Neutrosophic Sets and Systems, Vol. 61, 2023
University of New Mexico

Neutrosophic Algebraic Mathematical Morphology

A. A. Salama[1], Hewayda ElGhawalby[2], Huda E. Khalid[3], Ahmed K. Essa[4], Ahmed A. Mohammed[5]

[1] Department of Mathematics and Computer Science, Faculty of Sciences, Port Said University, Egypt
drsalama44@gmail.com
[2] Physics and Engineering Mathematics Department, Faculty of Engineering, Port Said University, Egypt
hewayda2011@eng.psu.edu.eg
[3] Telafer University, The Administration Assistant for the President of the Telafer University, Iraq
dr.huda-ismael@uotelafer.edu.iq
[4] Telafer University, Statistics Division, Telafer, Mosul, Iraq
ahmed.k.essa@uotelafer.edu.iq
[5] Telafer University, Manager of Construction and Projects Department, Telafer, Iraq; E-mail
ahmed.icap3@uotelafer.edu.iq

ABSTRACT

In this paper, we introduce and study the NeutroAlgebra structure and many of operations and properties of the mathematical morphology. This is a generalization of the operations of fuzzy and classical mathematical morphology. An explanation of the new given operations is provided through several examples and experimental results. Since mathematical morphology deals with forms and is used in image processing, we consider in this research the Indeterminate Image (i.e. image with missing, unclear, or overlapping pixels), whose basic morphological operator's dilation, erosion, opening and closing transform an indeterminate image into another indeterminate image. Therefore, in fact, we deal with neutro-dilation, neutro-erosion, neutro-opening and neutro-closing. For a determinate image (i.e. image with no indeterminacy), the classical morphological operators transform it also into a determinate image, while the neutro-morphological operators into an indeterminate image. All work from below is available for both the indeterminate and determinate image.

Keywords: Neutrosophic Fuzzy Set, Neutrosophic Crisp Sets, Mathematical Morphology, Neutrosophic Fuzzy Mathematical Morphology, Neutrosophic Crisp Mathematical Morphology, Neutro-Morphological Operators.

1. Introduction

In classical algebraic structures for mathematical morphology, all axioms are 100%, and all operations are 100% well defined, but in real life, in many cases these restrictions are too harsh, since in our world we have things that only partially verify some laws or some operations. Neutrosophy introduces a new concept, which represent indeterminacy with respect to some event, which can solve certain problems that cannot be solved by fuzzy logic and crisp logic. .In 1995, Smarandache initiated the theory of NFS as a new mathematical tool for handling problems involving imprecise indeterminacy, and inconsistent data. Several researchers dealing with the concept of NFS such as Bhowmik and Pal in [14] and Salama et al. introduced many applications in [6-13]. In [6] Salama introduced the concept of neutrosophic crisp sets, to represent any event by a triple crisp structure. A crisp structure is a structure whose all elements are characterized by the same given Relationships and Attributes. A NeutroStructure is a structure that has at least one NeutroRelation or one NeutroAttribute, and neither AntiRelation nor AntiAttribute. In 2019 and 2020, Smarandache [1, 2, 3, 4] generalized the classical Algebraic Structures to NeutroAlgebraic Structures. Neutrosophic mathematical morphology is most commonly applied to digital images, but it can be employed as well on graphs, surface meshes, solids, and many other spatial structures. Established in 1964, mathematical morphology was firstly introduced by Georges Matheron and Jean Serra, as a branch of image processing [29]. As morphology is the study of shape, mathematical morphology mostly deals with the mathematical theory of describing shapes using set theory. In image processing, the basic morphological operator's dilation, erosion, opening and closing form the fundamentals of this theory [29]. A morphological operator transforms an image into another image, using some structuring element, which can be chosen by the user. Mathematical morphology stands somewhat apart from traditional linear image processing, since the basic operations of morphology are nonlinear in nature, and thus make use of a totally different type of algebra than the linear algebra. At first, the theory was purely based on set theory and operators, which defined for binary cases only. Later on the theory was extended to grayscale images also, where the theory of lattices was introduced by Petros Maragos, who also gave a representation theory for image processing as a scientific branch, Mathematical Morphology expanded worldwide during the 1990s. It is also during that period, different models based on fuzzy set theory were introduced [16, 1717]. Today, mathematical morphology remains a challenging research field, e.g. [14 - 44].

2. Terminologies

We recall some relevant basic preliminaries, and in particular, the work of Smarandache in [1-5], Salama et al. [6-13], and some references in [14-53].

2.1 Abbreviations

1. Crisp Mathematical Morphology (CMM)
2. Fuzzy Mathematical Morphology (FMM).
3. Neutrosophic Fuzzy Set (NFS)
4. Neutrosophic Crisp Set (NCS)
5. Neutrosophic Fuzzy Morphological (NFM)
6. Neutrosophic Fuzzy Dilation (NFD)
7. Neutrosophic Fuzzy Erosion (NFE)
8. Neutrosophic Fuzzy Opening (NFO)
9. Neutrosophic Fuzzy Closing (NFC)
10. Neutrosophic Fuzzy Filters (NFF).
11. Neutrosophic Fuzzy Gradient Boundary (NFGB)
12. Neutrosophic Fuzzy External Boundary (NFEB)
13. Neutrosophic Fuzzy Internal Boundary (NFIB)
14. Neutrosophic Fuzzy Outline Boundary (NFOB)
15. Neutrosophic Fuzzy Mathematical Relation (NFMR)

16. Neutrosophic Crisp Mathematical Morphology (NCMM)
17. Neutrosophic Crisp Dilation (NCD).
18. Neutrosophic Crisp Erosion (NCE).
19. Neutrosophic Crisp Opening (NCO)
20. Neutrosophic Crisp Closing (NCC).
21. Neutrosophic Crisp External Boundary (NCEB).

2.2 Neutrosophic Intensity Image:

To transform the Image from its Spatial (Cartesian) Domain into Neutrosophic Domain, we should investigate the necessary mathematical tools as follow:

The image as a mathematical object (Spatial Domain) is an image mathematically represented by an $m\times\ n$ matrix $I=\left[g_{ij}\right]_{m\times n}$, with entities $g\ \ (i,j)$ corresponding to the intensity to the given pixel located at the node (i,j) The image in the Neutrosophic Domain (ND) where each pixel of the image is represented by P_{ij} having three components $P_{ij}=(T_{ij},I_{ij},F_{ij})$; Where,

T (i, j) $=\frac{\bar{g}(i,j)-\bar{g}_{min}}{\bar{g}_{max}-\bar{g}_{min}}$, I (i, j) = $\frac{\delta(i,j)-\delta_{min}}{\delta_{max}-\delta_{min}}$, F (i, j) =$1-\mathrm{T}\ \ (\mathrm{i,j})=\ \frac{\bar{g}_{\max}-\bar{g}(\mathrm{i,j})}{\bar{\bar{g}}_{\max}-\bar{g}_{\min}}$

$\bar{g}(i,j)$ is the mean intensity in some neighborhood w of the pixel given by:

$\bar{g}(i,j)=\frac{1}{w^2}\sum_{k=i-\frac{w}{2}}^{m=i+\frac{w}{2}}\sum_{l=j-\frac{w}{2}}^{n=j+\frac{w}{2}}g(k,l)$. Also, $\bar{g}_{max}=\ \ max\,\bar{g}(i,j),\ \bar{g}_{min}=min\,\bar{g}(i,j)$,

$\delta(i,j)=abs(g(i,j)-\bar{g}(i,j)),\ \delta_{max}=\max\delta(i,j),\ \delta_{min}=\min\delta(i,j)$. Hence, the image in the neutrosophic domain becomes a 3D matrix $I_{ND}=[T_{ij}\quad I_{ij}\quad F_{ij}]$, of $(m\times n\times 3)$ dimension.

2.3 Implementation and Experimental Results:

In this section, the following suggested algorithm has been used to transform the cartesian image domain into the neutrosophic image domain.

Step 1: Read the grayscale image.

Step 2: Compute the local mean intensity for each pixel in the image.

Step 3: Compute the maximum and minimum values of the local mean intensities.

Step 4: Compute the divergence between the intensity of each pixel and its local mean intensity.

Step 5: Compute the maximum and the minimum values of the divergence induced in the previous step.

Step 6: Construct the truth, indeterminacy, and falseness matrices T, I, F for each pixel.

Most neutrosophic morphological operations can be obtained by combining theoretical operations of the neutrosophic set with two traditional and basic image operations, dilation and erosion, the following section has been dedicated to this issue.

3. NFM Operations:

In this section, we introduce and study the neutrosophic algebraic structures and many operations and properties of mathematical neutrosophic morphology. This is a generalization of the classical mathematical morphological operations. An explanation of the new given operations is provided through several examples with giving experimental results. "Lena" image has been used to investigate the effect of each of the given operators on the image. Basic definitions for neutrosophic morphological operations are extracted and a study of its algebraic properties is presented. In our work, we demonstrate that neutrosophic morphological operations inherit properties and restrictions of fuzzy mathematical morphology. The operations of NFD, NFE, NFO, and NFC of the neutrosophic image by neutrosophic structuring element, are defined in terms of their membership, indeterminacy, and non-membership functions; which are defined for the first time as far as we know.

3.1. NFD and NFE:

The two basic operations for the construction of neutrosophic fuzzy morphological operators, namely,NFD and NFE. are based on the two Minkowski set operations, the Minkowski addition and subtraction of two NFS;respectively. We may define them as follows:

The process of structuring element B on image A and moving it across the image in a way like convolution is defined as a dilation operation. The two main inputs for the dilation operator [21] are the image, which is to be dilated, and a set of coordinate points known as a structuring element, which is can be defined also as a kernel. The exact effect of the dilation on the input image is determined by this structuring element [20]. Its dilation is defined as a set operation. A is dilated by B, written as A⊕B.

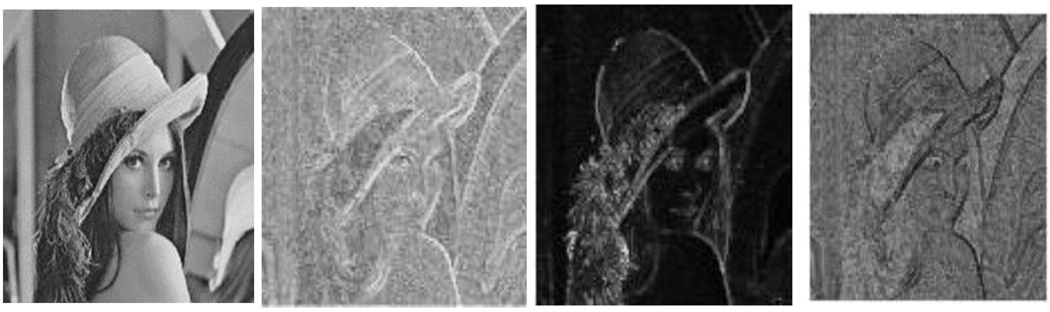
3.1.1 Definition

NFD of Type I:

Let A and B be two NFSs; then the NFD of type I is given as

$\left(A \widetilde{\oplus} B\right) = \langle T_{A\widetilde{\oplus}B}, I_{A\widetilde{\oplus}B}, F_{A\widetilde{\oplus}B}\rangle$; where for each $u, v \in Z^2$

$T_{A\widetilde{\oplus}B}(v) = \sup\limits_{u\in Z^2} min(T_A(v+u), T_B(u)),\ I_{A\widetilde{\oplus}B}(v) = \sup\limits_{u\in Z^2} min(I_A(v+u), I_B(u)), F_{A\widetilde{\oplus}B}(v) = \inf\limits_{u\in Z^2} max(1 - F_A(v+u), 1 - F_B(u))$.

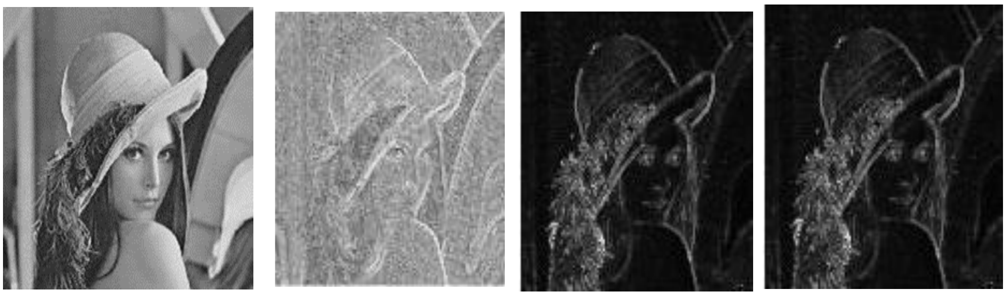


(a) **(b)** $T_{A\widetilde{\oplus}B}$ **(b)** $I_{A\widetilde{\oplus}B}$ (b) $F_{A\widetilde{\oplus}B}$

Fig 3.1.1 (I): Applying the NFD operator: (a) Original image , (b) Neutrosophic Fuzzy components of the dilated image in type I $\langle T_{A\widetilde{\oplus}B},\ I_{A\widetilde{\oplus}B}\ ,\ F_{A\widetilde{\oplus}B}\rangle$ respectively.

NFD of Type II:

$T_{A\widetilde{\oplus}B}(v) = \sup\limits_{u\in Z^2} min(T_A(v+u), T_B(u)),\ I_{A\widetilde{\oplus}B} = \inf\limits_{u\in Z^2} max(I_A(v+u), 1 - I_B(u)),\ F_{A\widetilde{\oplus}B} = \inf\limits_{u\in Z^2} max(F_A(v+u), 1 - F_B(u))$



(a) (b) $T_{A\widetilde{\oplus}B}$ (b) $I_{A\widetilde{\oplus}B}$ (b) $F_{A\widetilde{\oplus}B}$

Fig.3.1.1 (II): Applying the neutrosophic dilation operator: a) Original image b) Neutrosophic Fuzzy components of the dilated image in type II $\langle T_{A\widetilde{\oplus}B}, I_{A\widetilde{\oplus}B}, F_{A\widetilde{\oplus}B}\rangle$ respectively.

3.2 NFE Operation:

The erosion process is as same as dilation, but the pixels are converted to 'white', not 'black'. The two main inputs for the erosion operator are the image that is to be eroded and a set of coordinate points known as a structuring element, which is defined also as a kernel. The exact effect of the erosion on the input image is determined by this structuring element. The followings are the mathematical definitions of erosion type I and erosion type II for grey-scale images.

3.2.1 **Definition (**NFE of Type I, II):

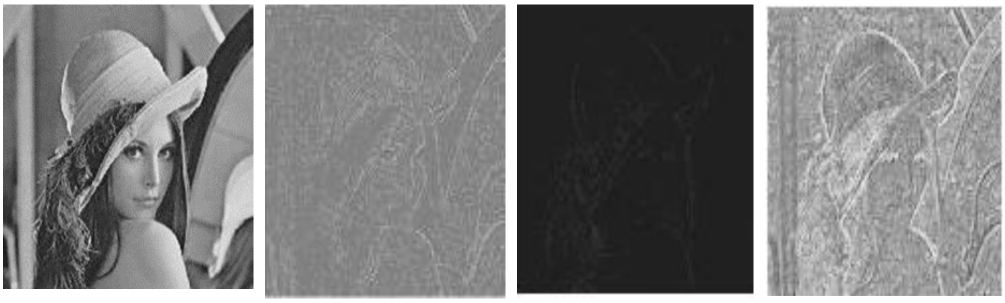
Let A and B be two neutrosophic sets, The neutrosophic fuzzy erosion of a neutrosophic set B from a neutrosophic set A is defined as $(A \widetilde{\ominus} B) = \langle T_{A\widetilde{\ominus}B}, I_{A\widetilde{\ominus}B}, F_{A\widetilde{\ominus}B}\rangle$; where for each $u, v \in Z^2$. The three components, $T_{A\widetilde{\ominus}B}, I_{A\widetilde{\ominus}B}, F_{A\widetilde{\ominus}B}$ are to be defined in different types as follows:

NFE of Type I:

Let A and B be two NFS, then the NFE is given by

$(\mathrm{A} \widetilde{\ominus} \mathrm{B}) = \langle \mathrm{T}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}, \mathrm{I}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}, \mathrm{F}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}\rangle$; where for each $u, v \in Z^2$, $\mathrm{T}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}(\mathrm{v}) = \inf_{\mathrm{u}\in \mathrm{Z}^2} \max(\mathrm{T}_{\mathrm{A}}(\mathrm{v}+\mathrm{u}), 1-\mathrm{T}_{\mathrm{B}}(\mathrm{u}))$,

$\mathrm{I}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}(\mathrm{v}) = \inf_{\mathrm{u}\in \mathrm{Z}^2} \max(\mathrm{I}_{\mathrm{A}}(\mathrm{v}+\mathrm{u}), 1-\mathrm{I}_{\mathrm{B}}(\mathrm{u}))$, $\mathrm{F}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}(\mathrm{v}) = \sup_{\mathrm{u}\in \mathrm{Z}^2} \min(1-\mathrm{F}_{\mathrm{A}}(\mathrm{v}+\mathrm{u}), \mathrm{F}_{\mathrm{B}}(\mathrm{u}))$.

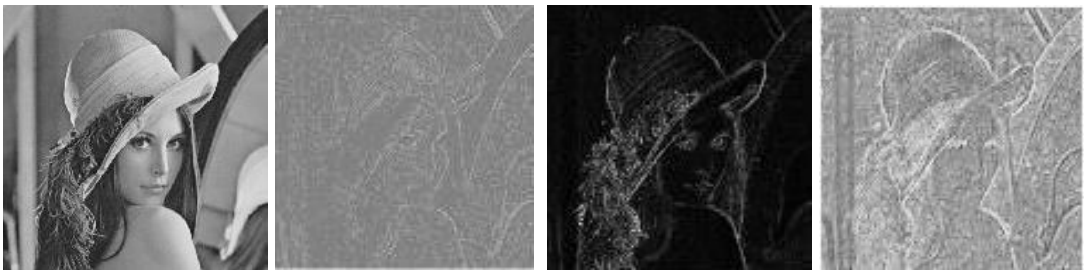


(a) (b) $T_{A\tilde{\ominus}B}$ (b) $I_{A\tilde{\ominus}B}$ (b) $F_{A\tilde{\ominus}B}$

Fig. 3.2.1 (I): Applying the NFE operator: (a) original image (b) neutrosophic components of the eroded in type I $\langle T_{A\ominus B}, I_{A\ominus B}, F_{A\ominus B}\rangle$ respectively.

NFE of Type II:

$$T_{A\tilde{\ominus}B}(v) = \inf_{u\in Z^2} max(T_A(v+u), 1-T_B(u)) \quad , \quad I_{A\tilde{\ominus}B}(v) = \sup_{u\in Z^2} min(I_A(v+u), I_B(u)) \quad , \quad F_{A\tilde{\ominus}B}(v) = \sup_{u\in Z^2} min\big(F_A(v+u), F_B(u)\big).$$



(a) (b) $T_{A\tilde{\ominus}B}$ (b) $I_{A\tilde{\ominus}B}$ (b) $F_{A\tilde{\ominus}B}$

Fig. 3.2.1 (II): Applying the neutrosophic erosion operator: (a) original image (b) neutrosophic components of the eroded in type II $\langle T_{A\tilde{\ominus}B}(v),\ I_{A\tilde{\ominus}B}(v),\ F_{A\tilde{\ominus}B}\rangle$ respectively.

3.3 NFO and NFC Operations:

The combination of the two main neutrosophic fuzzy operations, dilation and erosion, can produce more complex sequences. Opening and closing are the most useful of these for morphological filtering. An opening operation is defined as an erosion followed by dilation using the same structuring element for both operations. The basic two inputs for the opening operator are an image to be opened and a structuring element. The grey-level opening consists simply of grey-level erosion followed by grey-level dilation. The morphological opening $\circ$ and closing $\bullet$ are defined by:

$A \circ B = (A \,\widetilde{\ominus}\, B) \,\widetilde{\oplus}\, B$, $\quad A \bullet B = (A \,\widetilde{\oplus}\, B) \,\widetilde{\ominus}\, B$.

From a granularity perspective, opening and closing provide coarser descriptions of the neutrosophic fuzzy set A. The opening describes A as closely as possible without using the individual pixels but by fitting (possibly overlapping) copies of E within A. The closing describes the complement of A by fitting copies of E^* outside A. The actual set is always contained within these two extremes: $A \circ B \subseteq A \subseteq A \bullet B$ and the informal notion of fitting copies of E, or of E^*, within a set is made precisely in these equations:

The operator $P(E) \to P(E): A \to A \circ B$ is called the opening by B; it is the composition of the erosion $\ominus$, followed by the dilation $\oplus$. On the other hand, the operator $P(E) \to P(E): A \to A \bullet B$ is called the closing. To understand what a closing operation does: imagine the closing applied to a set; the dilation will expand object boundaries, which will be partly undone by the following erosion. Small, (i.e., smaller than the structuring element) holes and thin tube-like structures in the interior or at the boundaries of objects will be filled up by the dilation, and not reconstructed by the erosion, in as much as these structures no longer have a boundary for the erosion to act upon. In this sense, the term 'closing' is a well-chosen one, as the operation removes holes and thin cavities. In the same sense, the opening opens up holes that are near (with respect to the size of the structuring element) a boundary and removes small object protuberances.

Definition 3.3.1 (NFO of Type I, II):

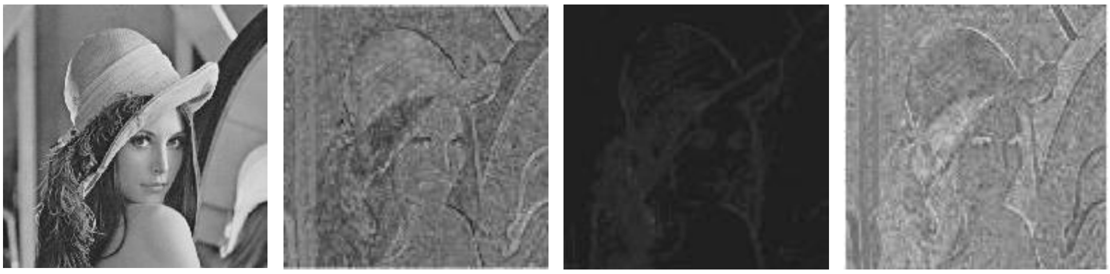
Two types of neutrosophic fuzzy opening operations NFO may defined as $NOF: (A \,\tilde{\circ}\, B) = \langle T_{A\tilde{\circ}B}, I_{A\tilde{\circ}B}, F_{A\tilde{\circ}B} \rangle$ where $u, v, w \in Z^2$.

NFO of Type I

$$T_{A\tilde{\circ}B}(v) = \sup_{w,v,u\in Z^2} \min\left[\inf_{z\in R^n} \max\big(A(v-u+w), 1-B(w)\big), B(u)\right],$$

$$I_{A\tilde{\circ}B}(v) = \sup_{w,v,u\in Z^2} \min\left[\inf_{z\in R^n} \max\big(A(v-u+w), 1-B(w)\big), B(u)\right],$$

$$F_{A\tilde{\circ}B}(v) = \inf_{w,v,u\in Z^2} \max\left[\sup_{z\in R^n} \min\big(1-A(v-u+w), B(w)\big), 1-B(u)\right].$$



(a) (b) $T_{A\tilde{\circ}B}$ (b) $I_{A\tilde{\circ}B}$ (b) $F_{A\tilde{\circ}B}$

Fig.3.3.1 (I): Applying the neutrosophic fuzzy opening operator: (a) Original image.

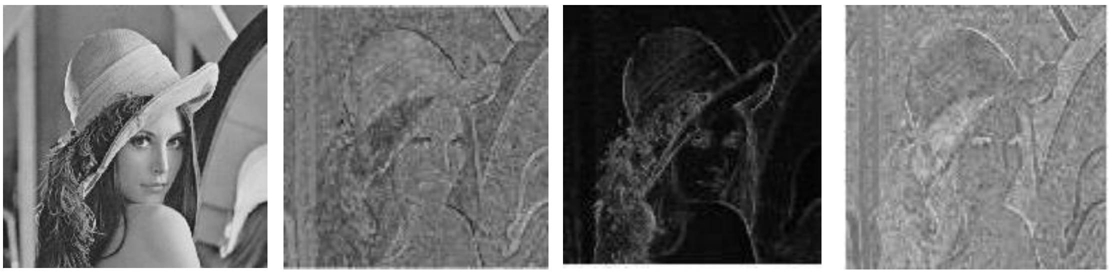
(b) Neutrosophic fuzzy opening components in type I $\langle T_{A\tilde{\circ}B}, I_{A\tilde{\circ}B}, F_{A\tilde{\circ}B}\rangle$ respectively.

NFO of Type II:

$$T_{A\tilde{\circ}B}(v) = \sup_{w,v,u\in Z^2} \min\ \left[\left(\inf \max\big(T_A(v-u+w), 1-T_B(w)\big), T_B(u)\right]\right.,$$

$$I_{A\tilde{\circ}B}(v) = \sup_{w,v,u\in Z^2} \min\ \left[\ \inf \max\big(I_A(v-u+w), I_B(w)\big), 1-I_B(u)\right],$$

$$F_{A\tilde{\circ}B}(v) = \inf_{w,v,u\in Z^2} \max\ \left[\ \sup \min\big(F_A(v-u+w), F_B(w)\big), 1-F_B(u)\right].$$



(a) (b) $T_{A\tilde{\circ}B}(v)$ (b) $I_{A\tilde{\circ}B}(v)$ (b) $F_{A\tilde{\circ}B}(v)$

Fig.3.3.1 (II): Applying the neutrosophic opening operator: (a) Original image (b) Neutrosophic opening components in type II $\langle T_{A\tilde{\circ}B}(v), I_{A\tilde{\circ}B}(v), F_{A\tilde{\circ}B}(v)\rangle$ respectively.

Definition 3.3.2 (NFC of Type I, II)

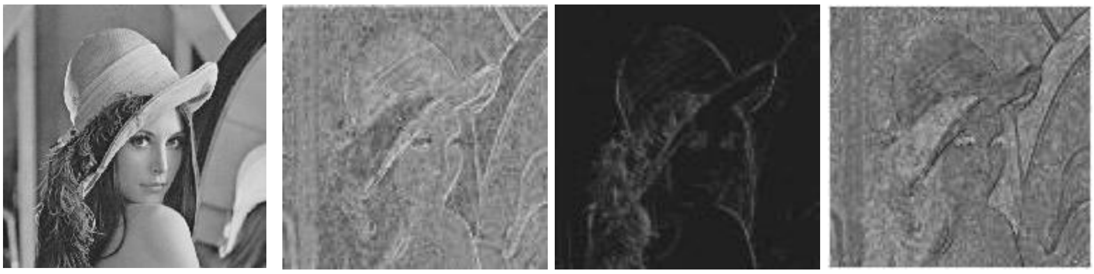
Let A and B be two types as the following may define two neutrosophic sets: $(A \tilde{\bullet} B) = \langle T_{A\tilde{\bullet}B}, I_{A\tilde{\bullet}B}, F_{A\tilde{\bullet}B} \rangle$ where

Type I

$$T_{A\tilde{\bullet}B}(v) = \inf_{w,v,u \in Z^2} max\big[sup\, min\big(T_A(v-u+w), T_B(w)\big), 1 - T_B(u)\big],$$

$$I_{A\tilde{\bullet}B}(v) = \inf_{w,v,u \in Z^2} max\big[sup\, min\big(I_A(v-u+w), I_B(w)\big), 1 - I_B(u)\big],$$

$$F_{A\tilde{\bullet}B}(v) = \sup_{w,v,u \in Z^2} min\big[inf\, max\big(1 - F_A(v-u+w), 1 - F_B(w)\big), F_B(u)\big]$$



(a) (b) $T_{A\tilde{\bullet}B}$ (b) $I_{A\tilde{\bullet}B}$ (b) $F_{A\tilde{\bullet}B}$

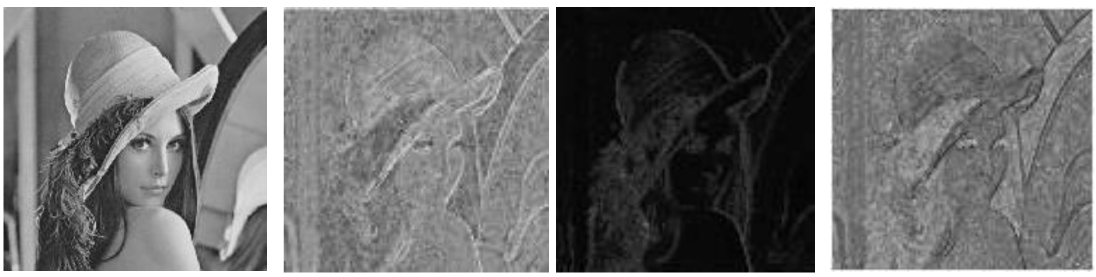
Fig. 3.3.2 (I): Applying the neutrosophic closing operator: (a) Original image (b) Neutrosophic closing components in type I $\langle T_{A\tilde{\bullet}B}, I_{A\tilde{\bullet}B}, F_{A\tilde{\bullet}B} \rangle$ respectively.

Neutrosophic Closing Type II:

$$T_{A\tilde{\bullet}B}(v) = \inf_{w,v,u \in Z^2} max\ [\sup min\ \big(T_A(v-u+w), T_B(w)\big), 1 - T_B(u)]\ ,$$

$$I_{A\tilde{\bullet}B}(v) = \sup_{w,v,u \in Z^2} min\ [\inf\ max\ \big(I_A(v-u+w), 1 - I_B(w)\big), I_B(u)],$$

$$F_{A\tilde{\bullet}B}(v) = \sup_{w,v,u \in Z^2} min\ [\inf max\ \big(F_A(v-u+w), 1 - F_B(w)\big), F_B(u)].$$



(a) (b) $T_{A\tilde{\bullet}B}(v)$ (b) $I_{A\tilde{\bullet}B}(v)$ (b) $F_{A\tilde{\bullet}B}(v)$

Fig.3.3.2 (II): Applying the neutrosophic closing operator: a) Original image b) Neutrosophic closing components in type II $\langle T_{A\tilde{\bullet}B}(v),\ I_{A\tilde{\bullet}B}(v),\ F_{A\tilde{\bullet}B}(v)\rangle$ respectively.

3.4 **Algebraic Properties of Neutrosophic Fuzzy Morphological Operations:**

This part has been dedicated to investigate some of the algebraic properties of the neutrosophic fuzzy morphological operations; i.e. NFD, NFE, NFO and neutrosophic fuzzy closing. The algebraic properties for neutrosophic fuzzy mathematical morphology erosion and dilation, as well as for NFO and closing operations are now considered.

3.4.1 Duality Theorem of NFD:

Let A and B be two NFSs. Then the NFE and the NFD both are dual operations, i.e. $(A^c \tilde{\oplus} B)^c = \langle T_{(A^c\tilde{\oplus}B)^c}\ , I_{(A^c\tilde{\oplus}B)^c}\ , F_{(A^cA\tilde{\oplus}B)^c}\rangle$; where for each $u\ \&\ v\ \in Z^2$ we have:

$$T_{(A^c\tilde{\oplus}B)^c}(v) = 1 - T_{(A^c\tilde{\oplus}B)}(v) = 1 - \sup_{v,u\in Z^2} \min(T_{A^c}(v+u), T_B(u)) = \inf_{v,u\in Z^2}[1-\min(T_{A^c}(v+u), T_B(u))] = \inf_{v,u\in Z^2}[\max(1-T_{A^c}(v+u), 1- T_B(u))] = \inf_{v,u\in Z^2}[\max(T_A(v+u), 1- T_B(u))] = T_{A\ominus B}(v)$$

$$I_{(A^c\tilde{\oplus}B)^c}(v) = 1 - I_{(A^c\tilde{\oplus}B)}(v) = 1 - \sup_{v,u\in Z^2} \min(I_{A^c}(v+u), I_B(u)) = \inf_{v,u\in Z^2}[1-\min(I_{A^c}(v+u), I_B(u))] = \inf_{v,u\in Z^2}[\max(1-I_{A^c}(v+u), 1- I_B(u))] = \inf_{v,u\in Z^2}[\max(I_A(v+u), 1- I_B(u))] = I_{A\ominus B}(v)$$

$$F_{(A^c\tilde{\oplus}B)^c}(v) = 1 - F_{(A^c\tilde{\oplus}B)}(v)$$
$$= 1 - \inf_{v,u\in Z^2}\max(1-F_{A^c}(v+u), 1-F_B(u)) = \sup_{v,u\in Z^2}[1 - \max(1-F_{A^c}(v+u), 1-F_B(u))] = \sup_{v,u\in Z^2}[\min(1-F_A(v+u), F_B(u))] = F_{A\ominus B}(v)$$

Suppose the set A is the image under processing and the set B is the structuring element, the NFO and the NFC are defined respectively, as define the neutrosophic fuzzy binary operation $\circ$ and $\bullet$ by setting for any A and B $\in\mathcal{N}(E)$.

3.4.2. Duality Theorem of NFC:

Let A and B are two NFSs, then the NFO and the NFC are also dual operation i.e:$T(A^c \tilde{\bullet} B)^c = \langle T_{(A^c \tilde{\bullet} B)^c}, I_{(A^c \tilde{\bullet} B)^c}, F_{(A^c \tilde{\bullet} B)^c} \rangle$, where for all v, u$\in Z^2$

$$T_{(A^c \tilde{\bullet} B)^c}(v) = 1 - T_{(A^c \tilde{\bullet} B)}(v) = 1 - \inf_{v,u,w \in Z^2} \max\left[\sup_{v,u,w \in Z^2} \min\left(T_{A^c}(v-u+w), T_{B(w)}\right), 1 - T_B(u)\right]$$

$$= \sup_{v,u,w \in Z^2} \min\left[1 - \sup_{v,u,w \in Z^2} \min\left(T_{A^c}(v-u+w), T_{B(w)}\right), 1 - \left(1 - T_B(u)\right)\right] = \sup_{v,u,w \in Z^2} \min\left[\inf_{v,u,w \in Z^2} \max\left(1 - T_{A^c}(v-u+w), 1 - T_{B(w)}\right), T_B(u)\right]$$

$$= \sup_{v,u,w \in Z^2} \min\left[\inf_{v,u,w \in Z^2} \max\left(T_{A^c}(v-u+w), 1 - T_{B(w)}\right), T_B(u)\right] = T_{A \circ B}$$

$$I_{(A^c \tilde{\bullet} B)^c}(v) = 1 - I_{(A^c \tilde{\bullet} B)}(v) = 1 - \inf_{v,u,w \in Z^2} \max\left[\sup_{v,u,w \in Z^2} \min\left(I_{A^c}(v-u+w), I_{B(w)}\right), 1 - I_B(u)\right]$$

$$= \sup_{v,u,w \in Z^2} \min\left[1 - \sup_{v,u,w \in Z^2} \min\left(I_{A^c}(v-u+w), I_{B(w)}\right), 1 - \left(1 - I_B(u)\right)\right] = \sup_{v,u,w \in Z^2} \min\left[\inf_{v,u,w \in Z^2} \max\left(1 - I_{A^c}(v-u+w), 1 - I_{B(w)}\right), I_B(u)\right]$$

$$= \sup_{v,u,w \in Z^2} \min\left[\inf_{v,u,w \in Z^2} \max\left(I_{A^c}(v-u+w), 1 - I_{B(w)}\right), I_B(u)\right] = I_{A \circ B}$$

$$F_{(A^c \tilde{\bullet} B)^c}(v) = 1 - F_{(A^c \tilde{\bullet} B)}(v) = 1 - \sup_{v,u,w \in Z^2} \min\left[\inf_{v,u,w \in Z^2} \max\left(1 - F_A(v-u+w), 1 - F_{B(w)}\right), F_B(u)\right]$$

$$= \inf_{v,u,w \in Z^2} \max\left[1 - \inf_{v,u,w \in Z^2} \max\left(1 - F_A(v-u+w), 1 - F_{B(w)}\right), F_B(u)\right] = \inf_{v,u,w \in Z^2} \max\left[\sup_{v,u,w \in Z^2} \min\left(1 - F_A(v-u+w), F_{B(w)}\right), 1 - F_B(u)\right] = F_{A \circ B}$$

3.5. Neutrosophic Fuzzy Mathematical Relation

3.5.1. Definition: Let A is a NFS and R be a neutrosophic relation on nonempty crisp set X then $A \oplus R$ my be defined by two types:

$$\mathbf{A} \tilde{\oplus} \mathbf{R} = \langle \mathbf{T}_{\mathbf{A} \tilde{\oplus} \mathbf{R}}, \mathbf{I}_{\mathbf{A} \tilde{\oplus} \mathbf{R}}, \mathbf{F}_{\mathbf{A} \tilde{\oplus} \mathbf{R}} \rangle$$

$$T_{A \tilde{\oplus} R}(v) = \sup_{v,u \in Z^2} \min(T_A(v+u), F_R(u)), \; I_{A \oplus R}(v) = \sup_{v,u \in Z^2} \min(I_A(v+u), F_R(u)) ,$$

$$F_{A \oplus R}(y) = \inf_{u \in Z^2} \max(1 - F_A(v+u), 1 - F_R(u)).$$

Given two relations on X, say R and S, the places you can get to following an arrow in R and then following an arrow in S are exactly the places you can get to by following an arrow in R; S. Formally, we have the right monoid action:

Lemma 3.5.1: The operation just defined, $\oplus$, provides a right action for the monoid of relations on the non-empty crisp set X on the power set. Specifically, for all $A \in PX$ and for all $R\&S \in RX$,

$$\mathbf{A} \widetilde{\oplus} \mathbf{1} = \langle \mathbf{T}_{\mathbf{A}\oplus\mathbf{1}}, \mathbf{I}_{\mathbf{A}\oplus\mathbf{1}}, \mathbf{F}_{\mathbf{A}\oplus\mathbf{1}} \rangle$$

$$T_{A\oplus 1}(v) = \sup_{v,u\in Z^2} \min(T_A(v+u), 1) = \sup_{v,u\in Z^2} \left(T_A(u+v)\right) = T_A$$

$$I_{A\oplus 1}(v) = \sup_{v,u\in Z^2} \min(I_A(v+u), 1) = \sup_{v,u\in Z^2} \left(I_A(u+v)\right) = I_A$$

$$F_{A\oplus 1}(v) = \inf_{v,u\in Z^2} \max(1 - F_A(v+u), 1-1) = \inf_{v,u\in Z^2} \left(1 - F_A(v+u)\right) = F_{A^c}$$

Lemma 3.5. 2 : Let A_i are indexed set neutrosophic fuzzy subsets on the non-empty crisp set X, then $\bigvee_i A_i \widetilde{\oplus} R = \left\langle T_{\bigvee_i A_i \oplus R}, I_{\bigvee_i A_i \oplus R}, F_{\left(\bigvee_i A_i \oplus R\right)^c} \right\rangle$, where $T_{\bigvee_i A_i \oplus R} = T_{\bigvee_i (A_i \oplus R)}$, $I_{\bigvee_i A_i \oplus R} = I_{\bigvee_i (A_i \oplus R)}$, $F_{\left(\bigvee_i A_i \oplus R\right)^c} = F_{\bigwedge_i (A_i \oplus R)^c}$.

3.6. Basic Properties of the Neutrosophic Fuzzy Morphological Operations:

3.6.1 Properties of the NFD Operation:

3.6.1.1 Proposition

The neutrosophic Minkowski-addition satisfies the following properties

i. **Commutativity:** $(\forall A, B \in \mathcal{N}(Z^2))\left(\langle T_{A\widetilde{\oplus}B}, I_{A\widetilde{\oplus}B}, F_{A\widetilde{\oplus}B} \rangle = \langle T_{B\widetilde{\oplus}A}, I_{B\widetilde{\oplus}A}, F_{B\widetilde{\oplus}A} \rangle\right)$;

ii. **Associativity:** $(\forall A, B, C \in \mathcal{N}(Z^2))$

$$\left(\langle T_{(A\widetilde{\oplus}B)\widetilde{\oplus}C}, I_{(A\widetilde{\oplus}B)\widetilde{\oplus}C}, F_{(A\widetilde{\oplus}B)\widetilde{\oplus}C} \rangle = \langle T_{A\widetilde{\oplus}(B\widetilde{\oplus}C)}, I_{A\widetilde{\oplus}(B\widetilde{\oplus}C)}, F_{A\widetilde{\oplus}(B\widetilde{\oplus}C)} \rangle\right)$$

Proof Straight forward

Notice that the property $\langle T_{-(A\widetilde{\oplus}B)}, I_{-(A\widetilde{\oplus}B)}, F_{-(A\widetilde{\oplus}B)} \rangle = \langle T_{(-A)\widetilde{\oplus}(-B)}, I_{(-A)\widetilde{\oplus}(-B)}, F_{(-A)\widetilde{\oplus}(-B)} \rangle$

3.6.1.2. Proposition

The neutrosophic dilation satisfies the following properties

i. **Neutrosophic Monotonicity** (increasing in both arguments):

$(\forall A, B, C \in \mathcal{N}(Z^2))(A \subseteq B \Longrightarrow \langle T_{A\widetilde{\oplus}C}, I_{A\widetilde{\oplus}C}, F_{A\widetilde{\oplus}C}\rangle \subseteq \langle T_{B\widetilde{\oplus}C}, I_{B\widetilde{\oplus}C}, F_{B\widetilde{\oplus}C}\rangle)$, here we have

$T_{A\widetilde{\oplus}C} \subseteq T_{B\widetilde{\oplus}C}$, $I_{A\widetilde{\oplus}C} \subseteq I_{B\widetilde{\oplus}C}$, $F_{A\widetilde{\oplus}C} \supseteq F_{B\widetilde{\oplus}C}$

$(\forall A, B, C \in \mathcal{N}(Z^2))(A \subseteq B \Longrightarrow \langle T_{C\widetilde{\oplus}A}, I_{C\widetilde{\oplus}A}, F_{C\widetilde{\oplus}A}\rangle \subseteq \langle T_{C\widetilde{\oplus}B}, I_{C\widetilde{\oplus}B}, F_{C\widetilde{\oplus}B}\rangle)$, here we have

$T_{C\widetilde{\oplus}A} \subseteq T_{C\widetilde{\oplus}B}$, $I_{C\widetilde{\oplus}A} \subseteq I_{C\widetilde{\oplus}B}$, $F_{C\widetilde{\oplus}A} \supseteq F_{C\widetilde{\oplus}B}$

ii. **Interaction with Zadeh's intersection;**

For any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$and $B \in \mathcal{N}(Z^2)$, we have:

$\langle T_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}, I_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}, F_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}\rangle \subseteq \langle T_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, I_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, F_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}\rangle$, where

$T_{\bigcap_{i\in I} A_i\widetilde{\oplus}B} \subseteq T_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}$, $I_{\bigcap_{i\in I} A_i\widetilde{\oplus}B} \subseteq I_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}$, $F_{\bigcap_{i\in I} A_i\widetilde{\oplus}B} \subseteq F_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}$.

Also we have, $\langle T_{B\widetilde{\oplus}\bigcap_{i\in I} A_i}, I_{B\widetilde{\oplus}\bigcap_{i\in I} A_i} F_{B\widetilde{\oplus}\bigcap_{i\in I} A_i}\rangle \subseteq \langle T_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}, I_{T_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}}, F_{T_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}}\rangle$, where

$T_{B\widetilde{\oplus}\bigcap_{i\in I} A_i} \subseteq T_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}$, $I_{B\widetilde{\oplus}\bigcap_{i\in I} A_i} \subseteq I_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}$, $F_{B\widetilde{\oplus}\bigcap_{i\in I} A_i} \subseteq F_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}$.

iii. **Interaction with Zadeh's union:**

For any family$(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\bigcup_{i\in I} A_i\widetilde{\oplus}B}, I_{\bigcup_{i\in I} A_i\widetilde{\oplus}B}, F_{\bigcup_{i\in I} A_i\widetilde{\oplus}B}\rangle \supseteq \langle T_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}, I_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}, F_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}\rangle$, where

$T_{\bigcup_{i\in I} A_i\widetilde{\oplus}B} \supseteq T_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}$, $I_{\bigcup_{i\in I} A_i\widetilde{\oplus}B} \supseteq I_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}$, $F_{\bigcup_{i\in I} A_i\widetilde{\oplus}B} \supseteq F_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}$.

Also we have, $\langle T_{B\widetilde{\oplus}\bigcup_{i\in I} A_i}, I_{B\widetilde{\oplus}\bigcup_{i\in I} A_i}, F_{B\widetilde{\oplus}\bigcup_{i\in I} A_i}\rangle \supseteq \langle T_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}, I_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}, F_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}\rangle$, where

$T_{B\widetilde{\oplus}\bigcup_{i\in I} A_i} \supseteq T_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}$, $I_{B\widetilde{\oplus}\bigcup_{i\in I} A_i} \supseteq I_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}$, $F_{B\widetilde{\oplus}\bigcup_{i\in I} A_i} \supseteq F_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}$.

Proof.

The proof of the first property is straightforward.

i. $\langle T_{A\widetilde{\oplus}C}, I_{A\widetilde{\oplus}C}, F_{A\widetilde{\oplus}C}\rangle \subseteq \langle T_{B\widetilde{\oplus}C}, I_{B\widetilde{\oplus}C}, F_{B\widetilde{\oplus}C}\rangle$.

ii. $\langle T_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}, I_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}, F_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}\rangle \subseteq \langle T_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, I_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, F_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}\rangle$,

$$T_{\underset{i\in I}{\cap} A_i \widetilde{\oplus} B}(v) = \sup_{u\in Z^n} \min\left(T_{\underset{i\in I}{\cap} A_i \widetilde{\oplus} B}(v+u), T_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\min_{i\in I} T_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v\right.$$

$$\left. + u), T_B(u)\right) = \sup_{v,u\in Z^2} \min_{i\in I}\left(\min T_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v+u), T_B(u)\right) \leq \bigcap_{\mathbf{i\in I}} \sup_{v,u\in Z^2} \left(\min T_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v\right.$$

$$\left. + u), T_B(u)\right) \leq \bigcap_{i\in I} T_{(A_i\widetilde{\oplus}B)}(v+u) \leq T_{\underset{i\in I}{\cap}(A_i\widetilde{\oplus}B)}$$

$$I_{\underset{i\in I}{\cap} A_i \widetilde{\oplus} B}(v) = \sup_{v,u\in Z^2} \min\left(I_{\underset{i\in I}{\cap} A_i \widetilde{\oplus} B}(v+u), I_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\min_{i\in I}\ I_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v\right.$$

$$\left. + u), I_B(u)\right) = \sup_{v,u\in Z^2} \min_{i\in I}\left(\min\ I_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v+u), I_B(u)\right) \leq \bigcap_{\mathbf{i\in I}} \sup_{u\in Z^2}\left(\min\ I_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v\right.$$

$$\left. + u), I_B(u)\right) \leq \bigcap_{i\in I} I_{(A_i\widetilde{\oplus}B)}(v+u) \leq I_{\underset{i\in I}{\cap}(A_i\widetilde{\oplus}B)}$$

$$F_{\underset{i\in I}{\cap} A_i \widetilde{\oplus} B}(v) = \inf_{v,u\in Z^n} \max\left(1 - F_{\underset{i\in I}{\cap} A_i \widetilde{\oplus} B}(v+u), 1 - F_B(u)\right) = \inf_{v,u\in Z^n} \max\left(\min_{i\in I}\ I_{A_i{}^c\widetilde{\oplus}B}(v+u), 1 - F_B(u)\right)$$

$$\leq \inf_{v,u\in Z^n} \min_{i\in I}\left(\max\ F_{\mathbf{A_i}^c\widetilde{\oplus}\mathbf{B}}(v+u), 1 - F_B(u)\right)$$

$$\leq \min_{i\in I} \inf_{v,u\in Z^n}\left(\max\ F_{\mathbf{A_i}^c\widetilde{\oplus}\mathbf{B}}(v+u), 1 - F_B(u)\right)$$

$$\leq \bigcap_{i\in I} \inf_{v,u\in Z^n}\left(\max\ F_{\mathbf{A_i}^c\widetilde{\oplus}\mathbf{B}}(v+u), 1 - F_B(u)\right) \leq F_{\underset{i\in I}{\cap}(A_i\widetilde{\oplus}B)}$$

iii. $\langle T_{\underset{i\in I}{\cup} A_i\widetilde{\oplus}B}, I_{\underset{i\in I}{\cup} A_i\widetilde{\oplus}B}, F_{\underset{i\in I}{\cup} A_i\widetilde{\oplus}B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i\widetilde{\oplus}B)}, I_{\underset{i\in I}{\cup}(A_i\widetilde{\oplus}B)}, F_{\underset{i\in I}{\cup}(A_i\widetilde{\oplus}B)}\rangle$

$$T_{\underset{i\in I}{\cup} A_i \widetilde{\oplus} B}(v) = \sup_{v,u\in Z^2} \min\left(T_{\underset{i\in I}{\cup} A_i \widetilde{\oplus} B}(v+u), T_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\max_{i\in I} T_{A_i\widetilde{\oplus}B}(v\right.$$

$$\left. + u), T_B(u)\right) \geq \sup_{v,u\in Z^2} \max_{i\in I}\left(\min T_{A_i\widetilde{\oplus}B}(v+u), T_B(u)\right) \geq \bigcup_{i\in I} \sup_{v,u\in Z^2}\left(\min T_{A_i\widetilde{\oplus}B}(v\right.$$

$$\left. + u), T_B(u)\right) \geq \bigcup_{i\in I} T_{(A_i\widetilde{\oplus}B)}(v+u) \geq T_{\underset{i\in I}{\cup}(A_i\widetilde{\oplus}B)}$$

$$I_{\underset{i\in I}{\cup} A_i \widetilde{\oplus} B}(v) = \sup_{v,u\in Z^2} \min\left(I_{\underset{i\in I}{\cup} A_i \widetilde{\oplus} B}(v+u), I_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\max_{i\in I}\ I_{A_i\widetilde{\oplus}B}(v\right.$$

$$\left. + u), I_B(u)\right) \geq \sup_{v,u\in Z^2} \max_{i\in I}\left(\min\ I_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v+u), I_B(u)\right) \geq \bigcup_{\mathbf{i\in I}} \sup_{v,u\in Z^2}\left(\min I_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v\right.$$

$$\left. + u), I_B(u)\right) \geq \bigcup_{i\in I} I_{(A_i\widetilde{\oplus}B)}(v+u) \geq I_{\underset{i\in I}{\cup}(A_i\widetilde{\oplus}B)}$$

$$F_{\underset{i\in I}{\cup}A_i\widetilde{\oplus}B}(v)=\inf_{v,u\in Z^n}\max\left(1-F_{\underset{i\in I}{\cup}A_i\widetilde{\oplus}B}(v+u),1-F_B(u)\right)=\inf_{v,u\in Z^n}\max\left(\max_{i\in I}\ I_{\mathbf{A_i}^c\widetilde{\oplus}\mathbf{B}}(v+u),1-F_B(u)\right)$$

$$=\inf_{v,u\in Z^n}\max_{i\in I}\left(\max\ F_{\mathbf{A_i}^c\widetilde{\oplus}\mathbf{B}}(v+u),1-F_B(u)\right)$$

$$\geq\max_{i\in I}\inf_{v,u\in Z^n}\left(\max\ F_{\mathbf{A_i}^c\widetilde{\oplus}\mathbf{B}}(v+u),1-F_B(u)\right)$$

$$\geq\underset{i\in I}{\cup}\inf_{v,u\in Z^n}\left(\max\ F_{\mathbf{A_i}^c\widetilde{\oplus}\mathbf{B}}(v+u),1-F_B(u)\right)$$

3.6.2 Properties of the NFE Operation:

3.6.2.1 Proposition

The NFE satisfies the following properties:

i. **Monotonicity** (increasing in the first argument and decreasing in the second argument):

$(\forall A,B,C\ \in\mathcal{N}(Z^2))(A\subseteq B\Longrightarrow\langle T_{A\widetilde{\ominus}C}\ ,I_{A\widetilde{\ominus}C}\ ,F_{A\widetilde{\ominus}C}\rangle\subseteq\langle T_{B\widetilde{\ominus}C}\ ,I_{B\widetilde{\ominus}C}\ ,F_{B\widetilde{\ominus}C}\rangle)$, where

$T_{A\widetilde{\ominus}C}\subseteq T_{B\widetilde{\ominus}C}\ ,I_{A\widetilde{\ominus}C}\subseteq I_{B\widetilde{\ominus}C}\ ,\ F_{A\widetilde{\ominus}C}\supseteq F_{B\widetilde{\ominus}C}$.

$(\forall A,B,C\ \in\mathcal{N}(Z^2))(A\subseteq B\Longrightarrow\langle T_{C\widetilde{\ominus}A}\ ,I_{C\widetilde{\ominus}A}\ ,F_{C\widetilde{\ominus}A}\rangle\supseteq\langle T_{C\widetilde{\ominus}B}\ ,I_{C\widetilde{\ominus}B}\ ,F_{C\widetilde{\ominus}B}\rangle)$, where

$T_{C\widetilde{\ominus}A}\subseteq T_{C\widetilde{\ominus}B}\ ,\ I_{C\widetilde{\ominus}A}\subseteq I_{C\widetilde{\ominus}B}\ ,\ F_{C\widetilde{\ominus}A}\subseteq F_{C\widetilde{\ominus}B}$.

ii. **Interaction with Zadeh's intersection:**

For any family $(A_i|i\in I)$ in $\mathcal{N}(Z^2)$and $B\in\mathcal{N}(Z^2)$, we have

$\langle T_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B},I_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}\ ,F_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}\rangle\subseteq\langle T_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)},I_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}\ ,F_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}\rangle$, where

$T_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}\subseteq T_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}\ ,\ I_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}\subseteq I_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)},F_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}\subseteq F_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}$.

$\langle T_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i}\ ,I_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i}F_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i}\rangle\supseteq\langle T_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)},I_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)},F_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}\rangle$, where

$T_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i}\supseteq T_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}\ ,I_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i}\supseteq I_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)},F_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i}\supseteq F_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}$.

iii. **Interaction with Zadeh's union.**

For any family$(A_i|i\in I)$ in $\mathcal{N}(Z^2)$ and $B\in\mathcal{N}(Z^2)$, we have

$\langle T_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B},I_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}\ ,F_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}\rangle\supseteq\langle T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)},I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\ ,F_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\rangle$, where

$T_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}\supseteq T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\ ,\ I_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}\supseteq I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\ ,F_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}\supseteq\ F_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}$.

$\langle T_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i},I_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\ ,F_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\rangle\subseteq\langle T_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\ ,I_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\ ,F_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\rangle$, where

$T_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i} \subseteq T_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}$, $I_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i} \subseteq I_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}$, $F_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i} \supseteq F_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}$.

Proof.

The proof of the first property is straightforward.

i. $\langle T_{A\widetilde{\ominus}C}, I_{A\widetilde{\ominus}C}, F_{A\widetilde{\ominus}C}\rangle \subseteq \langle T_{B\widetilde{\ominus}C}, I_{B\widetilde{\ominus}C}, F_{B\widetilde{\ominus}C}\rangle$.

ii. $\langle T_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}, I_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}, F_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}\rangle \subseteq \langle T_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}, I_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}, F_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}\rangle$

$T_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v) = \inf_{v,u\in Z^2} \max\left(T_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v+u), T_B(u)\right) = \inf_{v,u\in Z^2} \max\left(\min_{i\in I} T_{\mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+u), T_B(u)\right) \leq \inf_{v,u\in Z^2} \min_{i\in I}\left(\max T_{\mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+u), T_B(u)\right) \leq \underset{\mathbf{i\in I}}{\cap} \inf_{v,u\in Z^2}\left(\max T_{A_i\widetilde{\ominus}B}(v+u), T_B(u)\right) \leq \underset{i\in I}{\cap} T_{(A_i\widetilde{\ominus}B)}(v+u) \leq T_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}$.

$I_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v) = \inf_{v,u\in Z^2} \max\left(I_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v+u), I_B(u)\right) = \inf_{v,u\in Z^2} \max\left(\min_{i\in I} I_{\mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+u), I_B(u)\right) \leq \inf_{v,u\in Z^2} \min_{i\in I}\left(\max I_{\mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+u), I_B(u)\right) \leq \underset{\mathbf{i\in I}}{\cap} \inf_{v,u\in Z^2}\left(\max I_{\mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+u), I_B(u)\right) \leq \underset{i\in I}{\cap} I_{(A_i\widetilde{\ominus}B)}(v+u) \leq I_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}$.

$F_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v) = \sup_{v,u\in Z^2} \min\left(1 - F_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\min_{i\in I} I_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) = \sup_{v,u\in Z^2} \min_{i\in I}\left(\min F_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) \leq \min_{i\in I} \sup_{u\in Z^2}\left(\min F_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) \leq \underset{i\in I}{\cap} \sup_{v,u\in Z^2}\left(\min F_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) \leq F_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}$.

iii. $\langle T_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}, I_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}, F_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}, I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}, F_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\rangle$

$T_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}(v) = \inf_{v,u\in Z^2} \max\left(T_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}(v+u), T_B(u)\right) = \inf_{v,u\in Z^2} \max\left(\max_{i\in I} T_{A_i\widetilde{\ominus}B}(v+u), T_B(u)\right) = \inf_{v,u\in Z^2} \max_{i\in I}\left(\max T_{A_i\widetilde{\ominus}B}(v+u), T_B(u)\right) \geq \underset{i\in I}{\cup} \inf_{v,u\in Z^2}\left(\max T_{A_i\widetilde{\ominus}B}(v+u), T_B(u)\right) \geq \underset{i\in I}{\cup} T_{(A_i\widetilde{\ominus}B)}(v+u) \geq T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}$.

$$I_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}(v) = \inf_{v,u\in Z^2} \max\left(I_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}(v+u), I_B(u)\right) = \inf_{v,u\in Z^2} \max\left(\max_{i\in I}\ I_{\mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+u), I_B(u)\right) = \inf_{v,u\in Z^2} \max_{i\in I}\left(\max\ I_{\mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+u), I_B(u)\right) \geq \underset{\mathbf{i\in I}}{\cup} \inf_{v,u\in Z^n}\left(\max\ I_{\ \mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+u), I_B(u)\right) \geq \underset{i\in I}{\cup} I_{(A_i\widetilde{\ominus}B)}(v+u) \geq I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}.$$

$$F_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}(v) = \sup_{v,u\in Z^2} \min\left(1 - F_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\max_{i\in I}\ I_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) = \sup_{v,u\in Z^2} \max_{i\in I}\left(\min\ F_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) \geq \max_{i\in I} \sup_{v,u\in Z^2}\left(\min\ F_{A_i{}^c\widetilde{\oplus}B}(v+u), 1 - F_B(u)\right) \geq \underset{i\in I}{\cup} \sup_{v,u\in Z^2}\left(\min\ F_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right).$$

3.6.3 Properties of the NFC Operation:

3.6.3.1 Proposition

The NFC satisfies the following properties

i. Monotonicity (Increasing in the first argument):

$\left(\forall A, B, C\ \in \mathcal{N}(Z^2)\right)(A \subseteq B \Longrightarrow \langle T_{A\tilde{\bullet}C}\ , I_{A\tilde{\bullet}C}\ , F_{A\tilde{\bullet}C}\rangle \subseteq \langle T_{B\tilde{\bullet}C}\ , I_{B\tilde{\bullet}C}\ , F_{B\tilde{\bullet}C}\rangle)$, where $T_{A\tilde{\bullet}C} \subseteq T_{B\tilde{\bullet}C}$, $I_{A\tilde{\bullet}C} \subseteq I_{B\tilde{\bullet}C}$, $F_{A\tilde{\bullet}C} \subseteq F_{B\tilde{\bullet}C}$.

ii. Interaction with Zadeh's intersection:

For any family $(A_i|i \in I)$ in $\mathcal{N}(Z^2)$and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\underset{i\in I}{\cap}A_i\tilde{\bullet}B}, I_{\underset{i\in I}{\cap}A_i\tilde{\bullet}B}\ , F_{\underset{i\in I}{\cap}A_i\tilde{\bullet}B}\rangle \subseteq \langle T_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}, I_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}\ , F_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}\rangle$, where $T_{\underset{i\in I}{\cap}A_i\tilde{\bullet}B} \subseteq T_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}$, $I_{\underset{i\in I}{\cap}A_i\tilde{\bullet}B} \subseteq I_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}$, $F_{\underset{i\in I}{\cap}A_i\tilde{\bullet}B} \subseteq F_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}$.

iii. Interaction with Zadeh's union:

For any family$(A_i|i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\underset{i\in I}{\cup}A_i\tilde{\bullet}B}, I_{\underset{i\in I}{\cup}A_i\tilde{\bullet}B}\ , F_{\underset{i\in I}{\cup}A_i\tilde{\bullet}B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}, I_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}\ , F_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}\rangle$, where

$T_{\underset{i\in I}{\cup}A_i\tilde{\bullet}B} \supseteq T_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}$, $I_{\underset{i\in I}{\cup}A_i\tilde{\bullet}B} \supseteq I_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}$, $F_{\underset{i\in I}{\cup}A_i\tilde{\bullet}B} \supseteq F_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}$.

Proof (i) The first property (i.e. the monotonicity properties of the neutrosophic dilation and neutrosophic erosion have been satisfied from the fact that $A \subseteq B$, it follows that $\langle T_{C\widetilde{\oplus}A}\ , I_{C\widetilde{\oplus}A}\ , F_{C\widetilde{\oplus}A}\rangle \subseteq \langle T_{C\widetilde{\oplus}B}\ , I_{C\widetilde{\oplus}B}\ , F_{C\widetilde{\oplus}B}\rangle$.

The proof of **(ii) &** (**iii**) can be inherited from the property that $(A_i|i \in I)\left(\min_{i\in I} A_i \subseteq A_i \subseteq \max_{i\in I} A_i\right)$.

3.6.4 Properties of the NFO Operation:

3.6.4.1 Proposition

The neutrosophic opening satisfies the following properties

i. Monotonicity (increasing in the first argument):

$(\forall A, B, C \in \mathcal{N}(Z^2))(A \subseteq B \Rightarrow \langle T_{A \tilde{\circ} C}, I_{A \tilde{\circ} C}, F_{A \tilde{\circ} C}\rangle \subseteq \langle T_{B \tilde{\circ} C}, I_{B \tilde{\circ} C}, F_{B \tilde{\circ} C}\rangle)$, where

$T_{A\tilde{\circ} C} \subseteq T_{B\tilde{\circ} C}$, $I_{A\tilde{\circ} C} \subseteq I_{B\tilde{\circ} C}$, $F_{A\tilde{\circ} C} \subseteq F_{B\tilde{\circ} C}$.

ii. Interaction with Zadeh's intersection:

For any family $(A_i|i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\bigcap_{i\in I} A_i \tilde{\circ} B}, I_{\bigcap_{i\in I} A_i \tilde{\circ} B}, F_{\bigcap_{i\in I} A_i \tilde{\circ} B}\rangle \subseteq \langle T_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}, I_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}, F_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}\rangle$, where

$T_{\bigcap_{i\in I} A_i \tilde{\circ} B} \subseteq T_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}$, $I_{\bigcap_{i\in I} A_i \tilde{\circ} B} \subseteq I_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}$, $F_{\bigcap_{i\in I} A_i \tilde{\circ} B} \subseteq F_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}$.

iii. Interaction with Zadeh's union:

For any family$(A_i|i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\bigcup_{i\in I} A_i \tilde{\circ} B}, I_{\bigcup_{i\in I} A_i \tilde{\circ} B}, F_{\bigcup_{i\in I} A_i \tilde{\circ} B}\rangle \supseteq \langle T_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}, I_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}, F_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}\rangle$, where

$T_{\bigcup_{i\in I} A_i \tilde{\circ} B} \supseteq T_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}$, $I_{\bigcup_{i\in I} A_i \tilde{\circ} B} \supseteq I_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}$, $F_{\bigcup_{i\in I} A_i \tilde{\circ} B} \supseteq F_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}$.

The proofs are Similar to the proofs of the foregoing proposition.

3.7 Neutrosophic Fuzzy Mathematical Morphological Filters:

This section is considering the differences between two or more of the basic neutrosophic fuzzy morphological operators.

3.7.1 Some Types of Boundary Extraction Filters Using NFD and NFE:

As the neutrosophic dilation thickens the regions in the true level of the image, and the neutrosophic erosion shrinks them, the neutrosophic differences between the image and its neutrosophic dilation or its neutrosophic erosion may emphasize the boundaries between regions included in the image. Therefore, several boundary filters may be obtained as follows:

3.7.1.1 Neutrosophic Fuzzy Gradient Boundary:

To commence, we will investigate the neutrosophic fuzzy gradient filter, which is the mean value of the three components of the neutrosophic difference between the neutrosophic dilation of some images and its neutrosophic erosion. We get the neutrosophic gradient of the image by applying the mean of these boundaries. If the structure element is relatively small, the homogeneous areas will not be affected by NFD and NFE, and then the subtraction tends to eliminate them. The effect of neutrosophic morphological gradient operation is shown in Fig. 3.7.1.1, Type I, II, and to be defined in two different types as follows:

Type I:

$\tilde{\partial} gradient = (1/3)\ \ [min(TA \widetilde{\oplus} B(v), 1 - TA \widetilde{\ominus} B(v)), min(IA \widetilde{\oplus} B(v), 1 -$
$IA \widetilde{\ominus} B(v)), max(FA \widetilde{\oplus} B(v), 1 - FA \widetilde{\ominus} B(v))].$

In the following figure (fig.3.7.1.1 (I)), we present the results obtained when applying neutrosophic gradient boundary filter on some grayscale image.

(a) (b)

Fig. 3.7.1.1 (I): Applying the Neutrosophic Gradient Boundary: (a) Original Image (b) Neutrosophic Gradient Boundary Filtered.

Type II:

$\tilde{\partial} gradient = (1/3)[min(TA \widetilde{\oplus} B(v), 1 - TA \widetilde{\ominus} B(v)), max(IA \widetilde{\oplus} B(v), 1 - TA \widetilde{\ominus} B(v)),$
$max(FA \widetilde{\oplus} B(v), 1 - TA \widetilde{\ominus} B(v))].$

In the following figure (fig.3.7.1.1 (II)), we present the results obtained when applying neutrosophic gradient boundary filter on some grayscale images.

(a) **(b)**

Fig.3.7.1.1 (II): Applying the Neutrosophic Gradient Boundary: (a) Original Image (b) Neutrosophic Gradient Boundary Filtered.

3.7.1.2 Neutrosophic Fuzzy External Boundary:

In this filter, a neutrosophic dilation is firstly applied to the neutrosophic image (a) by some neutrosophic structure elements (b); hence, the output filtered image will be the neutrosophic difference between neutrosophic dilated image and the neutrosophic image (a). That is, the neutrosophic external boundary of (a) is to be defined in two different types as follows:

Type I: $\tilde{\partial}ext \ = (1/3)\ [min\ (TA \widetilde{\oplus} B(v), 1\ -\ TA(v)), min\ (IA \widetilde{\oplus} B(v), 1\ - IA(v)),$

$max\ (FA \widetilde{\oplus} B(v))\ , 1\ -\ FA(v)\ ].$

In the following figure (fig.3.7.1.2 (I)), we present the results obtained when applying neutrosophic external boundary filter on some grayscale images.

(a) **(b)**

Fig. 3.7.1.2 (I): Applying the Neutrosophic Fuzzy External Boundary: a) Original Image

a) Neutrosophic Fuzzy External Boundary Filtered Image

Type II:

$\tilde{\partial}ext \ = \left(\frac{1}{3}\right)[min(TA \widetilde{\oplus} B(v), 1 - TA(v))\ , max\ (IA \widetilde{\oplus} B(v), 1\ -$

$IA(v))\ , max\ (FA \widetilde{\oplus} B(v))\ , 1\ -\ FA(v)\ ]\ .$

In the following figure (fig 3.7.1.2 (II)), we present the results obtained when applying the Neutrosophic Fuzzy External Boundary Filter on Some Grayscale Images.


(a) (b)

Fig.3.7.1.2 (II): Applying the Neutrosophic Fuzzy Eternal Boundary: a) Original Image

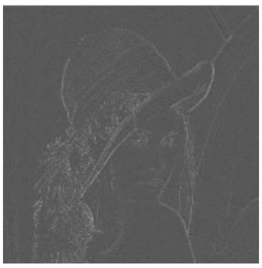
b) Neutrosophic External Boundary Filtered Image

3.7.1.3 Neutrosophic Fuzzy Internal Boundary:

The main step of the neutrosophic internal boundary filter, is to get the neutrosophic erosion of the neutrosophic image, hence, the output filtered image will be the neutrosophic difference between the original image in the neutrosophic domain and the neutrosophic eroded image that is the neutrosophic internal boundary of the neutrosophic image (a) is to be defined in two different types as follows:

Type I:

In the following figure (fig.3.7.1.3 (I)), we present the results obtained when applying neutrosophic internal boundary filter on some grayscale images.

a) b)

Fig. 3.7.1.3 (I): Applying the Neutrosophic Internal Boundary: a) Original Image

b) Neutrosophic Internal Boundary Filtered Image

Type II:

$$\tilde{\partial} int = (1/3)[min\ (T(v), 1 - (TA \mathbin{\widetilde{\ominus}} B(v)))\ , min\ (IA(v), 1 - IA \mathbin{\widetilde{\ominus}} B(v))\ , max\ (FA(v), 1 - FA \mathbin{\widetilde{\ominus}} B(v))\ ].$$

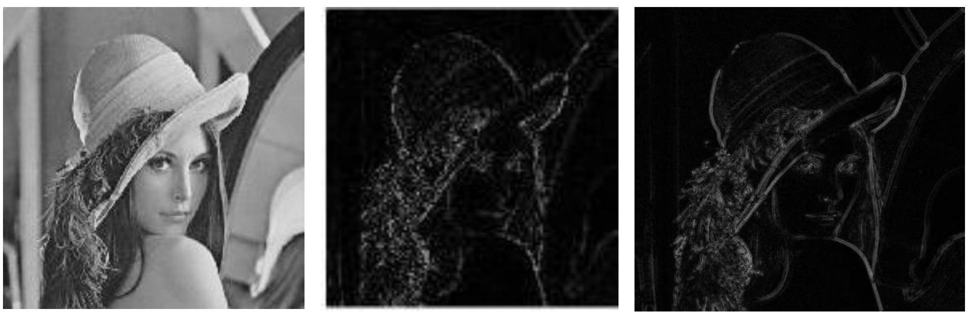
(a) **(b)**

Fig.3.7.1.3 (II): Applying the Neutrosophic Internal Boundary: a) Original Image

b) Neutrosophic Internal Boundary Filtered Image

3.7.2. Neutrosophic Fuzzy Outline Boundary:

The main step of the neutrosophic outline boundary filter, is to get the complement of the neutrosophic erosion of the neutrosophic image, hence, the output filtered image will be the neutrosophic difference between the original image in neutrosophic domain and the neutrosophic eroded image that is the neutrosophic outline boundary of the neutrosophic image A is to be defined as follows: $\tilde{\partial} outline\ (A) = (\partial 1A1 \cup \partial 3A3) \cap A2$, where; $\partial 1(A1) = co(A1 \ominus B1) \cap A1, \partial 3(A3) = co(A3 \oplus B3) \cup A3$. In the following figure (fig.3.7.2), we present the results obtained when applying the neutrosophic outline boundary filter on some grayscale images.

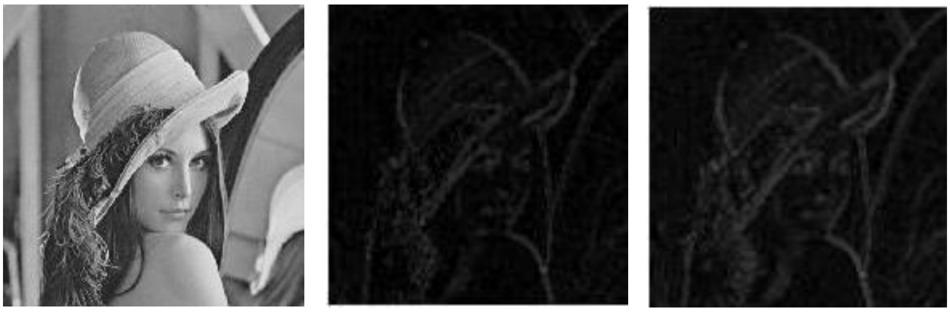


a) b) c)

Fig. 3.7.2: Neutrosophic Outline Boundary: a) Original Image, b) Neutrosophic Outline Boundary Filtered Image, c) Neutrosophic Outline Boundary Filtered Image

3.7.2.1 Some Combinations of the Neutrosophic Fuzzy External and Internal Boundary Filters

In the following figure (fig.3.7.2.1), we present the results obtained when applying the neutrosophic fuzzy sup. boundary filter on some grayscale images.



a) b) c)

Fig.3.7.2.1 Neutrosophic fuzzy sup. boundary: a) Original Image, b) Neutrosophic fuzzy sup. boundary filtered image, c) Neutrosophic fuzzy sup. boundary filtered image

4.1. Neutrosophic Crisp Mathematical Morphology:

As a generalization of the classical mathematical morphology, we present in this section the basic operations for the neutrosophic crisp mathematical morphology. To commence, we need to define the translation of a neutrosophic crisp set.

4.1.1. Definition:

Consider the space $X = R^n$ or $X = Z^n$, with origin $0 = (0,\ldots,0)$, given that the reflection of the structuring element B mirrored in its origin is defined as:
$-B = \langle -B^1, -B^2, -B^3 \rangle$

4.1.2. Definition:

For every $p \in A$, the translation by p is the map $p: X \to X, a \to a + p$; it transforms any subset A of X into its translate by $p \in Z^2$, $A = \langle A_p^1, A_p^2, A_p^3 \rangle$. Where $A_p^1 = \{u + p: u \in A^1, p \in B^1\}$, $A_p^2 = \{u + p: u \in A^2, p \in B^2\}$, $A_p^3 = \{u + p: u \in A^3, p \in B^3\}$.

4.2. Neutrosophic Crisp Mathematical Morphological Operations:

4.2.1. Neutrosophic Crisp Dilation Operator:

Let $A, B \in \mathcal{NC}(X)$, and then we define two types of the neutrosophic crisp dilation as follows:

Type I:

$A \,\widetilde{\oplus}\, B = \langle A^1 \oplus B^1, A^2 \oplus B^2, A^3 \ominus B^3 \rangle$, where for each $u, v \in Z^2$, we have $A^1 \oplus B^1 = \bigcup_{b \in B^1} A^1_b$, $A^2 \oplus B^2 = \bigcup_{b \in B^2} A^2_b$, $A^3 \ominus B^3 = \bigcap_{-b \in B^3} A^3_{-b}$.

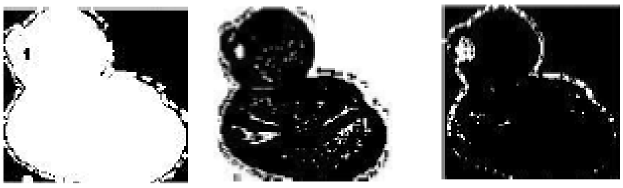


Fig. 4.2.1. (I): Neutrosophic Crisp Dilation Components in type I for $\langle A^1, A^2, A^3 \rangle$respectively.

Type II:

$A \,\overset{\approx}{\oplus}\, B = \langle A^1 \oplus B^1, A^2 \ominus B^2, A^3 \ominus B^3 \rangle$, where for each $u, v \in Z^2$, we have $A^1 \oplus B^1 = \bigcup_{b \in B^1} A^1_b$,
$A^2 \ominus B^2 = \bigcap_{-b \in B^2} A^2_{-b}$, $A^3 \ominus B^3 = \bigcap_{-b \in B^3} A^3_{-b}$.

Fig. 4.2.1. (II): Neutrosophic Crisp Dilation Components in type II for $\langle A^1, A^2, A^3 \rangle$ respectively.

4.2.2. Neutrosophic Crisp Erosion Operation:

Let $A, B \in \mathcal{NC}(X)$; then the neutrosophic erosion is given as two types:

Type I:

$A \widetilde{\ominus} B = \langle A^1 \ominus B^1, A^2 \ominus B^2, A^3 \oplus B^3 \rangle$, where for each $u, v \in Z^2$, we have $A^1 \ominus B^1 = \bigcap_{-b \in B^1} A^1_{-b}$,
$A^2 \ominus B^2 = \bigcap_{-b \in B^2} A^2_{-b}$, $A^3 \oplus B^3 = \bigcup_{b \in B^3} A^3_b$.

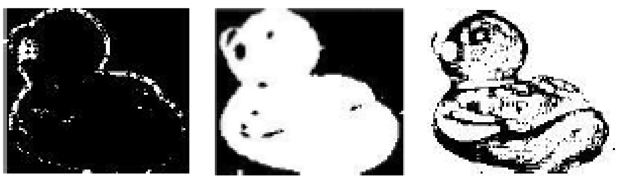


Fig. 4.2.2. (I): Neutrosophic Crisp Erosion Components in type I for $\langle A^1, A^2, A^3 \rangle$ respectively.

Type II:

$A\widetilde{\Theta}B = \langle A^1 \Theta B^1, A^2 \oplus B^2, A^3 \oplus B^3 \rangle$, where for each $u, v \in Z^2$, we have $A^1 \Theta B^1 = \bigcap_{-b \in B^1} A^1_{-b}$,

$A^2 \oplus B^2 = \bigcup_{b \in B^2} A^2_b$, $A^3 \oplus B^3 = \bigcup_{b \in B^3} A^3_b$.

Fig.4.2.2. (II): Neutrosophic Crisp Erosion Components in type II $\langle A^1, A^2, A^3 \rangle$ respectively.

4.2.3 Neutrosophic Crisp Opening Operation:

Let A, B∈ $\mathcal{N}$C(X); then we define two types of the neutrosophic crisp opening operator as follows:

Type I:

$\mathrm{A} \tilde{\circ} \mathrm{B} = \langle \mathrm{A}^1 \circ \mathrm{B}^1, \mathrm{A}^2 \circ \mathrm{B}^2, \mathrm{A}^3 \bullet \mathrm{B}^3 \rangle$, $A^1 \circ B^1 = (A^1 \Theta B^1) \oplus B^1$, $A^2 \circ B^2 = (A^2 \Theta B^2) \oplus B^2$, $A^3 \bullet B^3 = (A^3 \oplus B^3) \Theta B^3$.

Fig. 4.2.3. (I): Neutrosophic Crisp Opening Components in Type I $\langle A^1, A^2, A^3 \rangle$ respectively.

Type II:

$\mathrm{A} \tilde{\circ} \mathrm{B} = \langle \mathrm{A}^1 \circ \mathrm{B}^1, \mathrm{A}^2 \bullet \mathrm{B}^2, \mathrm{A}^3 \bullet \mathrm{B}^3 \rangle$, $A^1 \circ B^1 = (A^1 \Theta B^1) \oplus B^1$, $A^2 \bullet B^2 = (A^2 \oplus B^2) \Theta B^2$, $A^3 \bullet B^3 = (A^3 \oplus B^3) \Theta B^3$.

Fig. 4.2.3. (II): Neutrosophic Crisp Opening Components in type II $\langle A^1, A^2, A^3 \rangle$ respectively.

4.2.4. Neutrosophic Crisp Closing Operation:

Let A and B $\in$ $\mathcal{N}$C(X); then the neutrosophic closing is given as two types:

Type I:

$A \tilde{\bullet} B = \langle A^1 \bullet B^1, A^2 \bullet B^2, A^3 \circ B^3 \rangle$, $A^1 \bullet B^1 = (A^1 \oplus B^1) \Theta B^1$, $A^2 \bullet B^2 = (A^2 \oplus B^2) \Theta B^2$, $A^3 \circ B^3 = (A^3 \Theta B^3) \oplus B^3$.

Fig.4.2.4 (I): Neutrosophic Crisp Closing Components in type I for $\langle A^1, A^2, A^3 \rangle$ respectively.

Type II:

$A \tilde{\bullet} B = \langle A^1 \bullet B^1, A^2 \circ B^2, A^3 \circ B^3 \rangle$, $A^1 \bullet B^1 = (A^1 \oplus B^1) \Theta B^1$, $A^2 \circ B^2 = (A^2 \Theta B^2) \oplus B^2$, $A^3 \circ B^3 = (A^3 \Theta B^3) \oplus B^3$.

Fig.4.2.4. (II): Neutrosophic Crisp Closing Components in type II for $\langle A^1, A^2, A^3 \rangle$ respectively.

5. Algebraic Properties in Neutrosophic Crisp:

In this section, we investigate some of the algebraic properties of the neutrosophic crisp erosion and dilation, as well as the neutrosophic crisp opening and closing operator [15].

5. 1 Properties of the Neutrosophic Crisp Erosion Operator:

5.1.1 Proposition:

The Neutrosophic erosion satisfies the monotonicity for all A, B $\in \mathcal{NC}(Z^2)$.

Type I:

a) $A \subseteq B \Longrightarrow \langle A^1 \Theta C^1, A^2 \Theta C^2, A^3 \Theta C^3 \rangle \subseteq \langle B^1 \Theta C^1, B^2 \Theta C^2, B^3 \Theta C^3 \rangle$:

$A^1 \Theta C^1 \subseteq B^1 \Theta C^1, A^2 \Theta C^2 \subseteq B^2 \Theta C^2, A^3 \Theta C^3 \supseteq B^3 \Theta C^3$.

b) $A \subseteq B \Longrightarrow \langle C^1 \Theta A^1, C^2 \Theta A^2, C^3 \Theta A^3 \rangle \subseteq \langle C^1 \Theta B^1, C^2 \Theta B^2, C^3 \Theta B^3 \rangle$:

$C^1 \Theta A^1 \subseteq C^1 \Theta B^1, C^2 \Theta A^2 \subseteq C^2 \Theta B^2, C^3 \Theta A^3 \supseteq C^3 \Theta B^3$.

Type II:

a) $A \subseteq B \Longrightarrow \langle A^1 \Theta C^1, A^2 \Theta C^2, A^3 \Theta C^3 \rangle \subseteq \langle B^1 \Theta C^1, B^2 \Theta C^2, B^3 \Theta C^3 \rangle$:

$A^1 \Theta C^1 \subseteq B^1 \Theta C^1, A^2 \Theta C^2 \supseteq B^2 \Theta C^2, A^3 \Theta C^3 \supseteq B^3 \Theta C^3$.

b) $A \subseteq B \Longrightarrow \langle C^1 \Theta A^1, C^2 \Theta A^2, C^3 \Theta A^3 \rangle \subseteq \langle C^1 \Theta B^1, C^2 \Theta B^2, C^3 \Theta B^3 \rangle$:

$C^1 \Theta A^1 \subseteq C^1 \Theta B^1, C^2 \Theta A^2 \supseteq C^2 \Theta B^2, C^3 \Theta A^3 \supseteq C^3 \Theta B^3$.

Note that: Dislike the Neutrosophic crisp dilation operator, the Neutrosophic crisp erosion does not satisfy commutativity and the associativity properties.

5.1.2 Proposition: for any family $A_i \in \mathcal{NC}(Z^2), i \in I$, and $B \in \mathcal{NC}(Z^2)$.

Type I:

a) $\bigcap_{i\in I} A_i \,\widetilde{\ominus}\, B = \bigcap_{i\in I} (A_i \,\widetilde{\ominus}\, B) = \langle \cap A_i^1 \Theta B^1, \cap A_i^2 \Theta B^2, \cap A_i^3 \oplus B^3 \rangle = \langle \cap(A_i^1 \Theta B^1), \cap(A_i^2 \Theta B^2), \cap(A_i^3 \oplus B^3) \rangle$,

b) $B \,\widetilde{\ominus} \bigcap_{i\in I} A_i = \bigcap_{i\in I} (B \,\widetilde{\ominus}\, A_i) = \langle B^1 \Theta \cap A_i^1, B^2 \Theta \cap A_i^2, B^3 \oplus \cap A_i^3 \rangle = \langle \cap(B^1 \Theta A_i^1), \cap(B^2 \Theta A_i^2), \cap(B^3 \oplus A_i^3) \rangle$.

Type II:

a) $\bigcap_{i\in I} A_i \,\widetilde{\ominus}\, B = \bigcap_{i\in I} (A_i \,\widetilde{\ominus}\, B) \Longrightarrow$

$\langle \cap_{i\in I} A_i^1 \Theta B^1, \cap_{i\in I} A_i^2 \oplus B^2, \cap_{i\in I} A_i^3 \oplus B^3 \rangle = \langle \cap(A_i^1 \Theta B^1), \cap(A_i^2 \oplus B^2), \cap(A_i^3 \oplus B^3) \rangle$,

b) $B \,\widetilde{\ominus} \bigcap_{i\in I} A_i = \bigcap_{i\in I} (B \,\widetilde{\ominus}\, A_i) \Longrightarrow$

$\langle B^1 \Theta \cap A_i^1, B^2 \oplus \cap A_i^2, B^3 \oplus \cap A_i^3 \rangle = \langle \cap(B^1 \Theta A_i^1), \cap(B^2 \oplus A_i^2), \cap(B^3 \oplus A_i^3) \rangle$.

Proof: a) In two types:

Type I:

$\bigcap_{i\in I} A_i \,\widetilde{\ominus}\, B = \langle \cap_{i\in I}(\cap_{b\in B} A_{ib}^1), \cup_{i\in I}(\cap_{b\in B} A_{ib}^2), \cup_{i\in I}(\cap_{b\in B} A_{i(-b)}^3) \rangle =$

$\langle \cap_{i\in I}(\cap_{b\in B} A_{i(-b)}^1), \cap_{i\in I}(\cup_{b\in B} A_{i(-b)}^2), \cap_{i\in I}(\cup_{b\in B} A_{ib}^3) \rangle = \bigcap_{i\in I} (A_i \,\widetilde{\ominus}\, B)$

Type II:

Similarity, we can show that it is true in type 2,

b) The proof is similar to the (a).

5.1.3 Proposition: for any family $A_i \in \mathcal{NC}(Z^2), i \in I$, and $B \in \mathcal{NC}(Z^2)$

Type I:

a) $\cup_{i\in I} A_i \,\widetilde{\ominus}\, B = \cup_{i\in I}(A_i \,\widetilde{\ominus}\, B) \Longrightarrow$

$\langle \cup_{i\in I} A_i^1 \Theta B^1, \cup_{i\in I} A_i^2 \Theta B^2, \cup_{i\in I} A_i^3 \oplus B^3 \rangle = \langle \cup_{i\in I}(A_i^1 \Theta B^1), \cup_{i\in I}(A_i^2 \Theta B^2), \cup_{i\in I}(A_i^3 \oplus B^3) \rangle$,

b) $B \,\widetilde{\ominus} \cup_{i\in I} A_i = \cup_{i\in I}(B \,\widetilde{\ominus}\, A_i) \Longrightarrow$

$\langle B^1 \Theta \cup_{i\in I} A_i^1, B^2 \Theta \cup_{i\in I} A_i^2, B^3 \oplus \cup_{i\in I} A_i^3 \rangle = \langle \cup_{i\in I}(B^1 \Theta A_i^1), \cup_{i\in I}(B^2 \Theta A_i^2), \cup_{i\in I}(B^3 \oplus A_i^3) \rangle$.

Type II:

a) $\cup_{i\in I} A_i \,\widetilde{\ominus}\, B = \cup_{i\in I}(A_i \,\widetilde{\ominus}\, B) \Longrightarrow$

$\langle \cup_{i\in I} A_i^1 \Theta B^1, \cup_{i\in I} A_i^2 \oplus B^2, \cup_{i\in I} A_i^3 \oplus B^3 \rangle = \langle \cup_{i\in I}(A_i^1 \Theta B^1), \cup_{i\in I}(A_i^2 \oplus B^2), \cup_{i\in I}(A_i^3 \oplus B^3) \rangle$,

b) $B \,\widetilde{\ominus} \cup_{i\in I} A_i = \cup_{i\in I}(B \,\widetilde{\ominus}\, A_i) \Longrightarrow$

$\langle B^1 \Theta \cup_{i\in I} A_i^1, B^2 \oplus \cup_{i\in I} A_i^2, B^3 \oplus \cup_{i\in I} A_i^3 \rangle = \langle \cup_{i\in I}(B^1 \Theta A_i^1), \cup_{i\in I}(B^2 \oplus A_i^2), \cup_{i\in I}(B^3 \oplus A_i^3) \rangle$.

Proof: a) for both two types

Type I: $\bigcup_{i\in I} A_i \,\widetilde{\ominus}\, B = \langle \bigcap_{b\in B}\left(\bigcup_{i\in I} A^1_{i(-b)}\right), \bigcap_{b\in B}\left(\bigcup_{i\in I} A^2_{i(-b)}\right), \bigcup_{b\in B}\left(\bigcup_{i\in I} A^3_{ib}\right)\rangle$

$= \langle \bigcup_{i\in I}(\bigcap_{b\in B} A^1_{i(-b)}), \bigcup_{i\in I}(\bigcap_{b\in B} A^2_{i(-b)}), \bigcup_{i\in I}(\bigcup_{b\in B} A^3_{ib})\rangle = \underset{i\in I}{\cap}(A_i \,\widetilde{\ominus}\, B).$

Type II: can be verified in a similar way as in type 1.

b) The proof is similar to the (a).

5.2 Properties of the Neutrosophic Crisp Dilation Operator):

5.2.1 Proposition:

The neutrosophic dilation satisfies the following properties: $\forall A, B \in \mathcal{NC}(Z^2)$

i) Commutativity: $A \,\widetilde{\oplus}\, B = B \,\widetilde{\oplus}\, A$.

ii) Associativity: $(A \,\widetilde{\oplus}\, B) \,\widetilde{\oplus}\, C = A \,\widetilde{\oplus}\, (B \,\widetilde{\oplus}\, C)$.

iii) Monotonicity: (increasing in both arguments):

Type I:

a) $A \subseteq B \Longrightarrow \langle A^1 \oplus C^1, A^2 \oplus C^2, A^3 \oplus C^3\rangle \subseteq \langle B^1 \oplus C^1, B^2 \oplus C^2, B^3 \oplus C^3\rangle$:

$A^1 \oplus C^1 \subseteq B^1 \oplus C^1, A^2 \oplus C^2 \subseteq B^2 \oplus C^2, A^3 \oplus C^3 \supseteq B^3 \oplus C^3$.

b) $A \subseteq B \Longrightarrow \langle C^1 \oplus A^1, C^2 \oplus A^2, C^3 \oplus A^3\rangle \subseteq \langle C^1 \oplus B^1, C^2 \oplus B^2, C^3 \oplus B^3\rangle$:

$C^1 \oplus A^1 \subseteq C^1 \oplus B^1, C^2 \oplus A^2 \subseteq C^2 \oplus B^2, C^3 \oplus A^3 \supseteq C^3 \oplus B^3$.

Type II:

a) $A \subseteq B \Longrightarrow \langle A^1 \oplus C^1, A^2 \oplus C^2, A^3 \oplus C^3\rangle \subseteq \langle B^1 \oplus C^1, B^2 \oplus C^2, B^3 \oplus C^3\rangle$:

$A^1 \oplus C^1 \subseteq B^1 \oplus C^1, A^2 \oplus C^2 \supseteq B^2 \oplus C^2, A^3 \oplus C^3 \supseteq B^3 \oplus C^3$.

b) $A \subseteq B \Longrightarrow \langle C^1 \oplus A^1, C^2 \oplus A^2, C^3 \oplus A^3\rangle \subseteq \langle C^1 \oplus B^1, C^2 \oplus B^2, C^3 \oplus B^3\rangle$:

$C^1 \oplus A^1 \subseteq C^1 \oplus B^1, C^2 \oplus A^2 \supseteq C^2 \oplus B^2, C^3 \oplus A^3 \supseteq C^3 \oplus B^3$.

Proof: i), ii), iii) are obvious in two types I and II.

5.2.2 Proposition: for any family$(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$

Type I: $a)$ $\underset{i\in I}{\cap} \mathrm{A_i} \,\widetilde{\oplus}\, \mathrm{B} = \underset{i\in I}{\cap}(\mathrm{A_i} \,\widetilde{\oplus}\, \mathrm{B}) \Longrightarrow$

$\langle \bigcap_{i\in I} A^1_i \oplus B^1, \bigcap_{i\in I} A^2_i \oplus B^2, \bigcap_{i\in I} A^3_i \Theta B^3\rangle = \langle \bigcap_{i\in I}(A^1_i \oplus B^1), \bigcap_{i\in I}(A^2_i \oplus B^2), \bigcap_{i\in I}(A^3_i \Theta B^3)\rangle$.

b) $\mathrm{B} \,\widetilde{\oplus}\, \underset{i\in I}{\cap} \mathrm{A_i} = \underset{i\in I}{\cap}(\mathrm{B} \,\widetilde{\oplus}\, \mathrm{A_i}) \Longrightarrow$

$\langle B^1 \oplus \bigcap_{i\in I} A^1_i, B^2 \oplus \bigcap_{i\in I} A^2_i, B^3 \Theta \bigcap_{i\in I} A^3_i\rangle = \langle \bigcap_{i\in I}(B^1 \oplus A^1_i), \bigcap_{i\in I}(B^2 \oplus A^2_i), \bigcap_{i\in I}(B^3 \Theta A^3_i)\rangle$.

Type II: a) $\underset{i\in I}{\cap} \mathrm{A_i} \,\widetilde{\oplus}\, \mathrm{B} = \underset{i\in I}{\cap}(\mathrm{A_i} \,\widetilde{\oplus}\, \mathrm{B}) \Longrightarrow$

$\langle \bigcap_{i\in I} A^1_i \oplus B^1, \bigcap_{i\in I} A^2_i \Theta B^2, \bigcap_{i\in I} A^3_i \Theta B^3\rangle = \langle \bigcap_{i\in I}(A^1_i \oplus B^1), \bigcap_{i\in I}(A^2_i \Theta B^2), \bigcap_{i\in I}(A^3_i \Theta B^3)\rangle$.

b) $\mathrm{B} \widetilde{\oplus} \underset{\mathrm{i} \in \mathrm{I}}{\cap} \mathrm{A_i} = \underset{\mathrm{i} \in \mathrm{I}}{\cap} (\mathrm{B} \widetilde{\oplus} \mathrm{A_i}) \Longrightarrow$

$\langle B^1 \oplus \cap_{i \in I} A_i^1, B^2 \Theta \cap_{i \in I} A_i^2, B^3 \Theta \cap_{i \in I} A_i^3 \rangle = \langle \cap_{i \in I}(B^1 \oplus A_i^1), \cap_{i \in I}(B^2 \Theta A_i^2), \cap_{i \in I}(B^3 \Theta A_i^3) \rangle.$

Proof: we will prove this property for the two types of the neutrosophic crisp intersection operator:

Type I: a) $\cap_{i \in I} A_i \widetilde{\oplus} B = \underset{\mathrm{i} \in \mathrm{I}}{\cap} (\mathrm{A_i} \widetilde{\oplus} \mathrm{B}) \Longrightarrow$

$\langle \cup_{b \in B}(\cap_{i \in I} A_{ib}^1), \cup_{b \in B}(\cap_{i \in I} A_{ib}^2), \cap_{b \in B}(\cap_{i \in I} A_{i(-b)}^3) \rangle =$

$\langle \cap_{i \in I}(\cup_{b \in B} A_{ib}^1), \cap_{i \in I}(\cup_{b \in B} A_{ib}^2), \cap_{i \in I}(\cap_{b \in B} A_{i(-b)}^3) \rangle.$

Type II: a) $\underset{\mathrm{i} \in \mathrm{I}}{\cap} \mathrm{A_i} \widetilde{\oplus} \mathrm{B} = \underset{\mathrm{i} \in \mathrm{I}}{\cap} (\mathrm{A_i} \widetilde{\oplus} \mathrm{B}) \Longrightarrow$

$\langle \cup_{b \in B}(\cap_{i \in I} A_{ib}^1), \cap_{b \in B}(\cap_{i \in I} A_{ib}^2), \cap_{b \in B}(\cap_{i \in I} A_{i(-b)}^3) \rangle =$

$\langle \cap_{i \in I}(\cup_{b \in B} A_{ib}^1), \cap_{i \in I}(\cap_{b \in B} A_{ib}^2), \cap_{i \in I}(\cap_{b \in B} A_{i(-b)}^3) \rangle.$

The proof of (b) is similar to (a).

5.2.3 Proposition: for any family of $(\mathrm{A_i} | \mathrm{i} \in \mathrm{I})$ in $\mathcal{N}\mathrm{C}(\mathrm{Z}^2)$**,** and $B \in \mathcal{N}\mathrm{C}(\mathrm{Z}^2)$

Type I:

a) $\cup_{i \in I} A_i \widetilde{\oplus} B = \cup_{i \in I}(A_i \widetilde{\oplus} B) \Longrightarrow$

$\langle \cup_{i \in I} A_i^1 \oplus B^1, \cup_{i \in I} A_i^2 \oplus B^2, \cup_{i \in I} A_i^3 \Theta B^3 \rangle = \langle \cup_{i \in I}(A_i^1 \oplus B^1), \cup_{i \in I}(A_i^2 \oplus B^2), \cup_{i \in I}(A_i^3 \Theta B^3) \rangle.$

b) $B \widetilde{\oplus} \cup_{i \in I} A_i = \cup_{i \in I}(B \widetilde{\oplus} A_i) \Longrightarrow$

$\langle B^1 \oplus \cup_{i \in I} A_i^1, B^2 \oplus \cup_{i \in I} A_i^2, B^3 \Theta \cup_{i \in I} A_i^3 \rangle = \langle \cup_{i \in I}(B^1 \oplus A_i^1), \cup_{i \in I}(B^2 \oplus A_i^2), \cup_{i \in I}(B^3 \Theta A_i^3) \rangle.$

Type II:

a) $\cup_{i \in I} A_i \widetilde{\oplus} B = \cup_{i \in I}(A_i \widetilde{\oplus} B) \Longrightarrow$

$\langle \cup_{i \in I} A_i^1 \oplus B^1, \cup_{i \in I} A_i^2 \Theta B^2, \cup_{i \in I} A_i^3 \Theta B^3 \rangle = \langle \cup_{i \in I}(A_i^1 \oplus B^1), \cup_{i \in I}(A_i^2 \Theta B^2), \cup_{i \in I}(A_i^3 \Theta B^3) \rangle.$

Proof: a) we will prove this property for the two types of the neutrosophic crisp union operator:

Type I:

$\cup_{i \in I} A_i \widetilde{\oplus} B = \langle \cup_{b \in B}(\cup_{i \in I} A_{ib}^1), \cup_{b \in B}(\cup_{i \in I} A_{ib}^2), \cap_{b \in B}\big(\cup_{i \in I} A_{i(-b)}^3\big) \rangle =$

$\cup_{i \in I}(A_i \widetilde{\oplus} B) = \langle \cup_{i \in I}(\cup_{b \in B} A_{ib}^1), \cup_{i \in I}(\cup_{b \in B} A_{ib}^2), \cup_{i \in I}(\cap_{b \in B} A_{i(-b)}^3) \rangle$

Type II:

$\cup_{i \in I} A_i \widetilde{\oplus} B = \langle \cup_{b \in B}(\cup_{i \in I} A_{ib}^1), \cap_{b \in B}\big(\cup_{i \in I} A_{i(-b)}^2\big), \cap_{b \in B}\big(\cup_{i \in I} A_{i(-b)}^3\big) \rangle =$

$\cup_{i \in I}(A_i \widetilde{\oplus} B) = \langle \cup_{i \in I}(\cup_{b \in B} A_{ib}^1), \cup_{i \in I}(\cap_{b \in B} A_{i(-b)}^2, \cup_{i \in I}(\cap_{b \in B} A_{i(-b)}^3) \rangle.$

The proof of (b) is similar to (a)

5.2.4 Proposition (Duality Theorem of Neutrosophic Crisp Dilation):

Let $A, B \in \mathcal{N}\mathrm{C}(\mathrm{Z}^2)$ Neutrosophic Crisp Erosion and Dilation are Dual Operations i.e.

Type I:

$co\big(co\ \ A \widetilde{\oplus} B\big) = \langle co(coA^1 \oplus B^1), co(coA^2 \oplus B^2), co(coA^3 \Theta B^3) \rangle = \langle A^1 \Theta B^1, A^2 \Theta B^2, A^3 \oplus B^3 \rangle = A \widetilde{\Theta} B.$

Type II:

$co\big(co\ \ A \widetilde{\oplus} B\big) = \langle co(coA^1 \oplus B^1), co(coA^2 \Theta B^2), co(coA^3 \Theta B^3) \rangle = \langle A^1 \Theta B^1, A^2 \oplus B^2, A^3 \oplus B^3 \rangle = A \widetilde{\Theta} B.$

5.3 Properties of the Neutrosophic Crisp Opening Operator:

5.3.1 Proposition:

The neutrosophic crisp opening satisfies the monotonicity

$\forall \ A, B \in \mathcal{N}C(Z^2)$

Type I:

$A \subseteq B \Longrightarrow \langle A^1 \circ C^1, A^2 \circ C^2, A^3 \circ C^3 \rangle \subseteq \langle B^1 \circ C^1, B^2 \circ C^2, B^3 \circ C^3 \rangle$,

$A^1 \circ C^1 \subseteq B^1 \circ C^1, A^2 \circ C^2 \subseteq B^2 \circ C^2, A^3 \circ C^3 \supseteq B^3 \circ C^3.$

Type II:

$A \subseteq B \Longrightarrow \langle A^1 \circ C^1, A^2 \circ C^2, A^3 \circ C^3 \rangle \subseteq \langle B^1 \circ C^1, B^2 \circ C^2, B^3 \circ C^3 \rangle$,

$A^1 \circ C^1 \subseteq B^1 \circ C^1, A^2 \circ C^2 \supseteq B^2 \circ C^2, A^3 \circ C^3 \supseteq B^3 \circ C^3$

5.3.2 Proposition: for any family $(A_i | i \in I)$ in $\mathcal{N}C(Z^2)$, and $B \in \mathcal{N}C(Z^2)$

Type I:

$\bigcap_{i \in I} A_i \tilde{\circ} B = \bigcap_{i \in I}(A_i \tilde{\circ} B) \Longrightarrow$

$\langle \bigcap_{i \in I} A_i^1 \circ B^1, \bigcap_{i \in I} A_i^2 \circ B^2, \bigcap_{i \in I} A_i^3 \bullet B^3 \rangle = \langle \bigcap_{i \in I}(A_i^1 \circ B^1), \bigcap_{i \in I}(A_i^2 \circ B^2), \bigcap_{i \in I}(A_i^3 \bullet B^3) \rangle$

Type II:

$\bigcap_{i \in I} A_i \tilde{\circ} B = \bigcap_{i \in I}(A_i \tilde{\circ} B) \Longrightarrow$

$\langle \bigcap_{i \in I} A_i^1 \circ B^1, \bigcap_{i \in I} A_i^2 \bullet B^2, \bigcap_{i \in I} A_i^3 \bullet B^3 \rangle = \langle \bigcap_{i \in I}(A_i^1 \circ B^1), \bigcap_{i \in I}(A_i^2 \bullet B^2), \bigcap_{i \in I}(A_i^3 \bullet B^3) \rangle$

5.3.3 Proposition: for any family $(A_i | i \in I)$ in $\mathcal{N}C(Z^2)$ and $B \in \mathcal{N}C(Z^2)$

Type I:

$\bigcup_{i \in I} A_i \tilde{\circ} B = \bigcup_{i \in I}(A_i \tilde{\circ} B) \Longrightarrow$

$\langle \bigcup_{i \in I} A_i^1 \circ B^1, \bigcup_{i \in I} A_i^2 \circ B^2, \bigcup_{i \in I} A_i^3 \bullet B^3 \rangle = \langle \bigcup_{i \in I}(A_i^1 \circ B^1), \bigcup_{i \in I}(A_i^2 \circ B^2), \bigcup_{i \in I}(A_i^3 \bullet B^3) \rangle.$

Type II:

$\bigcup_{i \in I} A_i \tilde{\circ} B = \bigcup_{i \in I}(A_i \tilde{\circ} B) \Longrightarrow$

$\langle \bigcup_{i \in I} A_i^1 \circ B^1, \bigcup_{i \in I} A_i^2 \bullet B^2, \bigcup_{i \in I} A_i^3 \bullet B^3 \rangle = \langle \bigcup_{i \in I}(A_i^1 \circ B^1), \bigcup_{i \in I}(A_i^2 \bullet B^2), \bigcup_{i \in I}(A_i^3 \bullet B^3) \rangle.$

Proof: Is like the procedure used to prove the propositions given previously.

5.4 Properties of the Neutrosophic Crisp Closing

5.4.1 Proposition:

The neutrosophic closing satisfies the monotonicity

$\forall A, B \in \mathcal{N}C(Z^2)$

Type I:

a) $A \subseteq B \Longrightarrow \langle A^1 \bullet C^1, A^2 \bullet C^2, A^3 \bullet C^3\rangle \subseteq \langle B^1 \bullet C^1, B^2 \bullet C^2, B^3 \bullet C^3\rangle$

$A^1 \bullet C^1 \subseteq B^1 \bullet C^1, A^2 \bullet C^2 \subseteq B^2 \bullet C^2, A^3 \bullet C^3 \supseteq B^3 \bullet C^3$

Type II:

a) $A \subseteq B \Longrightarrow \langle A^1 \bullet C^1, A^2 \bullet C^2, A^3 \bullet C^3\rangle \subseteq \langle B^1 \bullet C^1, B^2 \bullet C^2, B^3 \bullet C^3\rangle$

$A^1 \bullet C^1 \subseteq B^1 \bullet C^1, A^2 \bullet C^2 \supseteq B^2 \bullet C^2, A^3 \bullet C^3 \supseteq B^3 \bullet C^3$

5.4.2 Proposition: for any family $(A_i|i \in I)$, in $\mathcal{N}C(Z^2)$, and $B \in \mathcal{N}C(Z^2)$

Type I:

$\bigcap_{i\in I} A_i \tilde{\bullet} B = \bigcap_{i\in I}(A_i \tilde{\bullet} B) \Longrightarrow$

$\langle \bigcap_{i\in I} A_i^1 \bullet B^1, \bigcap_{i\in I} A_i^2 \bullet B^2, \bigcap_{i\in I} A_i^3 \bullet B^3\rangle = \langle \bigcap_{i\in I}(A_i^1 \bullet B^1), \bigcap_{i\in I}(A_i^2 \bullet B^2), \bigcap_{i\in I}(A_i^3 \bullet B^3)\rangle$

Type II:

$\bigcap_{i\in I} A_i \tilde{\bullet} B = \bigcap_{i\in I}(A_i \tilde{\bullet} B) \Longrightarrow$

$\langle \bigcap_{i\in I} A_i^1 \bullet B^1, \bigcap_{i\in I} A_i^2 \circ B^2, \bigcap_{i\in I} A_i^3 \circ B^3\rangle = \langle \bigcap_{i\in I}(A_i^1 \bullet B^1), \bigcap_{i\in I}(A_i^2 \circ B^2), \bigcap_{i\in I}(A_i^3 \circ B^3)\rangle$.

5.4.3 Proposition: for any family$(A_i|i \in I)$, in $\mathcal{N}C(Z^2)$, and $B \in \mathcal{N}C(Z^2)$

Type I:

$\bigcup_{i\in I} A_i \tilde{\bullet} B = \bigcup_{i\in I}(A_i \tilde{\bullet} B) \Longrightarrow$

$\langle \bigcup_{i\in I} A_i^1 \bullet B^1, \bigcup_{i\in I} A_i^2 \bullet B^2, \bigcup_{i\in I} A_i^3 \circ B^3\rangle = \langle \bigcup_{i\in I}(A_i^1 \bullet B^1), \bigcup_{i\in I}(A_i^2 \bullet B^2), \bigcup_{i\in I}(A_i^3 \circ B^3)\rangle$.

Type II:

$\bigcup_{i\in I} A_i \tilde{\bullet} B = \bigcup_{i\in I}(A_i \tilde{\bullet} B) \Longrightarrow$

$\langle \bigcup_{i\in I} A_i^1 \bullet B^1, \bigcup_{i\in I} A_i^2 \circ B^2, \bigcup_{i\in I} A_i^3 \circ B^3\rangle = \langle \bigcup_{i\in I}(A_i^1 \bullet B^1), \bigcup_{i\in I}(A_i^2 \circ B^2), \bigcup_{i\in I}(A_i^3 \circ B^3)\rangle$.

Proof: Is similar to the procedure used to prove the propositions given previously.

5.4.4 Proposition (Duality theorem of Neutrosophic Crisp Closing**):**

Let A, B$A, B \in \mathcal{N}C(Z^2)$; Neutrosophic crisp erosion and dilation are dual operations i.e.

Type I:

$co(co\ \ A \tilde{\bullet} B) = \langle co(co\ \ A^1 \bullet B^1), co(co\ \ A^2 \bullet B^2), co(co\ \ A^3 \circ B^3)\rangle = \langle A^1 \circ B^1, A^2 \circ B^2, A^3 \bullet B^3\rangle = A \tilde{\circ} B.$

Type II:

$co(co\ \ A \tilde{\bullet} B) = \langle co(co\ \ A^1 \bullet B^1), co(co\ \ A^2 \circ B^2), co(co\ \ A^3 \circ B^3)\rangle = \langle A^1 \circ B^1, A^2 \bullet B^2, A^3 \bullet B^3\rangle = A \tilde{\circ} B.$

6. Neutrosophic Crisp Mathematical Morphological Filters:

6.1 Neutrosophic Crisp External Boundary:

Where A^1 is the set of all pixels that belong to the foreground of the picture, A^3 contains the pixels that belong to the background while contains those pixel which do not belong to neither A^2 nor A^1.

Let $\mathrm{A}, \mathrm{B} \in \mathcal{N}\mathrm{C}(\mathrm{Z}^2)$, such that $A = \langle A^1, A^2, A^3 \rangle$, and B is some structure element of the form $B = \langle B^1, B^2, B^3 \rangle$; then the NC boundary extraction filter is defined to be:

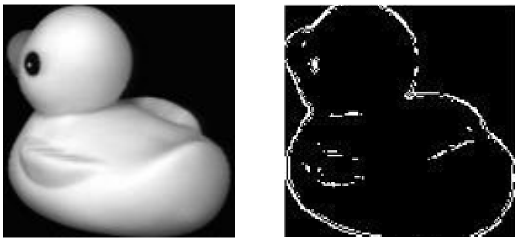
$\partial_1 A^1 = A^1 - (A^1 \Theta B^1)$,

$\partial_3 A^3 = (A^3 \Theta B^3) - A^3$,

$\partial(A) = A^2 - (\partial_1 A^1 \cup \partial_3 A^3)$,

$\partial^*(A) = A^2 - [(A^3 \oplus B^3) - (A^1 \Theta B^1)]$,

$b(A) = \partial^*(A) \cap \partial(A)$.



a) b)

Fig. 6.1: Applying the Neutrosophic Crisp External Boundary: a) the Original Image b) Neutrosophic Crisp Boundary.

6.2 Neutrosophic Crisp Top-Hat Filter:

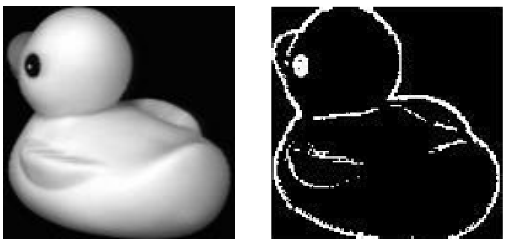
$B_1(A^1) = A^1 - (A^1 \circ B^1)$,

$B_3(A^3) = (A^3 \bullet B^3) - A^1$,

$B(A) = A^2 - (B_1(A^1) \cup B_3(A^3))$,

$B^*(A) = A^2 - [(A^1 \circ B^1) - (A^3 \bullet B^3)]$,

$T\tilde{o}p_{hat}(A) = B(A) \cap B^*(A)$.



a) b)

Fig. 6.2.: Applying the Neutrosophic Crisp Top-Hat Filter: a) Original Image b) Neutrosophic Crisp Components $\langle A^1, A^2, A^3 \rangle$ Respectively.

6.3 Bottom-Hat Filter:

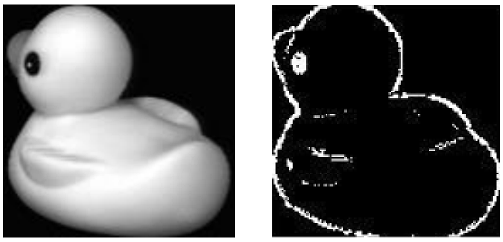
$B_1(A^1) = (A^1 \bullet B^1) - A^1,$

$B_3(A^3) = A^3 - (A^3 \circ B^3),$

$B(A) = A^2 - (B_1(A^1) \cup B_3(A^3)),$

$B^*(A) = A^2 - [(A^1 \bullet B^1) - (A^3 \circ B^3)],$

$Bo\widetilde{tt}omp_{hat}(A) = B(A) \cap B^*(A).$



a) b)

Fig. 6.3.: Applying the Neutrosophic Crisp Bottom-Hat Filter: Neutrosophic Crisp Components $\langle A^1, A^2, A^3 \rangle$ Respectively.

The following diagram represents the relationship between all types of mathematical morphology

Crisp Mathematical Morphology ⟶ **Neutrosophic Crisp Mathematical Morphology**

↓ ↓

Fuzzy Mathematical Morphology ⟶ **Neutrosophic Fuzzy Mathematical Morphology**

7. Conclusion:

In our work, we have proposed a new technique for analyzing and processing images; either grayscale or binary. The technique is a generalization for the fuzzy and crisp mathematical morphology; it handles the image in the neutrosophic domain.in such domain the image is analyzed into three different layers; the first layer describes how much each pixel belongs to the white set, and the third layer describes how much each pixel belongs to the non-white (black) set. In contrast, the second layer describes how much the pixel is neither white nor black. The properties of each layer were used to define the basic operations for what we called "Neutrosophic Mathematical Morphology". mainly, we introduced four basic operations; namely, the neutrosophic dilation, the neutrosophic erosion, the neutrosophic closing and the neutrosophic opening. The algebraic properties of the proposed operation were discussed. Furthermore, we introduced some advanced and generalized concepts of classical and fuzzy mathematical morphology. For this purpose, we developed serval neutrosophic crisp and fuzzy morphological operators; namely, the neutrosophic fuzzy and crisp dilation, the neutrosophic fuzzy and crisp erosion, the neutrosophic crisp opening and the neutrosophic crisp closing operators. These operators were presented in two different types, each type is determined according to the behaviour of the second component of the triple structure of the operator. Furthermore, we developed three neutrosophic crisp morphological filters; namely, the neutrosophic fuzzy and crisp boundary extraction. Some promising experimental results were presented to visualise the effect of the newly introduced operators and filters on the image in the neutrosophic fuzzy and crisp domain instead of the spatial domain.

Acknowledgement

This research is supported by the Neutrosophic Science International Association (NSIA) in both of its headquarters at New Mexico University and its Iraqi branch at Telafer University, for more details about (NSIA), see the URL http://neutrosophicassociation.org/.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: Aug 2, 2023. Accepted: Dec. 20, 2023