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An Enhanced Generalized Neutrosophic Number and its role in Multi-Criteria Decision-Making Challenges

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Abstract. In this article, we have proposed an ordering technique for Neutrosophic numbers with non-linear functions. Consequently, the non-linear functions overcome the limitations of linear function approaches by giving an enhanced framework for handling and modeling uncertainty. Hence, this study presents the Generalized Parabolic Single Valued Neutrosophic Number (GPSVNN) to address the uncertainties in Multi-Criteria Decision-Making (MCDM) circumstances. GPSVNN can handle uncertainty and perform arithmetic operations to deal with MCDM through the (α, β, γ)-cut technique. The computation of the (α, β, γ)-cut of the neutrosophic number is reduced by defining the "value" and "ambiguity." As a result, it becomes more systematic when the complicated computations using the (α, β, γ)-cut approach are carried out. A novel ordering approach has been developed in this study by incorporating the "value" and "ambiguity" of GPSVNN. Finally, we have given an example using GPSVNN in a life satisfaction survey to show its applicability.

Keywords: Generalized Parabolic Single Valued Neutrosophic Number(GPSVNN); Arithmetic Operators of GPSVNN; Values and Ambiguities of GPSVNN & Mutli-criteria Decision making problem.

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1. Introduction

Handling data that contain uncertainty and dealing with nonlinearity has become vital in numerous applications such as facial pattern recognition, transmission systems, knowledge-based models for risk assessment, stock trading, etc. Information derived from computational perception and cognition, which is unclear, imprecise, ambiguous, partially true, or lacking specific limits, can be dealt with the help of fuzzy logic. Lotfi A. Zadeh [1] initially proposed the concept of fuzzy sets in 1965. In [2, 3] fuzzy logic in multi-criteria decision-making using the concept of fuzzy numbers have been applied. The arithmetic operations for generalized parabolic fuzzy numbers and its applications were explored in [4]. In [5], the authors address the mF Dombi weighted averaging and geometric operators to solve multi-criteria decision-making problems that utilize mF information under M-polar fuzzy sets.

Chakraborty et al. [6] demonstrated the hexagonal fuzzy number and its characteristic representation, ranking, defuzzification method, and application in the manufacturing inventory management problem. By expanding the concept of evidence theory, Krishankumar [7] has suggested a unique ranking mechanism under the probabilistic hesitant fuzzy set.The fuzzy set gives one index to represent both membership and non-membership degrees.

The fuzzy set cannot express its independence. Atanassov [8] proposed the idea of intuitionistic fuzzy sets to solve this problem. Employing intuitionistic fuzzy logic enables the resolution of challenging decision-making problems. Many researchers [9–13] have applied various intuitionistic fuzzy numbers to multi-criteria decision-making problems. The study of modelling uncertainty is evolving rapidly. Researchers have previously conducted different significant and progressive investigations, and there are numerous approaches, including fuzzy and intuitionistic fuzzy sets, to handle these uncertainties in modelling. These problems, which apply to real-world problems, cannot deal with all forms of uncertainty, such as ambiguous and inconsistent information. Smarandache [14, 15] initiated neutrosophic theory, which further generalizes fuzzy and intuitionistic fuzzy sets. In [16], the authors defined a particular case of a neutrosophic set called a single-valued neutrosophic set and set-theoretic operators. Chen and Jiqian [17] introduced the Dombi operations of t-norm and t-conorm, and they benefit from being very flexible concerning the operational parameters. and to solve multi-criteria decision-making problems, in [18] a new tool have been developed, that considers the bipolar trapezoidal neutrosophic and the Dombi operators. In [19–21], investigated trapezoidal neutrosophic numbers and its applicability. Chakraborty [22–24] developed the de-neutrosophication approach using the elimination area method as a manifestation of the linear pentagonal neutrosophic number. In [25], a decision-making strategy is described by applying similarity measures based on distance measures. Paulraj S. [26] presented an expansion of single-valued trapezoidal neutrosophic ordered weighted harmonic averaging. Researchers widely use a proactive green supply chain management strategy in [27]. Janani [28] and Ramya [29] presented a perceptive investigation that expands on Bipolar Pythagorean refined set and Pythagorean Neutrosophic Hypersoft Sets, emphasizing the essential features. Many researchers [30–40] have used different neutrosophic numbers to deal with various multi-criteria decision-making problems.

The increasing complexity and unpredictability of decision-making situations in several fields need innovative mathematical frameworks which could effectively handle these challenges. However, there may be some restrictions due to insufficient or lacking quality of the currently available data. Sometimes, using linear functions is inadequate for the consideration of uncertainty. Therefore, the non-linear functions provide an improved framework for managing and modeling uncertainty. Hence, the non-linearity in the Neutrosophic numbers enhances its applicability range. In this study, we explore the Generalized Parabolic Single Valued Neutrosophic Number (GPSVNN), a novel form of non-linear neutrosophic number.

Contributions:

- We have introduced a new type of non-linear neutrosophic number called Generalized Parabolic Single Valued Neutrosophic Number.
- This study develops a novel ordering method by incorporating the ”value” and ”ambiguity” of these Neutrosophic numbers.
- By defining the ”value” and ”ambiguity” of these Neutrosophic numbers, significantly reduces the requirement to compute the (α, β, γ) -cut of the neutrosophic number. Consequently, it becomes more systematic when the tedious calculations employing the (α, β, γ)-cut approach are performed.
- Instead of computing over the complete integration range, the value and ambiguity are computed at (α, β, γ)- levels. These levels are referred to as flexibility parameters because they enable decision-makers to act at different stages of the decision-making process.

The paper is structured as follows. A detailed literature study and introduction are discussed in the first section, and essential preliminary remarks are presented in the second section. The definition of Generalized Parabolic Single-Valued Neutrosophic Numbers (GPSVNN), along with their arithmetic operators, values, and ambiguities, are provided in the following part. In [41], the study addressed the ranking of inhabitants' satisfaction levels with municipal services. Twenty municipal services from the Life Satisfaction Survey (LSS), conducted annually by the Turkish Statistical Institution, are considered possibilities for this purpose. Additionally, the 2014–2019 period was used as a set of criteria when evaluating the inhabitants' contentment, in addition to the previous year. The researchers transformed the participant responses in the dataset into Picture Fuzzy Numbers (PFNs) with four parameters to analyze the impact of all opinions on the decision-making process. (positive, neutral, negative, and refusal). Finally, they used PFNs arithmetic operators and evaluated the results using the VIKOR (VIseKriterijumska Optimizacija I Kompromisno Resenje) technique. In this scenario, we offered the choice within the neutrosophic environment for the same problem, indicating that the opinion type is expressed in GPSVNN. Using its values and ambiguity, we have ranked the alternative from 2014 to 2017.

2. Preliminaries

Definition 2.1. [14] A neutrosophic set A on a universal set X is defined as $A = \{\langle T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where T_A, I_A ,F_A : $X \to]0^-, 1[^+$, represents the degree of membership, degree of indeterministic, and degree of non-membership respectively of the element $x \in X$, such that $0^- \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$.

Definition 2.2. [16] A single valued neutrosophic set A on a universal set X is defined as $A = \{\langle T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where T_A, I_A ,$F_A : X \to [0,1]$, represents the degree of membership, degree of indeterministic, and degree of non-membership respectively of the element $x \in X$, such that $0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3$.

Definition 2.3. [24] A Single Valued Neutrosophic Number (SVNN) $\tilde{a} = \langle T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}\rangle$, in the set of real numbers R with truth-membership function $T_{\tilde{a}}$, indeterminacy-membership function $I_{\tilde{a}}$ and falsity-membership function $F_{\tilde{a}}$, is defined as

$$T_{\tilde{a}}(x) = \begin{cases} f_{\tilde{a}}(x) & , \text{ if } a_1 \leq x < b_1 \\ 1 & , \text{ if } b_1 \leq x < c_1 \\ g_{\tilde{a}}(x) & , \text{ if } c_1 \leq x < d_1 \\ 0 & , \text{ otherwise} \end{cases}, I_{\tilde{a}}(x) = \begin{cases} l_{\tilde{a}}(x) & , \text{ if } a_2 \leq x < b_2 \\ 0 & , \text{ if } b_2 \leq x < c_2 \\ m_{\tilde{x}}(x) & , \text{ if } c_2 \leq x < d_2 \\ 1 & , \text{ otherwise} \end{cases} \text{ and}$$

$$F_{\tilde{a}}(x) = \begin{cases} h_{\tilde{a}}(x) & , \text{ if } a_3 \leq x < b_3 \\ 0 & , \text{ if } b_3 \leq x < c_3 \\ k_{\tilde{x}}(x) & , \text{ if } c_3 \leq x < d_3 \\ 1 & , \text{ otherwise} \end{cases}$$ respectively, where $0 \leq T_{\tilde{a}} + I_{\tilde{a}} + F_{\tilde{a}} \leq 3$ and $a_i, b_i, c_i, di \in \mathbb{R}$, $a_i \leq b_i \leq c_i \leq d_i$ where $i = 1, 2, 3$ and the functions $f_{\tilde{a}}, g_{\tilde{a}}, l_{\tilde{a}}, m_{\tilde{a}}, h_{\tilde{a}}, k_{\tilde{a}}$:$\mathbb{R} \to [0, 1]$.

The functions, $f_{\tilde{a}}, m_{\tilde{a}}, k_{\tilde{a}}$ are non-decreasing continuous function and $g_{\tilde{a}}, l_{\tilde{a}}, h_{\tilde{a}}$ are non-increasing continuous function. SVNN is also denoted by $\tilde{a} = \langle (a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), (a_3, b_3, c_3, d_3)\rangle$

Definition 2.4. A Single Valued Neutrosophic Number defined on the set of real numbers $\mathbb{R}$ is said to be Generalized Single Valued Neutrosophic Number (GSVNN) $G_{\tilde{a}} = \langle T_{G\tilde{a}}, I_{G\tilde{a}}, F_{G\tilde{a}}; \omega, \rho, \delta\rangle$, with truth-membership function $T_{G\tilde{a}}(x)$, indeterminacy-membership function $I_{G\tilde{a}}(x)$ and falsity-membership function $F_{G\tilde{a}}(x)$ has the following characteristics.

(1) $T_{G\tilde{a}}, I_{G\tilde{a}}, F_{G\tilde{a}}\ \mathbb{R} \to [0, 1]$.
(2) $T_{G\tilde{a}} = 0, I_{G\tilde{a}} = 1, F_{G\tilde{a}} = 1 for\ \ all$ x $\in (-\infty, a_i] \cup [d_i, \infty)$.
(3) $T_{G\tilde{a}}(x)$ is strictly increasing on $[a_1, b_1]$ and $T_{G\tilde{a}}(x)$ is strictly decreasing on $[c_1, d_1]$.
$I_{G\tilde{a}}(x)$ is strictly decreasing on $[a_2, b_2]$ and $I_{G\tilde{a}}(x)$ is strictly increasing on $[c_2, d_2]$.
$F_{G\tilde{a}}(x)$ is strictly decreasing on $[a_3, b_3]$ and $F_{G\tilde{a}}(x)$ is strictly increasing on $[c_3, d_3]$.
(4) $T_{G\tilde{a}}(x) = \omega$ for all $x \in [b_1, c_1]$ where $0 < \omega \leq 1$. $I_{G\tilde{a}}(x) = \rho$ for all $x \in [b_2, c_2]$ where $0 \leq \rho < 1$. $F_{G\tilde{a}}(x) = \delta$ for all $x \in [b_2, c_2]$ where $0 \leq \delta < 1$.

3. A Generalized Parabolic Single Valued Neutrosophic Number(GPSVNN)

Definition 3.1. A Generalized Parabolic Single Valued Neutrosophic Number (GPSVNN) ,
$\tilde{A} = \langle (T_{\tilde{A}}; \omega), (I_{\tilde{A}}; \rho), (F_{\tilde{A}}; \delta)\rangle$, is a Neutrosophic set on real number $\mathbb{R}$ with truth-membership function $T_{\tilde{A}}$, indeterminacy-membership function $I_{\tilde{A}}$ and falsity-membership function $F_{\tilde{A}}$, is defined as

$$T_{\tilde{A}}(x) = \begin{cases} \omega(\frac{x-a_1}{b_1-a_1})^2 & ; x \in [a_1, b_1) \\ \omega & ; x \in [b_1, c_1) \\ \omega(\frac{d_1-x}{d_1-c_1})^2 & ; x \in [c_1, d_1) \\ 0 & ; otherwise \end{cases}, I_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x-a_2}{b_2-a_2})^2(1-\rho) & ; x \in [a_2, b_2) \\ \rho & ; x \in [b_2, c_2) \\ 1-(\frac{d_2-x}{d_2-c_2})^2(1-\rho) & ; x \in [c_2, d_2) \\ 1 & ; otherwise \end{cases} \text{ and}$$

$$F_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x-a_3}{b_3-a_3}{}^2)(1-\delta) & ; x \in [a_3, b_3) \\ \delta & ; x \in [b_3, c_3) \\ 1-(\frac{d_3-x}{d_3-c_3})^2(1-\delta) & ; x \in [c_3, d_3) \\ 1 & ; otherwise \end{cases}$$

where $0 \leq T_{\tilde{A}} + I_{\tilde{A}} + F_{\tilde{A}} \leq 3$, $0 < \omega \leq 1$, $0 \leq \rho < 1$, $0 \leq \delta < 1$ and $a_i, b_i, c_i, di \in \mathbb{R}$, $a_i \leq b_i \leq c_i \leq d_i$ where $i = 1, 2, 3$.

Note:1 GPSVNN is also denoted by

(1) $\tilde{A} = \langle (a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta) \rangle$

(2) $\tilde{A} = \langle (a, b, c, d)\ ; \omega, \rho, \delta \rangle$(If we consider the same values for the truth,falsity and indeterminacy membership).

Definition 3.2. The (α, β, γ) - cut of GPSVNN defined as $\tilde{A}^{(\alpha,\beta,\gamma)} = \{x | T_{\tilde{A}}(x) \geq \alpha, I_{\tilde{A}}(x) \leq \beta, F_{\tilde{A}}(x) \leq \gamma\}$, where $\alpha \in [0, \omega], \beta \in [\rho, 1], \gamma \in [\delta, 1]$ such that $\alpha + \beta + \gamma \leq 3$, ie., $\tilde{A}^{\alpha,\beta,\gamma} = \langle \tilde{A}^{\alpha}, \tilde{A}^{\beta}, \tilde{A}^{\gamma} \rangle$, where $\tilde{A}^{\alpha} = [\, a_1 + (b_1 - a_1)\sqrt{\alpha/\omega}, d_1 - (d_1 - c_1)\sqrt{\alpha/\omega}\,] = [L^{\alpha}, U^{\alpha}]$

$\tilde{A}^{\beta} = [\, a_2 + (b_2 - a_2)\sqrt{(1-\beta)/(1-\rho)}, d_2 - (d_2 - c_2)\sqrt{(1-\beta)/(1-\rho)}\,] = [L'^{\alpha}, U'^{\alpha}]$

$\tilde{A}^{\gamma} = [\, a_3 + (b_3 - a_3)\sqrt{(1-\gamma)/(1-\delta)}, d_3 - (d_3 - c_3)\sqrt{(1-\beta)/(1-\delta)}\,] = [L''^{\alpha}, U''^{\alpha}]$

Definition 3.3. A Parabolic Single Valued Neutrosophic Number (PSVNN) , $\tilde{A} = \langle T_{\tilde{A}}), I_{\tilde{A}}, F_{\tilde{A}} \rangle$, is a Neutrosophic set on real number $\mathbb{R}$ with truth-membership function $T_{\tilde{A}}$, indeterminacy-membership function $I_{\tilde{A}}$ and falsity-membership function $F_{\tilde{A}}$, is defined as,

$$T_{\tilde{A}}(x) = \begin{cases} (\frac{x-a_1}{b_1-a_1})^2 & ; x \in [a_1, b_1) \\ 1 & ; x \in [b_1, c_1) \\ (\frac{d_1-x}{d_1-c_1})^2 & ; x \in [c_1, d_1) \\ 0 & ; otherwise \end{cases}, I_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x-a_2}{b_2-a_2})^2 & ; x \in [a_2, b_2) \\ 0 & ; x \in [b_2, c_2) \\ 1-(\frac{d_2-x}{d_2-c_2})^2 & ; x \in [c_2, d_2) \\ 1 & ; otherwise \end{cases} \text{ and}$$

$$F_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x-a_3}{b_3-a_3}{}^2) & ; x \in [a_3, b_3) \\ 0 & ; x \in [b_3, c_3) \\ 1-(\frac{d_3-x}{d_3-c_3})^2 & ; x \in [c_3, d_3) \\ 1 & ; otherwise \end{cases}$$

where $0 \leq T_{\tilde{A}} + I_{\tilde{A}} + F_{\tilde{A}} \leq 3$, $a_i, b_i, c_i, di \in \mathbb{R}$, $a_i \leq b_i \leq c_i \leq d_i$ where $i = 1, 2, 3$.

Arithmetic Operators of GPSVNN

Definition 3.4. Let $\tilde{A}$ and $\tilde{B}$ are the two GPSVNN , then we define the arithmetic operators for
$\tilde{A} = \langle(a_1, b_1, c_1, d_1; \omega_1), (a_2, b_2, c_2, d_2; \rho_1), (a_3, b_3, c_3, d_3; \delta_1)\rangle$ and
$\tilde{B} = \langle(a'_1, b'_1, c'_1, d'_1; \omega_2), (a'_2, b'_2, c'_2, d'_2; \rho_2), (a'_3, b'_3, c'_3, d'_3; \delta_2)\rangle$ as follows.
where, $\omega = min\{\omega_1, \omega_2\}$, $\rho = max\{\rho_1, \rho_2\}$ and $\delta = max\{\delta_1, \delta_2\}$
(We denote $\wedge$ for min and $\vee$ for max.)

1. The addition of GPSVNN's $\tilde{A} + \tilde{B} = \tilde{C}$ is

$$T_{\tilde{C}}(x) = \begin{cases} \omega\left(\frac{x-(a_1+a'_1)}{(b_1+b'_1)-(a_1+a'_1)}\right)^2 & ; x \in [a_1+a'_1, b_1+b'_1) \\ \omega & ; x \in [b_1+b'_1, c_1+c'_1) \\ \omega\left(\frac{(d_1+d'_1)-x}{(d_1+d'_1)-(c_1+c'_1)}\right)^2 & ; x \in [c_1+c'_1, d_1+d'_1) \\ 0 & ; otherwise \end{cases},$$

$$I_{\tilde{C}}(x) = \begin{cases} 1-\left(\frac{x-(a_2+a'_2)}{(b_2+b'_2)-(a_2+a'_2)}\right)^2(1-\rho) & ; x \in [a_2+a'_2, b_2+b'_2) \\ \rho & ; x \in [b_2+b'_2, c_2+c'_2) \\ 1-\left(\frac{(d_2+d'_2)-x}{(d_2+d'_2)-(c_2+c'_2)}\right)^2(1-\rho) & ; x \in [c_2+c'_2, d_2+d'_2) \\ 1 & ; otherwise \end{cases}$$

$$\text{and } F_{\tilde{C}}(x) = \begin{cases} 1-\left(\frac{x-(a_3+a'_2)}{(b_3+b'_3)-(a_3+a'_3)}\right)^2(1-\delta) & ; x \in [a_3+a'_3, b_3+b'_3) \\ \delta & ; x \in [b_3+b'_3, c_3+c'_3) \\ 1-\left(\frac{(d_3+d'_3)-x}{(d_3+d'_3)-(c_3+c'_3)}\right)^2(1-\delta) & ; x \in [c_3+c'_3, d_3+d'_3) \\ 1 & ; otherwise \end{cases}$$

2.The subtraction of GPSVNN's $\tilde{A} - \tilde{B} = \tilde{C}$ is

$$T_{\tilde{C}}(x) = \begin{cases} \omega\left(\frac{x-(a_1-d'_1)}{(b_1-c'_1)-(a_1-d'_1}\right)^2 & ; x \in [a_1-d'_1, b_1-c'_1) \\ \omega & ; x \in [b_1-c'_1, c_1-b'_1) \\ \omega\left(\frac{(d_1-a'_1)-x}{(d_1-a'_1)-(c_1-b'_1)}\right)^2 & ; x \in [c_1-b'_1, d_1-a'_1) \\ 0 & ; otherwise \end{cases},$$

$$I_{\tilde{C}}(x) = \begin{cases} 1-\left(\frac{x-(a_2-d'_2)}{(b_2-c'_2)-(a_2-d'_2)}\right)^2(1-\rho) & ; x \in [a_2-d'_2, b_2-c'_2) \\ \rho & ; x \in [b_2-c'_2, c_2-b'_2) \\ 1-\left(\frac{(d_2-a'_2)-x}{(d_2-a'_2)-(c_2-b'_2)}\right)^2(1-\rho) & ; x \in [c_2-b'_2, d_2-a'_2) \\ 1 & ; otherwise \end{cases}$$

$$F_{\tilde{C}}(x)=\begin{cases}1-\left(\frac{x-(a_3-d_3')}{(b_3-c_3')-(a_3-d_3')}\right)^2(1-\delta) & ;x\in[a_3-d_3',b_3-c_3')\\ \delta & ;x\in[b_3-c_3',c_3-b_3')\\ 1-\left(\frac{(d_3-a_3')-x}{(d_3-a_3')-(c_3-b_3')}\right)^2(1-\delta) & ;x\in[c_3-b_3',d_3-a_3')\\ 1 & ;otherwise\end{cases}$$

3. The multiplication of GPSVNN's $\tilde{A}*\tilde{B}=\tilde{C}$ is

$$T_{\tilde{C}}(x)=\begin{cases}\omega(\frac{x-p_1}{p_2-p_1})^2 & ;x\in[p_1,p_2)\\ \omega & ;x\in[p_2,p_3)\\ \omega(\frac{p_4-x}{p_4-p_3})^2 & ;x\in[p_3,p_4)\\ 0 & ;otherwise\end{cases},\ I_{\tilde{C}}(x)=\begin{cases}1-(\frac{x-q_1}{q_2-q_1})^2(1-\rho) & ;x\in[q_1,q_2)\\ \rho & ;x\in[q_2,q_3)\\ 1-(\frac{q_4-x}{q_4-q_3})^2(1-\rho) & ;x\in[q_3,q_4)\\ 1 & ;otherwise\end{cases}$$

and $F_{\tilde{C}}(x)=\begin{cases}1-(\frac{x-r_1}{r_2-r_1})^2(1-\delta) & ;x\in[r_1,r_2)\\ \delta & ;x\in[r_2,r_3)\\ 1-(\frac{r_4-x}{r_4-r_3})^2(1-\delta) & ;x\in[r_3,r_4)\\ 1 & ;otherwise\end{cases}$

where $p_1=min\{a_1*a_1',a_1*d_1',d_1*a_1',d_1*d_1'\}$, $p_2=min\{b_1*b_1',b_1*c_1',c_1*b_1',c_1*c_1'\}$
$p_3=max\{b_1*b_1',b_1*c_1',c_1*b_1',c_1*c_1'\}$, $p_4=max\{a_1*a_1',a_1*d_1',d_1*a_1',d_1*d_1'\}$
$q_1=min\{a_2*a_2',a_2*d_2',d_2*a_2',d_2*d_2'\}$, $q_2=min\{b_2*b_2',b_2*c_2',c_2*b_2',c_2*c_2'\}$
$q_3=max\{b_2*b_2',b_2*c_2',c_2*b_2',c_2*c_2'\}$, $q_4=max\{a_2*a_2',a_2*d_2',d_2*a_2',d_2*d_2'\}$
$r_1=min\{a_3*a_3',a_3*d_3',d_3*a_3',d_3*d_3'\}$, $r_2=min\{b_3*b_3',b_3*c_3',c_3*b_3',c_3*c_3'\}$
$r_3=max\{b_3*b_3',b_3*c_3',c_3*b_3',c_3*c_3'\}$, $r_4=max\{a_3*a_3',a_3*d_3',d_3*a_3',d_3*d_3'\}$.

4. Inverse of GPSVNN

Consider the GPSVN-number, $\tilde{B}=\langle(a_1',b_1',c_1',d_1';\omega),(a_2',b_2',c_2',d_2';\rho),(a_3',b_3',c_3',d_3';\delta)\rangle$.

The inverse of this GPSVN-number is,

$\frac{1}{\tilde{B}}=\langle(\frac{1}{d_1'},\frac{1}{c_1'},\frac{1}{b_1'},\frac{1}{a_1'};\omega),(\frac{1}{d_2'},\frac{1}{c_2'},\frac{1}{b_2'},\frac{1}{a_2'};\rho),(\frac{1}{d_3'},\frac{1}{c_3'},\frac{1}{b_3'},\frac{1}{a_3'};\delta)\rangle$, $0\notin[a_i',d_i']$, where i = 1,2,3.

5. Division of GPSVNN The division of $\tilde{A}/\tilde{B}$ can be defined as the multiplication of two GPSVNN $\tilde{A}*\frac{1}{\tilde{B}}=\tilde{C}$,

$$T_{\tilde{C}}(x)=\begin{cases}\omega(\frac{x-p_1}{p_2-p_1})^2 & ;x\in[p_1,p_2)\\ \omega & ;x\in[p_2,p_3)\\ \omega(\frac{p_4-x}{p_4-p_3})^2 & ;x\in[p_3,p_4)\\ 0 & ;otherwise\end{cases},$$

$$I_{\tilde{C}}(x)=\begin{cases}1-(\frac{x-q_1}{q_2-q_1})^2(1-\rho) & ;x\in[q_1,q_2)\\ \rho & ;x\in[q_2,q_3)\\ 1-(\frac{q_4-x}{q_4-q_3})^2(1-\rho) & ;x\in[q_3,q_4)\\ 1 & ;otherwise\end{cases}$$

and $F_{\tilde{C}}(x)=\begin{cases}1-(\frac{x-r_1}{r_2-r_1})^2(1-\delta) & ;x\in[r_1,r_2)\\ \delta & ;x\in[r_2,r_3)\\ 1-(\frac{r_4-x}{r_4-r_3})^2(1-\delta) & ;x\in[r_3,r_4)\\ 1 & ;otherwise\end{cases}$

where $p_1=min\{\frac{a_1}{d_1'},\frac{a_1}{a_1'},\frac{d_1}{d_1'},\frac{d_1}{a_1'}\}$, $p_2=min\{\frac{b_1}{c_1'},\frac{b_1}{b_1'},\frac{c_1}{c_1'},\frac{c_1}{b_1'}\}$, $p_3=max\{\frac{b_1}{c_1'},\frac{b_1}{b_1'},\frac{c_1}{c_1'},\frac{c_1}{b_1'}\}$, $p_4=max\{\frac{a_1}{d_1'},\frac{a_1}{a_1'},\frac{d_1}{d_1'},\frac{d_1}{a_1'}\}$, $q_1=min\{\frac{a_2}{d_2'},\frac{a_2}{a_2'},\frac{d_2}{d_2'},\frac{d_2}{a_2'}\}$, $q_2=min\{\frac{b_2}{c_2'},\frac{b_2}{b_2'},\frac{c_2}{c_2'},\frac{c_2}{b_2'}\}$, $q_3=max\{\frac{b_2}{c_2'},\frac{b_2}{b_2'},\frac{c_2}{c_2'},\frac{c_2}{b_2'}\}$, $q_4=max\{\frac{a_2}{d_2'},\frac{a_2}{a_2'},\frac{d_2}{d_2'},\frac{d_2}{a_2'}\}$, $r_1=min\{\frac{a_3}{d_3'},\frac{a_3}{a_3'},\frac{d_3}{d_3'},\frac{d_3}{a_3'}\}$, $r_2=min\{\frac{b_3}{c_3'},\frac{b_3}{b_3'},\frac{c_3}{c_3'},\frac{c_3}{b_3'}\}$, $r_3=max\{\frac{b_3}{c_3'},\frac{b_3}{b_3'},\frac{c_3}{c_3'},\frac{c_3}{b_3'}\}$, $r_4=max\{\frac{a_3}{d_3'},\frac{a_3}{a_3'},\frac{d_3}{d_3'},\frac{d_3}{a_3'}\}$.

Example 3.5. $\tilde{A}=\langle(3,5,8,12);0.2,0.3,0.5\rangle$ and $\tilde{B}=\langle(-7,-5,6,7);0.2,0.3,0.5\rangle$ then

(1) $\tilde{A}+\tilde{B}$ **is**

$$T_{\tilde{A}}(x)=\begin{cases}0.2(\frac{x+4}{4})^2 & ;x\in[-4,0)\\ 0.2 & ;x\in[0,14)\\ 0.2(\frac{19-x}{5})^2 & ;x\in[14,19)\\ 0 & ;otherwise\end{cases},\ I_{\tilde{A}}(x)=\begin{cases}1-(\frac{x+4}{4})^2(0.7) & ;x\in[-4,0)\\ 0.3 & ;x\in[0,14)\\ 1-(\frac{19-x}{5})^2(0.7) & ;x\in[14,19)\\ 1 & ;otherwise\end{cases}$$

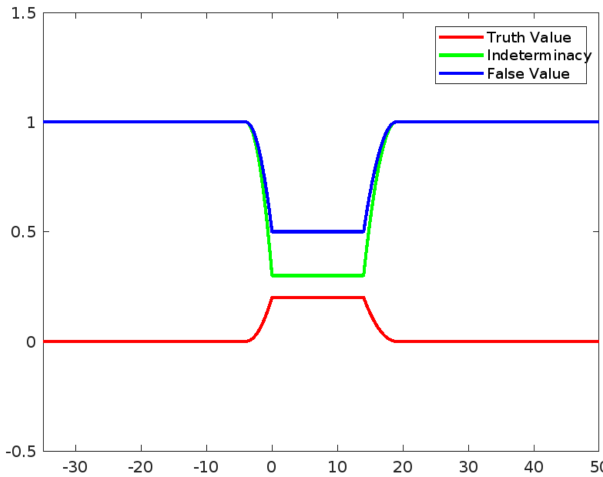
and $F_{\tilde{A}}(x)=\begin{cases}1-(\frac{x+4}{4})^2(0.5) & ;x\in[-4,0)\\ 0.5 & ;x\in[0,14)\\ 1-(\frac{19-x}{5})^2(0.5) & ;x\in[14,19)\\ 1 & ;otherwise\end{cases}$

Example 3.6. $\tilde{A}=\langle(3,5,8,12);0.2,0.3,0.5\rangle$ and $\tilde{B}=\langle(1,2,3,4);0.2,0.3,0.5\rangle$ then $\tilde{A}/\tilde{B}$ is

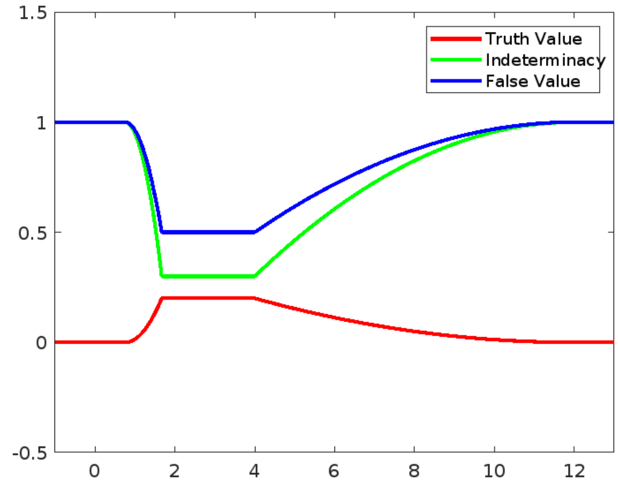
$$T_{\tilde{A}}(x)=\begin{cases}0.2(\frac{x-0.75}{0.92})^2 & ;x\in[0.75,1.67)\\ 0.2 & ;x\in[1.67,4)\\ 0.2(\frac{12-x}{8})^2 & ;x\in[4,12)\\ 0 & ;otherwise\end{cases},\ I_{\tilde{A}}(x)=\begin{cases}1-(\frac{x-0.75}{0.92})^2(0.7) & ;x\in[0.75,1.67)\\ 0.3 & ;x\in[1.67,4)\\ 1-(\frac{12-x}{8})^2(0.7) & ;x\in[4,12)\\ 1 & ;otherwise\end{cases}$$

and $F_{\tilde{A}}(x)=\begin{cases}1-(\frac{x-0.75}{0.92})^2(0.5) & ;x\in[0.75,1.67)\\ 0.5 & ;x\in[1.67,4)\\ 1-(\frac{12-x}{8})^2(0.5) & ;x\in[4,12)\\ 1 & ;otherwise\end{cases}$

The graphical interpretation of Example 3.5 and 3.6 are given below.



(A) Addition



(B) Division

3.1. *Value and Ambiguity*

Definition 3.7. Let $\tilde{A} = \langle (a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta) \rangle$ be a GPSVNN . Then (α, β, γ)-cut set of the GPSVNN are $\tilde{A}^{\alpha} = [L^{\alpha}, U^{\alpha}]$, $\tilde{A}^{\beta} = [L'^{\alpha}, U'^{\alpha}]$ and $\tilde{A}^{\gamma} = [L''^{\alpha}, U''^{\alpha}]$ respectively. Then the **Values** of GPSVNN are defined as,
$\mathcal{V}(\tilde{A}^{\alpha}) = \int_0^{\omega} (L^{\alpha} + U^{\alpha}) f(\alpha)\, d\alpha$ where, $f(\alpha) \in [0,1] (\alpha \in [0, \omega])$, $f(0) = 0$ and $f(\alpha)$ is monotonic and non-decreasing of $\alpha \in [0, \omega]$.
$\mathcal{V}(\tilde{A}^{\beta}) = \int_{\rho}^{1} (L'^{\alpha} + U'^{\alpha}) g(\beta)\, d\beta$ where , $g(\beta) \in [0,1] (\beta \in [\rho, 1])$, $g(1) = 0$ and $g(\beta)$ is monotonic and non-increasing of $\beta \in [\rho, 1]$.
$\mathcal{V}(\tilde{A}^{\gamma}) = \int_{\delta}^{1} (L''^{\alpha} + U''^{\alpha}) h(\gamma)\, d\gamma$ where , $h(\gamma) \in [0,1] (\gamma \in [\delta, 1])$, $h(1) = 0$ and $h(\gamma)$ is monotonic and non-increasing of $\gamma \in [\delta, 1]$.

Definition 3.8. Let $\tilde{A} = \langle (a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta) \rangle$ be a GPSVNN . Then (α, β, γ)-cut set of the GPSVNN $\tilde{A}^{\alpha} = [L^{\alpha}, U^{\alpha}]$, $\tilde{A}^{\beta} = [L'^{\alpha}, U'^{\alpha}]$ and $\tilde{A}^{\gamma} = [L''^{\alpha}, U''^{\alpha}]$ are respectively. Then the **Ambiguties** of GPSVNN are defined as,
$\mathcal{A}(\tilde{A}^{\alpha}) = \int_0^{\omega} (U^{\alpha} - L^{\alpha}) f(\alpha)\, d\alpha$ where , $f(\alpha) \in [0,1] (\alpha \in [0, \omega])$, $f(0) = 0$ and $f(\alpha)$ is monotonic and non-decreasing of $\alpha \in [0, \omega]$
$\mathcal{A}(\tilde{A}^{\beta}) = \int_{\rho}^{1} (U'^{\alpha} - L'^{\alpha}) g(\beta)\, d\beta$ where , $g(\beta) \in [0,1] (\beta \in [\rho, 1])$, $g(1) = 0$ and $g(\beta)$ is monotonic and non-increasing of $\beta \in [\rho, 1]$
$\mathcal{A}(\tilde{a}^{\gamma}) = \int_{\delta}^{1} (U''^{\alpha} - L''^{\alpha}) h(\gamma)\, d\gamma$ where , $h(\gamma) \in [0,1] (\gamma \in [\delta, 1])$, $h(1) = 0$ and $h(\gamma)$ is monotonic and non-increasing of $\gamma \in [\delta, 1]$

Result 3.9. *Let $\tilde{A} = \langle (a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta) \rangle$ be a GPSVNN . Then (α, β, γ)-cut set of the GPSVNN $\tilde{A}^{\alpha} = [L^{\alpha}, U^{\alpha}]$, $\tilde{A}^{\beta} = [L'^{\alpha}, U'^{\alpha}]$ and $\tilde{A}^{\gamma} = [L''^{\alpha}, U''^{\alpha}]$ are respectively. Then, for the truth membership,*
$\tilde{A}^{\alpha} = [L^{\alpha}, U^{\alpha}] = [a_1 + (b_1 - a_1)\sqrt{\alpha/\omega}, d_1 - (d_1 - c_1)\sqrt{\alpha/\omega}]$ *where* $\alpha \in [0, \omega]$.

If $f(\alpha) = \alpha$,we obtain value and ambiguity as,

$$\mathcal{V}(\tilde{A}^{\alpha}) = \int_0^{\omega} \left[a_1 + (b_1 - a_1)\sqrt{\alpha/\omega} + d_1 - (d_1 - c_1)\sqrt{\alpha/\omega}\right]\alpha d\alpha$$
$$= \left[\frac{\omega^2}{2}(a_1 + d_1) + \frac{2\omega^2}{5}(b_1 - a_1 - d_1 + c_1)\right]$$
$$= \frac{\omega^2}{10}(a_1 + d_1 + 4b_1 + 4c_1).$$

$$\mathcal{A}(\tilde{A}^{\alpha}) = \int_0^{\omega} \left[d_1 - (d_1 - c_1)\sqrt{\alpha/\omega} - a_1 - (b_1 - a_1)\sqrt{\alpha/\omega}\right]\alpha d\alpha$$
$$= \left[\frac{\omega^2}{2}(d_1 - a_1) - \frac{2\omega^2}{5}(d_1 - c_1 + b_1 - a_1)\right]$$
$$= \frac{\omega^2}{10}(d_1 - a_1 - 4b_1 + 4c_1).$$

For the indeterminancy membership ,
$\tilde{A}^{\beta} = [L'^{\alpha}, U'^{\alpha}] = [a_2 + (b_2 - a_2)\sqrt{(1-\beta)/(1-\rho)}, d_2 - (d_2 - c)_2\sqrt{(1-\beta)/(1-\rho)}]$ *where* $\beta \in [\rho, 1]$. *If $g(\rho) = (1 - \rho)$, we obtain value and ambiguity as,*

$$\mathcal{V}(\tilde{A}^{\beta}) = [\int_{\rho}^{1} \left[a_2 + (b_2 - a_2)\sqrt{(1-\beta)/(1-\rho)} + d_2 - (d_2 - c_2)\sqrt{(1-\beta)/(1-\rho)}\right](1-\beta)d\beta$$
$$= [\left[\frac{(1-\rho)^2}{2}(a_2 + d_2) + \frac{2(1-\rho)^2}{5}(b_2 - a_2 - d_2 + c_2)\right] = \frac{(1-\rho)^2}{10}(a_2 + d_2 + 4b_2 + 4c_2).$$

$$\mathcal{A}(\tilde{A}^{\beta}) = [\int_{\rho}^{1} \left[d_2 - (d_2 - c_2)\sqrt{(1-\beta)/(1-\rho)} - a_2 - (b_2 - a_2)\sqrt{(1-\beta)/(1-\rho)}\right](1-\beta)d\beta$$
$$= [\left[\frac{(1-\rho)^2}{2}(d_2 - a_2) - \frac{2\omega^2}{5}(d_2 - c_2 + b_2 - a_2)\right] = \frac{(1-\rho)^2}{10}(d_2 - a_2 - 4b_2 + 4c_2).$$

For the falsity membership ,
$\tilde{A}^{\gamma} = [L''^{\alpha}, U''^{\alpha}] = [a_3 + (b_3 - a_3)\sqrt{(1-\gamma)/(1-\delta)}, d_3 - (d_3 - c_3)\sqrt{(1-\gamma)/(1-\delta)}\Big]$ *where* $\gamma \in [\delta, 1]$. *If $h(\delta) = (1 - \delta)$,we obtain value and ambiguity as,*

$$\mathcal{V}(\tilde{A}^{\gamma}) = [\int_{\delta}^{1} \left[a_3 + (b_3 - a_3)\sqrt{(1-\gamma)/(1-\delta)} + d_3 - (d_3 - c_3)\sqrt{(1-\gamma)/(1-\delta)}\right](1-\gamma)d\gamma$$
$$= [\left[\frac{(1-\delta)^2}{2}(a_3 + d_3) + \frac{2(1-\delta)^2}{5}(b_3 - a_3 - d_3 + c_3)\right] = \frac{(1-\delta)^2}{10}(a_3 + d_3 + 4b_3 + 4c_3).$$

$$\mathcal{A}(\tilde{A}^{\gamma}) = [\int_{\delta}^{1} \left[d_3 - (d_3 - c_3)\sqrt{(1-\gamma)/(1-\delta)} - a_3 - (b_3 - a_3)\sqrt{(1-\gamma)/(1-\delta)}\right](1-\gamma)d\gamma$$
$$= [\left[\frac{(1-\delta)^2}{2}(d_3 - a_3) - \frac{2\omega^2}{5}(d_3 - c_3 + b_3 - a_3)\right] = \frac{(1-\delta)^2}{10}(d_3 - a_3 - 4b_3 + 4c_3).$$

Definition 3.10. Let $\tilde{A} = \langle(a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta)\rangle$ be a GPSVNN. The weighted value and ambiguity for $\lambda \in [0, 1]$ are,

$\mathcal{V}_\lambda(\tilde{A}) = \lambda\mathcal{V}(\tilde{A}^\alpha) + (1-\lambda)\mathcal{V}(\tilde{A}^\beta) + (1-\lambda)\mathcal{V}(\tilde{A}^\gamma)$.
$\mathcal{A}_\lambda(\tilde{A}) = \lambda\mathcal{A}(\tilde{A}^\alpha) + (1-\lambda)\mathcal{A}(\tilde{A}^\beta) + (1-\lambda)\mathcal{A}(\tilde{A}^\gamma)$.

Note: When $\lambda = 0$, it marks the preference for uncertainty, On the other hand,when $\lambda = 1$ it is considered with strongly preferred certainty.

Definition 3.11. (Ranking Order) Let $\tilde{A}$ and $\tilde{B}$ be two GPSVNN and $\lambda \in [0,1]$. For weighted values and ambiguities of the GPSVNN $\tilde{A}$ and $\tilde{B}$. The ranking order of $\tilde{A}$ and $\tilde{B}$ is defined as,

(1) If $\mathcal{V}_\lambda(\tilde{A}) > \mathcal{V}_\lambda(\tilde{B})$, then $\tilde{A} > \tilde{B}$.
(2) If $\mathcal{V}_\lambda(\tilde{A}) < \mathcal{V}_\lambda(\tilde{B})$, then $\tilde{A} < \tilde{B}$
(3) If $\mathcal{V}_\lambda(\tilde{A}) = \mathcal{V}_\lambda(\tilde{B})$, then

- If $\mathcal{A}_\lambda(\tilde{A}) = \mathcal{A}_\lambda(\tilde{B})$, then $\tilde{A} = \tilde{B}$.
- If $\mathcal{A}_\lambda(\tilde{A}) > \mathcal{A}_\lambda(\tilde{B})$, then $\tilde{A} > \tilde{B}$.
- If $\mathcal{A}_\lambda(\tilde{A}) < \mathcal{A}_\lambda(\tilde{B})$, then $\tilde{A} < \tilde{B}$.

Example : $\tilde{A} = \langle(3,5,8,12); 0.2, 0.3, 0.4\rangle$ and $\tilde{B} = \langle(-7,-5,6,7); 0.5, 0.4, 0.3\rangle$. Then the ranking for between these two numbers are .

$$\mathcal{V}_\lambda(\tilde{A}) = \frac{3+12+20+32}{10}[\lambda(0.2^2) + (1-\lambda)(1-0.3)^2 + (1-\lambda)(1-0.4)^2]$$
$$= 6.7[0.85 - 0.81\lambda] = 5.70 - 5.43\lambda.$$

$$\mathcal{V}_\lambda(\tilde{B}) = \frac{-7+7-20+24}{10}[\lambda(0.5^2) + (1-\lambda)(1-0.4)^2 + (1-\lambda)(1-0.3)^2]$$
$$= 0.4[0.85 - 0.60\lambda] = 0.34 - 0.24\lambda.$$

When $\lambda = 0, \mathcal{V}_\lambda(\tilde{A}) = 5.70$ and $\mathcal{V}_\lambda(\tilde{B}) = 0.34$
When $\lambda = 1, \mathcal{V}_\lambda(\tilde{A}) = 0.27$ and $\mathcal{V}_\lambda(\tilde{B}) = 0.10$.
Also for all values of λ between 0 and 1 $\mathcal{V}_\lambda(\tilde{A}) > \mathcal{V}_\lambda(\tilde{B})$. then the ranking order of the numbers $\tilde{A}$ and $\tilde{B}$ is $\tilde{A} > \tilde{B}$.

Theorem 3.12. *Let* $\tilde{A} = \langle(a_1,b_1,c_1,d_1;\omega),(a_2,b_2,c_2,d_2;\rho),(a_3,b_3,c_3,d_3;\delta)\rangle$ *and* $\tilde{B} = \langle(a'_1,b'_1,c'_1,d'_1;\omega_2),(a'_2,b'_2,c'_2,d'_2;\rho),(a'_3,b'_3,c'_3,d'_3;\delta_2)\rangle$ *be the two GPSVNN ,* $\lambda \in [0,1]$ *and* $k \in \mathbb{R}$ *then (i)* $\mathcal{V}_\lambda(\tilde{A}+\tilde{B}) = \mathcal{V}_\lambda(\tilde{A}) + \mathcal{V}_\lambda(\tilde{B})$ *(ii)*$V_\lambda(k\tilde{A}) = kV_\lambda(\tilde{A})$.

Proof:

$$(i)\mathcal{V}_\lambda(\tilde{A}+\tilde{B}) = \lambda\mathcal{V}(\tilde{A}^\alpha+\tilde{B}^\alpha) + (1-\lambda)\mathcal{V}(\tilde{A}^\beta+\tilde{B}^\beta) + (1-\lambda)\mathcal{V}(\tilde{A}^\gamma+\tilde{B}^\gamma)$$
$$= \lambda\mathcal{V}(\tilde{A}^\alpha) + \lambda\mathcal{V}(\tilde{B}^\alpha) + (1-\lambda)\mathcal{V}(\tilde{A}^\beta) + (1-\lambda)\mathcal{V}(\tilde{B}^\beta) + (1-\lambda)\mathcal{V}(\tilde{A}^\gamma) + (1-\lambda)\mathcal{V}(\tilde{B}^\gamma)$$
$$= \mathcal{V}_\lambda(\tilde{A}) + \mathcal{V}_\lambda(\tilde{B}).$$

$$(ii)\mathcal{V}_\lambda(k\tilde{A}) = \lambda\mathcal{V}(k\tilde{A}^\alpha) + (1-\lambda)\mathcal{V}(k\tilde{A}^\beta) + (1-\lambda)\mathcal{V}(k\tilde{A}^\gamma) = k[\lambda\mathcal{V}(\tilde{A}^\alpha) + (1-\lambda)\mathcal{V}(\tilde{A}^\beta) + (1-\lambda)\mathcal{V}(\tilde{A}^\gamma)]$$
$$= k\mathcal{V}_\lambda(\tilde{A}).$$

Theorem 3.13. *Let* $\tilde{A} = \langle(a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta)\rangle$ *and* $\tilde{B} = \langle(a'_1, b'_1, c'_1, d'_1; \omega_2), (a'_2, b'_2, c'_2, d'_2; \rho), (a'_3, b'_3, c'_3, d'_3; \delta_2)\rangle$ *be the two GPSVNN ,* $\lambda \in [0, 1]$ *and* $k \in \mathbb{R}$ *then* *(i)*$\mathcal{A}_\lambda(\tilde{A} + \tilde{B}) = \mathcal{A}_\lambda(\tilde{B}) + \mathcal{A}_\lambda(\tilde{B})$ *(ii)* $\mathcal{A}_\lambda(k\tilde{A}) = k\mathcal{A}_\lambda(\tilde{A})$.

Proof:

$$(i)\mathcal{A}_\lambda(\tilde{A} + \tilde{B}) = \lambda\mathcal{A}(\tilde{A}^\alpha + \tilde{B}^\alpha) + (1-\lambda)\mathcal{A}(\tilde{A}^\beta + \tilde{B}^\beta) + (1-\lambda)\mathcal{A}(\tilde{A}^\gamma + \tilde{B}^\gamma)$$
$$= \lambda\mathcal{A}(\tilde{A}^\alpha) + \lambda\mathcal{A}(\tilde{B}^\alpha) + (1-\lambda)\mathcal{A}(\tilde{A}^\beta) + (1-\lambda)\mathcal{A}(\tilde{B}^\beta) + (1-\lambda)\mathcal{A}(\tilde{A}^\gamma) + (1-\lambda)\mathcal{A}(\tilde{B}^\gamma)$$
$$= \mathcal{A}_\lambda(\tilde{A}) + \mathcal{A}_\lambda(\tilde{B}).$$

$$(ii)\mathcal{A}_\lambda(k\tilde{A}) = \lambda\mathcal{A}(k\tilde{A}^\alpha) + (1-\lambda)\mathcal{A}(k\tilde{A}^\beta) + (1-\lambda)\mathcal{A}(k\tilde{A}^\gamma) = k[\lambda\mathcal{A}(\tilde{A}^\alpha) + (1-\lambda)\mathcal{A}(\tilde{A}^\beta) + (1-\lambda)\mathcal{A}(\tilde{A}^\gamma)]$$
$$= kA_\lambda(\tilde{A}).$$

Theorem 3.14. *Suppose* $\tilde{A}, \tilde{B}$ *and* $\tilde{C}$ *are any GPSVNN , where* $\omega_1 = \omega_2$ *,* $\rho_1 = \rho_2$ *and* $\delta_1 = \delta_2$ *. If* $\tilde{A} > \tilde{B}$*, then* $(\tilde{A} + \tilde{C}) > (\tilde{B} + \tilde{C})$.

Proof:

$$\mathcal{V}(\tilde{A}^\alpha + \tilde{C}^\alpha) = \int_0^{\omega_1\wedge\omega_3}[L^\alpha_{\tilde{A}} + U^\alpha_{\tilde{A}} + L^\alpha_{\tilde{C}} + U^\alpha_{\tilde{C}}]f(\alpha)d\alpha$$
$$= \int_0^{\omega_1\wedge\omega_3}[L^\alpha_{\tilde{A}} + U^\alpha_{\tilde{A}}]f(\alpha)d\alpha + \int_0^{\omega_1\wedge\omega_3}[L^\alpha_{\tilde{C}} + U^\alpha_{\tilde{C}}]f(\alpha)d\alpha$$
$$\mathcal{V}(\tilde{B}^\alpha + \tilde{C}^\alpha) = \int_0^{\omega_2\wedge\omega_3}[L^\alpha_{\tilde{B}} + U^\alpha_{\tilde{B}} + L^\alpha_{\tilde{C}} + U^\alpha_{\tilde{C}}]f(\alpha)d\alpha$$
$$= \int_0^{\omega_2\wedge\omega_3}[L^\alpha_{\tilde{B}} + U^\alpha_{\tilde{B}}]f(\alpha)d\alpha + \int_0^{\omega_2\wedge\omega_3}[L^\alpha_{\tilde{C}} + U^\alpha_{\tilde{C}}]f(\alpha)d\alpha$$

From the conditions , $\tilde{A} > \tilde{B}$ *and* $\omega_1 = \omega_2$*, we have,*
$\int_0^{\omega_1\wedge\omega_3}[L^\alpha_{\tilde{A}} + U^\alpha_{\tilde{A}}]f(\alpha)d\alpha > \int_0^{\omega_2\wedge\omega_3}[L^\alpha_{\tilde{B}} + U^\alpha_{\tilde{B}}]f(\alpha)d\alpha$

$$\implies \mathcal{V}(\tilde{A}^\alpha + \tilde{C}^\alpha) > \mathcal{V}(\tilde{B}^\alpha + \tilde{C}^\alpha) \quad (1)$$

$$\mathcal{V}(\tilde{A}^\beta + \tilde{C}^\beta) = \int_{\rho_1\vee\rho_3}^1[L'^\beta_{\tilde{A}} + U'^\beta_{\tilde{A}} + L'^\beta_{\tilde{C}} + U'^\beta_{\tilde{C}}]g'^\beta d\beta$$
$$= \int_{\rho_1\vee\rho_3}^1[L'^\beta_{\tilde{A}} + U'^\beta_{\tilde{A}}]g'^\beta d\beta + \int_{\rho_1\vee\rho_3}^1[L'^\beta_{\tilde{C}} + U'^\beta_{\tilde{C}}]g'^\beta d\beta$$
$$\mathcal{V}(\tilde{B}^\beta + \tilde{C}^\beta) = \int_{\rho_2\vee\rho_3}^1[L'^\beta_{\tilde{B}} + U'^\beta_{\tilde{B}} + L'^\beta_{\tilde{C}} + U'^\beta_{\tilde{C}}]g'^\beta d\beta$$
$$= \int_{\rho_2\vee\rho_3}^1[L'^\beta_{\tilde{B}} + U'^\beta_{\tilde{B}}]g'^\beta d\beta + \int_{\rho_2\vee\rho_3}^1[L'^\beta_{\tilde{C}} + U'^\beta_{\tilde{C}}]g'^\beta d\beta$$

From the conditions , $\tilde{A} > \tilde{B}$ *and* $\rho_1 = \rho_2$*, we have,*
$\int_{\rho_1 \vee \rho_3}^{1} [L'^{\beta}_{\tilde{A}} + U'^{\beta}_{\tilde{A}}] g'^{\beta} d\beta > \int_{\rho_2 \vee \rho_3}^{1} [L'^{\beta}_{\tilde{B}} + U'^{\beta}_{\tilde{B}}] g'^{\beta} d\beta$

$$\implies \mathcal{V}(\tilde{A}^{\beta} + \tilde{C}^{\beta}) > \mathcal{V}(\tilde{B}^{\beta} + \tilde{C}^{\beta}) \tag{2}$$

$$\begin{aligned}
\mathcal{V}(\tilde{A}^{\gamma} + \tilde{C}^{\gamma}) &= \int_{\delta_1 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{A}} + U''^{\gamma}_{\tilde{A}} + L''^{\gamma}_{\tilde{C}} + U''^{\gamma}_{\tilde{C}}] h(\gamma) d\gamma \\
&= \int_{\delta_1 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{A}} + U''^{\gamma}_{\tilde{A}}] h(\gamma) d\gamma + \int_{\delta_1 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{C}} + U''^{\gamma}_{\tilde{C}}] h(\gamma) d\gamma \\
\mathcal{V}(\tilde{B}^{\gamma} + \tilde{C}^{\gamma}) &= \int_{\delta_2 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{B}} + U''^{\gamma}_{\tilde{B}} + L''^{\gamma}_{\tilde{C}} + U''^{\gamma}_{\tilde{C}}] h(\gamma) d\gamma \\
&= \int_{\delta_2 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{B}} + U''^{\gamma}_{\tilde{B}}] h(\gamma) d\gamma + \int_{\delta_2 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{C}} + U''^{\gamma}_{\tilde{C}}] h(\gamma) d\gamma
\end{aligned}$$

From the conditions , $\tilde{A} > \tilde{B}$ *and* $\gamma_1 = \gamma_2$*, we have,*
$\int_{\gamma_1 \vee \gamma_3}^{1} [L''^{\gamma}_{\tilde{A}} + U''^{\gamma}_{\tilde{A}}] h(\gamma) d\gamma > \int_{\gamma_2 \vee \gamma_3}^{1} [L''^{\gamma}_{\tilde{B}} + U''^{\gamma}_{\tilde{B}}] h(\gamma) d\gamma$

$$\implies \mathcal{V}(\tilde{A}^{\gamma} + \tilde{C}^{\gamma}) > \mathcal{V}(\tilde{B}^{\gamma} + \tilde{C}^{\gamma}) \tag{3}$$

by the combining equation (1), (2) and (3) the following inequality is always valid for any $\lambda \in [0, 1]$,

$$\lambda \mathcal{V}(\tilde{A}^{\alpha} + \tilde{C}^{\alpha}) + (1 - \lambda) \mathcal{V}(\tilde{A}^{\beta} + \tilde{C}^{\beta}) + (1 - \lambda) \mathcal{V}(\tilde{A}^{\gamma} + \tilde{C}^{\gamma}) >$$
$$\lambda \mathcal{V}(\tilde{B}^{\alpha} + \tilde{C}^{\alpha}) + (1 - \lambda) \mathcal{V}(\tilde{B}^{\beta} + \tilde{C}^{\beta}) + (1 - \lambda) \mathcal{V}(\tilde{B}^{\gamma} + \tilde{C}^{\gamma}).$$

Therefore $\mathcal{V}_{\lambda}(\tilde{A} + \tilde{C}) > \mathcal{V}_{\lambda}(\tilde{B} + \tilde{C})$*, and from the definition,* $(\tilde{A} + \tilde{C}) > (\tilde{B} + \tilde{C})$

Theorem 3.15. *Suppose that* $\tilde{A} = \langle (a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta) \rangle$ *and* $\tilde{B} = \langle (a'_1, b'_1, c'_1, d'_1; \omega_2), (a'_2, b'_2, c'_2, d'_2; \rho), (a'_3, b'_3, c'_3, d'_3; \delta_2) \rangle$ *be the two GPSVNN with* $\omega_1 = \omega_2$, $\rho_1 = \rho_2$ *and* $\delta_1 = \delta_2$*. If* $a_i > d'_i$*, where i=1,2,3, then* $\tilde{A} > \tilde{B}$*.*

Proof: *As we have ,* $\omega_1 = \omega_2$ *and* $a_1 > d'_1$ *,*

$$\mathcal{V}(\tilde{A}^{\alpha}) = \int_0^{\omega_1} [L^{\alpha} + U^{\alpha}] f(\alpha) d\alpha \geq 2a_1 \int_0^{\omega_1} f(\alpha) d\alpha$$

$$\mathcal{V}(\tilde{B}^{\alpha}) = \int_0^{\omega_2} [L^{\alpha} + U^{\alpha}] f(\alpha) d\alpha \leq 2d_1 \int_0^{\omega_2} f(\alpha) d\alpha$$

$\int_0^{\omega_1} f(\alpha) d\alpha = \int_0^{\omega_2} f(\alpha) d\alpha$*, we have,* $a_1 > d'_1$

$$\mathcal{V}(\tilde{A}^{\alpha}) \geq 2a_1 \geq 2d'_1 \geq \mathcal{V}(\tilde{B}^{\alpha}). \tag{4}$$

As we have , $\rho_1 = \rho_2$ *and* $a_2 > d'_2$ *,*

$$\mathcal{V}(\tilde{A}^{\beta}) = \int_{\rho_1}^{1} [L'^{\alpha} + U'^{\alpha}] g d\beta \geq 2a_2 \int_{\rho_2}^{1} g(\beta) d\beta$$

$$\mathcal{V}(\tilde{B}^{\beta}) = \int_{\rho_2}^{1} [L'^{\alpha} + U'^{\alpha}] g(\beta) d\beta \leq 2d'_2 \int_{\rho_2}^{1} g(\beta) d\beta$$

$\int_{\rho_1}^{1} g(\beta)d\beta = \int_{\rho_2}^{1} g(\beta)d\beta$, *we have,* $a_2 > d'_2$.

$$\mathcal{V}(\tilde{A}^{\beta}) \geq 2a_2 \geq 2d'_2 \geq \mathcal{V}(\tilde{B}^{\beta}). \tag{5}$$

As we have , $\delta_1 = \delta_2$ *and* $a_3 > d'_3$ *,*

$$\mathcal{V}(\tilde{A}^{\gamma}) = [\int_{\delta_1}^{1} [L''^{\alpha} + U''^{\alpha}] h(\gamma) d\gamma \geq 2a_3 [\int_{\delta_2}^{1} h(\gamma) d\gamma$$

$$\mathcal{V}(\tilde{B}^{\gamma}) = \int_{\delta_2}^{1} [L''^{\alpha} + U''^{\alpha}] h(\gamma) d\gamma \leq 2d'_3 \int_{\delta_2}^{1} h(\gamma) d\gamma$$

$\int_{\delta_1}^{1} h(\gamma)d\gamma = \int_{\delta_2}^{1} h(\gamma)d\gamma$ *we have ,* $a_3 > d'_3$

$$\mathcal{V}(\tilde{A}^{\gamma}) \geq 2a_3 \geq 2d'_3 \geq \mathcal{V}(\tilde{B}^{\gamma}) \tag{6}$$

According to the definition, from the equations (4), (5) and (6), we have ,
$\lambda\mathcal{V}(\tilde{A}^{\alpha}) + (1-\lambda)\mathcal{V}(\tilde{A}^{\beta}) + (1-\lambda)\mathcal{V}(\tilde{A}^{\gamma}) > \lambda\mathcal{V}(\tilde{B}^{\alpha}) + (1-\lambda)\mathcal{V}(\tilde{B}^{\beta}) + (1-\lambda)\mathcal{V}(\tilde{B}^{\gamma})$.
Therefore, from the definition, $\tilde{A} > \tilde{B}$.

4. Application of GPSVNN

An algorithm for the GPSVN-numbers multi-criteria decision-making method as follows; Let $S_i = \{S_1, S_2, ...S_m\}$ be the set of alternatives , $T_j = \{T_1, T_2, ...T_n\}$ be the set of criteria and $\{[\tilde{A_{ij}}] = \langle (a_{1ij}, b_{1ij}, c_{1ij}, d_{1ij}; \omega_{ij}), (a_{2ij}, b_{2ij}, c_{2ij}, d_{2ij}; \rho_{ij}), (a_{3ij}, b_{3ij}, c_{3ij}, d_{3ij}; \delta_{ij}) \rangle$ be the GPSVN-numbers.

Step 1: Construct the decision-making matrix, G=$[\tilde{A_{ij}}]_{m*n}$ using GPSVNN.

Step 2: Compute the normalised decision-making matrix, N = $[\tilde{n_{ij}}]_{m*n}$ of G, for $[\tilde{n_{ij}}]_{m*n} = \langle \big(\frac{a_{1ij}}{d_1{}^+}, \frac{b_{1ij}}{d_1{}^+}, \frac{c_{1ij}}{d_1{}^+}, \frac{d_{1ij}}{d_1{}^+}; \omega_{ij}\big), \big(\frac{a_{2ij}}{d_2{}^+}, \frac{b_{2ij}}{d_2{}^+}, \frac{c_{2ij}}{d_2{}^+}, \frac{d_{2ij}}{d_2{}^+}; \rho_{ij}\big), \big(\frac{a_{3ij}}{d_3{}^+}, \frac{b_{3ij}}{d_3{}^+}, \frac{c_{3ij}}{d_3{}^+}, \frac{d_{3ij}}{d_3{}^+}; \delta_{ij}\big) \rangle$, where d^+ = $max\{d_{ij}\}$.

Step 3: Compute the T = $[t_{ij}]_{m*n}$ of N,where $[t_{ij}]_{m*n}$ = $w_i * \tilde{r_{ij}}$,(should satisfy the normalized condition , $w_i = [0, 1], \sum_{i=1}^{\infty} w_i = 1$).

Step 4: Compute the comprehensive values $\tilde{C_i}$ as, $\tilde{C_i} = \sum_{j=1}^{\infty} [t_{ij}]$.

Step 5: Determine the increasing order of $\tilde{C_i}$.

Step 6: Rank the alternatives s_i according to the C_i and select the best and worst alternatives.

Application of GPSVNN in Multi-Criteria Decision Making:

Since 2003, the Life Satisfaction Survey (LSS) has been performed by the Turkish Statistics Institute (TUIK). LSS is essential to gauge how happy people feel in general, how they regard other people, how satisfied they are with their primary living conditions, and how comfortable they are with public services. Evaluation of the surveys using statistical techniques has been the primary focus of studies to ascertain the quality of municipal service in Turkey. They used the image fuzzy vikor approach to evaluate it in that regard. In this case, GPSVNN was employed. Additionally, the twenty choices' weight vector may be expressed as follows:

$w = (0.03, 0.08, 0.04, 0.02, 0.06, 0.05, 0.01, 0.07, 0.09, 0.04, 0.06, 0.07, 0.05, 0.04, 0.01, 0.02, 0.03, 0.1, 0.08, 0.05)^T$

The below numbers are choosed randomly in between the interval [0,10], which are GPSVN-numbers for the twenty municipal service alternatives $(S_1, S_2, ..., S_{20})$ with the four criteria as (T_1, T_2, T_3, T_4) . To find the best and worst alternatives in those municipal services, to improve the society and to award which have given it's best service.

TABLE 1. Alternative Sets.

S_i	**Service Alternative**	S_i	**Service Alternative**
S_1	Garbage and environmental cleanliness	S_2	Drainage
S_3	Drinking water	S_4	Public transport
S_5	Municipal police	S_6	Road and pavement construction
S_7	Parks and gardens	S_8	Minimization of noise and air pollution
S_9	Health, fitness center facilities	S_{10}	Zoning and city planning
S_{11}	Arrangements for the disabled	S_{12}	Social aids
S_{13}	Cultural activities	S_{14}	Public education centers
S_{15}	Street and road lighting	S_{16}	Cleanliness
S_{17}	Fire-fighting	S_{18}	Graveyard
S_{19}	Address information systems	S_{20}	Control of food producing facilities

TABLE 2. Decision-Matrix using GPSVNN

G	2014	2015	2016	2017
S_1	<(1.8,3.0, 4.2,7.1;0.4), (2.6,2.9,5.2,6.7;0.7), (4.6,5.5,6.9,7.2;0.2)>	<(1.5,3.6,4.2,7.3;0.5), (1.4,3.1,4.4,5.6;0.8), (0.7,3.2,4.7,8.9;0.3)>	<(3.5,4.3,6.0,7.4;0.8), (0.1,0.9,3.3,5.1;0.6), (1.8,3.0,4.2,5.7;0.3)>	<(0.1,0.9,1.7,7.3;0.5), (0.6,2.5,3.9,4.3;0.2), (5.4,6.9,8.5,9.7;0.7)>
S_2	<(1.8,2.5,2.9,6.6;0.6), (1.0,1.9,2.6,3.1;0.2), (1.9,3.3,5.0,8.1;0.6)>	<(4.5,6.7,8.3,9.1;0.7), (0.9,1,1.5,2.7;0.5), (0.6,1.2,1.8,3.1;0.4)>	<(1.5,3.6,5.8,8.1;0.5), (6.2,7.7, 9.1,10;0.2), (0.2,0.4,1.1,1.8;0.7)>	<(5.1,6.6,8.4,9.0;0.5), (1.7,2.9,4.7,6.6;0.8), (1.1,1.2,2.3,5.5;0.3)>
S_3	<(2.0,3.9,7.0,8.8;0.3), (0.7,3.6,4.3,9.0;0.5), (5.5,6.6,7.7,8.8;0.2)>	<(2.3,2.7,3.4,5.1;0.8), (1.0,3.6,6.2,7.2;0.6), (2.0,3.9,4.4,5.7;0.3)>	<(3.7, 5.0,6.2,7.5;0.4), (0.2,1.7,3.0,3.1;0.7), (0.9,1.5,3.4,4.7;0.5)>	<(3.0,4.5,6.9,7.5;0.2), (3.1,6.3,7.3,9.5;0.7), (0.1,0.3,1.1,7.5;0.6)>
S_4	<(0.9,1.2,2.1,5.9;0.7), (1.8,3.0,4.2,7.1;0.2), (0.6,2.5,2.9,5.9;0.3)>	<(0.1,0.7,1.1,1.4;0.3), (5.3,7.3,8.7,10;0.1), (4.4,4.5,4.7,4.9;0.3)>	<(6.3,7.5,8.0,9.9;0.3), (0.2,0.3,0.4,0.5;0.7), (0.1,0.2,0.3,0.4;0.5)>	<(6.2,6.9,7.5,9.9;0.7), (1.8,3.0,4.2,5.1;0.4), (0.2,0.7,8.1 10;0.3)>
S_5	<(1.1,1.9,2.6,5.4;0.6), (5.4,5.9,6.6,7.1;0.2), (3.1,6.7,7.1,7.9;0.2)>	<(0.5,1.8,3.9,5.5;0.7), (1.1,2.9,5.2,7.7;0.2), (5.1,6.7,7.1,7.9;0.3)>	<(0.8,1.1,2.2,2.6;0.6), (5.4,6.2,7.9,8.3;0.9), (4.6,5.5,6.9,7.2;0.8)>	<(2.9,3.7,5.9,8.1;0.3), (0.3,1.1,3.4,6.9;0.4), (1.1,2.0,2.8,3.0;0.2)>
S_6	<(0.5,1.0,2.9,5.6;0.4), (0.2,0.5,0.7,2.3;0.1), (1.2,4.3,5.0,6.7;0.3)>	<(0.7,1.2,2.7,5.6;0.4), (2.3,2.7,3.4,5.1;0.3), (3.4,5.2,6.2,8.7;0.5)>	<(0.3,1.5,4.3,7.3;0.4), (4.7,6.9,7.3,8.9;0.1), (3.4,5.2,6.6,7.7;0.3)>	<(0.4,1.2,3.0,5.4;0.6), (0.4,1.8,4.7,5.7;0.4), (2.3,5.6,8.5,9.8;0.3)>
S_7	<(0.9,1.0,2.7,5.4;0.5), (2.0,3.9,7.0,8.8;0.2), (1.8,3.0,4.2,7.1;0.4)>	<(4.7,6.9,7.3,8.5;0.7), (0.6,2.5,2.9,3.3;0.8), (0.2,,0.4,1.8,2.6;0.9)>	<(2.4,3.5,5.8,6.3;0.3), (4.4,5.2,6.7,7.8;0.6), (1.5,3.6,4.8,6.1;0.5)>	<(0.9,1.2,2.1,3.9;0.5), (2.7,5.7,6.2,6.9;0.2), (0.6,1.2,2.8,5.4;0.7)>
S_8	<(0.6,2.2,2.6,4.2;0.6), (0.2,1.2,2.0,5.4;0.3), (0.4,1.7,3.3,9.0;0.5)>	<(1.2,3.0,4.6,5.9;0.5), (2.7,5.2,6.7,7.9;0.3), (0.7,3.9,4.3,6.2;0.2)>	<(5.7,6.3,7.1,9.5;0.2), (0.6,1.4,1.8,2.3;0.4), (2.6,3.9,4.5,5.6;0.5)>	<(4.1,7.3,8.8,9.5;0.3), (0.5,1.1,2.6,3.5;0.5), (5.5,7.7,8.3,9.9;0.2)>
S_9	<(1.7,2.8,4.5,8.5;0.5), (0.4,1.2,3.0,5.4;0.2), (1.4,3.1,4.4,7.6;0.7)>	<(1.1,1.6,2.7,4.6;0.6), (3.4,6.9,7.3,9.3;0.9), (4.1,6.1,7.3,8.1;0.8)>	<(1.0,1.5,2.4,3.1;0.2), (4.5,6.7,8.3,9.4;0.6), (1.4,3.1,4.4,5.9;0.2)>	<(7.4,8.5,9.6,9.9;0.6), (0.9,1.0,1.5,2.7;0.2), (0.1,0.5,0.7,2.3;0.3)>
S_{10}	<(1.1,3.6,3.9,8.0;0.7), (1.8,3.9,5.7,9.0;0.4), (4.4,6.9,8.5,9.7;0.6)>	<(7.6,8.1,9.0,9.7;0.2), (0.9,1.5,3.4,4.3;0.4), (0.2,0.5,0.7,0.8;0.4)>	<(2.4,3.5,4.3,6.0;0.5), (3.4,5.5,7.8,9.0;0.2), (1.2,1.8,2.2,2.3;0.7)>	<(0.2,3.1,7.1,9.2;0.8), (1.5,3.7,3.8,4.2;0.6), (1.0,1.9,2.6,3.1;0.3)>
S_{11}	<(2.0,2.7,5.4,9.4;0.8), (1.1,1.9,2.6,5.4;0.5), (0.7,3.9,4.3,9.0;0.4)>	<(6.2,6.9,7.8,9.1;0.4), (2.8,3.0,4.2,5.3;0.7), (0.4,0.9,1.7,4.4;0.3)>	<(5.1,6.6,8.3,9.3;0.1), (1.5,1.5,3.5,6.3;0.3), (2.0,3.9,4.4,5.7;0.6)>	<(1.2,3.0,4.6,5.9;0.4), (0.8,4.4,6.2,8.1;0.5), (1.3,3.9,7.4,8.9;0.4)>
S_{12}	<(0.5,1.6,2.6,8.5;0.8), (1.8,2.5,2.9,6.6;0.1), (0.9,5.5,7.7,8.1;0.2)>	<(2.3,4.4,5.6,6.7;0.5), (2.0,3.4,5.7,8.4;0.2), (0.2,0.4,1.1,1.8;0.7)>	<(0.9,1.3,2.2,5.6;0.4), (3.1,6.3,7.3,9.5;0.1), (1.4,5.2,6.2,6.9;0.3)>	<(1.1,1.3,2.2,5.4;0.2), (2.6,3.9,4.5,5.6;0.4), (5.9,6.7,7.7,8.8;0.5)>
S_{13}	<(1.0,4.2,5.7,10;0.7), (5.3,7.3,8.7,9.1;0.2), (1.1,3.6,3.9,8.0;0.6)>	<(3.0,4.1,5.5,8.3;0.3), (1.8,3.0,4.2,5.1;0.5), (2.3,2.7,3.4,5.1;0.2)>	<(0.1,0.2,2.2,5.3;0.8), (5.9,6.7,7.9,8.8;0.5), (0.6,1.1,2.3,2.7;0.4)>	<(0.8,1.1,2.2,2.6;0.9), (2.8,3.1,5.3,5.6;0.1), (3.4,5.5,7.8,9.0;0.2)>
S_{14}	<(2.7,2.9,3.3,5.1;0.6), (1.2,4.3,5.0,7.1;0.2), (4.4,6.9,8.5,9.7;0.6)>	<(0.8,1.8,3.2,4.5;0.8), (5.4,5.9,6.6,7.1;0.5), (5.3,7.3,8.7,9.1;0.4)>	<(0.8,1.8,2.7,3.5;0.1), (2.3,7.8,8.3,8.9;0.2), (3.1,3.6,5.0,6.2;0.2)>	<(0.9,1.8,2.8,5.5;0.5), (0.8,1.0,2.7,5.4;0.2), (0.9,1.0,2.8,5.3;0.2)>
S_{15}	<(3.6,5.0,6.7,7.1;0.7), (0.5,1.6,2.6,3.5;0.2), (0.9,1.2,2.1,3.9;0.8)>	<(1.4,3.7,5.6,7.3;0.7), (0.9,1.0,2.7,5.4;0.3), (1.8,2.5,4.3,5.1;0.6)>	<(3.0,4.3,4.9,5.7;0.7), (0.3,1.1,3.4,6.9;0.2), (2.8,3.0,4.2,5.3;0.6)>	<(1.4,3.3,4.7,8.2;0.3), (5.3,6.2,7.7,9.9;0.3), (3.1,3.6,5.0,6.2;0.4)>
S_{16}	<(2.8,4.8,5.9,9.4;0.3), (1.9,2.7,3.9,5.6;0.4), (0.4,1.2,3.0,5.4;0.5)>	<(0.3,1.2,1.5,2.7;0.5), (4.4,6.9,8.5,9.0;0.3), (3.6,3.9,5.5,6.9;0.2)>	<(1.4,2.9,4.3,4.8;0.6), (0.3,1 1.4,7.4;0.2), (5.3,7.3,8.5,9.9;0.5)>	<(4.6,5.7,9.5,9.7;0.8), (1.2,1.8,2.2,2.3;0.5), (0.3,2.0,3.1,3.3;0.5)>
S_{17}	<(3.7,4.6,7.3,9.4;0.2), (0.5,1.0,2.9,3.6;0.7), (0.7,2.2,3.6,4.3;0.8)>	<(0.4,1.2,2.9,3.3;0.8), (2.0,3.4,5.5,7.1;0.5), (7.6,8.1,9.0,9.7;0.4)>	<(0.5,0.6,1.8,7.1;0.7), (1.3,1.6,2.3,4.8;0.4), (5.6,5.9,6.1,7.7;0.5)>	<(0.5,0.6,1.7,7.0;0.5), (0.4,0.7,2.0,6.9;0.1), (0.5,0.7,1.8,7.0;0.4)>
S_{18}	<(0.6,1.4,1.8,3.7;0.5), (5.4,5.9,6.6,7.1;0.3), (1.0,1.9,2.6,3.1;0.3)>	<(0.6,2.7,4.5,7.4;0.1), (1.2,4.3,5.0,6.7;0.2), (1.7,2.8,4.5,7.3;0.2)>	<(5.6,5.9,6.0,6.1;0.5), (0.3,1.4,5.0,7.8;0.2), (3.1,4.9,5.7,7.3;0.7)>	<(2.4,3.5,4.8,5.0;0.4), (0.6,1.4,1.8,2.3;0.3), (3.1,3.6,5.0,6.2;0.1)>
S_{19}	<(1.2,3.7,7.3,8.0;0.4), (0.2,0.4,1.1,1.8;0.2), (1.2,3.1,5.5,5.9;0.7)>	<(1.0,3.9,4.2,5.3;0.7), (2.7,4.6,7.8,9.0;0.4), (3.4,4.2,5.7,7.3;0.8)>	<(0.2,1.3,1.7,3.6;0.6), (0.2,0.5,0.7,2.3;0.5), (1.2,3.6,5.5,6.9;0.7)>	<(2.3,7.8,8.3,8.9;0.5), (0.8,1.8,7.3,8.5;0.1), (0.6,1.1,2.3,2.7;0.9)>
S_{20}	<(2.9,3.5,3.9,4.7;0.8), (1.9,3.0,5.4,9.2;0.6), (3.5,4.3,6.1,7.7;0.7)>	<(4.2,5.3,6.7,7.1;0.8), (1.8,3.9,4.7,5.0;0.1), (0.2,0.7,1.8,2.0;0.4)>	<(1.5,3.5,4.2,8.0;0.6), (1.5,2.3,3.1,5.9;0.8), (6.3,9.0,9.4,9.5;0.3)>	<(1.0,1.5,3.6,3.9;0.1), (1.1,1.8,3.5,3.6;0.3), (1.0,1.7,3.5,3.8;0.8)>

TABLE 3. Normalized-Matrix

N	2014	2015	2016	2017
S_1	<(0.18,0.30,0.42,0.71;0.4), (0.26,0.29,0.52,0.67;0.7), (0.46,0.55,0.69,0.72;0.2)>	<(0.15,0.36,0.42,0.73;0.5), (0.14,0.31,0.44,0.56;0.8), (0.07,0.32,0.47,0.89;0.3)>	<(0.35,0.43,0.60,0.74;0.8), (0.01,0.09,0.33,0.51;0.6), (0.18,0.30,0.42,0.57;0.3)>	<(0.01,0.09,0.17,0.73;0.5), (0.06,0.25,0.39,0.43;0.2), (0.54,0.69,0.85,0.97;0.7)>
S_2	<(0.18,0.25,0.29,0.66;0.6), (0.10,0.19,0.26,0.31;0.2), (0.19,0.33,0.50,0.81;0.6)>	<(0.45,0.67,0.83,0.91;0.7), (0.09,0.10,0.15,0.27;0.5), (0.06,0.12,0.18,0.31;0.4)>	<(0.15,0.36,0.58,0.81;0.5), (0.62,0.77,0.91,1.00;0.2), (0.02,0.04,0.11,0.18;0.7)>	<(0.51,0.66,0.84,0.90;0.5), (0.17,0.29,0.47,0.66;0.8), (0.11,0.12,0.23,0.55;0.3)>
S_3	<(0.20,,0.39,0.70,0.88;0.3), (0.07,0.36,0.43,0.90;0.5), (0.55,,0.66,0.77,0.88;0.2)>	<(0.23,0.27,0.34,0.51;0.8), (0.10,0.36,0.62,0.72;0.6), (0.20,0.39,0.44,0.57;0.3)>	<(0.37,0.50,0.62,0.75;0.4), (0.02,0.17,0.30,0.31;0.7), (0.09,0.15,0.34,0.47;0.5)>	<(0.30,0.45,0.69,0.75;0.2), (0.31,0.63,0.73,0.95;0.7), (0.01,0.03,0.11,0.75;0.6)>
S_4	<(0.09,0.12,0.21,0.59;0.7), (0.18,0.30,0.42,0.71;0.2), (0.06,0.25,0.29,0.59;0.3)>	<(0.01,0.07,0.11,0.14;0.3), (0.53,0.73,0.87,1.00;0.1), (0.44,0.45,0.47,0.49;0.3)>	<(0.63,0.75,0.80,0.99;0.3), (0.02,0.03,0.04,0.05;0.7), (0.01,0.02,0.03,0.04;0.5)>	<(0.62,0.69,0.75,0.99;0.7), (0.18,0.30,0.42,0.51;0.4), (0.02,0.07,0.81,1.00;0.3)>
S_5	<(0.11,0.19,0.26,0.54;0.6), (0.54,0.59,0.66,0.71;0.2), (0.31,0.67,0.71,0.79;0.2)>	<(0.05,0.18,0.39,0.55;0.7), (0.11,0.29,0.52,0.77;0.2), (0.51,0.67,0.71,0.79;0.3)>	<(0.08,0.11,0.22,0.26;0.6), (0.54,0.62,0.79,0.83;0.9), (0.46,0.55,0.69,0.72;0.8)>	<(0.29,0.37,0.59,0.81;0.3), (0.03,0.11,0.34,0.69;0.4), (0.11,0.20,0.28,0.30;0.2)>
S_6	<(0.05,0.10,0.29,0.56;0.4), (0.02,0.05,0.07,0.23;0.1), (0.12,0.43,0.50,0.67;0.3)>	<(0.07,0.12,0.27,0.56;0.4), (0.23,0.27,0.34,0.51;0.3), (0.34,0.52,0.62,0.87;0.5)>	<(0.03,0.15,0.43,0.73;0.4), (0.47,0.69,0.73,0.89;0.1), (0.34,0.52,0.66,0.77;0.3)>	<(0.04,0.12,0.30,0.54;0.6), (0.04,0.18,0.47,0.57;0.4), (0.23,0.56,0.85,0.98;0.3)>
S_7	<(0.09,0.10,0.27,0.54;0.5), (0.20,0.39,0.70,0.88;0.2), (0.18,0.30,0.42,0.71;0.4)>	<(0.47,0.69,0.73,0.85;0.7), (0.06,0.25,0.29,0.33;0.8), (0.02,0.04,0.18,0.26;0.9)>	<(0.24,0.35,0.58,0.63;0.3), (0.44,0.52,0.67,0.78;0.6), (0.15,0.36,0.48,0.61;0.5)>	<(0.09,0.12,0.21,0.39;0.5), (0.27,0.57,0.62,0.69;0.2), (0.06,0.12,0.28,0.54;0.7)>
S_8	<(0.06,0.22,0.26,0.42;0.6), (0.02,0.12,0.20,0.54;0.3), (0.04,0.17,0.33,0.90;0.5)>	<(0.12,0.30,0.46,0.59;0.5), (0.27,0.52,0.67,0.79;0.3), (0.07,0.39,0.43,0.62;0.2)>	<(0.57,0.63,0.71,0.95;0.2), (0.06,0.14,0.18,0.23;0.4), (0.26,0.39,0.45,0.56;0.5)>	<(0.41,0.73,0.88,0.95;0.3), (0.05,0.11,0.26,0.35;0.5), (0.55,0.77,0.83,0.99;0.2)>
S_9	<(0.17,0.28,0.45,0.85;0.5), (0.04,0.12,0.30,0.54;0.2), (0.14,0.31,0.44,0.76;0.7)>	<(0.11,0.16,0.27,0.46;0.6), (0.34,0.69,0.73,0.93;0.9), (0.41,0.61,0.73,0.81;0.8)>	<(0.10,0.15,0.24,0.31;0.2), (0.45,0.67,0.83,0.94;0.6), (0.14,0.31,0.44,0.59;0.2)>	<(0.74,0.85,0.96,0.99;0.6), (0.09,0.10,0.15,0.27;0.2), (0.01,0.05,0.07,0.23;0.3)>
S_{10}	<(0.11,0.36,0.39,0.80;0.7), (0.18,0.39,0.57,0.90;0.4), (0.44,0.69,0.85,0.97;0.6)>	<(0.76,0.81,0.90,0.97;0.2), (0.09,0.15,0.34,0.43;0.4), (0.02,0.05,0.07,0.08;0.4)>	<(0.24,0.35,0.43,0.60;0.5), (0.34,0.55,0.78,0.90;0.2), (0.12,0.18,0.22,0.23;0.7)>	<(0.02,0.31,0.71,0.92;0.8), (0.15,0.37,0.38,0.42;0.6), (0.10,0.19,0.26,0.31;0.3)>
S_{11}	<(0.20,0.27,0.54,0.94;0.8), (0.11,0.19,0.26,0.54;0.5), (0.07,0.39,0.43,0.90;0.4)>	<(0.62,0.69,0.78,0.91;0.4), (0.28,0.30,0.42,0.53;0.7), (0.04,0.09,0.17,0.44;0.3)>	<(0.51,0.66,0.83,0.93;0.1), (0.15,0.15,0.35,0.63;0.3), (0.20,0.39,0.44,0.57;0.6)>	<(0.12,0.30,0.46,0.59;0.4), (0.08,0.44,0.62,0.81;0.5), (0.13,0.39,0.74,0.89;0.4)>
S_{12}	<(0.05,0.16,0.26,0.85;0.8), (0.18,0.25,0.29,0.66;0.1), (0.09,0.55,0.77,0.81;0.2)>	<(0.23,0.44,0.56,0.67;0.5), (0.20,0.34,0.57,0.84;0.2), (0.02,0.04,0.11,0.18;0.7)>	<(0.09,0.13,0.22,0.56;0.4), (0.31,0.63,0.73,0.95;0.1), (0.14,0.52,0.62,0.69;0.3)>	<(0.11,0.13,0.22,0.54;0.2), (0.26,0.39,0.45,0.56;0.4), (0.59,0.67,0.77,0.88;0.5)>
S_{13}	<(0.10,0.42,0.57,1.00;0.7), (0.53,0.73,0.87,0.91;0.2), (0.11,0.36,0.39,0.80;0.6)>	<(0.30,0.41,0.55,0.83;0.3), (0.18,0.30,0.42,0.51;0.5), (0.23,0.27,0.34,0.51;0.2)>	<(0.01,0.02,0.22,0.53;0.8), (0.59,0.67,0.79,0.88;0.5), (0.06,0.11,0.23,0.27;0.4)>	<(0.08,0.11,0.22,0.26;0.9), (0.28,0.31,0.53,0.56;0.1), (0.34,0.55,0.78,0.90;0.2)>
S_{14}	<(0.27,0.29,0.33,0.51;0.6), (0.12,0.43,0.50,0.71;0.2), (0.44,0.69,0.85,0.97;0.6)>	<(0.08,0.18,0.32,0.45;0.8), (0.54,0.59,0.66,0.71;0.5), (0.53,0.73,0.87,0.91;0.4)>	<(0.08,0.18,0.27,0.35;0.1), (0.23,0.78,0.83,0.89;0.2), (0.31,0.36,0.50 0.62;0.2)>	<(0.09,0.18,0.28,0.55;0.5), (0.08,0.10,0.27,0.54;0.2), (0.09,0.10,0.28,0.53;0.2)>
S_{15}	<(0.36,0.50,0.67,0.71;0.7), (0.05,0.16,0.26,0.35;0.2), (0.09,0.12,0.21,0.39;0.8)>	<(0.14,0.37,0.56,0.73;0.7), (0.09,0.10,0.27,0.54;0.3), (0.18,0.25,0.43,0.51;0.6)>	<(0.30,0.43,0.49,0.57;0.7), (0.03,0.11,0.34,0.69;0.2), (0.28,0.30,0.42,0.53;0.6)>	<(0.14,0.33,0.47,0.82;0.3), (0.53,0.62,0.77,0.99;0.3), (0.31,0.36,0.50,0.62;0.4)>
S_{16}	<(0.28,0.48,0.59,0.94;0.3), (0.19,0.27,0.39,0.56;0.4), (0.04,0.12,0.30,0.54;0.5)>	<(0.03,0.12,0.15,0.27;0.5), (0.44,0.69,0.85,0.90;0.3), (0.36,0.39,0.55,0.69;0.2)>	<(0.14,0.29,0.43,0.48;0.6), (0.03,0.10,0.14,0.74;0.2), (0.53,0.73,0.85,0.99;0.5)>	<(0.46,0.57,0.95,0.97;0.8), (0.12,0.18,0.22,0.23;0.5), (0.03,0.20,0.31,0.33;0.5)>
S_{17}	<(0.37,0.46,0.73,0.94;0.2), (0.05,0.10,0.29,0.36;0.7), (0.07,0.22,0.36,0.43;0.8)>	<(0.04,0.12,0.29,0.33;0.8), (0.20,0.34,0.55,0.71;0.5), (0.76,0.81,0.90,0.97;0.4)>	<(0.05,0.06,0.18,0.71;0.7), (0.13,0.16,0.23,0.48;0.4), (0.56,0.59,0.61,0.77;0.5)>	<(0.05,0.06,0.17,0.70;0.5), (0.04,0.07,0.20,0.69;0.1), (0.05,0.07,0.18,0.70;0.4)>
S_{18}	<(0.06,0.14,0.18,0.37;0.5), (0.54,0.59,0.66,0.71;0.3), (0.10,0.19,0.26,0.31;0.3)>	<(0.06,0.27,0.45,0.74;0.1), (0.12,0.43,0.50,0.67;0.2), (0.17,0.28,0.45,0.73;0.2)>	<(0.56,0.59,0.60,0.61;0.5), (0.03,0.14,0.50,0.78;0.2), (0.31,0.49,0.57,0.73;0.7)>	<(0.24,0.35,0.48,0.50;0.4), (0.06,0.14,0.18,0.23;0.3), (0.31,0.36,0.50,0.62;0.1)>
S_{19}	<(0.12,0.37,0.73,0.80;0.4), (0.02,0.04,0.11,0.18;0.2), (0.12,0.31,0.55,0.59;0.7)>	<(0.10,0.39,0.42,0.53;0.7), (0.27,0.46,0.78,0.90;0.4), (0.34,0.42,0.57,0.73;0.8)>	<(0.02,0.13,0.17,0.36;0.6), (0.02,0.05,0.07,0.23;0.5), (0.12,0.36,0.55,0.69;0.7)>	<(0.23,0.78,0.83,0.89;0.5), (0.08,0.18,0.73,0.85;0.1), (0.06,0.11,0.23,0.27;0.9)>
S_{20}	<(0.29,0.35,0.39,0.47;0.8), (0.19,0.30,0.54,0.92;0.6), (0.35,0.43,0.61,0.77;0.7)>	<(0.42,0.53,0.67,0.71;0.8), (0.18,0.39,0.47,0.50;0.1), (0.02,0.07,0.18,0.20;0.4)>	<(0.15,0.35,0.42,0.80;0.6), (0.15,0.23,0.31,0.59;0.8), (0.63,0.90,0.94,0.95;0.3)>	<(0.10,0.15,0.36,0.39;0.1), (0.11,0.18,0.35,0.36;0.3), (0.10,0.17,0.35,0.38;0.8)>

TABLE 4. $T = w_i * r_{ij}$

T	2014	2015	2016	2017
S_1	<(0.0054,0.0090,0.0126,0.0213;0.4), (0.0078,0.0087,0.0156,0.0201;0.7), (0.0138,0.0165,0.0207,0.0216;0.2)>	<(0.0045,0.0108,0.0126,0.0219;0.5), (0.0042,0.0093,0.0132,0.0168;0.8), (0.0021,0.0096,0.0141,0.0267;0.3)>	<(0.0105,0.0129,0.0180,0.0222;0.8), (0.0003,0.0027,0.0099,0.0153;0.6), (0.0054,0.0090,0.0126,0.0171;0.3)>	<(0.0003,0.0027,0.0051,0.0219;0.5), (0.0018,0.0075,0.0117,0.0129;0.2), (0.0162,0.0207,0.0255,0.0291;0.7)>
S_2	<(0.0144,0.0200,0.0232, 0.0528;0.6), (0.0080,0.0152,0.0208,0.0248;0.2), (0.0152,0.0264,0.0400,0.0648;0.6)>	<(0.0360,0.0536,0.0664,0.0728;0.7), (0.0072,0.0080,0.0120,0.0216;0.5), (0.0048,0.0096,0.0144,0.0248;0.4)>	<(0.0120,0.0288,0.0464,0.0648;0.5), (0.0496,0.0616,0.0728,0.0800;0.2), (0.0016,0.0032,0.0088,0.0144;0.7)>	<(0.0408,0.0528,0.0672,0.0720;0.5), (0.0136,0.0232,0.0376,0.0528;0.8), (0.0088,0.0096,0.0184,0.0440;0.3)>
S_3	<(0.0080,0.0156,0.0280,0.0352;0.3), (0.0028,0.0144,0.0172,0.0360;0.5), (0.0220,0.0264,0.0308,0.0352;0.2)>	<(0.0092,0.0108,0.0136,0.0204;0.8), (0.0040,0.0144,0.0248,0.0288;0.6), (0.0080,0.0156,0.0176,0.0228;0.3)>	<(0.0148,0.0200,0.0248,0.0300;0.4), (0.0008,0.0068,0.0120,0.0124;0.7), (0.0036,0.0060,0.0136,0.0188;0.5)>	<(0.0120,0.0180,0.0276,0.0300;0.2), (0.0124,0.0252,0.0292,0.0380;0.7), (0.0004,0.0012,0.0044,0.0300;0.6)>
S_4	<(0.0018,0.0024,0.0042,0.0118;0.7), (0.0036,0.0060,0.0084,0.0142;0.2), (0.0012,0.0050,0.0058,0.0118;0.3)>	<(0.0002,0.0014,0.0022,0.0028;0.3), (0.0106,0.0146,0.0174,0.0200;0.1), (0.0088,0.0090,0.0094,0.0098;0.3)>	<(0.0126,0.0150,0.0160,0.0198;0.3), (0.0004,0.0006,0.0008,0.001;0.7), (0.0002, 0.0004, 0.0006, 0.0008;0.5)>	<(0.0124,0.0138,0.0150,0.0198;0.7), (0.0036,0.0060,0.0084,0.0102;0.4), (0.0004,0.0014,0.0162,0.0200;0.3)>
S_5	<(0.0066,0.0114,0.0156,0.0324;0.6), (0.0324,0.0354,0.0396,0.0426;0.2), (0.0186,0.0402,0.0426,0.0474;0.2)>	<(0.0030,0.0108,0.0234,0.0330;0.7), (0.0066,0.0174,0.0312,0.0462;0.2), (0.0306,0.0402,0.0426,0.0474;0.3)>	<(0.0048,0.0066,0.0132,0.0156;0.6), (0.0324,0.0372,0.0474,0.0498;0.9), (0.0276,0.0330,0.0414,0.0432;0.8)>	<(0.0174,0.0222,0.0354,0.0486;0.3), (0.0018,0.0066,0.0204,0.0414;0.4), (0.0066,0.0120,0.0168,0.0180;0.2)>
S_6	<(0.0025,0.0050,0.0145,0.0280;0.4), (0.0010,0.0025,0.0035,0.0115;0.1), (0.0060,0.0215,0.0250,0.0335;0.3)>	<(0.0035,0.0060,0.0135,0.0280;0.4), (0.0115,0.0135,0.0170,0.0255;0.3), (0.0170,0.0260,0.0310,0.0435;0.5)>	<(0.0015,0.0075,0.0215,0.0365;0.4), (0.0235,0.0345,0.0365,0.0445;0.1), (0.0170,0.0260,0.0330,0.0385;0.3)>	<(0.0020,0.0060,0.0150,0.0270;0.6), (0.0020,0.0090,0.0235,0.0285;0.4), (0.0115,0.0280,0.0425,0.0490;0.3)>
S_7	<(0.0009,0.0010,0.0027,0.0054;0.5), (0.0020,0.0039,0.0070,0.0088;0.2), (0.0018,0.0030,0.0042,0.0071;0.4)>	<(0.0047,0.0069,0.0073,0.0085;0.7), (0.0006,0.0025,0.0029,0.0033;0.8), (0.0002,0.0004,0.0018,0.0026;0.9)>	<(0.0024,0.0035,0.0058,0.0063;0.3), (0.0044,0.0052,0.0067,0.0078;0.6), (0.0015,0.0036,0.0048,0.0061;0.5)>	<(0.0009,0.0012,0.0021,0.0039;0.5), (0.0027,0.0057,0.0062,0.0069;0.2), (0.0006,0.0012,0.0028,0.0054;0.7)>
S_8	<(0.0042,0.0154,0.0182,0.0294;0.6), (0.0014,0.0084,0.0140,0.0378;0.3), (0.0028,0.0119,0.0231,0.0630;0.5)>	<(0.0084,0.0210,0.0322,0.0413;0.5), (0.0189,0.0364,0.0469,0.0553;0.3), (0.0049,0.0273,0.0301,0.0434;0.2)>	<(0.0399,0.0441,0.0497,0.0665;0.2), (0.0042,0.0098,0.0126,0.0161;0.4), (0.0182,0.0273,0.0315,0.0392;0.5)>	<(0.0287,0.0511,0.0616,0.0665;0.3), (0.0035,0.0077,0.0182,0.0245;0.5), (0.0385,0.0539,0.0581,0.0693;0.2)>
S_9	<(0.0153,0.0252,0.0405,0.0765;0.5), (0.0036,0.0108,0.0270,0.0486;0.2), (0.0126,0.0279,0.0396,0.0684;0.7)>	<(0.0099,0.0144,0.0243,0.0414;0.6), (0.0306,0.0621,0.0657,0.0837;0.9), (0.0369,0.0549,0.0657,0.0729;0.8)>	<(0.0090,0.0135,0.0216,0.0279;0.2), (0.0405,0.0603,0.0747,0.0846;0.6), (0.0126,0.0279,0.0396,0.0531;0.2)>	<(0.0666,0.0765,0.0864,0.0891;0.6), (0.0081,0.0090,0.0135,0.0243;0.2), (0.0009,0.0045,0.0063,0.0207;0.3)>
S_{10}	<(0.0044,0.0144,0.0156,0.0320;0.7), (0.0072,0.0156,0.0228,0.0360;0.4), (0.0176,0.0276,0.0340,0.0388;0.6)>	<(0.0304,0.0324,0.0360,0.0388;0.2), (0.0036,0.0060,0.0136,0.0172;0.4), (0.0008,0.0020,0.0028,0.0032;0.4)>	<(0.0096,0.0140,0.0172,0.0240;0.5), (0.0136,0.0220,0.0312,0.0360;0.2), (0.0048,0.0072,0.0088,0.0092;0.7)>	<(0.0008,0.0124,0.0284,0.0368;0.8), (0.0060,0.0148,0.0152,0.0168;0.6), (0.0040,0.0076,0.0104,0.0124;0.3)>
S_{11}	<(0.0120,0.0162,0.0324,0.0564;0.8), (0.0066,0.0114,0.0156,0.0324;0.5), (0.0042,0.0234,0.0258,0.0540;0.4)>	<(0.0372,0.0414,0.0468,0.0546;0.4), (0.0168,0.0180,0.0252,0.0318;0.7), (0.0024,0.0054,0.0102,0.0264;0.3)>	<(0.0306,0.0396,0.0498,0.0558;0.1), (0.0090,0.0090,0.0210,0.0378;0.3), (0.0120,0.0234,0.0264,0.0342;0.6)>	<(0.0072,0.0180,0.0276,0.0354;0.4), (0.0048,0.0264,0.0372,0.0486;0.5), (0.0078,0.0234,0.0444,0.0534;0.4)>
S_{12}	<(0.0035,0.0112,0.0182,0.0595;0.8), (0.0126,0.0175,0.0203,0.0462;0.1), (0.0063,0.0385,0.0539,0.0567;0.2)>	<(0.0161,0.0308,0.0392,0.0469;0.5), (0.0140,0.0238,0.0399,0.0588;0.2), (0.0014,0.0028,0.0077,0.0126;0.7)>	<(0.0063,0.0091,0.0154,0.0392;0.4), (0.0217,0.0441,0.0511,0.0665;0.1), (0.0098,0.0364,0.0434,0.0483;0.3)>	<(0.0077,0.0091,0.0154,0.0378;0.2), (0.0182,0.0273,0.0315,0.0392;0.4), (0.0413,0.0469,0.0539,0.0616;0.5)>
S_{13}	<(0.0050,0.0210,0.0285,0.0500;0.7), (0.0265,0.0365,0.0435,0.0455;0.2), (0.0055,0.0180,0.0195,0.0400;0.6)>	<(0.0150,0.0205,0.0275,0.0415;0.3), (0.0090,0.0150,0.0210,0.0255;0.5), (0.0115,0.0135,0.0170,0.0255;0.2)>	<(0.0005,0.0010,0.0110,0.0265;0.8), (0.0295,0.0335,0.0395,0.0440;0.5), (0.0030,0.0055,0.0115,0.0135;0.4)>	<(0.0040,0.0055,0.0110,0.0130;0.9), (0.0140,0.0155,0.0265,0.0280;0.1), (0.0170,0.0275,0.0390,0.0450;0.2)>
S_{14}	<(0.0108,0.0116,0.0132,0.0204;0.6), (0.0048,0.0172,0.0200,0.0284;0.2), (0.0176,0.0276,0.0340,0.0388;0.6)>	<(0.0032,0.0072,0.0128,0.0180;0.8), (0.0216,0.0236,0.0264,0.0284;0.5), (0.0212,0.0292,0.0348,0.0364;0.4)>	<(0.0032,0.0072,0.0108,0.0140;0.1), (0.0092,0.0312,0.0332,0.0356;0.2), (0.0124,0.0144,0.0200,0.0248;0.2)>	<(0.0036,0.0072,0.0112,0.0220;0.5), (0.0032,0.0040,0.0108,0.0216;0.2), (0.0036,0.0040,0.0112,0.0212;0.2)>
S_{15}	<(0.0036,0.0050,0.0067,0.0071;0.7), (0.0005,0.0016,0.0026,0.0035;0.2), (0.0009,0.0012,0.0021,0.0039;0.8)>	<(0.0014,0.0037,0.0056,0.0073;0.7), (0.0009,0.0010,0.0027,0.0054;0.3), (0.0018,0.0025,0.0043,0.0051;0.6)>	<(0.0030,0.0043,0.0049,0.0057;0.7), (0.0003,0.0011,0.0034,0.0069;0.2), (0.0028,0.0030,0.0042,0.0053;0.6)>	<(0.0014,0.0033,0.0047,0.0082;0.3), (0.0053,0.0062,0.0077,0.0099;0.3), (0.0031,0.0036,0.0050,0.0062;0.4)>
S_{16}	<(0.0056,0.0096,0.0118,0.0188;0.3), (0.0038,0.0054,0.0078,0.0112;0.4), (0.0008,0.0024,0.0060,0.0108;0.5)>	<(0.0006,0.0024,0.0030,0.0054;0.5), (0.0088,0.0138,0.0170,0.0180;0.3), (0.0072,0.0078,0.0110,0.0138;0.2)>	<(0.0028,0.0058,0.0086,0.0096;0.6), (0.0006,0.0020,0.0028,0.0148;0.2), (0.0106,0.0146,0.0170,0.0198;0.5)>	<(0.0092,0.0114,0.0190,0.0194;0.8), (0.0024,0.0036,0.0044,0.0046;0.5), (0.0006,0.0040,0.0062,0.0066;0.5)>
S_{17}	<(0.0111,0.0138,0.0219,0.0282;0.2), (0.0015,0.0030,0.0087,0.0108;0.7), (0.0021,0.0066,0.0108,0.0129;0.8)>	<(0.0012,0.0036,0.0087,0.0099;0.8), (0.0060,0.0102,0.0165,0.0213;0.5), (0.0228,0.0243,0.0270,0.0291;0.4)>	<(0.0015,0.0018,0.0054,0.0213;0.7), (0.0039,0.0048,0.0069,0.0144;0.4), (0.0168,0.0177,0.0183,0.0231;0.5)>	<(0.0015,0.0018,0.0051,0.0210;0.5), (0.0012,0.0021,0.0060,0.0207;0.1), (0.0015,0.0021,0.0054,0.0210;0.4)>
S_{18}	<(0.0060,0.0140,0.0180,0.0370;0.5), (0.0540,0.0590,0.0660,0.0710;0.3), (0.0100,0.0190,0.0260,0.0310;0.3)>	<(0.0060,0.0270,0.0450,0.0740;0.1), (0.0120,0.0430,0.0500,0.0670;0.2), (0.0170,0.0280,0.0450,0.07300.2)>	<(0.0560,0.0590,0.0600,0.0610;0.5), (0.0030,0.0140,0.0500,0.0780;0.2), (0.0310,0.0490,0.0570,0.0730;0.7)>	<(0.0240,0.0350,0.0480,0.0500;0.4), (0.0060,0.0140,0.0180,0.0230;0.3), (0.0310,0.0360,0.0500,0.0620;0.1)>
S_{19}	<(0.0096,0.0296,0.0584,0.0640;0.4), (0.0016,0.0032,0.0088,0.0144;0.2), (0.0096,0.0248,0.0440,0.0472;0.7)>	<(0.0080,0.0312,0.0336,0.0424;0.7), (0.0216,0.0368,0.0624,0.0720;0.4), (0.0272,0.0336,0.0456,0.0584;0.8)>	<(0.0016,0.0104,0.0136,0.0288;0.6), (0.0016,0.0040,0.0056,0.0184;0.5), (0.0096,0.0288,0.0440,0.0552;0.7)>	<(0.0184,0.0624,0.0664,0.0712;0.5), (0.0064,0.0144,0.0584,0.0680;0.1), (0.0048,0.0088,0.0184,0.0216;0.9)>
S_{20}	<(0.0145,0.0175,0.0195,0.0235;0.8), (0.0095,0.0150,0.0270,0.0460;0.6), (0.0175,0.0215,0.0305,0.0385;0.7)>	<(0.0210,0.0265,0.0335,0.0355;0.8), (0.0090,0.0195,0.0235,0.0250;0.1), (0.0010,0.0035,0.0090,0.0100;0.4)>	<(0.0075,0.0175,0.0210,0.0400;0.6), (0.0075,0.0115,0.0155,0.0295;0.8), (0.0315,0.0450,0.0470,0.0475;0.3)>	<(0.0050,0.0075,0.0180,0.0195;0.1), (0.0055,0.0090,0.0175,0.0180;0.3), (0.0050,0.0085,0.0175,0.0190;0.8)>

TABLE 5. Comprehensive Values

C	Comprehensive Values
C_1	<(0.0207,0.0354,0.0483,0.0873;0.4),(0.0141,0.0282,0.0504,0.0651;0.8),(0.0375,0.0558,0.0729,0.0945;0.7)>
C_2	<(0.1032,0.1552,0.2032,0.2624;0.5),(0.0784,0.1080,0.1432,0.1792;0.8),(0.0304,0.0488,0.0816,0.1480;0.7)>
C_3	<(0.0440,0.0644,0.0940,0.1156;0.2),(0.0200,0.0608,0.0832,0.1152;0.7),(0.0340,0.0492,0.0664,0.1068;0.6)>
C_4	<(0.0270,0.0326,0.0374,0.0542;0.3),(0.0182,0.0272,0.0350,0.0454;0.7),(0.0106,0.0158,0.0320,0.0424;0.5)>
C_5	<(0.0318,0.0510,0.0876,0.1296;0.3),(0.0732,0.0966,0.1386,0.1800;0.9),(0.0834,0.1254,0.1434,0.1560;0.8)>
C_6	<(0.0095,0.0245,0.0645,0.1195;0.4),(0.0380,0.0595,0.0805,0.1100;0.4),(0.0515,0.1015,0.1315,0.1645;0.5)>
C_7	<(0.0089,0.0126,0.0179,0.0241;0.3),(0.0097,0.0173,0.0228,0.0268;0.8),(0.0041,0.0082,0.0136,0.0212;0.9)>
C_8	<(0.0812, 0.1316,0.1617,0.2037;0.2),(0.0280,0.0623,0.0917,0.1337;0.5),(0.0644,0.1204,0.1428,0.2149;0.4)>
C_9	<(0.1008,0.1296,0.1728,0.2349;0.2),(0.0828,0.1422,0.1809,0.2412;0.9),(0.0630,0.1152,0.1512,0.2151;0.8)>
C_{10}	<(0.0452,0.0732,0.0972,0.1316;0.2),(0.0304,0.0584,0.0828,0.1060;0.6),(0.0272,0.0444,0.0560,0.0636;0.7)>
C_{11}	<(0.0870,0.1152,0.1566,0.2022;0.1),(0.0372,0.0648,0.0990,0.1506;0.7),(0.0264,0.0756,0.1068,0.1680;0.6)>
C_{12}	<(0.0336,0.0602,0.0882,0.1834;0.2),(0.0665,0.1127,0.1428,0.2107;0.4),(0.0588,0.1246,0.1589,0.1792;0.7)>
C_{13}	<(0.0245,0.0480,0.0780,0.1310;0.3),(0.0790,0.1005,0.1305,0.1430;0.5),(0.0370,0.0645,0.0870,0.1240;0.6)>
C_{14}	<(0.0208,0.0332,0.0480,0.0744;0.1),(0.0388,0.0760,0.0904,0.1140;0.5),(0.0548,0.0752,0.1000,0.1212;0.6)>
C_{15}	<(0.0094,0.0163,0.0219,0.0283;0.3),(0.0070,0.0099,0.0164,0.0257;0.3),(0.0086,0.0103,0.0156,0.0205;0.8)>
C_{16}	<(0.0182,0.0292,0.0424,0.0532;0.3),(0.0156,0.0248,0.0320,0.0486;0.5),(0.0192,0.0288,0.0402,0.0510;0.5)>
C_{17}	<(0.0153,0.0210,0.0411,0.0804;0.2),(0.0126,0.0201,0.0381,0.0672;0.7),(0.0432,0.0507,0.0615,0.0861;0.8)>
C_{19}	<(0.0376,0.1336,0.1720,0.2064;0.4),(0.0312,0.0584,0.1352,0.1728;0.5),(0.0512,0.0960,0.1520,0.1824;0.9)>
C_{20}	<(0.0480,0.0690,0.0920,0.1185;0.1),(0.0315,0.0550,0.0835,0.1185;0.8),(0.0550,0.0785,0.1040,0.1150;0.8)>

TABLE 6. Values and Ambiguities of the alternatives

Values	Ambiguities
$V_1 = 0.0074 - 0.0003\lambda$	$A_1 = 0.0017 + .0002\lambda$
$V_2 = 0.0113 + 0.0337\lambda$	$A_2 = 0.0032 + 0.0056\lambda$
$V_3 = 0.0161 - 0.0129\lambda$	$A_3 = 0.004 - 0.0032\lambda$
$V_4 = 0.0089 - 0.0056\lambda$	$A_4 = 0.0029 - 0.0025\lambda$
$V_5 = 0.0065 - 0.0001\lambda$	$A_5 = 0.0009 + 0.0013\lambda$
$V_6 = 0.0542 - 0.0464\lambda$	$A_6 = 0.0114 - 0.0071\lambda$
$V_7 = 0.0009 + 0.0005\lambda$	$A_7 = 0.0002 + 0.0001\lambda$
$V_8 = 0.0674 - 0.0616\lambda$	$A_8 = 0.0142 - 0.0132\lambda$
$V_9 = 0.007 - 0.0008\lambda$	$A_9 = 0.0015 - 0.0003\lambda$
$V_{10} = 0.0156 - 0.0122\lambda$	$A_{10} = 0.0035 - 0.0028\lambda$
$V_{11} = 0.0224 - 0.0210\lambda$	$A_{11} = 0.0066 - 0.0063\lambda$
$V_{12} = 0.0591 - 0.0559\lambda$	$A_{12} = 0.0118 - 0.0108\lambda$
$V_{13} = 0.0410 - 0.0351\lambda$	$A_{13} = 0.0074 - 0.0054\lambda$
$V_{14} = 0.0345 - 0.0341\lambda$	$A_{14} = 0.0059 - 0.0058\lambda$
$V_{15} = 0.0073 - 0.0056\lambda$	$A_{15} = 0.0023 - 0.00196\lambda$
$V_{16} = 0.0160 - 0.0128\lambda$	$A_{16} = 0.0034 - 0.0026\lambda$
$V_{17} = 0.0051 - 0.0037\lambda$	$A_{17} = 0.0014 - 0.00086\lambda$
$V_{18} = 0.0910 - 0.0895\lambda$	$A_{18} = 0.0216 - 0.0213\lambda$
$V_{19} = 0.0257 - 0.0022\lambda$	$A_{19} = 0.0116 - 0.0064\lambda$
$V_{20} = 0.0064 - 0.0056\lambda$	$A_{20} = 0.0015 - 0.0013\lambda$

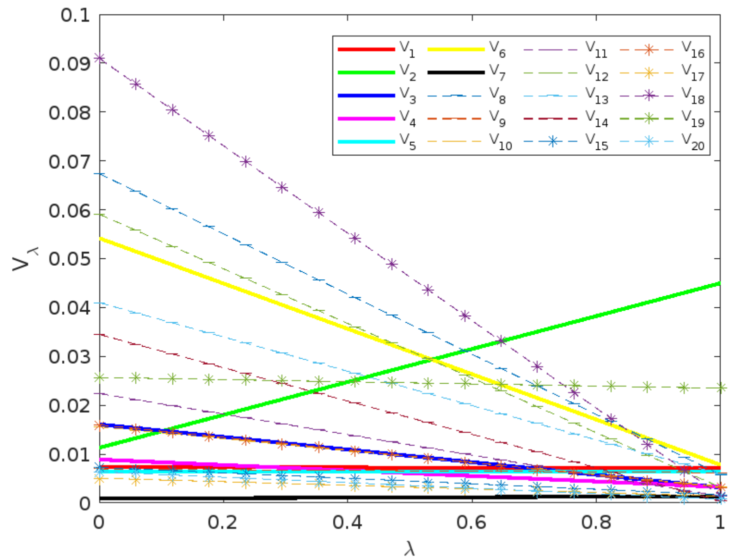


FIGURE 2. Value

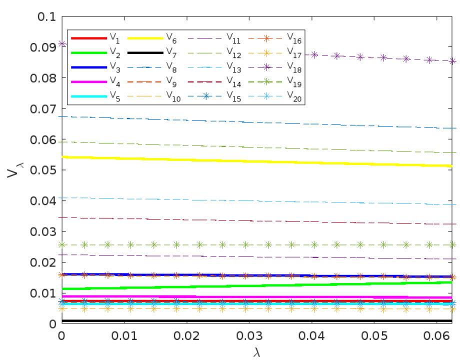


FIGURE 3. Value-$\lambda \in (0, 0.0625)$

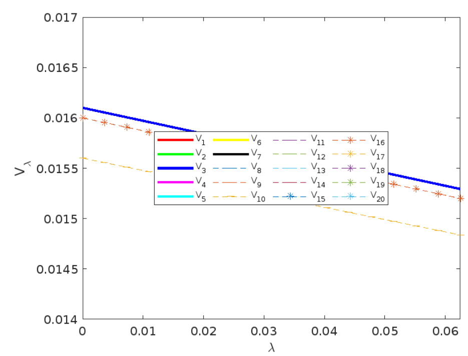


FIGURE 4. Value-$\lambda \in (0, 0.0625)$

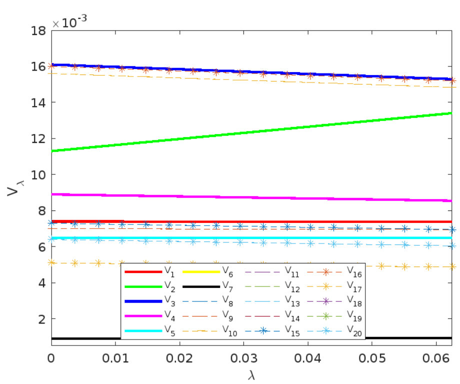


FIGURE 5. Value-$\lambda \in (0, 0.0625)$

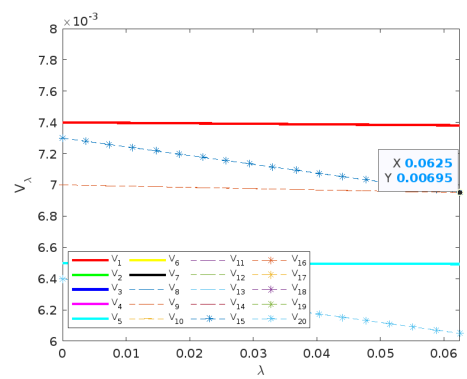


FIGURE 6. Value-$\lambda \in (0, 0.0625)$

The graphical representation of the values is shown in the figure 2 . The intersection of lines denotes that the values of GPSVNN are same at the value λ. First we compare the values for $\lambda \in [0, 0.0625]$, we split the graph by comparing values of the alternatives. Finally, we calculate the ambuiguity at the point of intersection for $\lambda = 0.0625$. The ranking of values are given in the graphs 3 ,4, 5 and 6 . At $\lambda = 0.0625$ we calculate the ambiguity $A_9 = 0.001494$ and $A_{15} = 0.002210$ in which V_9 and V_{15} are intersecting. Hence the ranking of the alternatives for $\lambda \in [0, 0.0625]$ are $S_{18} > S_8 > S_{12} > S_6 > S_{13} > S_{14} > S_{19} > S_{11} > S_3 > S_{16} > S_{10} > S_2 > S_4 > S_1 > S_{15} > S_9 > S_5 > S_{20} > S_{17} > S_7$. If $\lambda \in [0.6220, 0.6300]$, at $\lambda = 0.6220$ V_{12} and V_{19} are intersecting. Hence we calculate the ambiguity at 0.6220 for the alternatives. ie $A_{19} = 0.007602$ and $A_{12} = 0.0050824$. Then the order of the alternatives are $S_{18} > S_2 > S_8 > S_6 > S_{19} > S_{12} > S_{13} > S_{14} > S_{11} > S_3 > S_{16} > S_{10} > S_1 > S_9 > S_5 > S_4 > S_{15} > S_{20} > S_{17} > S_7$. If $\lambda \in [0.9976, 0.9980]$ at $\lambda = 0.9976$ V_{15} and V_{18} and at $\lambda = 0.9980$ V_4 and V_{12} are intersecting. Hence we calculate the ambiguity at λ=0.9976 for the alternatives, ie $A_{15} = 0.00037640568$ and $A_{18} = 0.0003251824$ and at λ=0.9980 for the alternatives, ie $A_4 = 0.000422616$ and $A_{12} = 0.0010216$. Then the ranking is $S_2 > S_{19} > S_6 > S_1 > S_5 > S_9 > S_{13} > S_8 > S_{10} > S_{12} > S_4 > S_3 > S_{16} > S_{15} > S_{18} > S_{11} > S_{17} > S_7 > S_{20} > S_{14}$. At $\lambda \in (0.9980, 1)$ the ranking is $S_2 > S_{19} > S_6 > S_1 > S_5 > S_9 > S_{13} > S_8 > S_{10} > S_4 > S_{12} > S_3 > S_{16} > S_{15} > S_{18} > S_{11} > S_{17} > S_7 > S_{20} > S_{14}$. At $\lambda = 1$ we calculate the ambiguties for the intersecting values and the ranking order is $S_2 > S_{19} > S_6 > S_1 > S_5 > S_9 > S_{13} > S_8 > S_{10} > S_4 > S_{12} > S_{16} > S_3 > S_{15} > S_{18} > S_{11} > S_7 > S_{17} > S_{20} > S_{14}$. The ranking order is related to the weight $\lambda \in [0, 1]$.

5. Conclusion

In this research article, the concept of Generalized Parabolic Single-Valued Neutroposophic Number (GPSVNN) has been developed. We have defined the (α, β, γ)-cut of GPSVNN. Also, the arithmetic operators of these numbers are discussed and illustrated using graphical representation. A demonstration of the De-Neutrosophication method utilising values and ambiguities has been introduced here for the conversion of a GPSVNN into a real number. Further, this result is applied in the ranking of the satisfaction levels of citizens in municipal services. For this purpose, 20 municipal services included in the Life Satisfaction Survey (LSS) that the Turkish Statistical Institution regularly applies every year are considered as alternatives. In addition, the satisfaction of citizens was evaluated for the period of 2014–2017. To analyse the effect of all opinion types on the decision process, the participant responses constituting the dataset of GPSVNN and these years were considered as a set of criteria. We have utilised the values and ambiguities to evaluate the citizens' satisfaction levels with the municipality's services. Finally, the best and worst alternatives were chosen by ranking the alternatives.

In the future, researchers can develop algorithms using GPSVNN in various fields like image processing problems, pattern recognition problems, cloud computing problems, and other mathematical modelling problems involving uncertainty and nonlinearity.

Appendix A

In this section, we have given the MATLAB code for calculating the Normalized values, Comprehensive values, Values and Ambiguities for the alternatives. The matrix A1 to A4 denotes the Truth-membership, A5-A8 represents the Indeterminacy membership and A9-A12 for Falsity membership for the four alternatives, respectively. W1 represents the weight of each criterion of the alternatives. $OMEGA = min(\omega_{ij})$, $RHO = max(\rho_{ij})$ $DELTA = max(\delta_{ij})$.

```
A1=input('Matrix A1');
A2=input('Matrix A2');
A3=input('Matrix A3');
A4=input('Matrix A4');
A5=input('Matrix A5');
A6=input('Matrix A6');
A7=input('Matrix A7');
A8=input('Matrix A8');
A9=input('Matrix A9');
A10=input('Matrix A10');
A11=input('Matrix A11');
A12=input('Matrix A12');
W1=input('Enter W1');
OMEGA=input('Enter OMEGA');
RHO=input('Enter RHO');
DELTA=input('Enter DELTA');
N1=A1/10
N2=A2/10
N3=A3/10
N4=A4/10
N5=A5/10
N6=A6/10
N7=A7/10
N8=A8/10
N9=A9/10
N10=A10/10
N11=A11/10
N12=A12/10
C1=N1*W1
C2=N2*W1
```

C3=N3*W1

C4=N4*W1

C5=N5*W1

C6=N6*W1

C7=N7*W1

C8=N8*W1

C9=N9*W1

C10=N10*W1

C11=N11*W1

C12=N12*W1

D1=C1+C2+C3+C4

D2=C5+C6+C7+C8

D3=C9+C10+C11+C12

VALUES1=OMEGA^2/10*(D1(1)+4*D1(2)+4*D1(3)+D1(4))

VALUES2=(1−RHO)^2/10*(D2(1)+4*D2(2)+4*D2(3)+D2(4))

VALUES3=(1−DELTA)^2/10*(D3(1)+4*D3(2)+4*D3(3)+D3(4))

AM1=OMEGA^2/10*(−D1(1)−4*D1(2)+4*D1(3)+D1(4))

AM2=(1−RHO)^2/10*(−D2(1)−4*D2(2)+4*D2(3)+D2(4))

AM3=(1−DELTA)^2/10*(−D3(1)−4*D3(2)+4*D3(3)+D3(4))

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