) Subtraction:

$$\tilde{a}^1_{SVNTN} \ominus \tilde{a}^2_{SVNTN} = \{(\zeta^1 - \theta^2, \eta^1 - \eta^2, \theta^1 - \zeta^2); min(\rho^1, \rho^2), max(\sigma^1, \sigma^2), max(\tau^1, \tau^2)\}$$

(4) Scalar Product:

$$\lambda\, \tilde{a}^1_{SVNTN} = \begin{cases} \{(\lambda\zeta^1, \lambda\eta^1, \lambda\theta^1); \rho^1, \sigma^1, \tau^1\} & \text{for } \lambda > 0 \\ \{(\lambda\theta^1, \lambda\eta^1, \lambda\zeta^1); \rho^1, \sigma^1, \tau^1\} & \text{for } \lambda < 0 \end{cases}$$

3. **Matrix Games Models**

3.1. *Crisp Matrix Game (CMG)*

A crisp zero-sum two person matrix game denoted by triplet (A, S_1, S_2), where $A = \{a^{jk}\}_{mxn}$ is a real payoff matrix and $S_1 = \{1, 2, ..., m\}$, $S_2 = \{1, 2, ..., n\}$ are pure strategies of player-1 and player-2 respectively. Player-1 is called the maximising player as he plays his pure strategy to maximise his minimum gain and player-2 is called the minimising player as he plays his pure strategy to minimise his maximum loss. This is known as maxmin and minmax principle of matrix game. If the saddle point of the game exist at $(rs)^{th}$ position in the payoff matrix, then a^{rs}, $1 \leq r \leq m$; $1 \leq s \leq n$, is the payoff value for player-1 and its negative is the payoff value for player-2 if they choose to play r^{th} and s^{th} pure strategy respectively. If matrix game (A, S_1, S_2) has no saddle point i.e. $\max_{j \in S_1} \left\{ \min_{k \in S_2} \{a^{jk}\} \right\} \neq \min_{k \in S_2} \left\{ \max_{j \in S_1} \{a^{jk}\} \right\}$, then mixed strategy sets

$$S_1 = \left\{ P = (p_1, p_2, ..., p_m) \in R^m, p_j \geq 0 \ \forall \ j = 1, 2, ..., m, and \ \sum_{j=1}^{m} p_j = 1 \right\} \ and$$

$$S_2 = \left\{ Q = (q_1, q_2, ..., q_n) \in R^n, q_j \geq 0 \ \forall \ k = 1, 2, ..., n, and \ \sum_{k=1}^{n} q_k = 1 \right\}$$

are adopted for player-1 and player-2 respectively.

Here if

$$\max_{P \in S_1} \left\{ \min_{Q \in S_2} \left\{ \sum_{j=1}^{m} \left(\sum_{k=1}^{n} p_j a^{jk} p_k \right) \right\} \right\} = \min_{Q \in S_2} \left\{ \max_{P \in S_1} \left\{ \sum_{k=1}^{n} \left(\sum_{j=1}^{m} p_j a^{jk} p_k \right) \right\} \right\} = v^*(say)$$

then v^* is called the value of the game, and $P = (p_1, p_2, ..., p_m) \in S_1, Q = (q_1, q_2, ..., q_n) \in S_2$ are optimal mixed strategies for player-1 and player-2 respectively.

3.2. *Neutrosophic Matrix Games (NMG)*

If the payoff matrix $\tilde{A} = \{\tilde{a}^{jk}_{SVNTN}\}_{mxn}$ is equiped with single valued neutrosophic triangular numbers $\tilde{a}^{jk}_{SVNTN}, \ j = 1, 2, ..., m; \ k = 1, 2, ..., n$, then the game $(\tilde{A}, S_1, S_2)$ is called Neutrosophic Triangular matrix game (NTMG). Thus employing the maxmin and minmax principle for NTMG, we get the following mathematical modals for two players respectively

For player-1:

$$\begin{cases} \max_{p_j \in S_1} \left\{ \min \left\{ \sum_{j=1}^{m} \tilde{a}^{j1}_{SVNTN} \ p_j, \sum_{j=1}^{m} \tilde{a}^{j2}_{SVNTN} \ p_j, ..., \sum_{j=1}^{m} \tilde{a}^{jn}_{SVNTN} \ p_j \right\} \right\} \\ s.t., \qquad \sum_{j=1}^{m} p_j = 1 \\ and \qquad p_j \geq 0, \forall \ j = 1, 2, ..., m. \end{cases} \tag{4}$$

For player-2:

$$\begin{cases} \min_{q_k \in S_2} \left\{ \max \left\{ \sum_{k=1}^{n} \tilde{a}^{1k}_{SVNTN} \ q_k, \sum_{k=1}^{n} \tilde{a}^{2k}_{SVNTN} \ q_k, ..., \sum_{k=1}^{n} \tilde{a}^{mk}_{SVNTN} \ q_k \right\} \right\} \\ s.t., \qquad \sum_{k=1}^{n} q_k = 1 \\ and \qquad q_k \geq 0, \forall \ k = 1, 2, ..., n. \end{cases} \tag{5}$$

Now, let $\min \left\{ \sum_{j=1}^{m} \tilde{a}^{j1}_{SVNTN} \ p_j, \sum_{j=1}^{m} \tilde{a}^{j2}_{SVNTN} \ p_j, ..., \sum_{j=1}^{m} \tilde{a}^{jn}_{SVNTN} \ p_j \right\} = \tilde{u}_{SVNTN}$ (say)

is the minimum expected gain for player-1 and

$\max \left\{ \sum_{k=1}^{n} \tilde{a}^{1k}_{SVNTN} \ q_k, \sum_{k=1}^{n} \tilde{a}^{2k}_{SVNTN} \ q_k, ..., \sum_{k=1}^{n} \tilde{a}^{mk}_{SVNTN} \ q_k \right\} = \tilde{v}_{SVNTN}$ (say)

is the maximum expected loss for player-2, we get the following two neutrosophic linear programming problem (NLPP) models for the two players:

For player-1: $(\text{NLPP})^{\text{I}}$

$$\begin{cases} Maximise & \tilde{u}_{SVNTN} \\ subject\ to & \sum_{j=1}^{m} \tilde{a}_{SVNTN}^{j1}\ p_j \succeq \tilde{u}_{SVNTN} \\ & \sum_{j=1}^{m} \tilde{a}_{SVNTN}^{j2}\ p_j \succeq \tilde{u}_{SVNTN} \\ & \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ & \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ & \sum_{j=1}^{m} \tilde{a}_{SVNTN}^{jn}\ p_j \succeq \tilde{u}_{SVNTN} \\ & \sum_{j=1}^{m} p_j = 1, \\ and & p_j \geq 0,\ \forall\ j = 1, 2, ..., m. \end{cases} \tag{6}$$

For player-2: $(\text{NLPP})^{\text{II}}$

$$\begin{cases} Minimise & \tilde{v}_{SVNTN} \\ subject\ to & \sum_{k=1}^{n} \tilde{a}_{SVNTN}^{1k}\ q_k \preceq \tilde{v}_{SVNTN} \\ & \sum_{k=1}^{n} \tilde{a}_{SVNTN}^{2k}\ q_k \preceq \tilde{v}_{SVNTN} \\ & \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ & \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ & \sum_{k=1}^{n} \tilde{a}_{SVNTN}^{mk}\ q_k \preceq \tilde{v}_{SVNTN} \\ & \sum_{k=1}^{n} q_k = 1, \\ and & q_k \geq 0,\ \forall\ k = 1, 2, ..., n. \end{cases} \tag{7}$$

Here $\tilde{u}_{SVNTN}$ and $\tilde{v}_{SVNTN}$ are SVNTNs representing expected minimum gain and expected maximum loss for player-1 and palyer-2 respectively. The symbols $\succeq$ and $\preceq$ represents the neutrosophic adaptations of order relation $\geq$ and $\leq$ respectively. The above NLPP models for the two players can be restructured as follows-
For player-1: $(\text{NLPP})^{\text{I}}$

$$\begin{cases} Maximise & \tilde{u}_{SVNTN} \\ s.t., & \sum_{j=1}^{m} \tilde{a}_{SVNTN}^{jk}\ p_j \succeq \tilde{u}_{SVNTN}\ \forall\ k = 1, 2, ..., n. \\ & \sum_{j=1}^{m} p_j = 1, \\ and & p_j \geq 0,\ \forall\ j = 1, 2, ..., m. \end{cases} \tag{8}$$

For player-2: $(\text{NLPP})^{\text{II}}$

$$\begin{cases} Minimise & \tilde{v}_{SVNTN} \\ s.t., & \sum_{k=1}^{n} \tilde{a}_{SVNTN}^{jk}\ q_k \preceq \tilde{v}_{SVNTN}\ \forall\ j = 1, 2, ..., m. \\ & \sum_{k=1}^{n} q_k = 1, \\ and & q_k \geq 0,\ \forall\ k = 1, 2, ..., n. \end{cases} \tag{9}$$

4. Value Index of SVNTN using Mellin's Transform

Let $\tilde{a}_{SVNTN} = \{(\zeta, \eta, \theta); \rho, \sigma, \tau\}$ be any SVNTN and $t_{\tilde{a}_{SVNTN}}(\omega),\ i_{\tilde{a}_{SVNTN}}(\omega),\ f_{\tilde{a}_{SVNTN}}(\omega)$ are associated truth, indeterminacy and falsify membership function respectively. Now we first define probability density function (p.d.f) from truth, indeterminacy and falsify membership function respectively as follows

$$\phi_1(\omega) = k_1 t_{\tilde{a}_{SVNTN}}(\omega),\ \phi_2(\omega) = k_2 i_{\tilde{a}_{SVNTN}}(\omega)\ and\ \phi_3(\omega) = k_3 f_{\tilde{a}_{SVNTN}}(\omega)$$

where $k_1,\ k_2$ and k_3 are constants to be obtained using the property of probability density function i.e.,
$\int_{-\infty}^{\infty} \phi_1(\omega) d\omega = 1, \int_{-\infty}^{\infty} \phi_2(\omega) d\omega = 1, \int_{-\infty}^{\infty} \phi_3(\omega) d\omega = 1$ respectively.

We get

$$k_1 = \frac{2}{(\theta - \zeta)\rho},\ k_2 = \frac{2}{(\theta - \zeta)(1 + \sigma)},\ k_3 = \frac{2}{(\theta - \zeta)(1 + \tau)}. \tag{10}$$

Using $k_1,\ k_2,\ k_3$ in $\phi_1(\omega),\ \phi_2(\omega),\ \phi_3(\omega)$ respectively, we now define $\phi(\omega)$ as the p.d.f corresponding to SVNTN $\tilde{a}_{SVNTN}$ as follows

$$\phi(\omega) = \lambda\phi_1(\omega) + (1 - \lambda)\phi_2(\omega) + (1 - \lambda)\phi_3(\omega),\ (0 \leq \lambda \leq 1) \tag{11}$$

where $\lambda \in [0, 1]$ represents the player's preference information. If $\lambda \in \left[0, \frac{1}{2}\right]$, it means that the player is pessimist i.e. he incurs negative feeling and prefer uncertainity. If $\lambda \in \left]\frac{1}{2}, 1\right[$ it means that the player is optimist i.e. he incurs positive feeling and prefer certainity. If $\lambda = \frac{1}{2}$ the player is indifferent of positive or negative feeling, he is moderate.

Now, the Mellin's Transform $M[\phi(\omega), s]$ of a p.d.f $\phi(\omega)$ is defined as $M[\phi(\omega), s] = \int_0^{\infty} \omega^{s-1} \phi(\omega) d\omega$, provided the integral exists. Using the function $\phi(\omega)$ (equation 11) we get

$$M[\phi(\omega), s] = \int_0^{\infty} \omega^{s-1} [\lambda\phi_1(\omega) + (1 - \lambda)\phi_2(\omega) + (1 - \lambda)\phi_3(\omega)] d\omega \tag{12}$$

Now by taking $s = 2$, Mellin's transform is converted into expected value of associated random variable. Hence we get the de-neutrosophicated value or the expected value of SVNTN $\tilde{a}_{SVNTN} = \{(\zeta, \eta, \theta); \rho, \sigma, \tau\}$.

We obtain for $s = 2$,

$$M[\phi(\omega), 2] = \lambda\left\{\frac{\zeta+\eta+\theta}{3}\right\} + (1-\lambda)\left\{\frac{\sigma(\zeta+\eta+\theta)+2(\zeta+\theta)-\eta}{3(1+\sigma)}\right\} + (1-\lambda)\left\{\frac{\tau(\zeta+\eta+\theta)+2(\zeta+\theta)-\eta}{3(1+\tau)}\right\} \quad (13)$$

Where, $\lambda \in [0,1]$ express the degree of optimism of the player. $\lambda \in \left[0, \frac{1}{2}\right[$ express the pessimist behaviour, $\lambda \in \left]\frac{1}{2}, 1\right[$ express the optimist behaviour and $\lambda = \frac{1}{2}$ express that the player is moderate.

Since $M[\phi(\omega), 2]$ depends on λ, let us denote it by $V(\tilde{a}_{SVNTN}, \lambda)$ and is called the 'Value Index' of single valued neutrosophic triangular number $\tilde{a}_{SVNTN} = \{(\zeta, \eta, \theta); \rho, \sigma, \tau\}$. We write

$$V(\tilde{a}_{SVNTN}, \lambda) = M[\phi(\omega), 2] = \lambda\left\{\frac{\zeta+\eta+\theta}{3}\right\} + (1-\lambda)\left\{\frac{\sigma(\zeta+\eta+\theta)+2(\zeta+\theta)-\eta}{3(1+\sigma)}\right\} + (1-\lambda)\left\{\frac{\tau(\zeta+\eta+\theta)+2(\zeta+\theta)-\eta}{3(1+\tau)}\right\} \quad (14)$$

Proposition-1: For a given $\lambda \in [0,1]$ and a given $\tilde{a}_{SVNTN} = \{(\zeta, \eta, \theta); \rho, \sigma, \tau\}$, $V(\tilde{a}_{SVNTN}, \lambda)$ is a unique real number.

4.1. *De-neutrosophication and Ranking of SVNTN:*

Let $\widetilde{neu}(\Re)$ be the set of all SVNTNs, and $\lambda \in [0,1]$ be a given number. We define a mapping $h_\lambda : \widetilde{neu}(\Re) \to \Re$ such that $h_\lambda(\tilde{a}_{SVNTN}) = V(\tilde{a}_{SVNTN}, \lambda) \quad \forall \quad \tilde{a}_{SVNTN} \in \widetilde{neu}(\Re)$; $\Re$ being the set of real numbers. The mapping h_λ is well-defined and associates each $\tilde{a}_{SVNTN} \in \widetilde{neu}(\Re)$ to a unique real number (proposition-1) where the order relations exist naturally. h_λ is called a de-neutrosophication function and used to rank SVNTNs as detailed out in proposition-2 below.

Proposition-2: Let $\tilde{a}_{SVNTN} = \{(\zeta^1, \eta^1, \theta^1); \rho^1, \sigma^1, \tau^1\}$ and $\tilde{b}_{SVNTN} = \{(\zeta^2, \eta^2, \theta^2); \rho^2, \sigma^2, \tau^2\}$ be SVNTNs and $\lambda \in [0,1]$, then the ranking order relation between the two SVNTNs are defined as follows

(1) $\tilde{a}_{SVNTN} \preceq \tilde{b}_{SVNTN} \Leftrightarrow h_\lambda(\tilde{a}_{SVNTN}, \lambda) \leq h_\lambda(\tilde{b}_{SVNTN}, \lambda)$
(2) $\tilde{a}_{SVNTN} \succeq \tilde{b}_{SVNTN} \Leftrightarrow h_\lambda(\tilde{a}_{SVNTN}, \lambda) \geq h_\lambda(\tilde{b}_{SVNTN}, \lambda)$
(3) $\tilde{a}_{SVNTN} \approx \tilde{b}_{SVNTN} \Leftrightarrow h_\lambda(\tilde{a}_{SVNTN}, \lambda) = h_\lambda(\tilde{b}_{SVNTN}, \lambda)$

Where the symbols $\preceq$, $\succeq$ and $\approx$ represents the neutrosophic adaptations of order relation $\leq$, $\geq$ and $=$ respectively.

5. **Proposed Solution Methodology**

In this section, we detail out a step-wise solution methodology we propose to solve any NTMG. The steps of our proposed solution methodology are as follows:

Step-1: Write the respective neutrosophic linear programming problem (NLPP), i.e. equations (8) and (9), for the two players respectively.

Step-2: Write the de-neutrosophic version of NLPPs obtained in step-1 by using the value index of all SVNTNs involved. We get the following respective crisp linear programming problems (CLPPs) for the two players.

For player-1:(CLPP)$^{\mathrm{I}}$

$$\begin{cases} Maximise & V(\tilde{u}_{SVNTN}, \lambda) \\ s.t., & \sum_{j=1}^{m} V(\tilde{a}_{SVNTN}^{jk}, \lambda)p_j \geq V(\tilde{u}_{SVNTN}, \lambda) \; \forall \; k = 1, 2, ..., n. \\ & \sum_{j=1}^{m} p_j = 1, \\ and & p_j \geq 0, \; \forall \; j = 1, 2, ..., m. \end{cases} \tag{15}$$

For player-2: (CLPP)$^{\mathrm{II}}$

$$\begin{cases} Minimise & V(\tilde{v}_{SVNTN}, \lambda) \\ s.t., & \sum_{k=1}^{n} V(\tilde{a}_{SVNTN}^{jk}, \lambda)q_k \leq V(\tilde{v}_{SVNTN}, \lambda) \; \forall \; j = 1, 2, ..., m. \\ & \sum_{k=1}^{n} q_k = 1, \\ and & q_k \geq 0, \; \forall \; k = 1, 2, ..., n. \end{cases} \tag{16}$$

Step-3: Use the formula for value index, i.e. equation-(14), and write the CLPPs for various values of $\lambda \in [0, 1]$ for both players.

Step-4: Solve these CLPPs by simplex method to get optimal mixed strategies and the optimal value of the game for both players.

5.1. *Flowchart*

For an easy understanding, a visual representation of the proposed solution methodology has been depicted by the flow-chart in Figure-2 below.

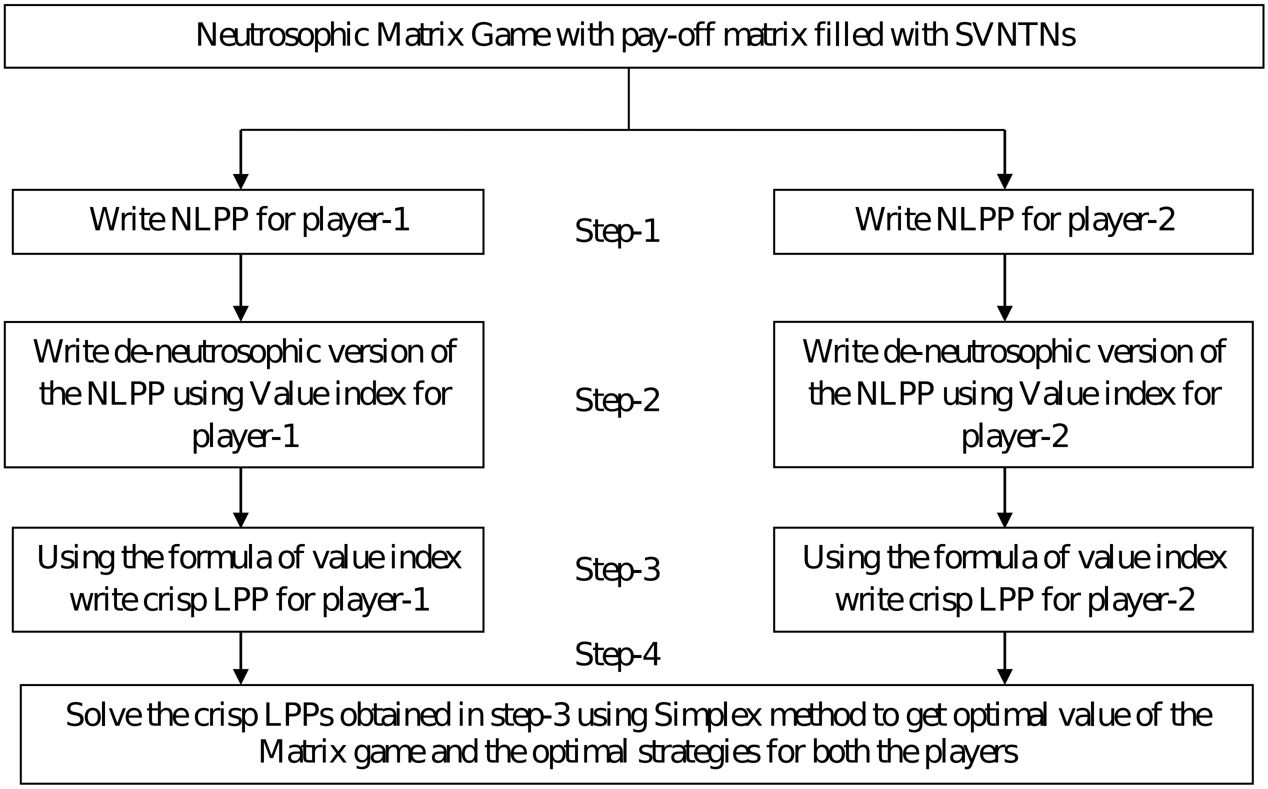


FIGURE 2. Flow Chart of the Proposed Solution Methodology

6. Numerical Models

In this section of our work, we show the validity and applicability of our solution methodology by giving the solution procedure of two NTMG examples. In example-1 we took a simple case of 2x2 pay-off matrix of a NTMG from the work of Jangid et al. [22]. We solve it by our method and then discuss, analyse, and compare their results with the our results using Tables-(1, 2) and a histogram (Figure-3).

In example-2 we consider a real-world case study from telecom sector by taking a 3x3 pay-off matrix of strategies adopted by the companies to capture the market share in the target area. Strategies floated by the companies are represented by SVNTNs. Results obtained are discussed and analysed by means of a graph in Figure-4 and Tables-(3, 4, 5).

6.1. *Example-1:(Jangid et al. [22])*

Let NTMG $(\tilde{A}, S_1, S_2) = \begin{bmatrix} \tilde{a}^{11}_{SVNTN} & \tilde{a}^{12}_{SVNTN} \\ \tilde{a}^{21}_{SVNTN} & \tilde{a}^{22}_{SVNTN} \end{bmatrix}$, where SVNTNs $\tilde{a}^{jk}_{SVNTN}$ are as follows-

$\tilde{a}^{11}_{SVNTN} = \widehat{180} = \{(175, 180, 190); 0.6, 0.4, 0.2\}$,
$\tilde{a}^{12}_{SVNTN} = \widehat{156} = \{(150, 156, 158); 0.6, 0.35, 0.1\}$,

$\tilde{a}^{21}_{SVNTN} = \widehat{90} = \{(80, 90, 100); 0.9, 0.5, 0.1\}$,
$\tilde{a}^{22}_{SVNTN} = \widehat{180} = \{(175, 180, 190); 0.6, 0.4, 0.2\}$

Solution Procedure: Let (p_1, p_2) and (q_1, q_2) are the optimal strategies and $\tilde{u}_{SVNTN}$, $\tilde{v}_{SVNTN}$ are optimal SVNTN values of the game for player-1 and player-2 respectively, then NLPPs for the two players are written as follows-

$$\text{For player-1 (NLPP)}^{\text{I}}: \begin{cases} Max & \tilde{u}_{SVNTN} \\ s.t., & \widehat{180}\ p_1 + \widehat{90}\ p_2 \succeq \tilde{u}_{SVNTN} \\ & \widehat{156}\ p_1 + \widehat{180}\ p_2 \succeq \tilde{u}_{SVNTN} \\ & p_1 + p_2 = 1, \\ and & p_1,\ p_2 \geq 0. \end{cases} \tag{17}$$

$$\text{For player-2 (NLPP)}^{\text{II}}: \begin{cases} Min & \tilde{v}_{SVNTN} \\ s.t., & \widehat{180}\ q_1 + \widehat{156}\ q_2 \preceq \tilde{v}_{SVNTN} \\ & \widehat{90}\ q_1 + \widehat{180}\ q_2 \preceq \tilde{u}_{SVNTN} \\ & q_1 + q_2 = 1, \\ and & q_1,\ q_2 \geq 0. \end{cases} \tag{18}$$

For de-neutrosophication of above NLPP models, we apply the value index of all the SVNTNs, we get the following CLPPs for both the players

For player-1 $(\text{CLPP})^{\text{I}}$:

$$\begin{cases} Max & V(\tilde{u}_{SVNTN}, \lambda) \\ s.t., & V(\widehat{180}, \lambda)p_1 + V(\widehat{90}, \lambda)p_2 \geq V(\tilde{u}_{SVNTN}, \lambda) \\ & V(\widehat{156}, \lambda)p_1 + V(\widehat{180}, \lambda)p_2 \geq V(\tilde{u}_{SVNTN}, \lambda) \\ & p_1 + p_2 = 1, \\ and & p_1,\ p_2 \geq 0. \end{cases} \tag{19}$$

For player-2 $(\text{CLPP})^{\text{II}}$:

$$\begin{cases} Min & V(\tilde{v}_{SVNTN}, \lambda) \\ s.t., & V(\widehat{180}, \lambda)q_1 + V(\widehat{156}, \lambda)q_2 \leq V(\tilde{v}_{SVNTN}, \lambda) \\ & V(\widehat{90}, \lambda)q_1 + V(\widehat{180}, \lambda)q_2 \leq V(\tilde{v}_{SVNTN}, \lambda) \\ & q_1 + q_2 = 1, \\ and & q_1,\ q_2 \geq 0. \end{cases} \tag{20}$$

The value index of all the different SVNTNs involved are calculated using the formulla (equation-14) explained in section-4. They are given as follows:

$V(\widehat{180}, \lambda) = (365.9126 - 184.246\lambda)$,

$V(\widehat{156}, \lambda) = (307.1335 - 152.4669\lambda)$, and

$V(\widehat{90}, \lambda) = (180 - 90\lambda)$

Using these value indexes we get the following crisp LPPs for the two players

For player-1 $(\text{CLPP})^{\text{I}}$:

$$\begin{cases} Max & V(\tilde{u}_{SVNTN}, \lambda) = u(say) \\ s.t., & (365.9126 - 184.246\lambda)p_1 + (180 - 90\lambda)p_2 \geq u \\ & (307.1335 - 152.4669\lambda)p_1 + (365.9126 - 184.246\lambda)p_2 \geq u \\ & p_1 + p_2 = 1, \\ and & p_1,\ p_2 \geq 0. \end{cases} \tag{21}$$

For player-2 $(\text{CLPP})^{\text{II}}$:

$$\begin{cases} Min & V(\tilde{v}_{SVNTN}, \lambda) = v(say) \\ s.t., & (365.9126 - 184.246\lambda)q_1 + (307.1335 - 152.4669\lambda)q_2 \leq v \\ & (180 - 90\lambda)q_1 + (365.9126 - 184.246\lambda)q_2 \leq v \\ & q_1 + q_2 = 1, \\ and & q_1,\ q_2 \geq 0. \end{cases} \tag{22}$$

For various values of optimism degree λ, the value index of SVNTNs are calculated using the formula explained in section-4 (equation 14), and are given in Table-1 below.

TABLE 1. Value index of SVNTNs for different values of optimism degree λ

λ	$V(\widehat{180}, \lambda)$	$V(\widehat{156}, \lambda)$	$V(\widehat{90}, \lambda)$
0.0	365.9126	307.1335	180
0.1	347.4880	291.8868	171
0.2	329.0634	276.6401	162
0.3	310.6388	261.3934	153
0.4	292.2142	246.1467	144
0.5	273.7896	230.9000	135
0.6	255.3650	215.6533	126
0.7	236.9404	200.4066	117
0.8	218.5158	185.1599	108
0.9	200.0912	169.9132	99
1.0	181.6666	154.6666	90

Using the values given in Table 1, optimal solutions for various degree of optimism λ are obtained by solving CLPPs for player-1 and player-2 equations 21 and 22 and are given in Table-2 below.

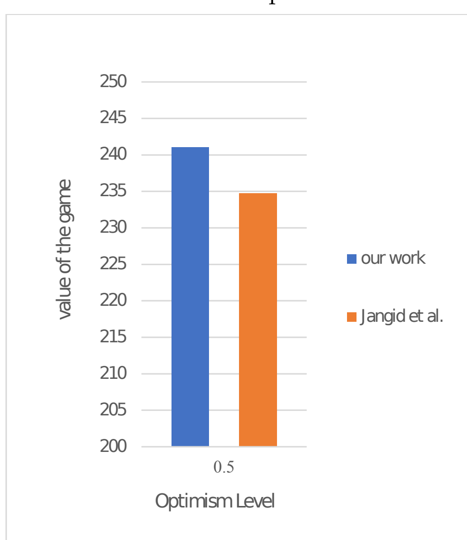
TABLE 2. Optimal Solution for Player-1 at different values of optimism degree λ

λ	p_1	p_2	q_1	q_1	$Max(u)$
0.0	0.7598	0.2402	0.2402	0.7598	305.2071
0.2	0.7612	0.2388	0.2388	0.7612	289.1611
0.3	0.7620	0.2380	0.2380	0.7620	273.1155
0.4	0.7629	0.2371	0.2371	0.7629	257.0701
0.5	0.7639	0.2361	0.2361	0.7639	241.0251
0.6	0.7651	0.2349	0.2349	0.7651	224.9805
0.7	0.7665	0.2335	0.2335	0.7665	208.9366
0.8	0.7682	0.2318	0.2318	0.7682	192.8933
0.9	0.7701	0.2299	0.2299	0.7701	176.8509
1.0	0.7725	0.2296	0.2296	0.7725	160.8099

6.1.1. *Discussion and Comparison of Results of Example-1*

Our solution results for various values of degree of optimism are given in Table-2 above. Results show that value of the game decreases from 321.2532 to 160.8099 as the degree of optimism increases from 0.0 to 1.0, it means the value of the game is inversely proportional to the degree of optimism of the incumbent player. It righty suggests that it is not wise to take decisions with high level of optimism. It is better to be more realistic than too much optimistic. For a better understanding and analysis, the graphical representation of obtained optimal values 'u' against different values of the degree of optimism 'λ' for player-1 is given in Figure-4.

FIGURE 3. Comparison of Works



Jangid et al. [22] with their solution methodology have solved only for $\lambda = \frac{1}{2}$ in their work. They have got SVNTN $\langle(152.22, 158.1312, 160.8416); 0.6, 0.4, 0.2\rangle$ as the value of the game for player-1, its value index can be calculated as 234.7706 using formula explained in section-4. Whereas our approach gives 241.0251 as the optimal value of the game for $\lambda = \frac{1}{2}$ (refer Table-2). So, our methodology yields better results for given optimism level, comparison of our result with Jangid et al. [22] can be seen in the histogram (Figure-3). Also, their method is difficult on calculations so they have calculated it only for $\lambda = \frac{1}{2}$, whereas our is an easy procedure, we have done it for various values of λ varying from 0 to 1.

6.2. *Example-2:(A telecom sector case-study)*

Nowadays it is impossible to think a life without a high-speed internet connection in your mobile handset or a high speed wi-fi internet connection at your home and at your workplace. In this regard the launch of fifth generation (5G) network recently has brought a revolution in India. Presently the number of mobile network subscribers in India is around 449 million and this number is surely going to increase with launch of 5G network. The two major telecom operators in India, Airtel (C_1) and Vodafone (C_2) (say), want to take advantage of this situation and each of them aim to increase the number of subscribers than the other. The two companies have fixed number of costumers and each of them want to add new costumers by porting to one from the other or by adding new subscribers. They make the following strategies to lure more costumers –

Strategy-I: 'Reducing the tariff of their data plans per GB'

Strategy-II: 'Giving free soaps like hotstar, amazon prime etc with their data plan'

Strategy-III: 'Advertising through print, electronic and social media'

The market research wing (MRW) experts of the two companies cannot precisely predict the increase in the number of costumers because of the uncertainty and indeterminacy that is always present in the large telecom market. They can only provide some estimated data with some amount of uncertainty and indeterminacy involved in it. This competitive situation between the two companies can be presented by means of a matrix game (MG) with payoff matrix equipped with SVNTNs. Supposing that the MRW experts of the two companies after analysing the collected data through some survey and their expertise, presented the following pay-off matrix on the number of costumers. (All numbers are supposed to be multiplied by 1000).

$$\tilde{E} = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{ccc} I & II & III \end{array} \\ \begin{bmatrix} \langle(176,180,183);.6,.5,.2\rangle & \langle(83,90,96);.8,.4,.2\rangle & \langle(110,120,133);.9,.5,.1\rangle \\ \langle(87,89,92);.6,.4,.2\rangle & \langle(176,180,183);.6,.5,.2\rangle & \langle(118,125,130);.7,.5,.3\rangle \\ \langle(118,125,130);.7,.5,.3\rangle & \langle(145,150,153);.8,.4,.1\rangle & \langle(83,90,96);.8,.4,.2\rangle \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{ccc} I & II & III \end{array} \\ \begin{bmatrix} \langle\widehat{180}\rangle & \langle\widehat{90}\rangle & \langle\widehat{120}\rangle \\ \langle\widehat{89}\rangle & \langle\widehat{180}\rangle & \langle\widehat{125}\rangle \\ \langle\widehat{125}\rangle & \langle\widehat{150}\rangle & \langle\widehat{90}\rangle \end{bmatrix} \end{array} = \begin{bmatrix} \tilde{a}^{jk}_{SVNTN} \end{bmatrix} \ (say)\ j = 1,2,3;\ k = 1,2,3.$$

Where, $\tilde{a}^{12}_{SVNTN} = \langle\widehat{90}\rangle = \langle(83,90,96);.8,.4,.2\rangle$ means that the company C_1 (Player-1) will get an increase of 90 units in its customer base if C_1 sticks to apply strategy-I (i.e., 'Reducing the tariff of their data plans per GB') and if company C_2 sticks to apply strategy-II (i.e., 'Giving free soaps like hotstar, amazon prime etc with their data'). MRW experts are 80% positive about it, 20% they are not positive and they remain indeterminate by 40% about the increase.

All other SVNTNs can be explained similarly.

Solution Procedure: let (p_1, p_2, p_3) and (q_1, q_2, q_3) are the optimal strategies and $\tilde{u}_{SVNTN}$, $\tilde{v}_{SVNTN}$ are optimal SVNTN values of the game for company C_1 and company C_2 respectively, then NLPPs for the two players (C_1 & C_2) are written as follows-

For player-1 $(\text{NLPP})^{\text{I}}$:

$$\begin{cases} Max & \tilde{u}_{SVNTN} \\ s.t., & \tilde{a}^{11}_{SVNTN}\ p_1 + \tilde{a}^{21}_{SVNTN}\ p_2 + \tilde{a}^{31}_{SVNTN}\ p_3 \succeq \tilde{u}_{SVNTN} \\ & \tilde{a}^{12}_{SVNTN}\ p_1 + \tilde{a}^{22}_{SVNTN}\ p_2 + \tilde{a}^{32}_{SVNTN}\ p_3 \succeq \tilde{u}_{SVNTN} \\ & \tilde{a}^{13}_{SVNTN}\ p_1 + \tilde{a}^{23}_{SVNTN}\ p_2 + \tilde{a}^{33}_{SVNTN}\ p_3 \succeq \tilde{u}_{SVNTN} \\ & p_1 + p_2 + p_3 = 1, \\ and & p_1,\ p_2,\ p_3 \geq 0. \end{cases} \tag{23}$$

For player-2 $(\text{NLPP})^{\text{II}}$:

$$\begin{cases} Min & \tilde{v}_{SVNTN} \\ s.t., & \tilde{a}^{11}_{SVNTN}\ q_1 + \tilde{a}^{12}_{SVNTN}\ q_2 + \tilde{a}^{13}_{SVNTN}\ q_3 \preceq \tilde{v}_{SVNTN} \\ & \tilde{a}^{21}_{SVNTN}\ q_1 + \tilde{a}^{22}_{SVNTN}\ q_2 + \tilde{a}^{23}_{SVNTN}\ q_3 \preceq \tilde{v}_{SVNTN} \\ & \tilde{a}^{31}_{SVNTN}\ q_1 + \tilde{a}^{32}_{SVNTN}\ q_2 + \tilde{a}^{33}_{SVNTN}\ q_3 \preceq \tilde{v}_{SVNTN} \\ & q_1 + q_2 + q_3 = 1, \\ and & q_1,\ q_2,\ q_3 \geq 0. \end{cases} \tag{24}$$

For de-neutrosophication of above NLPP models, we apply the value index of all the SVNTNs. we get the following CLPPs for the two players

For player-1 $(\text{CLPP})^{\text{I}}$:

$$\begin{cases} Max & V(\tilde{u}_{SVNTN}, \lambda) = u(say) \\ s.t., & V(\widehat{180}, \lambda)\ p_1 + V(\widehat{89}, \lambda)\ p_2 + V(\widehat{125}, \lambda)\ p_3 \geq u \\ & V(\widehat{90}, \lambda)\ p_1 + V(\widehat{180}, \lambda)\ p_2 + V(\widehat{150}, \lambda)\ p_3 \geq u \\ & V(\widehat{120}, \lambda)\ p_1 + V(\widehat{125}, \lambda)\ p_2 + V(\widehat{90}, \lambda)\ p_3 \geq u \\ & p_1 + p_2 + p_3 = 1, \\ and & p_1,\ p_2,\ p_3 \geq 0. \end{cases} \tag{25}$$

For player-2 $(\text{CLPP})^{\text{II}}$:

$$\begin{cases} Min & V(\tilde{v}_{SVNTN},\lambda)=v(say) \\ s.t., & V(\widehat{180},\lambda)\ q_1+V(\widehat{90},\lambda)\ q_2+V(\widehat{120},\lambda)\ q_3\leq v \\ & V(\widehat{89},\lambda)\ q_1+V(\widehat{180},\lambda)\ q_2+V(\widehat{125},\lambda)\ q_3\leq v \\ & V(\widehat{125},\lambda)\ q_1+V(\widehat{150},\lambda)\ q_2+V(\widehat{90},\lambda)\ q_3\leq v \\ & q_1+q_2+q_3=1, \\ and & q_1,\ q_2,\ q_3\geq 0. \end{cases} \tag{26}$$

The value index of all the distinct SVNTNs involved are calculated as
$V(\tilde{a}^{11}_{SVNTN},\lambda)=V(\widehat{180},\lambda)=(378.8332–199.1666\lambda)=V(\tilde{a}^{22}_{SVNTN},\lambda)$;
$V(\tilde{a}^{12}_{SVNTN},\lambda)=V(\widehat{90},\lambda)=(178.8173–89.1507\lambda)=V(\tilde{a}^{33}_{SVNTN},\lambda)$;
$V(\tilde{a}^{13}_{SVNTN},\lambda)=V(\widehat{120},\lambda)=(243.5756–122.5756\lambda)$;
$V(\tilde{a}^{21}_{SVNTN},\lambda)=V(\widehat{89},\lambda)=(179.1825–89.8492\lambda)$;
$V(\tilde{a}^{23}_{SVNTN},\lambda)=V(\widehat{125},\lambda)=(247.7093–123.3760\lambda)=V(\tilde{a}^{31}_{SVNTN},\lambda)$;
$V(\tilde{a}^{32}_{SVNTN},\lambda)=V(\widehat{150},\lambda)=(297.5843–148.2510\lambda)$;
Using these value indexes, we get the following CLPPs for the two players
For player-1 $(\text{CLPP})^{\text{I}}$:

$$\begin{cases} Max & V(\tilde{u}_{SVNTN},\lambda)=u(say) \\ s.t., & (378.8332–199.1666\lambda)\ p_1+(179.1825–89.8492\lambda)\ p_2+(247.7093–123.3760\lambda)\ p_3\geq u \\ & (178.8173–89.1507\lambda)\ p_1+(378.8332–199.1666\lambda)\ p_2+(297.5843–148.2510\lambda)\ p_3\geq u \\ & (243.5756–122.5756\lambda)\ p_1+(247.7093–123.3760\lambda)\ p_2+(178.8173–89.1507\lambda)\ p_3\geq u \\ & p_1+p_2+p_3=1, \\ and & p_1,\ p_2,\ p_3\geq 0. \end{cases} \tag{27}$$

For player-2 $(\text{CLPP})^{\text{II}}$:

$$\begin{cases} Min & V(\tilde{v}_{SVNTN},\lambda)=v(say) \\ s.t., & (378.8332–199.1666\lambda)\ q_1+(178.8173–89.1507\lambda)\ q_2+(243.5756–122.5756\lambda)\ q_3\leq v \\ & (179.1825–89.8492\lambda)\ q_1+(378.8332–199.1666\lambda)\ q_2+(247.7093–123.3760\lambda)\ q_3\leq v \\ & (247.7093–123.3760\lambda)\ q_1+(297.5843–148.2510\lambda)\ q_2+(178.8173–89.1507\lambda)\ q_3\leq v \\ & q_1+q_2+q_3=1, \\ and & q_1,\ q_2,\ q_3\geq 0. \end{cases} \tag{28}$$

The value index of distinct SVNTNs are calculated using the proposed method for various degree of optimism λ, and are given in table-3 below.

Table 3. Value index of SVNTNs for different values of optimism degree λ

λ	$V(\widehat{180},\lambda)$	$V(\widehat{90},\lambda)$	$V(\widehat{120},\lambda)$	$V(\widehat{89},\lambda)$	$V(\widehat{125},\lambda)$	$V(\widehat{150},\lambda)$
0.0	378.8332	178.8173	243.5756	179.1825	247.7093	297.5843
0.1	358.9165	169.9022	231.3180	170.1975	235.3717	282.7592
0.2	338.9998	160.9871	219.0604	161.2126	223.0341	267.9341
0.3	319.0832	152.0720	206.8029	152.2277	210.6965	253.1090
0.4	299.1665	143.1570	194.5453	143.2428	198.3589	238.2839
0.5	279.2499	134.2419	182.2878	134.2579	186.0213	223.4588
0.6	259.3332	125.3268	170.0302	125.2729	173.6837	208.6337
0.7	239.4165	116.4118	157.7726	116.2880	161.3461	193.8086
0.8	219.4999	107.4967	145.5151	107.3031	149.0085	178.9835
0.9	199.5832	98.5816	133.2575	98.3182	136.6709	164.1584
1.0	179.6666	89.6666	121.0000	89.3333	124.3333	149.3333

Using the values given in Table-3, optimal solutions for various degree of optimism λ are obtained by solving CLPP for Player-1 (equation-27) and are given in table-4 below.

Table 4. Optimal Solution For Player-1 for Different Values of Optimism Degree λ

λ	p_1	p_2	p_3	$Max(u)$
0.0	0.3363	0.6637	0.0000	246.3193
0.1	0.3381	0.6619	0.0000	234.0012
0.2	0.3401	0.6599	0.0000	221.6825
0.3	0.3424	0.6576	0.0000	209.3632
0.4	0.3450	0.6550	0.0000	197.0430
0.5	0.3480	0.6520	0.0000	184.7219
0.6	0.3515	0.6485	0.0000	172.3994
0.7	0.3556	0.6444	0.0000	160.0753
0.8	0.3605	0.6395	0.0000	147.7492
0.9	0.3664	0.6336	0.0000	135.4203
1.0	0.3737	0.6263	0.0000	123.0878

Using the values given in Table-3, optimal solutions for various degree of optimism λ are obtained by solving CLPP for Player-2 (equation-28) and are given in table-5 below.

TABLE 5. Optimal Solution For Player-2 for Different Values of Optimism Degree λ

λ	q_1	q_2	q_3	$Max(v)$
0.0	0.0203	0.0000	0.9797	246.3199
0.1	0.0210	0.0000	0.9790	234.0012
0.2	0.0219	0.0000	0.9781	221.6825
0.3	0.0228	0.0000	0.9772	209.3632
0.4	0.0239	0.0000	0.9761	197.0430
0.5	0.0251	0.0000	0.9749	184.7219
0.6	0.0265	0.0000	0.9735	172.3994
0.7	0.0282	0.0000	0.9718	160.0753
0.8	0.0302	0.0000	0.9698	147.7492
0.9	0.0326	0.0000	0.9674	135.4203
1.0	0.0356	0.0000	0.9644	123.0878

6.2.1. *Conclusive Words on the Results of Example-2:*

The solution results for various values of degree of optimism for the incumbent player i.e., company C_1 (Airtel), are given in Table-4 above. Results show that value of the game decreases from 246.3193 to 123.0878 as the degree of optimism increases from 0.0 to 1.0, it means the value of the game is inversely proportional to the degree of optimism of the incumbent player. The results of example-2 almost follow the same pattern as of example-1. This can be observed from the graphical representation in Figure-4 of obtained optimal values 'u' for incumbent player-1 against different values of the degree of optimism 'λ' for both example-1 & 2.

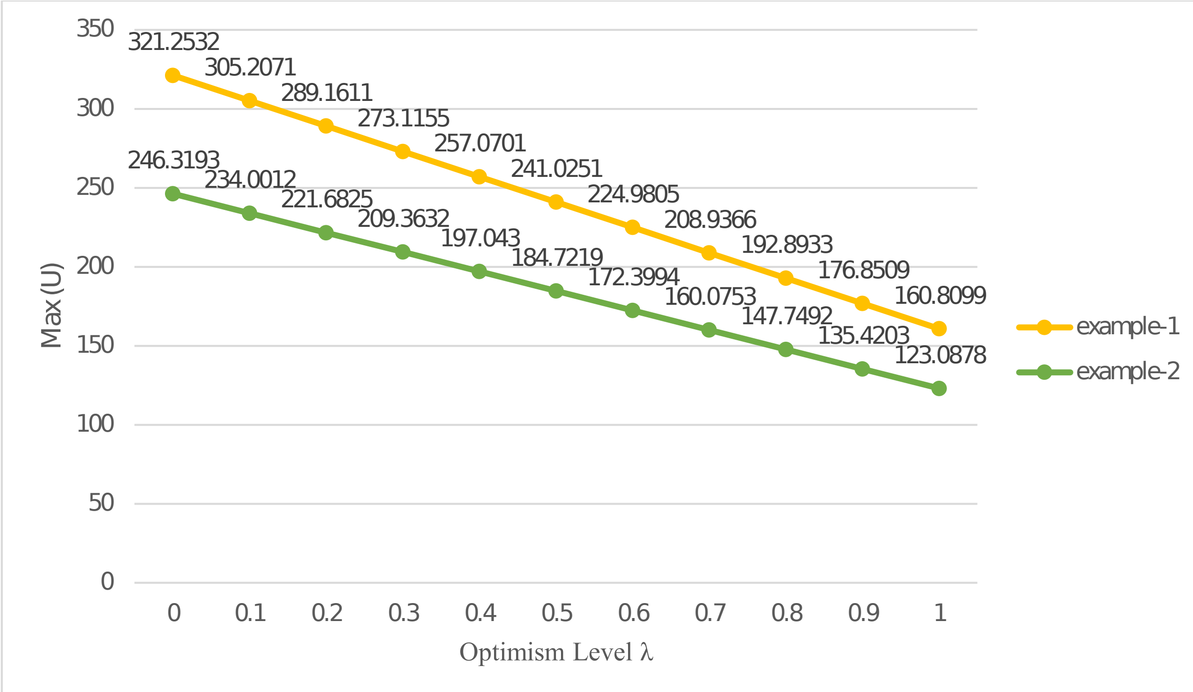


Figure 4. Value of the Game Against Degree of Optimism λ

7. Discussion:

In the zero sum NMG, the optimal expected loss of player-2 is equal to the optimal expected gain of player-1, it can be observed from Table-2 in Example-1 and Table-4 & 5 in Example-2. A graphical representation of optimal values against different values of degree of optimism λ is given in Figure-4 for both Example-1 and Example-2. As all the results obtained by our solution methodology are crisp, they are more reliable and trustworthy. Analysing the results of our work it can be summarised that the optimum value of game for player-1 decreases as degree of optimism 'λ' increases in interval [0, 1]. So we can say that optimal value obtained for player-1 is inversely proportional to his degree of optimism, i.e., the more optimistic you are, and the more you may lose. So, we can conclude that it good to be moderate rather than over optimistic.

8. Conclusion:

This study introduces an efficient de-neutrosophication method that leverages Mellin's transform to derive precise values from Single-Valued Neutrosophic Triangular Numbers (SVNTNs), significantly enhancing decision-making processes in Neutrosophic Matrix-Game strategies (NTMGs). We have demonstrated the efficacy of this technique through detailed numerical examples. By transforming Neutrosophic Linear Programming Problems (NLPPs) into Crisp Linear Programming Problems (CLPPs), and adjusting for varying degrees of optimism, we have utilized the TORA-2.0 software to achieve optimal solutions that promise to aid competitive players in the industrial sector in making more informed and economically beneficial decisions. We aim to broaden the scope of our research to encompass a wider array of neutrosophic numbers, including but not limited to trapezoidal, pentagonal, interval-valued, bi-polar, and spherical neutrosophic numbers. This expansion is anticipated to address more complex and diverse decision-making scenarios, offering a comprehensive toolkit for both theoretical exploration and practical application in the field of neutrosophic decision-making and optimization.

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