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Multi-Attribute Group Decision-Making Based on Aggregation Operator and Score Function of Bipolar Neutrosophic Hypersoft Environment

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Abstract. Hypersoft sets (HSSs) have gained popularity because of their ability to formulate data in the form of several trait-valued disjoint sets that blend various traits. Motivated by this idea, in this study, we present a new hyper-approach referred to as bipolar neutrosophic hypersoft sets (BNHSSs) by a generalization of neutrosophic hypersoft sets (NHSSs) and bipolar fuzzy hypersoft sets (BFHSSs) or by merging and subjecting both HSSs and neutrosophic sets (NSs) to a bipolarity property of real numbers. By utilizing positive and negative neutrosophic structures, we construct different notions and operations on the basis of BNHSSs, such as an absolute BNHSS, a null BNHSS, a complement, subset-hood, a restricted and extended union, and a restricted and extended intersection, along with their related properties. Also, some operations like OR and AND on BNHSS have been initiated. In addition, some properties are displayed paired together, and some numerical hypothetical examples are given to clarify the mechanism of using these instruments. Finally, to prove the efficiency and applicability of the proposed model, we established two novel algorithms based on mathematical techniques (aggregation operator and score function) applied to our model (BNHSS). The aforementioned methods have been utilized in the resolution of a multi-attribute group decision-making (MAGDM) problem. Some discussions and comparisons between the given techniques are also presented to demonstrate their effectiveness and applicability.

Keywords: Bipolar fuzzy Set, Bipolar Neutrosophic Set, Hypersoft Set, Neutrosophic Set, Neutrosophic Hypersoft Set, Soft Set.

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1. General Introduction

In the course of our daily routines, we frequently meet a multitude of circumstances that present dual perspectives or facets of information: the first is positive and the second is negative, and here the human mind tends towards two patterns of thinking: positive thinking and negative thinking in judging such situations. Where the positive character indicates that the information to be evaluated is satisfactory or desirable, while the negative character indicates rejection or negativity of the choice.

On the other hand, multi-attribute group decision-making (MAGDM) methods seek analysis and evaluation of real-life issues that face the human mind and contain full uncertainty nature, including a positive and negative nature, in order to help the decision maker (user) in selecting the best object. In order to handle MCDM issues that contain imprecise and two sides of information (positive and negative information), Zhang introduced a new mathematical approach named bipolar fuzzy sets (BFSs) as an extension of the range of fuzzy set memberships from positive degrees to positive and negative degrees. This concept is characterized by the bifurcation of the fuzzy memberships into two poles, positive membership $\mu^{+} : A \rightarrow [1, 0]$ correspond with positive preferences and desires and negative membership $\mu^{-} : A \rightarrow [0, -1]$ corresponds to a lack of preference and a rejection rate.

2. Literature Review

To handle the complicated MAGDM issues that contain uncertainty, indeterminacy, and consistency, Smarandache [1] proposed a new mathematical notion known as neutrosophic set (NS) by developing the ordinary fuzzy set [2] (FS) and the ordinary intuitionistic fuzzy set [3] (IFS). A neutrosophic set (NS) [4] structure is made up of three functions: truthness, indeterminacy, and falsity functions. Each element in the universal set corresponds to three membership functions, all of which lie in the closed interval [0, 1]. For decades, this novel notion has been used successfully to model uncertainty in several fields such as reasoning, control, pattern recognition, decision making, and computer vision. The NS has been extended and studied by many researchers in various fields, for example. Khalifa and Kumar [5, 6] presented novel approaches regarding trapezoidal neutrosophic numbers and linear fractional programming, respectively, under an interval-value neutrosophic environment. Sallam and Mohamed [7] utilized N-MCDM Methodology for the examination of onshore wind for electricity generation. Nishtar and Afzal [8] work on an analysis of a system for multiple combining schemes. Rodrigo and Maheswari [9] introduced properties and characterizations of a new idea of Ne-mapping namely Neu-open maps and Ne-closed maps in Ne-topological spaces. Researchers did not stop developing this concept at the real level, but rather creative works continued to the complex level, taking into account the importance and characteristics of the complex level.

Ali and Smarandache [10] have further extended the NS to the complex field by developing the notion of complex neutrosophic set (CNS), which is progressed rapidly to complex single-valued neutrosophic set (CSVNS) [11] and Q-complex neutrosophic set (Q-CNS) [12]. In addition to other studies dealing with supply chain (SC) networks [13], facing challenges for a sustainable future, and using logistic systems [14].

In 1999, Molodtsov [15] put forth the notion of a soft set (SS) as a new parametric form when he noticed a gap in the previous concepts, that is their inability to deal with real-world data in the parametric environment. The fertile hybrid environment provided by the SS provoked the attention of researchers and prompted them to create a great deal of contributions by merging the previous concepts with the properties of the SS. Maji [16] introduced and studied the basic definitions and operations of neutrosophic soft set (NSS). Deli and Broumi [17] introduced a preference relationships technique on NSSs that allows to amalgamate two NSSs. Deli [18]again developed a forecasting approach based interval-NSS. Ozturk et al. [19] introduced and studied some definitions and theorems on NS in topological spaces. Saeed et al. [20] applied similarity and distance measures on multi-polar neutrosophic soft set (mpNSS) and experimented it to handle some medical diagnosis and DM-problems. Broumi et al. [21] smelted both SS and NS to produce the idea of complex neutrosophic soft sets (CNSSs). Following them Al-Sharqi et al. [22]- [32] made a great effort to represent the idea of Bromi et al. in an interval manner. Abdel-Basset et al [33] developed a novel risk assessment framework, called RAF-CPWS, which works perfectly to estimate the risks of water and wastewater technologies. In addition, there are contributions in several fields see [34]- [43]. In some practical scenarios, traits that provide further elaboration of the choices should be separated into trait values to provide more clarity. In light of this intent , recently, Smarandache [44,45] has suggested the HSS as an upgraded structure of the SS. Also, he clarified the mechanism of performance of this idea with FS and its extension. According to this idea, Samarandache opened the doors to develop previous models that built on SS by rehashing it into multi-trait function. At present, scholars have released several studies on HSSs. Saeed et al. [46,47] developed fundamental HSS operations. Yolcu and Ozturk [48] prepared critical decision-making applications for fuzzy hypersoft (FHSS). Saeed et al. [49] conceptualized the notion of FHSS under interval form when they established the notion of interval-FHSS. More results were shown on IFHSS by Yolcu et. al. [50]. Some mathematical measures on neutrosophic hypersoft set (NHSS) were demonstrated by Saqlain et al. [51].

On the opposite side, the principle of bipolarity craeted to handel practical challenges encountered in everyday life, which are given by two distinct aspects. namily positive aspect and negative aspect, such as black and white, return and progress, profit and loss and et. Then, Zhang [52,53] is the first initiaed the idea of bipolar fuzzy set (BFS) when he extension of the range of fuzzy set memberships from positive degrees to positive and negative degrees. This concept is characterized by the bifurcation of the fuzzy memberships into two poles, positive membership $\mu^+ : A \to [1, 0]$ correspond with positive preferences and desires and negative membership $\mu^- : A \to [0, -1]$ corresponds to a lack of preference and a rejection rate. Naz and Shabir [54] built some algebraic structure on fuzzy bipolar soft set (BFSS).

The idea of bipolar soft set (BSS) has been redefined by Karaaslan and Karatas [55]. Mahmood [56] improved the previous definitions of BSS by establishing the notion of T-bipolar soft sets which is more close to the concept of bipolarity as compared to the previous ones. Jana and Pal [57] applied the bipolarity information on IFSS. Deli et al. [58] elaborated on the notion of bipolar-NS (BNS). Ali et al. [59] presented bipolar-NSS (BNSS) and trailed it to decipher decision-making problems. In complex space, Mahmood and Rehman [60] first proposed an approach to bipolar complex fuzzy sets (BCFSs), which is closer to bipolarity When comparing this model with other models. Then, Aczel-Alsina aggregation operators applied by Mahmood et al. [61] on bipolar complex fuzzy information to handle MCGDM issues. Following in this direction, Al-Quran et al. [62] established the concept of complex bipolar- valued NSS as a hybrid model of BNSS and complex fuzzy set (CFS).

Recently, the concept of BSS was expanded to the bipolar hypersoft set (BHSS) by Musa and Asaad [63], and they presented some basec algebraic properties. Following this direction, Al-Quran et al. [64] extended the notion of BHSS to BFHSSs. However, BFHSSs can only handle uncertain data but not be able todeal with ambiguous, contradictory, and indeterminate information which usually results in real-life problems. To adapt to such situations, we propose a new hybrid approach, namely BNHSS, By combining the qualities that distinguish BNS and HSS from each other. BNHSS is superior to BFHSS with its three independent membership functions, which play a role in increasing the accuracy of the end decision. Therefore, the advantages and benefits of the suggested method are shown as follows. Firstly, BNHSS exhibits a high level of applicability in real-life scenarios when decision-makers seek to address dualistic or dichotomous judgemental thinking, encompassing both positive and negative perspectives. Secondly, the purpose of this study is to include the concept of bipolarity into decision-making processes through the utilization of the HSS, the HSS is equipped with a parameterization tool that enables the portrayal of sub-divided features in a more comprehensive and thorough manner. Thirdly, another advantage is the inclusion of the neutrosophic set, which possesses the capacity to simultaneously analyze and handle truth, indeterminate, and false information in order to facilitate decision-making. Finally, the suggested model incorporates all of the aforementioned components into a single framework, rendering it more suitable for addressing decision-making challenges that are not amenable to other existing decision-making models.

This article is split into the following parts: Figure 1:

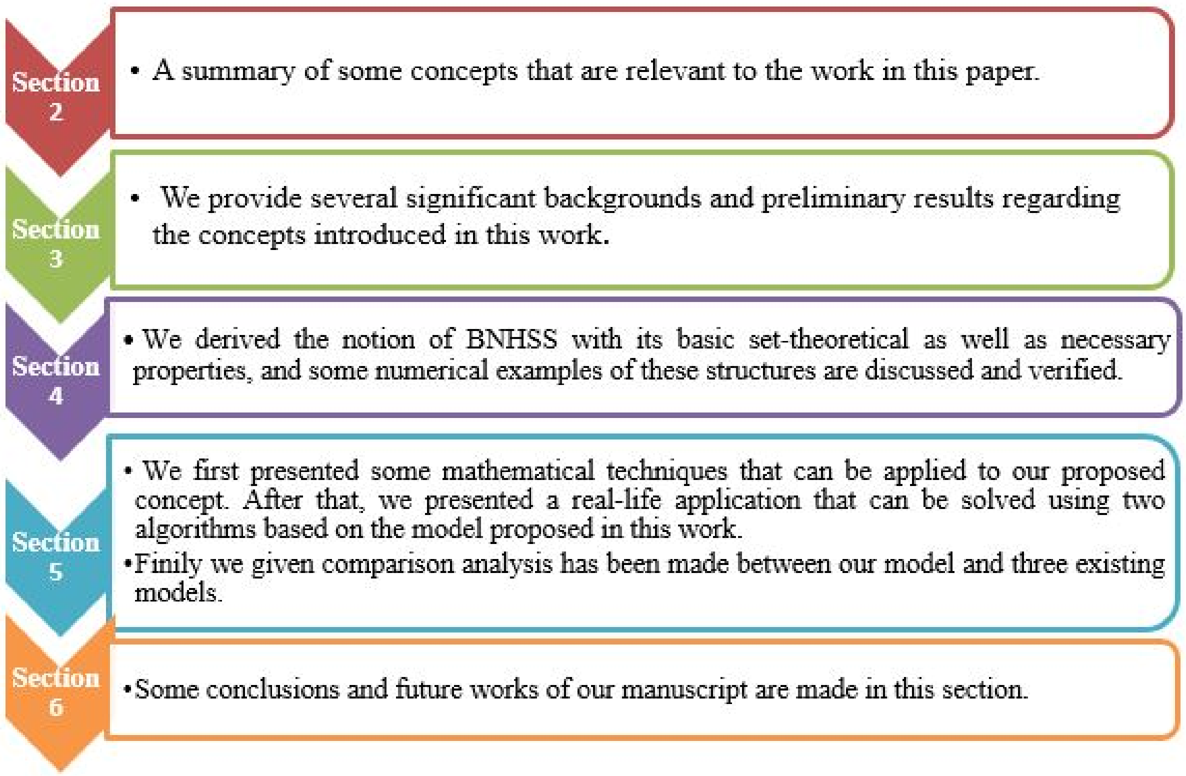

FIGURE 1. show how we organize our manuscript in a brief way.

3. Preliminaries

This part revised some ideas connected to the suggested work. We review SS, HSS, BNS, BNSS and NHSS.

Molodtsov [8] defined the idea of SS as a set-valued map that helps the user describe objects by utilizing many parameters.

Definition 3.1. [8] A SS $\left(\hat{\mathcal{G}}, \mathcal{A}\right)$ on $\hat{\mathfrak{C}}$ non-empty universal set is represented as a mapping as follows:

$$\hat{\mathcal{G}} : \mathcal{A} \to \widehat{\mathbb{P}}\left(\hat{\mathfrak{C}}\right)$$

where $\widehat{\mathbb{P}}\left(\hat{\mathfrak{C}}\right)$ is the power set of $\hat{\mathfrak{C}}$ and here both $\hat{\mathfrak{C}}$ and $\mathcal{A} \subseteq \mathfrak{M}$ refer to the non-empty universal set and the parameter family respectively.

Smarandache [44] expanded the idea of SS to HSS by modifying the function to incorporate many attributes.

Definition 3.2. [44] A HSS structures $\left(\hat{\mathcal{G}}, \mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{W}_3 \times ... \times \mathcal{W}_n\right)$ on the non-empty universal set $\hat{\mathfrak{C}}$ portrayed as follows:
$\{(\nu, \hat{\mathcal{G}}(\nu)) : \hat{\mathcal{G}}(\nu) \subseteq \hat{\mathfrak{C}}, \forall \nu \in \mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{W}_3 \times ... \times \mathcal{W}_n \subseteq \mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times ... \times \mathfrak{A}_n\}$, where $\mathfrak{A}_i : i = 1, 2, ..., n$ are separate sets of parameters terms and $W_i \subseteq \mathfrak{A}_i, \forall i = 1, 2, ..., n$.

Definition 3.3. [1] A NS structures

$$\hat{\mathcal{S}} = \{< \hat{\varkappa}; \mathcal{T}_{\mathcal{S}}(\hat{\varkappa}), \mathcal{I}_{\mathcal{S}}(\hat{\varkappa}), \mathcal{F}_{\mathcal{S}}(\hat{\varkappa}) >: \hat{\varkappa} \in \mathcal{X}\},$$

on non-empty universal set $\mathcal{X}$ called neutrosophic set (NS),

where $\mathcal{T}_{\mathcal{S}}$; $\mathcal{I}_{\mathcal{S}}$; $\mathcal{F}_{\mathcal{S}} : \mathcal{X} \rightarrow]^-0; 1^+[$ denoted to the TM,IM and FM of any object $\hat{\varkappa} \in \mathcal{X}$, respectively with $^-0 \leq \mathcal{T}_{\mathcal{S}} + \mathcal{I}_{\mathcal{S}} + \mathcal{F}_{\mathcal{S}} \leq 3^+$.

Deli et al. [59] generalized BFS by defining BNS as follows.

Definition 3.4. [59] The BNS $\mathbb{A}$ on the universe $\mathfrak{C}$ is signified as follows.
$\mathbb{A} = \{\langle \mathfrak{c}; \mathcal{T}^+_{\mathbb{A}}(\hat{\varkappa}), \mathcal{T}^-_{\mathbb{A}}(\hat{\varkappa}), \mathcal{I}^+_{\mathbb{A}}(\hat{\varkappa}), \mathcal{I}^-_{\mathbb{A}}(\hat{\varkappa}), \mathcal{F}^+_{\mathbb{A}}(\hat{\varkappa}), \mathcal{F}^-_{\mathbb{A}}(\hat{\varkappa})\rangle : \hat{\varkappa} \in \mathfrak{C}\}$, where, $\mathcal{T}^+, \mathcal{I}^+, \mathcal{F}^+ : \mathfrak{C} \rightarrow [0, 1]$ denote, respectively the positive-TM, positive-IM and positive-FM degrees of an element $\hat{\varkappa} \in \mathfrak{C}$ to the property in line with a BNS $\mathbb{A}$, and $\mathcal{T}^-, \mathcal{I}^-, \mathcal{F}^- : \mathfrak{C} \rightarrow [-1, 0]$ denote, respectively the negative-TM, negative-IM and negative-FM degrees of an object $\hat{\varkappa} \in \mathfrak{C}$.

Ali et al. [59] defined BNSS and its fundamental operations as in the following two definitions.

Definition 3.5. [59] A structures $(\mathbb{F}, \mathbb{A})$ is called a BNSS over the universe $\mathfrak{C}$, where $\mathbb{F}$ is a transformation given by $\mathbb{F} : \mathbb{A} \longrightarrow BN(\mathfrak{C})$ and $BN(\mathfrak{C})$ refers to the set of all bipolar neutrosophic subsets of $\mathfrak{C}$.

Definition 3.6. [59] Suppose $(\mathbb{F}, \mathbb{A})$ and $(\mathbb{G}, \mathbb{B})$ are two BNSSs over the non-empty universal set $\mathfrak{C}$, then $(\mathbb{F}, \mathbb{A})$ is given as:
$\mathbb{F}(\mathfrak{a}) = \Big\{\langle \hat{\varkappa}, \{\mathcal{T}^+_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}), \mathcal{T}^-_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}), \mathcal{I}^+_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}), \mathcal{I}^-_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}), \mathcal{F}^+_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}),$
$\mathcal{F}^-_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa})\}\rangle : \forall \hat{\varkappa} \in \mathfrak{C}, \mathfrak{a} \in \mathbb{A}\Big\}$ and the second BNSS $(\mathbb{G}, \mathbb{B})$ is given as $\mathbb{G}(\mathfrak{b}) = \Big\{\langle \hat{\varkappa}, \{\mathcal{T}^+_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{T}^-_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{I}^+_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{I}^-_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{F}^+_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{F}^-_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa})\}\rangle : \forall \hat{\varkappa} \in \mathfrak{C}, \mathfrak{b} \in \mathbb{B}\Big\}$. Then,

(1.) $\mathbb{F}^c(\mathfrak{a}) = \Big\{\langle \hat{\hat{\varkappa}}, \{\mathcal{F}^+_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}), \mathcal{F}^-_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}), 1 - \mathcal{I}^+_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}), -1 - \mathcal{I}^-_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}),$
$\mathcal{T}^+_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}), \mathcal{T}^-_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}})\}\rangle : \forall \hat{\varkappa} \in \mathfrak{C}, \mathfrak{a} \in \mathbb{A}\Big\}$,

(2.) $\mathbb{F}(\mathfrak{a}) \subseteq \mathbb{G}(\mathfrak{b})$ iff:
$\mathcal{T}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}) \leq \mathcal{T}^{+}_{\mathbb{G}(\mathfrak{b})}(\hat{\hat{\varkappa}}),\ \mathcal{T}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}) \geq \mathcal{T}^{-}_{\mathbb{G}(\mathfrak{b})}(\hat{\hat{\varkappa}}),\ \mathcal{I}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}) \geq \mathcal{I}^{+}_{\mathbb{G}(\mathfrak{b})}(\hat{\hat{\varkappa}}),$
$\mathcal{I}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}) \leq \mathcal{I}^{-}_{\mathbb{G}(\mathfrak{b})}(\hat{\hat{\varkappa}}),\ \mathcal{F}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}) \geq \mathcal{F}^{+}_{\mathbb{G}(\mathfrak{b})}(\hat{\hat{\varkappa}}), \mathcal{F}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}) \leq \mathcal{F}^{-}_{\mathbb{G}(\mathfrak{b})}(\hat{\hat{\varkappa}}),$

(3.) $\mathbb{F}(\mathfrak{a}) \cup \mathbb{G}(\mathfrak{b}) = \{\langle \hat{\hat{\varkappa}}, \mathcal{T}^{+}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \vee \mathcal{T}^{+}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}}),\ \mathcal{T}^{-}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \wedge \mathcal{T}^{-}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}}),$
$\mathcal{I}^{+}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \wedge \mathcal{I}^{+}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}}),\ \mathcal{I}^{-}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \vee \mathcal{I}^{-}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}}),\ \mathcal{F}^{+}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \wedge \mathcal{F}^{+}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}}),$
$\mathcal{F}^{-}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \vee \mathcal{F}^{-}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}})\rangle : \forall \hat{\hat{\varkappa}} \in \mathfrak{C}, \epsilon \in \mathbb{A} \cap \mathbb{B}\},$

(4.) $\mathbb{F}(\mathfrak{a}) \cap \mathbb{G}(\mathfrak{b}) = \{\langle \hat{\hat{\varkappa}}, \mathcal{T}^{+}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \wedge \mathcal{T}^{+}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}}),\ \mathcal{T}^{-}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \vee \mathcal{T}^{-}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}}),$
$\mathcal{I}^{+}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \vee \mathcal{I}^{+}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}}),\ \mathcal{I}^{-}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \wedge \mathcal{I}^{-}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}}),\ \mathcal{F}^{+}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \vee \mathcal{F}^{+}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}}),$
$\mathcal{F}^{-}_{\mathbb{F}(\epsilon)}(\hat{\hat{\varkappa}}) \wedge \mathcal{F}^{-}_{\mathbb{G}(\epsilon)}(\hat{\hat{\varkappa}})\rangle : \forall \hat{\hat{\varkappa}} \in \mathfrak{C}, \epsilon \in \mathbb{A} \cap \mathbb{B}\},$

where max $= \vee$ and min $= \wedge$.

NHSS is defined for the first time by Smarandache [50] in the following manner.

Definition 3.7. [50] A NHSS structures $\left(\hat{\mathcal{H}}, \mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{W}_3 \times ... \times \mathcal{W}_n\right)$ on the non-empty universal set $\hat{\mathfrak{C}}$ portrayed as a mapping as follows:

$$\hat{\mathcal{H}} : \mathcal{W} \longrightarrow NH(\hat{\mathfrak{C}})$$

where the component $NH(\hat{\mathfrak{C}})$ refer to a family of all NSs over non-empty universal set $\hat{\mathfrak{C}}$ such that $\hat{\mathcal{H}}(\nu) = \{(\hat{\varkappa}, \mathfrak{T}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa}), \mathfrak{I}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa}), \mathfrak{F}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa})) : \hat{\varkappa} \in \mathfrak{C}, \nu \in W = W_1 \times W_2 \times W_3 \times ... \times W_n \subseteq \mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times ... \times \mathfrak{A}_n\}$,
such that $\mathfrak{T}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa}), \mathfrak{I}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa})$ and $\mathfrak{F}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa})$ are the TM, IM and FM, respectively and $\mathfrak{A}_i : i = 1, 2, ..., n$ are pairwise disjoint sets of attribute values.

Recently, Al-Quran et al. [64] have extended the notions of HSS and BFSS by introducing the notion of BFHSS as in the following definition.

Definition 3.8. [64] A BFHSS structures (Φ, Λ) on the non-empty universal set $\hat{\mathfrak{C}}$ portrayed as a mapping as follows:

$$\Phi : \Lambda \to P\left(\hat{\mathfrak{C}}\right)$$

and written as: $(\Phi, \Lambda) = \{\langle \alpha, \{(\hat{m}, \mathcal{T}^{+}_{\Phi(\alpha)}(\hat{m}), \mathcal{T}^{-}_{\Phi(\alpha)}(\hat{m})) : \forall \hat{m} \in \hat{\mathfrak{C}}\}\rangle : \alpha \in \Lambda \subseteq \Delta\}$.
where $\Lambda = \mathcal{J}_{\nu_1} \times \mathcal{J}_{\nu_1} \times ... \times \mathcal{J}_{\nu_n}$, $\Delta = \mathcal{H}_{\nu_1} \times \mathcal{H}_{\nu_1} \times ... \times \mathcal{H}_{\nu_n}$,and $P\left(\hat{\mathfrak{C}}\right)$ indicated to power of non-empty universal set $\hat{\mathfrak{C}}$.

4. Bipolar Neutrosophic Hypersoft Set

This section of our work consists of presenting the primary definition of BNHSS along with some illustrations and hypothetical examples, basic set theory operations, and some rudimentary properties.

Definition 4.1. Let $\hat{\chi}$ be a universal set and $\widehat{p}\ (\hat{\chi})$ denotes to the powerful set of $\hat{\chi}$. Let $\mu_k : k = 1, 2, ..., m$ are m-well-defined qualities that are in line with characteristics and facets values respectively , the pairwise disjoint sets $\mathcal{C}_{\mu_k}$: $k = 1, 2, ..., m$. Let $\mathcal{D}_{\mu_k}$ be the nonempty subset of $\mathcal{C}_{\mu_k}$ $\forall k = 1, 2, ..., m$.
A BNHS (Ψ, Γ) is identified by the following mapping $\Psi : \Gamma \to \widehat{p}\ (\hat{\chi})$ whose functional value is the BNS
$\Psi(\nu) = \Big\{\langle \hat{\varkappa}, \{\mathcal{T}^+_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^-_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^+_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^-_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^+_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^-_{\Psi(\nu)}(\hat{\varkappa})\}\rangle : \forall \hat{\varkappa} \in \hat{\chi}, \nu \in \Gamma \subseteq \Omega\Big\}$,
where $\Gamma = \mathcal{D}_{\mu_1} \times \mathcal{D}_{\mu_2} \times ... \times \mathcal{D}_{\mu_m}$ and $\Omega = \mathcal{C}_{\mu_1} \times \mathcal{C}_{\mu_2} \times ... \times \mathcal{C}_{\mu_m}$ such that $\mathcal{T}^+, \mathcal{I}^+, \mathcal{F}^+ : \chi \to [0, 1]$ denote, respectively the positive-TM, positive-IM and positive-FM degrees of the attribute ν_* with regard to component $\hat{\varkappa}_*$ for the property in line with a BNHS (Ψ, Γ), while $\mathcal{T}^-, \mathcal{I}^-, \mathcal{F}^- :$ $\chi \to [-1, 0]$ denote, respectively the negative-TM, negative-IM and negative-FM degrees of some implicit counter-property of the attribute ν_* with regard to component $\hat{\varkappa}_*$ line with a a BNHS (Ψ, Γ).

We can view the BNHS (Ψ, Γ) as follows:

$$(\Psi, \Gamma) = \Big\{\Big\langle \nu, \big\{\big(\hat{\varkappa}, \mathcal{T}^+_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^-_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^+_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^-_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^+_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^-_{\Psi(\nu)}(\hat{\varkappa})\big) : \forall \hat{\varkappa} \in \hat{\chi}\big\}\Big\rangle : \nu \in \Gamma \subseteq \Omega\Big\}.$$

The following numerical example makes above definition clear.

Example 4.2. Suppose the alternatives set encompasses three mobile phones of the same brand $\hat{\chi} = \{\hat{\varkappa}_1, \hat{\varkappa}_2, \hat{\varkappa}_3\}$ and the attributes are μ_1 = Price, μ_2 = Camera resolution, μ_3 = RAM size.Suppose the attribute's values are
$\mathcal{C}_{\mu_1} = \{\alpha_1 = 1200, \alpha_2 = 1500, \alpha_3 = 2000\}$, $\mathcal{C}_{\mu_2} = \{\alpha_4 = 8MP, \alpha_5 = 12MP, \alpha_6 = 16MP\}$, $\mathcal{C}_{\mu_3} = \{\alpha_7 = 6GB, \alpha_8 = 8GB, \alpha_9 = 12GB\}$. If we take the subset $\mathcal{D}_{\mu_k}$ of $\mathcal{C}_{\mu_k}$ $\forall k = 1, 2, 3$ as follows.

$\mathcal{D}_{\mu_1} = \{\alpha_2 = 1500, \alpha_3 = 2000\}$,
$\mathcal{D}_{\mu_2} = \{\alpha_5 = 12MP\}$, $\mathcal{D}_{\mu_3} = \{\alpha_7 = 6GB, \alpha_8 = 8GB\}$. Then, we obtain the following BNHS (Ψ, Γ)

$$(\Psi, \Gamma) = \Big\{\Big\langle\big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\varkappa}_1, .6, -.1, .5, -.9, .8, -.1), (\hat{\varkappa}_2, .7, -.4, .1, -.2, .7, -.5), (\hat{\varkappa}_3, .6, -.4, .6, -.4, .5, -.7)\}\big), \big((\alpha_2, \alpha_5, \alpha_8), \{(\hat{\varkappa}_1, .5, -.2, .2, -.4, .5, -.6), (\hat{\varkappa}_2, .2, -.4, .1, -.5, .3, -.6), (\hat{\varkappa}_3, .6, -.2, .1, -.3, .9, -.8)\}\big), \big((\alpha_3, \alpha_5, \alpha_7), \{(\hat{\varkappa}_1, 0, -.4, .8, -.3, .4, -.5), (\hat{\varkappa}_2, .6, -1, .2, -.3, .4, -.5), (\hat{\varkappa}_3, .8, -.9, .4, -.8,$$

$.2, -.5)\}\big), \big((\alpha_3, \alpha_5, \alpha_8), \{(\hat{\varkappa}_1, .8, -.5, .3, -.4, .6, -.3), (\hat{\varkappa}_2, .7, -.1, .7, -.3, .8, -.9), (\hat{\varkappa}_3, 1, -.9, .2, -.5, .7, -.3)\}\big)\Big\rangle\Big\}.$

Definition 4.3. Let $\hat{\chi}$ be a non-empty universe. A BNHS denoted by $(\Psi, \Gamma)_0$, is called empty BNHS and defined as:

$(\Psi, \Gamma)_0 = \Big\{\big\langle \nu, \{\hat{\hat{\varkappa}}, 0, 0, 1, -1, 1, -1\}\big\rangle : \forall \hat{\varkappa} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega\Big\}$, where $\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}) = 0, \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = 1, \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}) = \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = -1,\ \forall \hat{\hat{\varkappa}} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega.$

Definition 4.4. Let $\hat{\chi}$ be a non-empty universe. A BNHS denoted by $(\Psi, \Gamma)_{\chi}$, is called absolute BNHS and defined as:

$(\Psi, \Gamma)_{\chi} = \Big\{\big\langle \nu, \{\hat{\hat{\varkappa}}, 1, -1, 0, 0, 0, 0\}\big\rangle : \forall \hat{\varkappa} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega\Big\}$, where $\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = 1, \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = -1$ and $\mathcal{I}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = 0,\ \forall \hat{\hat{\varkappa}} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega.$

The complement operator of the BNHS is defined in this part.

Definition 4.5. Let $(\Psi, \Gamma) = \Big\{\big\langle \nu, \big\{\big(\hat{\varkappa}, \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})\big) : \forall \hat{\varkappa} \in \hat{\chi}\big\}\big\rangle : \nu \in \Gamma \subseteq \Omega\Big\}$ be a BNHS. Then the complement of (Ψ, Γ) is denoted by $(\Psi, \Gamma)^c$ and is defined as:

$(\Psi, \Gamma)^c = (\Psi^c, \Gamma) = \Big\{\big\langle \nu, \big\{\big(\hat{\hat{\varkappa}}, \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}), 1 - \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}), -1 - \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}), \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa})\big) : \forall \hat{\hat{\varkappa}} \in \hat{\chi}\big\}\big\rangle : \nu \in \Gamma \subseteq \Omega\Big\}$

Now, we display the use of the complement operator through an example as follows:

Example 4.6. With reference to Example 3.2. The complement of the BNHS (Ψ, Γ) is

$(\Psi, \Gamma)^c = (\Psi^c, \Gamma) = \Big\{\Big\langle\big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\varkappa}_1, .8, -.1, .5, -.1, .6, -.1), (\hat{\varkappa}_2, .7, -.5, .9, -.8, .7, -.4), (\hat{\varkappa}_3, .5, -.7, .4, -.6, .6, -.4)\}\big), \big((\alpha_2, \alpha_5, \alpha_8), \{(\hat{\varkappa}_1, .5, -.6, .8, -.6, .5, -.2), (\hat{\varkappa}_2, .3, -.6, .9, -.5, .2, -.4), (\hat{\varkappa}_3, .9, -.8, .9, -.7, .6, -.2)\}\big), \big((\alpha_3, \alpha_5, \alpha_7), \{(\hat{\varkappa}_1, .4, -.5, .2, -.7, 0, -.4), (\hat{\varkappa}_2, .4, -.5, .8, -.7, .6, -1), (\hat{\varkappa}_3, .2, -.5, .6, -.2, .8, -.9)\}\big), \big((\alpha_3, \alpha_5, \alpha_8), \{(\hat{\varkappa}_1, .6, -.3, .7, -.6, .8, -.5), (\hat{\varkappa}_2, .8, -.9, .3, -.7, .7, -.1), (\hat{\varkappa}_3, .7, -.3, .8, -.5, 1, -.9)\}\big)\Big\rangle\Big\}.$

Proposition 4.7. *The complement of the complement of a BNHS* (Ψ, Γ) *is simply the BNHS* (Ψ, Γ) *itself. In symbols,* $((\Psi, \Gamma)^c)^c = (\Psi, \Gamma)$.

Proof: Suppose the BNHS $(\Psi, \Gamma) = \Big\{\langle \nu, \{(\hat{\varkappa}, \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})) : \forall \hat{\varkappa} \in \hat{\chi}\} \rangle : \nu \in \Gamma \subseteq \Omega\Big\}$. By Definition 3.5, $(\Psi, \Gamma)^c = (\Psi^c, \Gamma) =$
$\Big\{\langle \nu, \{(\hat{\varkappa}, \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa}), 1 - \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}), -1 - \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa})) : \forall \hat{\varkappa} \in \hat{\chi}\} \rangle : \nu \in \Gamma \subseteq \Omega\Big\}$. Using Definition 3.5 again, we obtain.

$((\Psi, \Gamma)^c)^c =$
$\Big\{\langle \nu, \{(\hat{\varkappa}, \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}), 1 - (1 - \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa})), -1 - (-1 - \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa})), \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})) : \forall \hat{\varkappa} \in \hat{\chi}\} \rangle : \nu \in \Gamma \subseteq \Omega\Big\}$,
$= \Big\{\langle \nu, \{(\hat{\varkappa}, \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})) : \forall \hat{\varkappa} \in \hat{\chi}\} \rangle : \nu \in \Gamma \subseteq \Omega\Big\}$,
$= (\Psi, \Gamma)$.

Proposition 4.8. *Assume, (Ψ, Γ) is a BNHS over $\hat{\chi}$. Then,*

1. $((\Psi, \Gamma)_0)^c = (\Psi, \Gamma)_\chi$,
2. $((\Psi, \Gamma)_\chi)^c = (\Psi, \Gamma)_0$.

1. Suppose $(\Psi, \Gamma)_0 = \Big\{\langle \nu, \{\hat{\varkappa}, 0, 0, 1, -1, 1, -1\} \rangle : \forall \hat{\varkappa} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega\Big\}$ is an empty BNHS. Based on Definition 11, $((\Psi, \Gamma)_0)^c = \Big\{\langle \nu, \{\hat{\varkappa}, 1, -1, 1 - 1, -1 - (-1), 0, 0\} \rangle : \forall \hat{\varkappa} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega\Big\}$,
$= \Big\{\langle \nu, \{\hat{\varkappa}, 1, -1, 0, 0, 0, 0\} \rangle : \forall \hat{\varkappa} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega\Big\} = (\Psi, \Gamma)_\chi$.
2. Proof of this item is similar to that of (1).

Now, we define subset-hood operator on two BNHSs.

Definition 4.9. Suppose (Ψ, Γ) and (Φ, Λ) are two BNHSs over $\hat{\chi}$. Where $(\Psi, \Gamma) = \Big\{\langle \nu, \{(\hat{\varkappa}, \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})) : \forall \hat{\varkappa} \in \hat{\chi}\} \rangle : \nu \in \Gamma \subseteq \Omega\Big\}$ and $(\Phi, \Lambda) = \Big\{\langle \nu, \{(\hat{\varkappa}, \mathcal{T}^{+}_{\Phi(\nu)}(\hat{\varkappa}), \mathcal{T}^{-}_{\Phi(\nu)}(\hat{\varkappa}), \mathcal{I}^{+}_{\Phi(\nu)}(\hat{\varkappa}), \mathcal{I}^{-}_{\Phi(\nu)}(\hat{\varkappa}), \mathcal{F}^{+}_{\Phi(\nu)}(\hat{\varkappa}), \mathcal{F}^{-}_{\Phi(\nu)}(\hat{\varkappa})) : \forall \hat{\varkappa} \in \hat{\chi}\} \rangle : \nu \in \Lambda \subseteq \Omega\Big\}$. We said that (Ψ, Γ) is a subset of (Φ, Λ), denoted as $(\Psi, \Gamma) \subseteq (\Phi, \Lambda)$, if:

1. $\Gamma \subseteq \Lambda$,
2. $\forall \nu \in \Gamma,\ \forall \hat{\varkappa} \in \hat{\chi}$, $\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}) \leq \mathcal{T}^{+}_{\Phi(\nu)}(\hat{\varkappa})$, $\mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}) \geq \mathcal{T}^{-}_{\Phi(\nu)}(\hat{\varkappa})$, $\mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}) \geq \mathcal{I}^{+}_{\Phi(\nu)}(\hat{\varkappa})$, $\mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}) \leq \mathcal{I}^{-}_{\Phi(\nu)}(\hat{\varkappa})$, $\mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}) \geq \mathcal{F}^{+}_{\Phi(\nu)}(\hat{\varkappa})$, $\mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa}) \leq \mathcal{F}^{-}_{\Phi(\nu)}(\hat{\varkappa})$.

Remark 4.10. From Definition 3.9, it is clear that $((\Psi, \Gamma)_0) \subseteq (\Psi, \Gamma)_\chi$.

The equality between two BNHSs (Ψ, Γ) and (Φ, Λ) can be defined as follows.

Definition 4.11. We said that (Ψ, Γ) is equal to (Φ, Λ), denoted as $(\Psi, \Gamma) = (\Phi, \Lambda)$, if:

1. $\Gamma = \Lambda$,
2. $\forall \nu \in \Gamma,\ \forall \hat{\hat{\varkappa}} \in \hat{\chi},\quad \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{T}^{+}_{\Phi(\nu)}(\hat{\hat{\varkappa}}),\quad \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{T}^{-}_{\Phi(\nu)}(\hat{\varkappa}),\quad \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{I}^{+}_{\Phi(\nu)}(\hat{\varkappa}),$ $\mathcal{I}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{I}^{-}_{\Phi(\nu)}(\hat{\hat{\varkappa}}),\quad \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{F}^{+}_{\Phi(\nu)}(\hat{\hat{\varkappa}}),\quad \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{F}^{-}_{\Phi(\nu)}(\hat{\hat{\varkappa}}).$

The following, is a numerical example clarifies Definition 3.9.

Example 4.12. Consider Example 1 and suppose that $\mathcal{E}_{\mu_1} = \{\alpha_3 = 2000\}$, $\mathcal{E}_{\mu_2} = \{\alpha_5 = 12MP\}$, $\mathcal{E}_{\mu_3} = \{\alpha_7 = 6GB, \alpha_8 = 8GB\}$, be another subsets of $\mathcal{C}_{\mu_k}\ \forall k = 1,2,3$ and $\Lambda = \mathcal{E}_{\mu_1} \times \mathcal{E}_{\mu_2} \times \mathcal{E}_{\mu_3}$. Then, we can obtain the following BNHS (Φ,Λ), where, $(\Phi,\Lambda) =$ $\Big\{\Big\langle\big((\alpha_3,\alpha_5,\alpha_7),\{(\hat{\varkappa}_1,0,-0.2,0.9,-0.5,0.6,-0.7),(\hat{\varkappa}_2,0.3,-0.8,0.3,-0.4,0.7,-0.7),(\hat{\varkappa}_3,0.6,-0.5,0.5,-0.9,0.6,-0.7)\}\big),$ $\big((\alpha_3,\alpha_5,\alpha_8),\{(\hat{\varkappa}_1,0.6,-0.3,0.5,-0.6,0.7,-0.4),(\hat{\varkappa}_2,0.6,0,0.8,-0.4,0.9,-0.9),(\hat{\varkappa}_3,1,-0.7,0.3,-0.6,0.8,-0.4)\}\big)\Big\rangle\Big\}$. Based on Definition 4.9, it is clear that $(\Phi,\Lambda) \subseteq (\Psi,\Gamma)$, where $(\Psi,\Gamma) =$ $\Big\{\Big\langle\big((\alpha_2,\alpha_5,\alpha_7),\{(\hat{\hat{\varkappa}}_1,.6,-.1,.5,-.9,.8,-.1),(\hat{\varkappa}_2,.7,-.4,.1,-.2,.7,-.5),(\hat{\hat{\varkappa}}_3,.6,-.4,.6,-.4,.5,-.7)\}\big),\big((\alpha_2,\alpha_5,\alpha_8),\{(\hat{\varkappa}_1,.5,-.2,.2,-.4,.5,-.6),(\hat{\varkappa}_2,.2,-.4,.1,-.5,.3,-.6),(\hat{\varkappa}_3,.6,-.2,.1,-.3,.9,-.8)\}\big),\big((\alpha_3,\alpha_5,\alpha_7),\{(\hat{\hat{\varkappa}}_1,0,-.4,.8,-.3,.4,-.5),(\hat{\hat{\varkappa}}_2,.6,-1,.2,-.3,.4,-.5),(\hat{\hat{\varkappa}}_3,.8,-.9,.4,-.8,.2,-.5)\}\big),\big((\alpha_3,\alpha_5,\alpha_8),\{(\hat{\hat{\varkappa}}_1,.8,-.5,.3,-.4,.6,-.3),(\hat{\hat{\varkappa}}_2,.7,-.1,.7,-.3,.8,-.9),(\hat{\varkappa}_3,1,-.9,.2,-.5,.7,-.3)\}\big)\Big\rangle\Big\}$.

To combine two BNHSs into a single BNHS, we will define the following fundamental operations on BNHSs.

Definition 4.13. The restricted union of two BNHSs (Ψ,Γ) and (Φ,Λ) over the universe $\hat{\chi}$ is signified by $(\Psi,\Gamma) \Cup_R (\Phi,\Lambda)$ and stated as: $(\Pi_R,\Upsilon) = (\Psi,\Gamma) \Cup_R (\Phi,\Lambda)$, where $\Upsilon = \Gamma \cap \Lambda$ and (Π_R,Υ) is characterized as:

$$(\Pi_R,\Upsilon) = \Big\{\big\langle\varepsilon,\big\{\big(\hat{\hat{\varkappa}},\mathcal{T}^{+}_{\Psi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big) \vee \mathcal{T}^{+}_{\Phi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big),\mathcal{T}^{-}_{\Psi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big) \wedge \mathcal{T}^{-}_{\Phi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big),\mathcal{I}^{+}_{\Psi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big) \wedge \mathcal{I}^{+}_{\Phi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big),\mathcal{I}^{-}_{\Psi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big) \vee \mathcal{I}^{-}_{\Phi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big),\mathcal{F}^{+}_{\Psi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big) \wedge \mathcal{F}^{+}_{\Phi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big),\mathcal{F}^{-}_{\Psi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big) \vee \mathcal{F}^{-}_{\Phi(\varepsilon)}\big(\hat{\hat{\varkappa}}\big) : \forall\hat{\hat{\varkappa}} \in \hat{\chi}\big\}\big\rangle : \varepsilon \in \Gamma \cap \Lambda\Big\}.$$

Where max $= \vee$ and min $= \wedge$.

To clarify Definition 3.13, we provide the following example.

Example 4.14. Consider the BNHS (Ψ, Γ) in Example 3.2, where $(\Psi, \Gamma) =$
$\Big\{\Big\langle\big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .6, -.1, .5, -.9, .8, -.1), (\hat{\hat{\varkappa}}_2, .7, -.4, .1, -.2, .7, -.5), (\hat{\hat{\varkappa}}_3, .6, -.4, .6, -.4, .5, -.7)\}\big), \big((\alpha_2, \alpha_5, \alpha_8), \{(\hat{\hat{\varkappa}}_1, .5, -.2, .2, -.4, .5, -.6), (\hat{\hat{\varkappa}}_2, .2, -.4, .1, -.5, .3, -.6), (\hat{\hat{\varkappa}}_3, .6, -.2, .1, -.3, .9, -.8)\}\big), \big((\alpha_3, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, 0, -.4, .8, -.3, .4, -.5), (\hat{\hat{\varkappa}}_2, .6, -1, .2, -.3, .4, -.5), (\hat{\varkappa}_3, .8, -.9, .4, -.8, .2, -.5)\}\big), \big((\alpha_3, \alpha_5, \alpha_8), \{(\hat{\hat{\varkappa}}_1, .8, -.5, .3, -.4, .6, -.3), (\hat{\hat{\varkappa}}_2, .7, -.1, .7, -.3, .8, -.9), (\hat{\hat{\varkappa}}_3, 1, -.9, .2, -.5, .7, -.3)\}\big)\Big\rangle\Big\}$.

Suppose that $\mathcal{H}_{\mu_1} = \{\alpha_1 = 1200, \alpha_2 = 1500\}$, $\mathcal{H}_{\mu_2} = \{\alpha_5 = 12MP, \alpha_6 = 16MP\}$, $\mathcal{H}_{\mu_3} = \{\alpha_7 = 6GB\}$, be another subsets of $\mathcal{C}_{\mu_k}$ $\forall k = 1, 2, 3$ and $\lambda = \mathcal{H}_{\mu_1} \times \mathcal{H}_{\mu_2} \times \mathcal{H}_{\mu_3}$. Then, we obtain the following BNHS (Θ, λ), where, $(\Theta, \lambda) =$
$\Big\{\Big\langle\big((\alpha_1, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .1, -.8, .3, -.2, .9, -.1), (\hat{\hat{\varkappa}}_2, .5, -.1, .1, -.6, .3, -.2), (\hat{\hat{\varkappa}}_3, .8, -.7, .9, -.2, .9, -.8)\}\big), \big((\alpha_1, \alpha_6, \alpha_7), \{(\hat{\varkappa}_1, .6, -.4, .7, -.3, .2, -.7), (\hat{\hat{\varkappa}}_2, .3, -.5, .1, -.6, .4, -.7), (\hat{\hat{\varkappa}}_3, .8, -.4, .1, -.3, .8, -.1)\}\big), \big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, 0, -1, .5, -.1, .2, -.7), (\hat{\hat{\varkappa}}_2, 0, -1, .8, -.5, 1, -.5), (\hat{\hat{\varkappa}}_3, .2, -1, .5, -.7, .1, -.2)\}\big), \big((\alpha_2, \alpha_6, \alpha_7), \{(\hat{\hat{\varkappa}}_1, 1, -.5, .2, -.8, .5, -.2), (\hat{\hat{\varkappa}}_2, 0, -.1, 1, -.3, .2, -.9), (\hat{\hat{\varkappa}}_3, 1, -.7, .5, -.3, .2, -.4)\}\big)\Big\rangle\Big\}$.

The restricted union of (Ψ, Γ) and (Θ, λ) can be calculated as follows.
$(\Psi, \Gamma) \Cup_R (\Theta, \lambda) =$

$\Big\{\Big\langle\big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .6, -1, .5, -.1, .2, -.1), (\hat{\hat{\varkappa}}_2, .7, -1, .1, -.2, .7, .5), (\hat{\hat{\varkappa}}_3, .6, -1, .5, -.4, .1, -.2)\}\big)\Big\rangle\Big\}$.

The following properties hold under the BNHS union.

Proposition 4.15. *Let (Ψ, Γ), (Φ, Λ) and (Θ, λ) be three BNHSs over $\hat{\chi}$. Then,*

1. $(\Psi, \Gamma) \Cup_R (\Psi, \Gamma)_0 = (\Psi, \Gamma)$,
2. $(\Psi, \Gamma) \Cup_R (\Psi, \Gamma)_{\hat{\chi}} = (\Psi, \Gamma)_{\hat{\chi}}$,
3. $(\Psi, \Gamma) \Cup_R (\Phi, \Lambda) = (\Phi, \Lambda) \Cup_R (\Psi, \Gamma)$,
4. $\big((\Psi, \Gamma) \Cup_R (\Phi, \Lambda)\big) \Cup_R (\Theta, \lambda) = (\Psi, \Gamma) \Cup_R \big((\Phi, \Lambda) \Cup_R (\Theta, \lambda)\big)$.

Definition 4.16. The extended union of two BNHSs (Ψ,Γ) and (Φ,Λ) over the universe $\hat{\chi}$ is signified by $(\Psi,\Gamma)\Cup_E(\Phi,\Lambda)$ and stated as: $(\Pi_E,\Upsilon)=(\Psi,\Gamma)\Cup_E(\Phi,\Lambda)$, where $\Upsilon=\Gamma\cup\Lambda$ and $\forall\varepsilon\in\Upsilon,\ \ \forall\hat{\hat{\varkappa}}\in\hat{\chi}$,

$$\mathcal{T}^{+}_{\Pi_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{T}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{T}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{T}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\vee\mathcal{T}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{T}^{-}_{\Pi_E(\varepsilon)}\left(\hat{\varkappa}\right)=\begin{cases}\mathcal{T}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{T}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{T}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right)\wedge\mathcal{T}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{I}^{+}_{\Pi_E(\varepsilon)}\left(\hat{\varkappa}\right)=\begin{cases}\mathcal{I}^{+}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{I}^{+}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{I}^{+}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right)\wedge\mathcal{I}^{+}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{I}^{-}_{\Pi_E(\varepsilon)}\left(\hat{\varkappa}\right)=\begin{cases}\mathcal{I}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{I}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{I}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right)\vee\mathcal{I}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{F}^{+}_{\Pi_E(\varepsilon)}\left(\hat{\varkappa}\right)=\begin{cases}\mathcal{F}^{+}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{F}^{+}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{F}^{+}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right)\wedge\mathcal{F}^{+}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{F}^{-}_{\Pi_E(\varepsilon)}\left(\hat{\varkappa}\right)=\begin{cases}\mathcal{F}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{F}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{F}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right)\vee\mathcal{F}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

Where max $=\vee$ and min $=\wedge$.

To clarify Definition 3.16, we provide the following hypothetical example.

Example 4.17. Consider Example 3.14. The extended union of (Ψ,Γ) and (Θ,λ) can be calculated as follows.

$(\Psi,\Gamma)\Cup_E(\Theta,\lambda)=$

$\Big\{\Big\langle\big((\alpha_2,\alpha_5,\alpha_7),\{(\hat{\hat{\varkappa}}_1,.6,-1,.5,-.1,.2,-.1),(\hat{\hat{\varkappa}}_2,$
$.7,-1,.1,-.2,.7,.5),(\hat{\hat{\varkappa}}_3,.6,-1,.5,-.4,.1,$
$-.2)\}\big),\big((\alpha_2,\alpha_5,\alpha_8),\{(\hat{\hat{\varkappa}}_1,.5,-.2,.2,-.4,.5,-.6),$
$(\hat{\hat{\varkappa}}_2,.2,-.4,.1,-.5,.3,-.6),(\hat{\hat{\varkappa}}_3,.6,-.2,.1,-.3,$
$.9,-.8)\}\big),\big((\alpha_3,\alpha_5,\alpha_7),\{(\hat{\hat{\varkappa}}_1,0,-.4,.8,-.3,.4,-.5$

$), (\hat{\hat{\varkappa}}_2, .6, -1, .2, -.3, .4, -.5), (\hat{\hat{\varkappa}}_3, .8, -.9, .4, -.8,$
$.2, -.5)\}\big), \big((\alpha_3, \alpha_5, \alpha_8), \{(\hat{\hat{\varkappa}}_1, .8, -.5, .3, -.4, .6,$
$-.3), (\hat{\varkappa}_2, .7, -.1, .7, -.3, .8, -.9), (\hat{\hat{\varkappa}}_3, 1, -.9, .2,$
$-.5, .7, -.3)\}\big), \big((\alpha_1, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .1, -.8, .3, -.2,$
$.9, -.1), (\hat{\hat{\varkappa}}_2, .5, -.1, .1, -.6, .3, -.2), (\hat{\hat{\varkappa}}_3, .8, -.7,$
$.9, -.2, .9, -.8)\}\big), \big((\alpha_1, \alpha_6, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .6, -.4, .7,$
$-.3, .2, -.7), (\hat{\hat{\varkappa}}_2, .3, -.5, .1, -.6, .4, -.7), (\hat{\hat{\varkappa}}_3, .8,$
$-.4, .1, -.3, .8, -.1)\}\big), \big((\alpha_2, \alpha_6, \alpha_7), \{(\hat{\hat{\varkappa}}_1, 1, -.5, .2,$
$-.8, .5, -.2), (\hat{\hat{\varkappa}}_2, 0, -.1, 1, -.3, .2, -.9), (\hat{\hat{\varkappa}}_3, 1, -.7,$
$.5, -.3, .2, -.4)\}\big)\Big\rangle\Big\}.$

Definition 4.18. The restricted intersection of two BNHSs (Ψ, Γ) and (Φ, Λ) over the non-empty universe $\hat{\chi}$ is signified by $(\Psi, \Gamma) \Cap_R (\Phi, \Lambda)$ and stated as: $(\Xi_R, \Upsilon) = (\Psi, \Gamma) \Cap_R (\Phi, \Lambda)$, where $\Upsilon = \Gamma \cap \Lambda$ and (Ξ_R, Υ) is characterized as:

$$(\Xi_R, \Upsilon) = \Big\{\langle \varepsilon, \big\{\big(\hat{\varkappa}, \mathcal{T}^+_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \wedge \mathcal{T}^+_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right), \mathcal{T}^-_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \vee \mathcal{T}^-_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right), \mathcal{I}^+_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \vee \mathcal{I}^+_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right), \mathcal{I}^-_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \wedge \mathcal{I}^-_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right), \mathcal{F}^+_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \vee \mathcal{F}^+_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right), \mathcal{F}^-_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \wedge \mathcal{F}^-_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) : \forall \hat{\varkappa} \in \hat{\chi}\big\}\rangle : \varepsilon \in \Gamma \cap \Lambda\Big\}.$$

Where max $= \vee$ and min $= \wedge$.

To clarify Definition 4.18, we provide the following hypothetical example.

Example 4.19. Consider Example 3.14. The restricted intersection of (Ψ, Γ) and (Θ, λ) can be calculated as follows.

$(\Psi, \Gamma) \Cap_R (\Theta, \lambda) =$

$$\Big\{\Big\langle\big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\varkappa}_1, 0, -.1, .5, -.9, .8, -.7), (\hat{\hat{\varkappa}}_2, 0, -.4, .8, -.5, 1, -.5), (\hat{\hat{\varkappa}}_3, .2, -.4, .6, -.7, .5, -.7)\}\big)\Big\rangle\Big\}.$$

The following properties hold under the BNHS intersection.

Proposition 4.20. *Let (Ψ, Γ), (Φ, Λ) and (Θ, λ) be three BNHSs over $\hat{\chi}$. Then,*

1. $(\Psi, \Gamma) \Cap_R (\Psi, \Gamma)_0 = (\Psi, \Gamma)_0$,
2. $(\Psi, \Gamma) \Cap_R (\Psi, \Gamma)_{\hat{\chi}} = (\Psi, \Gamma)$,
3. $(\Psi, \Gamma) \Cap_R (\Phi, \Lambda) = (\Phi, \Lambda) \Cap_R (\Psi, \Gamma)$,
4. $\big((\Psi, \Gamma) \Cap_R (\Phi, \Lambda)\big) \Cap_R (\Theta, \lambda) = (\Psi, \Gamma) \Cap_R \big((\Phi, \Lambda) \Cap_R (\Theta, \lambda)\big)$.

Definition 4.21. The extended intersection of two BNHSs (Ψ,Γ) and (Φ,Λ) over the universe $\hat{\chi}$ is signified by $(\Psi,\Gamma)\Cap_E(\Phi,\Lambda)$ and stated as: $(\Delta_E,\Upsilon)=(\Psi,\Gamma)\Cap_E(\Phi,\Lambda)$, where $\Upsilon=\Gamma\cup\Lambda$ and $\forall\varepsilon\in\Upsilon,\ \ \forall\hat{\varkappa}\in\hat{\chi}$,

$$\mathcal{T}^+_{\Delta_E(\varepsilon)}(\hat{\varkappa})=\begin{cases}\mathcal{T}^+_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{T}^+_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{T}^+_{\Psi(\varepsilon)}(\hat{\varkappa})\wedge\mathcal{T}^+_{\Phi(\varepsilon)}(\hat{\varkappa}) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{T}^-_{\Delta_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{T}^-_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{T}^-_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{T}^-_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\vee\mathcal{T}^-_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{I}^+_{\Delta_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{I}^+_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{I}^+_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{I}^+_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\vee\mathcal{I}^+_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{I}^-_{\Delta_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{I}^-_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{I}^-_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{I}^-_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\wedge\mathcal{I}^-_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{F}^+_{\Delta_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{F}^+_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{F}^+_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{F}^+_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\vee\mathcal{F}^+_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{F}^-_{\Delta_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{F}^-_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{F}^-_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{F}^-_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\wedge\mathcal{F}^-_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

Where max $=\vee$ and min $=\wedge$.

To clarify Definition 3.21, we provide the following hypothetical example.

Example 4.22. Take Example 3.14. The extended intersection of (Ψ,Γ) and (Θ,λ) can be calculated as follows.

$(\Psi,\Gamma)\Cap_E(\Theta,\lambda)=$
$\Big\{\Big\langle\big((\alpha_2,\alpha_5,\alpha_7),\{(\hat{\varkappa}_1,0,-.1,.5,-.9,.8,-.7),(\hat{\hat{\varkappa}}_2,$
$0,-.4,.8,-.5,1,-.5),(\hat{\hat{\varkappa}}_3,.2,-.4,.6,-.7,.5,$
$-.7)\}\big),\big((\alpha_2,\alpha_5,\alpha_8),\{(\hat{\hat{\varkappa}}_1,.5,-.2,.2,-.4,.5,-.6),$
$(\hat{\hat{\varkappa}}_2,.2,-.4,.1,-.5,.3,-.6),(\hat{\hat{\varkappa}}_3,.6,-.2,.1,-.3,$
$.9,-.8)\}\big),\big((\alpha_3,\alpha_5,\alpha_7),\{(\hat{\hat{\varkappa}}_1,0,-.4,.8,-.3,.4,-.5$

$), (\hat{\hat{\varkappa}}_2, .6, -1, .2, -.3, .4, -.5), (\hat{\hat{\varkappa}}_3, .8, -.9, .4, -.8, .2, -.5)\}\Big), \Big((\alpha_3, \alpha_5, \alpha_8), \{(\hat{\hat{\varkappa}}_1, .8, -.5, .3, -.4, .6, -.3), (\hat{\hat{\varkappa}}_2, .7, -.1, .7, -.3, .8, -.9), (\hat{\hat{\varkappa}}_3, 1, -.9, .2, -.5, .7, -.3)\}\Big), \Big((\alpha_1, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .1, -.8, .3, -.2, .9, -.1), (\hat{\hat{\varkappa}}_2, .5, -.1, .1, -.6, .3, -.2), (\hat{\hat{\varkappa}}_3, .8, -.7, .9, -.2, .9, -.8)\}\Big), \Big((\alpha_1, \alpha_6, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .6, -.4, .7, -.3, .2, -.7), (\hat{\hat{\varkappa}}_2, .3, -.5, .1, -.6, .4, -.7), (\hat{\hat{\varkappa}}_3, .8, -.4, .1, -.3, .8, -.1)\}\Big), \Big((\alpha_2, \alpha_6, \alpha_7), \{(\hat{\hat{\varkappa}}_1, 1, -.5, .2, -.8, .5, -.2), (\hat{\hat{\varkappa}}_2, 0, -.1, 1, -.3, .2, -.9), (\hat{\hat{\varkappa}}_3, 1, -.7, .5, -.3, .2, -.4)\}\Big)\Big\rangle\Big\}.$

In the following, we define AND and OR operations on BNHSs.

Definition 4.23. Let (Ψ, Γ) and (Φ, Λ) be two BNHSs over the universe $\hat{\chi}$. Then, AND operation is a BNHS over $\hat{\chi}$ and signified by

$(\Psi, \Gamma) \bigtriangledown (\Phi, \Lambda) = (\Re, \Gamma \times \Lambda)$, where, $\Re(\nu_i, \eta_j) = \Psi(\nu_i) \bar{\cap} \Phi(\eta_j)$, $\forall (\nu_i, \eta_j) \in \Gamma \times \Lambda$, where $\bar{\cap}$ is a BN-intersection.

The following is an example on AND operation.

Example 4.24. Consider Example 4, where $\nu_1 = (\alpha_2, \alpha_5, \alpha_7)$, $\nu_2 = (\alpha_2, \alpha_5, \alpha_8)$ are the hypersoft parameters(attributes) for the BNHS (Ψ, Γ) and $\eta_1 = (\alpha_1, \alpha_5, \alpha_7), \eta_2 = (\alpha_1, \alpha_6, \alpha_7)$ are the hypersoft parameters(attributes) for the BNHS (Θ, λ). Then $\Gamma \times \lambda = \{(\nu_1, \eta_1), (\nu_1, \eta_2), (\nu_2, \eta_1), (\nu_2, \eta_2)\}$. The values of $(\Psi, \Gamma) \bigtriangledown (\Theta, \lambda) = (\Re, \Gamma \times \lambda)$ is as follows.

$\{\langle((\nu_1 \times \eta_1), \{(\hat{\hat{\varkappa}}_1, .1, -.8, .5, -.2, .9, -.1), (\hat{\hat{\varkappa}}_2, .5, -.4, .1, -.2, .7, -.2), (\hat{\hat{\varkappa}}_3, .6, -.7, .9, -.2, .9, -.7)\}), ((\nu_1 \times \eta_2), \{(\hat{\hat{\varkappa}}_1, .6, -.4, .7, -.3, .8, -.1), (\hat{\hat{\varkappa}}_2, .2, -.5, .1, -.2, .7, -.5), (\hat{\hat{\varkappa}}_3, .6, -.4, .6, -.3, .8, -.1)\}), ((\nu_2 \times \eta_1), \{(\hat{\hat{\varkappa}}_1, .1, -.8, .3, -.2, .9, -.1), (\hat{\hat{\varkappa}}_2, .2, -.5, .1, -.5, .3, -.2), (\hat{\hat{\varkappa}}_3, .6, -.7, .1, -.3, .9, -.8)\}), ((\nu_2 \times \eta_2), \{(\hat{\hat{\varkappa}}_1, .5, -.4, .7, -.3, .5, -.6), (\hat{\hat{\varkappa}}_2, .3, -.1, .2, -.3, .4, -.5), (\hat{\hat{\varkappa}}_3, .8, -.9, .4, -.3, .8, -.1)\})\rangle\}$

Here, we provide the definition of OR operation.

Definition 4.25. Let (Ψ, Γ) and (Φ, Λ) be two BNHSs over the universe $\hat{\chi}$. Then, OR operation is a BNHS over $\hat{\chi}$ and signified by $(\Psi, \Gamma) \bigtriangleup (\Phi, \Lambda) = (\Sigma, \Gamma \times \Lambda)$, where, $\Sigma(\nu_i, \eta_j) = \Psi(\nu_i) \bar{\cup} \Phi(\eta_j), \forall (\nu_i, \eta_j) \in \Gamma \times \Lambda$, where $\bar{\cup}$ is a BN-union.

Example 4.26. Consider Example 3.24. Then, $(\Psi, \Gamma) \triangle (\Phi, \Lambda) = (\Sigma, \Gamma \times \Lambda)$ is calculated as follows.

$$= \{\langle((\nu_1 \times \eta_1), \left\{\left(\hat{\hat{\varkappa}}_1, .6, -.1, .3, -.9, .8, -.1\right),\right.$$
$$\left(\hat{\hat{\varkappa}}_2, .7, -.1, .1, -.6, .3, -.5\right),$$
$$\left.\left(\hat{\hat{\varkappa}}_3, .8, -.4, .6, -.4, .5, -.8\right)\right\}),$$
$$((\nu_1 \times \eta_2), \left\{\left(\hat{\hat{\varkappa}}_1, .6, -.1, .5, -.9, .2, -.7\right),\right.$$
$$\left(\hat{\hat{\varkappa}}_2, .7, -.4, .1, -.6, .4, -.7\right),$$
$$\left.\left(\hat{\hat{\varkappa}}_3, .8, -.4, .1, -.4, .5, -.7\right)\right\}),$$
$$((\nu_2 \times \eta_1), \left\{\left(\hat{\hat{\varkappa}}_1, .5, -.2, .2, -.4, .5, -.6\right),\right.$$
$$\left(\hat{\hat{\varkappa}}_2, .5, -.1, .1, -.6, .3, -.6\right),$$
$$\left.\left(\hat{\hat{\varkappa}}_3, .8, -.2, .1, -.3, .9, -.8\right)\right\}),$$
$$((\nu_2 \times \eta_2), \left\{\left(\hat{\hat{\varkappa}}_1, .6, -.2, .2, -.4, .2, -.7\right),\right.$$
$$\left(\hat{\hat{\varkappa}}_2, .3, -.4, .1, -.6, .3, -.7\right),$$
$$\left.\left(\hat{\hat{\varkappa}}_3, .8, -.4, .1, -.8, .2, -.5\right)\right\})\rangle\}$$

5. **Applicability of BNHSSs in MAGDM based on mathematical tools**

In this part, we will demonstrate the mechanism for applying our proposed approach to dealing with real-life problems that include uncertainty data with two sides (positive and negative) by proposing two algorithms based on some mathematical tools that can be adapted to our approach, such as the score function (SF) of BNHSS and the aggregation operator (AO) of BNHS. Therefore, we will first begin by presenting the mathematical definitions for each SF of BNHSS and AO of BNHS.

Definition 5.1. For BNHSN $\Psi = \left(\mathcal{T}^+_\Psi, \mathcal{T}^-_\Psi, \mathcal{I}^+_\Psi, \mathcal{I}^-_\Psi, \mathcal{F}^+_\Psi, \mathcal{F}^-_\Psi\right)$ then the SF value defined as $S(\Psi) = \frac{\left(\mathcal{T}^+_\Psi + 1 - \mathcal{I}^+_\Psi + 1 - \mathcal{F}^+_\Psi + 1 + \mathcal{T}^-_\Psi - \mathcal{I}^-_\Psi - \mathcal{F}^-_\Psi\right)}{6}$.

Definition 5.2. Assume that (Ψ, Γ) be a BNHS over $\hat{\chi}$. Where $(\Psi, \Gamma) = \left\{\left\langle \nu, \left\{\left(\hat{\hat{\varkappa}}, \mathcal{T}^+_{\Psi(\nu)}(\hat{\hat{\varkappa}}), \mathcal{T}^-_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^+_{\Psi(\nu)}(\hat{\hat{\varkappa}}), \mathcal{I}^-_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^+_{\Psi(\nu)}(\hat{\hat{\varkappa}}), \mathcal{F}^-_{\Psi(\nu)}(\hat{\hat{\varkappa}})\right) : \forall \hat{\hat{\varkappa}} \in \hat{\chi}\right\}\right\rangle : \nu \in \Gamma \subseteq \Omega\right\}$. Then AO of BNHS, denoted by $\widehat{\mathfrak{B}}_{agg}$ and defined as the following:

$$\widehat{\mathfrak{B}}_{agg} = \left\{\Xi_{\widehat{\mathfrak{B}}}\left(\hat{\hat{\varkappa}}\right) : \hat{\hat{\varkappa}} \in \hat{\chi}\right\}$$

Such that:

$$\Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa}) = \frac{1}{2\left|\Omega \times \hat{\chi}\right|} \sum_{\upsilon \in \Gamma \subseteq \Omega} \left(\left|1 - \mathcal{I}^+_{\Psi(\hat{\varkappa})}\left(\mathcal{T}^+_{\Psi(\hat{\varkappa})} - \mathcal{F}^+_{\Psi(\hat{\varkappa})}\right) + 1 - \mathcal{I}^-_{\Psi(\hat{\varkappa})}\left(\mathcal{T}^-_{\Psi(\hat{\varkappa})} - \mathcal{F}^-_{\Psi(\hat{\varkappa})}\right)\right|\right)$$

For the purpose of solving this problem, we will organize above two algorithms based on definitions 4.1 and 4.2, as shown in Figures 1 and 2.

5.1. *Numerical Example*

Choosing a professor to work at a private university: Private universities are always looking to improve their academic reputation, so they work to select teaching staff according to strict standards. Therefore, this selection process is classified as a multi-criteria selection problem. Here, in this partial section, we assume that a private university wants to choose a professor to teach genetics in the Department of Biological Sciences in the College of Science among a number of applicants according to multiple criteria, including academic qualification, scientific degree, and scientific experience. Also, these standards have sub-criteria that are compatible with HSSs. Accordingly, two committees were selected from the college deanship to undertake the task of interviewing each candidate individually in accordance with the criteria mentioned above. Based on this interview, the two committees formulate their opinions in accordance with our proposed model.

Assumptions:

(1) Let $\hat{\chi} = \{\hat{\varkappa}_1, \hat{\varkappa}_2, \hat{\varkappa}_3\}$ be the set of candidates to fill the job advertised.
(2) Let μ be a set of attributes include μ_1 = Academic Qualification, μ_2 = Scientific Degree,μ_3 =Scientific Experience : the criteria upon which selection is made.
(3) The attributes mentioned in (2) are categorized into the following:
$\mu_1 = \alpha_1$ =Phd ,α_2 = Post Doctorate
$\mu_2 = \alpha_3$ =Assistant Professor ,α_4 = Associate Professor
$\mu_3 = \alpha_5$ =3 years ,α_6 = 5 years ,α_7 = 10 years

Now, we can apply the two proposed algorithms 1 and 2 to help the committee choose suitable candidates as follows:

Algorithm 1. Using score function (SF) values $S(\Psi)$ to choose suitable candidate

Step 1. Put up BNHSSs $(\Psi,\Gamma)_{G_1}, (\Psi,\Gamma)_{G_2}$ respectively, based on expert opinions (two committees).

Step 2. Calculating the union value $(\Psi,\Gamma)_{G_1 \cup G_2}$ between two BNHSSs which given in step 1.

Step 3. Compute the value SF value of $(\Psi,\Gamma)_{G_1 \cup G_2}$ based on definition 4.1.

Step 4. Find the value $\mathcal{M}_i = \sum_{i=1}^{3} S(\Psi)_i$ for the candidate X_i,$i = 1, 2, 3$.

Step 5. Decision: Choose the highest value of $\mathcal{M}_i$.

Step 6. End algorithm 1.

In addition Figure 2 bellow representation of algorithm 1.

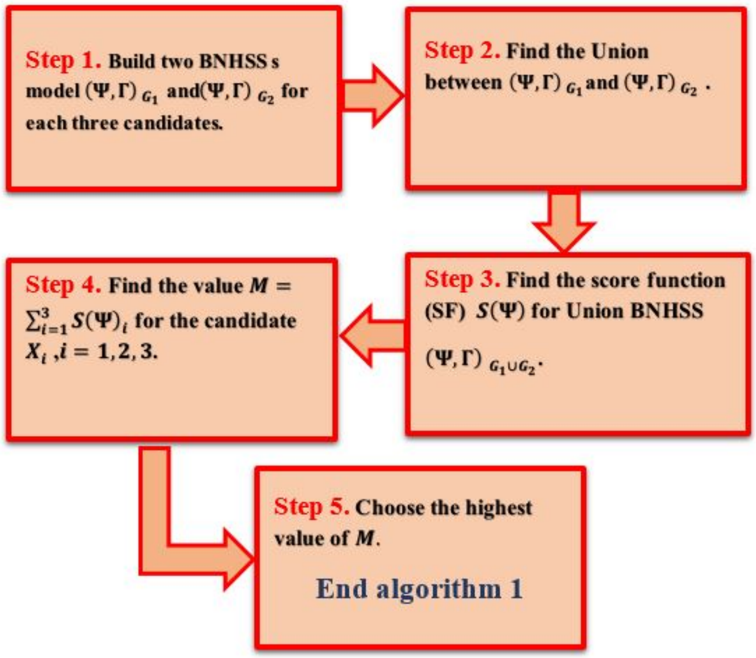

FIGURE 2. Algorithm 1. depends on score function (SF) values $S(\Psi)$

Step1. The jury members put their valuable opinions of each candidate in the form of two BNHSSs separately, as follows:

$(\Psi, \Gamma)_{G_1} =$
$\{(\alpha_1, \alpha_3, \alpha_5), [\hat{\varkappa}_1, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7], [\hat{\varkappa}_2, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7],$
$[\hat{\varkappa}_3, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7],$
$(\alpha_1, \alpha_4, \alpha_5), [\hat{\varkappa}_1, 0.4, -0.3, 0.1, -0.4, 0.3, -0.6], [\hat{\varkappa}_2, 0.3, -0.4, 0.2, -0.6, 0.8, -0.5],$
$[\hat{\varkappa}, 0.4, -0.3, 0.7, -0.4, 0.9, -0.2],$
$(\alpha_1, \alpha_4, \alpha_7), [\hat{\varkappa}_1, 0.3, -0.2, 0.6, -0.4, 0.2, -0.9], [\hat{\varkappa}_2, 0.2, -0.6, 0.8, -0.2, 0.9, -0.1],$
$[\hat{\varkappa}_3, 0.4, -0.2, 0.9, -0.3, 0.8, -0.8],$
$(\alpha_1, \alpha_3, \alpha_6), [\hat{\varkappa}_1, 0.3, -0.6, 0.3, -0.5, 0.8, -0.3], [\hat{\varkappa}_2, 0.6, -0.8, 0.3, -0.6, 0.3, -0.9],$
$[\hat{\varkappa}_3, 0.3, -0.5, 0.7, -0.7, 0.9, -0.1],$
$(\alpha_2, \alpha_3, \alpha_5), [\hat{\varkappa}_1, 0.3, -0.8, 0.2, -0.4, 0.3, -0.9], [\hat{\varkappa}_2, 0.3, -0.1, 0.2, -0.4, 0.8, -0.3],$

$[\hat{\varkappa}_3, 0.1, -0.4, 0.3, -0.8, 0.4, -0.7]$,
$(\alpha_2, \alpha_4, \alpha_5), [\hat{\varkappa}_1, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7], [\hat{\varkappa}_2, 0.6, -0.8, 0.3, -0.6, 0.3, -0.9]$,
$[\hat{\varkappa}_3, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7]$,
$(\alpha_2, \alpha_4, \alpha_7), [\hat{\varkappa}_1, 0.2, -0.2, 0.4, -0.2, 0.7, -0.3], [\hat{\varkappa}_2, 0.5, -0.3, 0.9, -0.4, 0.3, -0.5]$,
$[\hat{\varkappa}_3, 0.4, -0.5, 0.8, -0.7, 0.4, -0.3]$,
$(\alpha_2, \alpha_3, \alpha_6), [\hat{\varkappa}_1, 0.5, -0.4, 0.8, -0.3, 0.8, -0.2], [\hat{\varkappa}_2, 0.3, -0.6, 0.3, -0.5, 0.8, -0.3]$,
$[\hat{\varkappa}_3, 0.3, -0.5, 0.7, -0.7, 0.9, -0.1]\}$

$(\Psi, \Gamma)_{G_2} =$
$\{(\alpha_1, \alpha_3, \alpha_5), [x_1, 0.5, -0.8, 0.1, -0.5, 0.4, -0.3], [x_2, 0.5, -0.3, 0.2, -0.5, 0.3, -0.9]$,
$[x_3, 0.3, -0.5, 0.2, -0.9, 0.4, -0.2]$,
$(\alpha_1, \alpha_4, \alpha_5), [x_1, 0.3, -0.6, 0.3, -0.5, 0.8, -0.3], [x_2, 0.6, -0.8, 0.3, -0.6, 0.3, -0.9]$,
$[x_3, 0.3, -0.5, 0.7, -0.7, 0.9, -0.1]$,
$(\alpha_1, \alpha_4, \alpha_7), [x_1, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7], [x_2, 0.6, -0.8, 0.3, -0.6, 0.3, -0.9]$,
$[x_3, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7]$,
$(\alpha_1, \alpha_3, \alpha_6), [x_1, 0.6, -0.6, 0.2, -0.8, 0.5, -0.4], [x_2, 0.3, -0.9, 0.2, -0.9, 0.2, -0.7]$,
$[x_3, 0.7, -0.8, 0.9, -0.2, 0.8, -0.3]$,
$(\alpha_2, \alpha_3, \alpha_5), [x_1, 0.1, -0.8, 0.2, -0.7, 0.3, -0.4], [x_2, 0.3, -0.1, 0.8, -0.4, 0.5, -0.3]$,
$[x_3, 0.2, -0.5, 0.3, -0.6, 0.3, -0.7]$,
$(\alpha_2, \alpha_4, \alpha_5), [x_1, 0.8, -0.6, 0.3, -0.6, 0.4, -0.8], [x_2, 0.5, -0.9, 0.4, -0.8, 0.7, -0.9]$,
$[x_3, 0.9, -0.1, 0.5, -0.5, 0.8, -0.7]$,
$(\alpha_2, \alpha_4, \alpha_7), [x_1, 0.2, -0.2, 0.4, -0.2, 0.7, -0.3], [x_2, 0.5, -0.3, 0.9, -0.4, 0.3, -0.5]$,
$[x_3, 0.4, -0.5, 0.8, -0.7, 0.4, -0.3]$,
$(\alpha_2, \alpha_3, \alpha_6), [x_1, 0.7, -0.2, 0.8, -0.9, 0.8, -0.2], [x_2, 0.8, -0.6, 0.8, -0.6, 0.8, -0.8]$,
$[x_3, 0.4, -0.5, 0.7, -0.7, 0.4, -0.1]\}$

Step 2. We follow the implementation of the two algorithms, precisely the second step , by calculating the union value between two BNHSSs. $(\Psi, \Gamma)_{G_1 \cup G_2}$ as follows .
$(\Psi, \Gamma)_{G_1 \cup G_2} =$
$\{(\alpha_1, \alpha_3, \alpha_5), [x_1, 0.5, -0.8, 0.1, -0.5, 0.4, -0.3], [x_2, 0.5, -0.3, 0.2, -0.5, 0.3, -0.9]$,
$[x_3, 0.3, -0.5, 0.2, -0.9, 0.8, -0.2]$,
$(\alpha_1, \alpha_4, \alpha_5), [x_1, 0.3, -0.6, 0.1, -0.4, 0.3, -0.3], [x_2, 0.6, -0.8, 0.3, -0.6, 0.3, -0.5]$,
$[x_3, 0.4, -0.5, 0.7, -0.4, 0.9, -0.1]$,
$(\alpha_1, \alpha_4, \alpha_7), [x_1, 0.3, -0.1, 0.5, -0.4, 0.2, -0.7], [x_2, 0.6, -0.8, 0.8, -0.2, 0.3, -0.1]$,
$[x_3, 0.4, -0.2, 0.5, -0.3, 0.8, -0.7]$,

$(\alpha_1, \alpha_3, \alpha_6), [x_1, 0.6, -0.6, 0.2, -0.8, 0.5, -0.3], [x_2, 0.6, -0.8, 0.2, -0.6, 0.2, -0.9],$
$[x_3, 0.7, -0.5, 0.7, -0.2, 0.8, -0.1],$
$(\alpha_2, \alpha_3, \alpha_5), [x_1, 0.3, -0.8, 0.2, -0.4, 0.3, -0.4], [x_2, 0.3, -0.1, 0.2, -0.4, 0.5, -0.3],$
$[x_3, 0.7, -0.5, 0.7, -0.2, 0.8, -0.1],$
$(\alpha_2, \alpha_4, \alpha_5), [x_1, 0.8, -0.1, 0.3, -0.9, 0.4, -0.7], [x_2, 0.6, -0.8, 0.3, -0.8, 0.3, -0.9],$
$[x_3, 0.9, -0.1, 0.5, -0.5, 0.8, -0.7],$
$(\alpha_2, \alpha_4, \alpha_7), [x_1, 0.8, -0.6, 0.4, -0.2, 0.7, -0.3], [x_2, 0.5, -0.9, 0.4, -0.4, 0.3, -0.5],$
$[x_3, 0.9, -0.5, 0.5, -0.5, 0.4, -0.3],$
$(\alpha_2, \alpha_3, \alpha_6), [x_1, 0.7, -0.2, 0.8, -0.3, 0.8, -0.2], [x_2, 0.8, -0.3, 0.3, -0.5, 0.3, -0.3],$
$[x_3, 0.4, -0.5, 0.7, -0.7, 0.4, -0.1]\}$

Table 1. SF values of $\hat{\varkappa}_i$ for candidates

$\mathcal{K}_\pi$	**SF for value** $\hat{\varkappa}_1$	**SF value for** $\hat{\varkappa}_2$	**SF value for** $\hat{\varkappa}_3$
$(\alpha_1, \alpha_3, \alpha_5)$	0.50	0.65	0.48
$(\alpha_1, \alpha_4, \alpha_5)$	0.50	0.55	0.30
$(\alpha_1, \alpha_4, \alpha_7)$	0.60	0.33	0.48
$(\alpha_1, \alpha_3, \alpha_6)$	0.47	0.65	0.33
$(\alpha_2, \alpha_3, \alpha_5)$	0.47	0.63	0.56
$(\alpha_2, \alpha_4, \alpha_5)$	0.43	0.50	0.55
$(\alpha_2, \alpha_4, \alpha_7)$	0.50	0.61	0.43
$(\alpha_2, \alpha_3, \alpha_6)$	0.76	0.65	0.62
Total Values of $\mathcal{M}_i$	$\mathcal{M}_1$= 2.756	$\mathcal{M}_2$= 2.527	$\mathcal{M}_3$= 2.936
Final Decision	$\mathcal{M}_1$= ×	$\mathcal{M}_2$=×	$\mathcal{M}_3$= √

Step 3 . Table 1 collects the rest of the steps (3,4 and 5) mentioned in Algorithm 1, and the choice falls on the candidate $\hat{\varkappa}_3$.

Algorithm 2. Using the aggregation value $\widehat{\mathfrak{B}}_{agg}$ for candidates $\hat{\varkappa}_i$

Step 1. Put up BNHSSs $(\Psi, \Gamma)_{G_1}, (\Psi, \Gamma)_{G_2}$ respectively, based on expert opinions (two committees).

Step 2. Calculating the union value $(\Psi, \Gamma)_{G_1 \cup G_2}$ between two BNHSSs which given in step 1.

Step 3. Find the aggregation value $\widehat{\mathfrak{B}}_{agg}$ for Union BNHSS $(\Psi, \Gamma)_{G_1 \cup G_2}$ based on definition 4.2.

Step 4. Decision: Choose the highest value for the candidate X_i,$i = 1, 2, 3$. to choose a suitable candidate.

Step 5. End algorithm 2.

In addition Figure 3 bellow representation of algorithm 2.

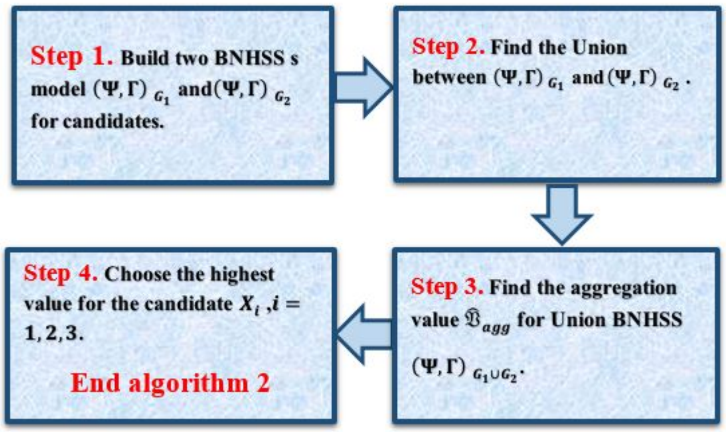

FIGURE 3. Algorithm 2. depends on aggregation values $\widehat{\mathfrak{B}}_{agg}$

Step 1 and Step 2 : These steps are the same as in steps 1 and 2 of algorithm 1. **Step 3 .**Table 2 collects the rest of the steps (3 and 4) mentioned in Algorithm 2, and the choice falls on the candidate $\hat{\varkappa}_3$.

TABLE 2. Aggregation value $\widehat{\mathfrak{B}}_{agg}$ of $\hat{\varkappa}_i$ for candidates

$\Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_i$	**Aggregation value** $\widehat{\mathfrak{B}}_{agg}$
$\Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_1$	0.870
$\Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_2$	0.838
$\Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_3$	0.896
Final Decision	$\mathcal{M}_1 = \Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_1 = \times$
	$\mathcal{M}_2 = \Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_2 = \times$
	$\mathcal{M}_3 = \Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_3 = \surd$

5.2. *Comparison analysis*

In this section, we prepared Table 3 and Figure 1 to compare the two algorithms presented in this part of the work. Both algorithms (algorithm 1 based on score function (SF) and algorithm 2 based on aggregation value) rely mainly on analyzing the data of the problem to be solved using our concept presented in this work.

In another instance of similar comparison, Table 4 provides another method of comparison with some of the previous works mentioned in the previous study in the first part of this

TABLE 3. Comparison between the values obtained from the two algorithms

Methods	$\hat{\varkappa}_1$	$\hat{\varkappa}_2$	$\hat{\varkappa}_3$	Ranking
SF for value $\hat{\varkappa}_i$	2.756	2.527	2.936	$\mathcal{W}_3 \succ \mathcal{W}_1 \succ \mathcal{W}_2$
Aggregation value $\widehat{\mathfrak{B}}_{agg}$	0.870	0.838	0.896	$\mathcal{W}_3 \succ \mathcal{W}_1 \succ \mathcal{W}_2$

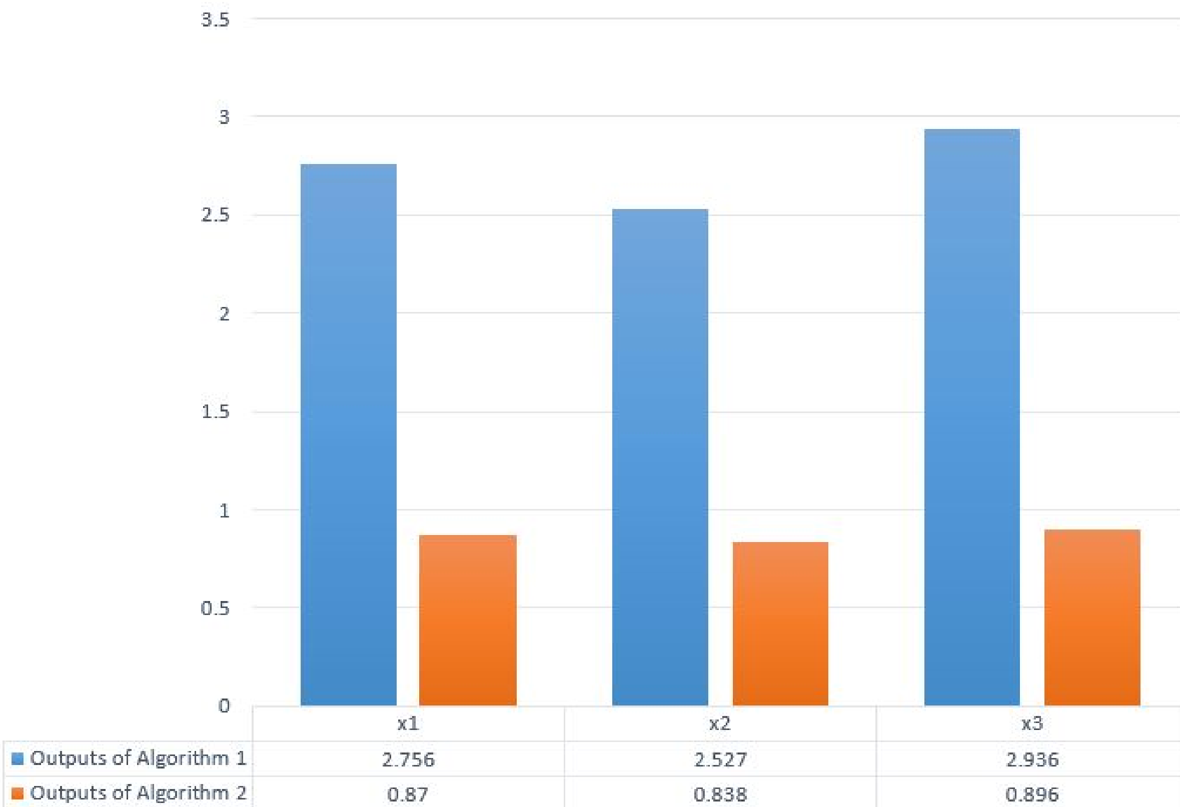

FIGURE 4. A statistical chart showing the vivid comparison between the numerical outputs of the two proposed algorithms

work. Our proposed concept is compared with some existing extensions of the fuzzy soft set under bipolarity, such as: bipolar fuzzy soft set(BFSS), bipolar intuitionistic fuzzy soft set (BIFSS), bipolar neutrosophic soft set (BNSS), and bipolar fuzzy hypersoft set(BFHSS) based on their structural composition, where TMD, IMD, FMD, SS, and HSS indicate to three NS memberships degree, soft set, and hypersoft set, respectively.

From Table 4, we notice that our concept is different from the previous concepts mentioned in the literature, and therefore it can be said that our proposed concept is more comprehensive than the previous concepts in covering ambiguous data of a positive and negative nature at the same time.

TABLE 4. The vivid comparison between the proposed structure and the existing structure.

Methods	Authors	TMD	IMD	FMD	SS	HSS
BFS	Zhang [52]	√	×	×	×	×
BFSS	Naz and Shabir [54]	√	×	×	√	×
BIFSS	Jana and Pal [57]	√	×	√	√	×
BNSS	Ali et al. [59]	√	√	√	√	×
BFHSS	Al-Quran et al. [64]	√	×	×	√	√
BNHSS	**Propose model**	√	√	√	√	√

6. Conclusions

In this work, the novel idea of a new hybrid model of BNHSS by merging both neutrosophic sets (NSs) and HSSs under the bipolarity property of real numbers is provided. Furthermore, we studied its properties and necessary operations, such as complements, subset, unions, and intersections. Subsequently, we describe some operations, like "AND" and "OR," as well as their properties and some numerical examples. Two algorithms are discussed that rely on some mathematical methods (aggregation operator and score function) to deal with MAGDM in the BNHSS environment. In this study we attempt to develop more sophisticated model which has the advantages of all the previous models, however,there are still certain challenges with the work that is being suggested. . In BNHSS, we have only taken into consideration the evaluation information given in one dimension, where the time dimension does not enter into determining its fate. For the purpose of dealing with such data, we recommend that future studies combine the tools presented in this work with complex numbers. Also, the proposed model could be investigated more by proposing some aggregation operators such as Heronian mean, power mean, Hamacher, Bonferroni mean and Dombis aggregation operators to solve the existing decision making problems.

Conflicts of Interest: The authors declare no conflict of interest.

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