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Some remarks on Δ^m-Cesàro summability in neutrosophic normed spaces

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Abstract. In this paper, we define the notion of a generalized summability, called Cesàro summability in neutrosophic normed spaces (briefly NNS). We obtain conditions under which ordinary summability follows from Cesàro summability. Later, we define a concept of slowly oscillating sequences in NNS and establish related Tauberian Theorems in neutrosophic normed spaces.

Keywords: Neutrosophic normed spaces, Cesàro summability, slow oscillation and Tauberian theorem.

MSC: 46S40, 11B39, 03E72 and 40G15.

1. **Introduction**

In Analysis, we usually face many situations where the analytic solutions to some problems seem difficult due to the divergence of an infinite series or a power series. Consequently, we look forward to a modified method of convergence that can sum up the divergence series in some sense and call it a method of summability. A well-known method is due to Cesàro for number sequences known as Cesàro summability and is defined as follows:
"A sequence $x = (x_n)$ of numbers is said to be Cesàro summable [or $(C, 1)-$ summable to x_0 if

$$\lim_{n\to\infty}\left(\frac{x_1 + x_2 + \ldots x_n}{n}\right) = x_0."$$

If $\lim_{n\to\infty}(x_n) = x_0$, then (x_n) is $(C, 1)-$summable to x_0 however, the reverse way implication may not be true. But by adding some additional conditions on sequence called "Tauberian conditions", we obtained the result in the reverse way too. These results obtained by imposing Tauberian conditions are known as Tauberian Theorems. In past years, many interesting works have been carried out in this direction and various kinds of Tauberian theorems have been proved. For some historical view on Cesàro summability and Tauberian Theorems, we refer to the reader [1], [13]-[15] and [24]-[26].

On the other side, Zadeh [28] observed first time that many real-life situations cannot be set in the framework of classical sets. Therefore, to deal with such situations, in 1965, he proposed the idea of fuzzy sets via introducing the membership function. Later, a revolutionary development on fuzzy sets has been started. Many existing ideas have been developed again by applying fuzzy logic. During this developmental phase, several intriguing generalizations of fuzzy sets have emerged in the literature. For example: intuitionistic fuzzy sets (IFS) [2], vague fuzzy sets [5], neutrosophic sets (NS) [12], interval-valued fuzzy sets [27], etc. Analogous to the classical set theory, these sets have also been employed to introduce novel spaces, including fuzzy normed spaces ([6], [11]), intuitionistic fuzzy normed spaces ([7], [8], [19], [21]), and neutrosophic normed spaces ([3], [4], [9], [10], [17], [18], [20], [22], [23]). To develop these spaces mathematically and topologically, we need to define the concept of limit as one of the fundamental concepts. Some interesting works in this direction can be found in [7] - [11], etc. Recently, Talo and Yavuz [25] studied Cesàro summability and proved some Tauberian theorems in an intuitionistic fuzzy normed space. As neutrosophic normed spaces are generalizations of intuitionistic fuzzy normed spaces so it is natural to extend Cesàro summability and related concepts in these spaces. In present paper, we define Cesàro summability, slowly oscillating sequences and prove some Tauberian theorems in neutrosophic normed spaces. We organize the paper as follows, the first and second sections are introductory and provide basic information needed in the sequel. In third section we define Δ^m-Cesàro summability in NNS and obtained certain results. Finally in last section we define Slowly oscillating sequences in NNS and establish related Tauberian Theorems in neutrosophic normed spaces.

2. Background and Preliminaries

This section begin with a short review on some definitions and results.

Throughout this work, I will denote the closed interval $[0, 1]$, and $\mathbb{N}$ and $\mathbb{R}^+$ denotes the set of positive integers and positive reals, respectively.

Definition 2.1 [8] "A map from $\circ : I \times I$ to I is said to be a continuous $t-$norm if, $\forall$ $f, g, h, i \in I$ we have:

(i) $f \circ g = g \circ f$;

(ii)$f \circ (g \circ h) = (f \circ g) \circ h$;

(iii) $\circ$ is continuous;

(iv) $f \circ 1 = f$ and

(v) $f \circ g \leq h \circ i$ whenever $f \leq h$ and $g \leq i$."

Definition 2.2 [8] “A map from $\diamond : I \times I$ to I is said to be a continuous triangular conorm or $t-$conorm if for all $f, g, h, i \in I$ we have:

(i) $f \diamond g = g \diamond f$;

(ii)$f \diamond (g \diamond h) = (f \diamond g) \circ h$;

(iii) $\diamond$ is continuous;

(iv) $f \diamond 0 = f$ for every $f \in [0,1]$

(v) $f \diamond g \leq h \diamond i$ whenever $f \leq h$ and $g \leq i$.”

Definition 2.3 [10] “A four tuple $V = (F, N, \ \circ, \diamond,)$ where F be a vector space, $N = \{\langle \vartheta, \mathcal{H}(\vartheta), \mathcal{I}(\vartheta), \mathcal{J}(\vartheta)\rangle : \vartheta \in F\}$ be a normed space with $N : F \times \mathbb{R}^+ \to I$ and $\circ$, $\diamond$ respectively are continuous $t-$norm and continuous $t-$conorm, is called a neutrosophic normed spaces (NNS) if the following conditions hold: For every $u, v \in F$ and $\mathfrak{y}_1, \mathfrak{y}_2 > 0$ and for every $\alpha \neq 0$ we have (i) $0 \leq \mathcal{H}(u, \ \mathfrak{y}_1) \leq 1, 0 \leq \ \mathcal{I}(u, \ \mathfrak{y}_1) \leq 1, 0 \leq \mathcal{J}(u, \ \mathfrak{y}_1) \leq 1$ for every $\mathfrak{y}_1 \in \mathbb{R}^+$;

(ii) $\mathcal{H}(u, \ \mathfrak{y}_1) + \ \mathcal{I}(u, \ \mathfrak{y}_1) + \mathcal{J}(u, \ \mathfrak{y}_1) \leq 3$ for $\mathfrak{y}_1 \in \mathbb{R}^+$;

(iii) $\mathcal{H}(u, \ \mathfrak{y}_1) = 1 \quad$ (for $\mathfrak{y}_1 > 0$) if and only if $u = \theta$;

(iv) $\mathcal{H}(\alpha u, \ \mathfrak{y}_1) = \mathcal{H}\left(u, \ \frac{\mathfrak{y}_1}{|\alpha|}\right)$; (v) $\mathcal{H}(u, \ \mathfrak{y}_1) \circ \mathcal{H}(v, \ \mathfrak{y}_2) \leq \mathcal{H}(u + v, \ \mathfrak{y}_1 + \mathfrak{y}_2)$;

(vi) $\mathcal{H}(u, \ .)$ is continuous non-decreasing function;

(vii) $\lim_{\mathfrak{y}_1 \to \infty} \mathcal{H}(u, \ \mathfrak{y}_1) = 1$;

(viii) $\mathcal{I}(u, \ \mathfrak{y}_1) = 0 \quad$ (for $\mathfrak{y}_1 > 0$) if and only if $u = \theta$;

(ix) $\mathcal{I}(\alpha u, \ \mathfrak{y}_1) = \mathcal{I}\left(u, \ \frac{\mathfrak{y}_1}{|\alpha|}\right)$;

(x) $\mathcal{I}(u, \ \mathfrak{y}_1) \diamond \mathcal{I}(v, \ \mathfrak{y}_2) \geq \mathcal{I}(u + v, \ \mathfrak{y}_1 + \mathfrak{y}_2)$;

(xi) $\mathcal{I}(u, \ .)$ is continuous non-decreasing function;

(xii) $\lim_{\mathfrak{y}_1 \to \infty} \mathcal{I}(u, \ \mathfrak{y}_1) = 0$;

(xiii) $\mathcal{J}(u, \ \mathfrak{y}_1) = 0 \quad$ (for $\mathfrak{y}_1 > 0$) if and only if $u = \theta$;

(xiv) $\mathcal{J}(\alpha u, \ \mathfrak{y}_1) = \mathcal{J}\left(u, \ \frac{\mathfrak{y}_1}{|\alpha|}\right)$;

(xv) $\mathcal{J}(u, \ \mathfrak{y}_1) \diamond \mathcal{J}(v, \ \mathfrak{y}_2) \geq \mathcal{J}(u + v, \ \mathfrak{y}_1 + \mathfrak{y}_2)$;

(xvi) $\mathcal{J}(u, \ .)$ is continuous non-decreasing function;

(xvii) $\lim_{\mathfrak{y}_1 \to \infty} \mathcal{J}(u, \ \mathfrak{y}_1) = 0$;

(xviii) If $\mathfrak{y}_1 \leq 0$, then $\mathcal{H}(u, \ \mathfrak{y}_1) = 0, \quad \mathcal{I}(u, \ \mathfrak{y}_1) = 1 \quad$ and $\quad \mathcal{J}(u, \ \mathfrak{y}_1) = 1$.

We call $N = (\mathcal{H}, \mathcal{I}, \mathcal{J})$, the neutrosophic norm and $V = (F, \mathcal{H}, \mathcal{I}, \mathcal{J}, \circ, \diamond)$, the neutrosophic normed space.”

For some examples on these spaces we refer [10].

“A sequence (u_n) in a neutrosophic normed spaces V is said to convergent if for each $\varepsilon > 0$ and $\mathfrak{y} > 0$, there exists a positive integer m and $u_0 \in F$ such that $\mathcal{H}(u_n - u_0, \mathfrak{y}) > 1 - \varepsilon$, $\mathcal{I}(u_n - u_0, \mathfrak{y}) < \varepsilon$ and $\quad \mathcal{J}(u_n - u_0, \mathfrak{y}) < \varepsilon$ for all $n \geq m$. This is equivalent to say that

$\lim_{n\to\infty} \mathcal{H}(u_n - u_0, \mathfrak{y}) = 1$, $\lim_{n\to\infty} \mathcal{I}(u_n - u_0, \mathfrak{y}) = 0$ and $\lim_{n\to\infty} \mathcal{J}(u_n - u_0, \mathfrak{y}) = 0$ and we write $N - \lim_{n\to\infty} u_n = u_0$."

"A sequence (u_n) is said to be Cauchy if for each $\varepsilon > 0$ and $\mathfrak{y} > 0$, there exists a positive integer p such that $\mathcal{H}(u_k - u_n,\ \mathfrak{y}) > 1 - \varepsilon$, $\mathcal{I}(u_k - u_n,\ \mathfrak{y}) < \varepsilon$ and $\mathcal{J}(u_k - u_n,\ \mathfrak{y}) < \varepsilon$ for all $k,\ n \geq p$."

"Let w denotes the set of all sequences in the neutrosophic normed space $V = (F, \mathcal{H}, \mathcal{I}, \mathcal{J}, \circ, \diamond)$. Define $\Delta^m : w \to w$ by

$$\Delta^0\ a_k = a_k;$$

$$\Delta^1\ a_k = a_k - a_{k+1};$$

$$\Delta^m\ a_k = \Delta^{m-1}(a_k - a_{k+1}) m \geq 2 \text{ and } \forall\ k \in \mathbb{N}.$$

We now demonstrate two important Lemmas of [24].

For $\mu > 0$ and $n \in \mathbb{N}$, let $\mu_n = \lfloor \mu n \rfloor$ i.e, the sequence of integral parts of the product μn."

If we define $\langle \mu \rangle = \mu - \lfloor \mu \rfloor$, then we have the following Lemmas.

Lemma 2.1 [24] "(i) If $\mu > 1$, then $\mu_n > n,\ \forall\ n \in \mathbb{N} - \{0\}$ along with $n > \langle \mu \rangle^{-1}$.

(ii) If $0 < \mu < 1$, then $\mu_n < n,\ \forall\ n \in \mathbb{N} - \{0\}$."

Lemma 2.2[24] "(i) If $\mu > 1$, then $\forall\ n \in \mathbb{N} - \{0\}$ along with $n \geq \frac{3\mu - 1}{\mu(\mu - 1)}$, we have

$$\frac{\mu}{(\mu - 1)} < \frac{\mu_n + 1}{\mu_n - n} < \frac{2\mu}{\mu - 1}.$$

(ii) If $0 < \mu < 1$, then $\forall\ n \in \mathbb{N} - \{0\}$ along with $n = \mu^{-1}$ we have

$$0 < \frac{\mu_n + 1}{n - \mu_n} < \frac{2\mu}{1 - \mu}.$$

We now turn towards our main section. Throughout the work, V denotes a neutrosophic normed space with neutrosophic norm N unless otherwise stated and θ, the $0-$th element in V."

3. Δ^m-Cesàro summability in NNS

Definition 3.1 A sequence $u = (u_n)$ in V is called Δ^m-Cesàro summable [or $(C, \Delta^m, 1)-$summable w.r.t. N] to u_0 if $N - \lim_{n\to\infty} \sigma_n = u_0$ where the sequence (σ_n) is precisely defined by

$$\sigma_n = \frac{v_1 + v_2 + \dots v_n}{n} = \frac{\sum_{k=1}^{n} v_k}{n}. \quad (n \in \mathbb{N}) \text{ and}$$

$$v_n = \Delta^m u_n = \sum_{p=0}^{m} (-1)^p \binom{m}{p} u_{n+p}.$$

This is similar to say, for $\epsilon > 0$ and $\mathfrak{y} > 0$ there exist $n_0 \in \mathbb{N}$ satisfying

$$\mathcal{H}(\sigma_n - u_0, \mathfrak{y}) > 1 - \epsilon \text{ and } \mathcal{I}(\sigma_n - u_0, \mathfrak{y}) < \epsilon, \mathcal{J}(\sigma_n - u_0, \mathfrak{y}) < \epsilon.$$

In this case, we abbreviate it as $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$.

Next Theorem gives the relationship between $N-$convergence and $N(C, \Delta^m, 1)-$summability.

Theorem 3.1 For any sequence $u = (u_n)$ in V, if $N - \lim_{n\to\infty} \Delta^m u_n = u_0$, then $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$.

Proof. Assume that $N - \lim_{n\to\infty} \Delta^m u_n = u_0$. We wish to prove that $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$. Let $\epsilon > 0$ be given and take $\mathfrak{y} > 0$. As $N - \lim_{n\to\infty} \Delta^m u_n = u_0$ so $\exists$ $n_1 \in \mathbb{N}$ satisfying, for all $n \geq n_1$

$$\mathcal{H}\left(\Delta^m u_n - u_0, \frac{\mathfrak{y}}{2}\right) > 1 - \epsilon \text{ and } \mathcal{I}\left(\Delta^m u_n - u_0, \frac{\mathfrak{y}}{2}\right) < \epsilon, \mathcal{J}\left(\Delta^m u_n - u_0, \frac{\mathfrak{y}}{2}\right) < \epsilon;$$

Moreover,

$$\lim_{n\to\infty} \mathcal{H}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) = 1 \text{ and } \lim_{n\to\infty} \mathcal{I}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) = 0,$$
$$\lim_{n\to\infty} \mathcal{J}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) = 0;$$

gives another $n_2 \in \mathbb{N}$ with $n \geq n_2$ such that

$$\mathcal{H}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) > 1 - \epsilon \text{ and } \mathcal{I}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) < \epsilon,$$
$$\mathcal{J}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) < \epsilon.$$

Now, for $n > \max\{n_1, n_2\}$ we have

$$\begin{aligned}
\mathcal{H}\left(\frac{1}{n}\sum_{k=1}^{n} \Delta^m u_k - u_0, \mathfrak{y}\right) &= \mathcal{H}\left(\frac{1}{n}\sum_{k=1}^{n} (\Delta^m u_k - u_0), \mathfrak{y}\right) = \mathcal{H}\left(\sum_{k=1}^{n} (\Delta^m u_k - u_0), n\mathfrak{y}\right) \\
&\geq \min\left\{\mathcal{H}\left(\sum_{k=1}^{n_1} (\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\right), \mathcal{H}\left(\sum_{k=n_1+1}^{n} (\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\right)\right\} \\
&\geq \min\left\{\mathcal{H}\left(\sum_{k=1}^{n_1} (\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\right), \mathcal{H}\left(\sum_{k=n_1+1}^{n} (\Delta^m u_k - u_0), (n - n_1).\frac{\mathfrak{y}}{2}\right)\right\} \\
&\geq \min\left\{\mathcal{H}\left(\sum_{k=1}^{n_1} (\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\right), \mathcal{H}\left((\Delta^m u_{n_1+1} - u_0), \frac{\mathfrak{y}}{2}\right), \mathcal{H}\left((\Delta^m u_{n_1+2} - u_0), \frac{\mathfrak{y}}{2}\right), \cdots \right. \\
&\qquad \left. \mathcal{H}\left((\Delta^m u_n - u_0), \frac{\mathfrak{y}}{2}\right)\right\} \\
&> (1 - \epsilon) \text{ and}
\end{aligned}$$

$$\mathcal{I}\bigg(\frac{1}{n}\sum_{k=1}^{n}\Delta^m u_k - u_0, \mathfrak{y}\bigg) = \mathcal{I}\bigg(\frac{1}{n}\sum_{k=1}^{n}(\Delta^m u_k - u_0), \mathfrak{y}\bigg) = \mathcal{I}\bigg(\sum_{k=1}^{n}(\Delta^m u_k - u_0), n\mathfrak{y}\bigg)$$
$$< \max\bigg\{\mathcal{I}\bigg(\sum_{k=1}^{n_1}(\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\bigg), \mathcal{I}\bigg(\sum_{k=n_1+1}^{n}(\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\bigg)\bigg\}$$
$$< \max\bigg\{\mathcal{I}\bigg(\sum_{k=1}^{n_1}(\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\bigg), \mathcal{I}\bigg(\sum_{k=n_1+1}^{n}(\Delta^m u_k - u_0), (n-n_1)\frac{\mathfrak{y}}{2}\bigg)\bigg\}$$
$$< \max\bigg\{\mathcal{I}\bigg(\sum_{k=1}^{n_1}(\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\bigg),$$
$$\mathcal{I}\bigg((\Delta^m u_{n_1+1} - u_0), \frac{\mathfrak{y}}{2}\bigg), \mathcal{I}\bigg((\Delta^m u_{n_1+2} - u_0), \frac{\mathfrak{y}}{2}\bigg), \cdots \mathcal{I}\bigg((\Delta^m u_n - u_0), \frac{\mathfrak{y}}{2}\bigg)\bigg\} < \epsilon.$$

Similarly one can show

$$\mathcal{J}\bigg(\frac{1}{n}\sum_{k=1}^{n}\Delta^m u_k - u_0, \mathfrak{y}\bigg) < \epsilon.$$

This implies that $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$, which completes the proof of the Theorem.□

Example 3.1 Let $(\mathbb{R}, |.|)$ denote the space of reals with the usual norm. For $a, b \in [0, 1]$, let the $t-$ norm and $t-$conorm are defined by

$$a \circ b = ab \text{ and } a \diamond b = a + b - ab$$

Let, $u \in \mathbb{R}$ and $\mathfrak{y} > 0$ with $\mathfrak{y} > |u|$. Define $\mathcal{H}, \mathcal{I}$ and $\mathcal{J}$ as follows:

$$\mathcal{H}(u, \mathfrak{y}) = \frac{\mathfrak{y}}{\mathfrak{y} + |u|}, \mathcal{I}(u, \mathfrak{y}) = \frac{|u|}{\mathfrak{y} + |u|} \text{ and } \mathcal{J}(u, \mathfrak{y}) = \frac{|u|}{\mathfrak{y}},$$

then $N(\mathcal{H}, \mathcal{I}, \mathcal{J})$ is a neutrosophic norm and $(\mathbb{R}, \circ, \diamond, \mathcal{H}, \mathcal{I}, \mathcal{J})$ is a NNS.

Define a sequence (u_n) by $u_n = (-1)^n$, then for $m = 1$, $\Delta^1 u_n = 2(-1)^n$ and therefore the sequence σ_n is given by

$$\sigma_n = \frac{2(-1)^1 + 2(-1)^2 + \cdots + 2(-1)^n}{n} = 0 \text{ or } \frac{-2}{n},$$

according as n is even or odd respectively.

Case-I: If n is even, then $\sigma_n = 0$, and therefore we have

$$\lim_{n\to\infty} \mathcal{H}(0, \mathfrak{y}) = 1 \text{ and } \lim_{n\to\infty} \mathcal{I}(0, \mathfrak{y}) = \lim_{n\to\infty} \mathcal{J}(0, \mathfrak{y}) = 0.$$
(by Definition NNS)

Case-II: If n is odd, then

$$\mathcal{H}(\sigma_n - 0, \mathfrak{y}) = \mathcal{H}(\sigma_n, \mathfrak{y}) = \frac{\mathfrak{y}}{\mathfrak{y} + |\sigma_n|} = \frac{\mathfrak{y}}{\mathfrak{y} + |\frac{-2}{n}|}$$

$$\text{so,} \quad \lim_{n\to\infty} \mathcal{H}(\sigma_n - 0, \mathfrak{y}) = \lim_{n\to\infty} \frac{\mathfrak{y}}{\mathfrak{y} + |\frac{-2}{n}|} = 1;$$

and

$$\mathcal{I}(\sigma_n - 0, \mathfrak{y}) = \mathcal{I}(\sigma_n, \mathfrak{y}) = \frac{|\sigma_n|}{\mathfrak{y} + |\sigma_n|} = \frac{|\frac{-2}{n}|}{\mathfrak{y} + |\frac{-2}{n}|} \text{ gives}$$

$$\lim_{n\to\infty} \mathcal{I}(\sigma_n - 0, \mathfrak{y}) = \lim_{n\to\infty} \frac{|\frac{-2}{n}|}{\mathfrak{y} + |\frac{-2}{n}|} = 0;$$

$$\mathcal{J}(\sigma_n - 0, \mathfrak{y}) = \mathcal{J}(\sigma_n, \mathfrak{y}) = \frac{\|\sigma_n\|}{\mathfrak{y}} = \frac{|\frac{-2}{n}|}{\mathfrak{y}} \text{ will imply}$$

$$\lim_{n\to\infty} \mathcal{J}(\sigma_n - \theta, \mathfrak{y}) = \lim_{n\to\infty} \frac{|\frac{-2}{n}|}{\mathfrak{y}} = 0.$$

Hence, in both cases,

$$\lim_{n\to\infty} \mathcal{H}(\sigma_n - 0, \mathfrak{y}) = 1, \lim_{n\to\infty} \mathcal{I}(\sigma_n - 0, \mathfrak{y}) = \lim_{n\to\infty} \mathcal{J}(\sigma_n - 0, \mathfrak{y}) = 0,$$

and therefore, $N - \lim_{n\to\infty} \sigma_n = 0$ i.e., $N(C, \Delta, 1) - \lim_{n\to\infty} u_n = 0$.

But clearly the sequence $(u_n) = 2(-1)^n$ is not $N-$convergent as

$$\mathcal{H}(u_n - u_0, \mathfrak{y}) = \frac{\mathfrak{y}}{\mathfrak{y} + \|u_n - u_0\|} = \frac{\mathfrak{y}}{\mathfrak{y} + |2(-1)^n - u_0|} = \begin{cases} \frac{\mathfrak{y}}{\mathfrak{y}+|-2-u_0|} & \text{if } n \text{ is odd;} \\ (\frac{\mathfrak{y}}{\mathfrak{y}+|2-u_0|}) & \text{if } n \text{ is even.} \end{cases}$$

Thus, if we choose $u_0 = -2$ when n is odd and $u_0 = 2$ when n is even, then we have

$$\lim_{n\to\infty} \mathcal{H}(u_n - u_0, \mathfrak{y}) = \begin{cases} 1 & \text{if } n \text{ is odd;} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Similarly one can show

$$\lim_{n\to\infty} \mathcal{I}(u_n - u_0, \mathfrak{y}) = \lim_{n\to\infty} \mathcal{J}(u_n - u_0, \mathfrak{y}) \begin{cases} 0 & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

In this way we obtain two subsequences of the sequence $u_n = (-1)^n$ corresponding to sets of even and odd integers and which are $N-$convergent to different limits. This shows that $(u_n) = (-1)^n$ is not $N-$convergent. □

The following Theorem gives the reverse way of Theorem 3.1 via applying some additional conditions.

Theorem 3.2 For any sequence $u = (u_n)$ in V, if $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$, then $N - \lim_{n\to\infty} \Delta^m u_n = u_0$ if and only if

$$(i) \sup_{\mu>1} \left[\liminf_{n\to\infty} \mathcal{H} \left(\frac{1}{\mu_n - n} \sum_{k=n+1}^{\mu_n} (\Delta^m u_k - \Delta^m u_n), \mathfrak{y} \right) \right] = 1;$$
$$(ii) \inf_{\mu>1} \left[\limsup_{n\to\infty} \mathcal{I} \left(\frac{1}{\mu_n - n} \sum_{k=n+1}^{\mu_n} (\Delta^m u_k - \Delta^m u_n), \mathfrak{y} \right) \right] = 0;$$
$$(iii) \inf_{\mu>1} \left[\limsup_{n\to\infty} \mathcal{J} \left(\frac{1}{\mu_n - n} \sum_{k=n+1}^{\mu_n} (\Delta^m u_k - \Delta^m u_n), \mathfrak{y} \right) \right] = 0.$$

(Here for $\mu > 0, \mu_n = \lfloor \mu n \rfloor$i.e., the integral part of μn)

Proof. Necessity: Let, $u = (u_n)$ be any sequence in V with $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$. We first assume that $N - \lim_{n\to\infty} \Delta^m u_n = u_0$ and obtain conditions $(i), (ii)$ and (iii). Let $\mathfrak{y} > 0$ and take $\mu > 1$. Then, by Lemma 2.1, for each $n \in \mathbb{N} - \{0\}$ we have $\mu_n > n$ and $n \geq \frac{1}{\langle \mu \rangle}$ where $\langle \mu \rangle = \mu - \lfloor \mu \rfloor$. Moreover, by Lemma 2.1, we can write the difference of $(\Delta^m u_n - \Delta^m \sigma_n)$ as

$$\Delta^m u_n - \sigma_n = \frac{\mu_n + 1}{\mu_n - n} \left[\sigma_{\mu_n} - \sigma_n \right] - \frac{1}{\mu_n - n} \sum_{k=n+1}^{\mu_n} (\Delta^m u_k - \Delta^m u_n) \, ;$$

and therefore by Lemma 2.2, we have for $n \geq \frac{3\mu - 1}{\mu(\mu-1)}$,

$$\mathcal{H} \left(\frac{\mu_n + 1}{\mu_n - n} \left[\sigma_{\mu_n} - \sigma_n \right], \mathfrak{y} \right) = \mathcal{H} \left(\left[\sigma_{\mu_n} - \sigma_n \right], \frac{\mathfrak{y}}{\frac{\mu_n + 1}{\mu_n - n}} \right)$$
$$\geq \mathcal{H} \left(\left[\sigma_{\mu_n} - \sigma_n \right], \frac{\mathfrak{y}}{\frac{2\mu}{\mu - 1}} \right)$$
$$\text{so we have, } \lim_{n\to\infty} \mathcal{H} \left(\frac{\mu_n + 1}{\mu_n - n} \left[\sigma_{\mu_n} - \sigma_n \right], \mathfrak{y} \right) = \mathcal{H} \left(0, \frac{\mathfrak{y}}{\frac{2\mu}{\mu - 1}} \right) = 1.$$

as (σ_n) is a Cauchy sequence.

Now,

$$\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=\mathcal{I}\left(\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{\frac{\mu_n+1}{\mu_n-n}}\right)$$
$$\leq \mathcal{I}\left(\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{\frac{2\mu}{\mu-1}}\right)$$

so we have, $$\lim_{n\to\infty}\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=\mathcal{I}\left(0,\frac{\mathfrak{y}}{\frac{2\mu}{\mu-1}}\right)=0;$$

similarly,

$$\mathcal{J}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=\mathcal{J}\left(\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{\frac{\mu_n+1}{\mu_n-n}}\right)$$
$$\leq \mathcal{J}\left(\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{\frac{2\mu}{\mu-1}}\right)$$

and therefore, $$\lim_{n\to\infty}\mathcal{J}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=\mathcal{J}\left(0,\frac{\mathfrak{y}}{\frac{2\mu}{\mu-1}}\right)=0.$$

Hence, we obtain

$$\lim_{n\to\infty}\mathcal{H}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=1,\lim_{n\to\infty}\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)$$
$$=\lim_{n\to\infty}\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=0,$$

which immediately imply (i), (ii) and (iii).

Sufficiency: Suppose (i), (ii) and (iii) holds. We shall show that $N-\lim_{n\to\infty}\Delta^m u_n=u_0$. For this, let $\epsilon>0$ be given and take $\mathfrak{y}>0$. By hypothesis, there exists a $\mu>1$ and a $m_1\in\mathbb{N}$ such that for $n>m_1$, we have

$$\mathcal{H}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k-\Delta^m u_n),\frac{\mathfrak{y}}{3}\right)>1-\epsilon \text{ and } \mathcal{I}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k-\Delta^m u_n),\frac{\mathfrak{y}}{3}\right)<\epsilon,$$
$$\mathcal{J}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k-\Delta^m u_n),\frac{\mathfrak{y}}{3}\right)<\epsilon;$$

Since, $N(C,\Delta^m,1)-\lim_{n\to\infty}u_n=u_0$, so we have another $m_2\in\mathbb{N}$ such that for all $n>m_2$ we have

$$\mathcal{H}\Big(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\Big)>1-\epsilon \text{ and } \mathcal{I}\Big(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\Big)<\epsilon,\mathcal{J}\Big(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\Big)<\epsilon.$$

Moreover, as $N-\lim_{n\to\infty}\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right]=0$, so there is $m_3\in\mathbb{N}$ such that for all $n>m_3$ we have

$$\mathcal{H}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right)>1-\epsilon \text{ and } \mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right)<\epsilon,$$
$$\mathcal{J}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right)<\epsilon.$$

Now,

$$\begin{aligned}\mathcal{H}(\Delta^m u_n-u_0,\mathfrak{y})&=\mathcal{H}(\Delta^m u_n-\sigma_n+\sigma_n-u_0,\mathfrak{y})\\&=\mathcal{H}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right]-\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right)+\sigma_n-u_0,\mathfrak{y}\right)\end{aligned}$$
$$\geq\min\left\{\mathcal{H}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right),\mathcal{H}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right),\frac{\mathfrak{y}}{3}\right),\mathcal{H}\left(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\right)\right\};$$

and

$$\begin{aligned}\mathcal{I}(\Delta^m u_n-u_0,\mathfrak{y})&=\mathcal{I}(\Delta^m u_n-\sigma_n+\sigma_n-u_0,\mathfrak{y})\\&=\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right]-\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right)+\sigma_n-u_0,\mathfrak{y}\right)\end{aligned}$$
$$\leq\max\left\{\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right),\mathcal{I}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right),\frac{\mathfrak{y}}{3}\right),\mathcal{I}\left(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\right)\right\},$$
$$\begin{aligned}\mathcal{J}(\Delta^m u_n-u_0,\mathfrak{y})&=\mathcal{J}(\Delta^m u_n-\sigma_n+\sigma_n-u_0,\mathfrak{y})\\&=\mathcal{J}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right]-\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right)+\sigma_n-u_0,\mathfrak{y}\right)\end{aligned}$$
$$\leq\max\left\{\mathcal{J}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right),\mathcal{J}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right),\frac{\mathfrak{y}}{3}\right),\mathcal{J}\left(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\right)\right\},$$

Thus, if we select $m=\max\{m_1,m_2,m_3\}$, then we have $\mathcal{H}(\Delta^m u_n-u_0,\mathfrak{y})>1-\epsilon$ and $\mathcal{I}(\Delta^m u_n-u_0,\mathfrak{y})<\epsilon$, $\mathcal{J}(\Delta^m u_n-u_0,\mathfrak{y})<\epsilon$ and therefore $N-\lim_{n\to\infty}\Delta^m u_n=u_0$. □

The case for $0<\mu<1$ follows similarly by using the expression

$$\Delta^m u_n-\sigma_n=\frac{\mu_n+1}{n-\mu_n}\left[\sigma_n-\sigma_{\mu_n}\right]-\frac{1}{n-\mu_n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_n-\Delta^m u_k\right).$$

Another similar result related to Cesàro summability and $N-$convergence is as follows.

Theorem 3.3 For any sequence $u=(u_n)$ in V, if $N(C,1)-\lim_{n\to\infty}u_n=u_0$, then $N-$

$\lim_{n\to\infty} u_n = u_0$ if and only if

$$(i) \sup_{0<\mu<1}\left[\liminf_{n\to\infty}\mathcal{H}\left(\frac{1}{n-\mu_n}\sum_{k=\mu_n+1}^{n}(\Delta^m u_n - \Delta^m u_k),\mathfrak{y}\right)\right] = 1;$$

$$(ii) \inf_{0<\mu<1}\left[\limsup_{n\to\infty}\mathcal{I}\left(\frac{1}{n-\mu_n}\sum_{k=\mu_n+1}^{n}(\Delta^m u_n - \Delta^m u_k),\mathfrak{y}\right)\right] = 0;$$

$$(iii) \inf_{0<\mu<1}\left[\limsup_{n\to\infty}\mathcal{J}\left(\frac{1}{n-\mu_n}\sum_{k=\mu_n+1}^{n}(\Delta^m u_n - \Delta^m u_k),\mathfrak{y}\right)\right] = 0.\square$$

4. Slowly oscillating sequences in NNS

For μ_n, the sequence of integer part of μn, the concept of Δ^m-slowly oscillating sequences in neutrosophic normed spaces is defined as follow.

Definition 4.1 A sequence $u = (u_n)$ in V is called slowly oscillating if for all $\mathfrak{y} > 0$

$$(i) \sup_{\mu>1}\left[\liminf_{n\to\infty}\left\{\min_{n<k\leq\ \mu_n}\mathcal{H}(\Delta^m u_k - \Delta^m u_n,\mathfrak{y})\right\}\right] = 1 \text{ and}$$

$$(ii) \inf_{\mu>1}\left[\limsup_{n\to\infty}\left\{\max_{n<k\leq\mu_n}\mathcal{I}(\Delta^m u_k - \Delta^m u_n,\mathfrak{y})\right\}\right] = 0,$$

$$(iii) \inf_{\mu>1}\left[\limsup_{n\to\infty}\left\{\max_{n<k\leq\mu_n}\mathcal{J}(\Delta^m u_k - \Delta^m u_n,\mathfrak{y})\right\}\right] = 0.$$

Above definition immediately gives the following remarks.

Remark 4.1 In Definition 3.2, $\sup_{\mu>1}$ and $\inf_{\mu>1}$ is equivalent to say $\lim_{\mu\to1^+}$.

Remark 4.2 A sequence $u = (u_n)$ in V is Δ^m-slowly oscillating, if and only if, for all $0 < \epsilon < 1$ and $\mathfrak{y} > 0$ there exists $\mu > 1$ and $n_0(\epsilon,\mathfrak{y}) \in \mathbb{N}$ such that

$$\mathcal{H}(\Delta^m u_k - \Delta^m u_n,\mathfrak{y}) > 1-\epsilon \text{ and } \mathcal{I}(\Delta^m u_k - \Delta^m u_n,\mathfrak{y}) < \epsilon, \mathcal{J}(\Delta^m u_k - \Delta^m u_n,\mathfrak{y}) < \epsilon.$$

holds for every $n_0 \leq n < k \leq \mu_n$.

Example 4.1 Let $(\mathbb{R}, |\cdot|)$ be a normed space. Let $a \circ b = ab$ and $a \diamond b = a+b-ab\ \forall\ a,b \in [0,1]$. For all $u \in \mathbb{R}$ and every $\mathfrak{y} > 0$, we consider $\mathcal{H}(u,\mathfrak{y}) = \frac{\mathfrak{y}}{\mathfrak{y}+|u|}, \mathcal{I}(u,\mathfrak{y}) = \frac{|u|}{\mathfrak{y}+|u|}, \mathcal{J}(u,\mathfrak{y}) = \frac{|u|}{\mathfrak{y}}$, then $(\mathbb{R},\circ,\diamond,\mathcal{H},\mathcal{I},\mathcal{J})$ is a NNS. Define a sequence (u_n) as follows:

$u_1 = 1,$

$u_2 = u_3 = 1 + \frac{1}{2},$

$u_4 = u_5 = u_6 = u_7 = 1 + \frac{1}{2} + \frac{1}{3},$

$\ldots$

$u_{2^n} = u_{2^n+1} = \ldots = u_{2^{n+1}-1} = \sum_{j=1}^{n+1} \frac{1}{j}$.

Given $\epsilon > 0$, let $\delta = 1$ and $m = 0$. Choose $n_0 \in \mathbb{N}$ s.t $\frac{1}{n_0} < \epsilon$. Then if $n > n_0$ and $n \leq k \leq 2n$, we have

$\mathcal{H}(u_k - u_n, \mathfrak{y}) = \frac{\mathfrak{y}}{\mathfrak{y}+|u_k-u_n|} > 1-\epsilon$ and $\mathcal{I}(u_k-u_n, \mathfrak{y}) = \frac{|u_k-u_n|}{\mathfrak{y}+|u_k-u_n|} < \epsilon$, $\mathcal{J}(u_k-u_n, \mathfrak{y}) = \frac{|u_k-u_n|}{\mathfrak{y}} < \epsilon$.

This shows that (u_n) is slowly oscillating sequence in $(\mathbb{R}, \circ, \diamond, \mathcal{H}, \mathcal{I}, \mathcal{J})$

Theorem 4.1 Let $u = (u_n)$ in V be a Δ^m-slowly oscillating sequence. Then for every $\mathfrak{y} > 0$, the conditions (i), (ii) and (iii) in Definition 4.1 are respectively equivalent to

$$(i) \sup_{0<\mu<1} \left[\liminf_{n\to\infty} \left\{ \min_{\mu n<k\leq n} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 1 \text{ and}$$

$$(ii) \inf_{0<\mu<1} \left[\limsup_{n\to\infty} \left\{ \max_{\mu n<k\leq n} \mathcal{I}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 0,$$

$$(iii) \inf_{0<\mu<1} \left[\limsup_{n\to\infty} \left\{ \max_{\mu n<k\leq n} \mathcal{J}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 0.$$

Proof. We first prove that the following conditions are equivalent:

$$\sup_{\mu>1} \left[\liminf_{n\to\infty} \left\{ \min_{n<k\leq \mu n} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 1,$$

$$\sup_{0<\mu<1} \left[\liminf_{n\to\infty} \left\{ \min_{\mu n<k\leq n} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 1.$$

Let $\mathfrak{y} > 0$ be given and for $\mu > 1$, we define

$$f_1(\mu) = \liminf_{n\to\infty} \left\{ \min_{n<k\leq\lfloor\mu n\rfloor} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \text{ and}$$

$$f_2\left(\frac{1}{\mu}\right) = \liminf_{k\to\infty} \left\{ \min_{\lfloor\frac{k}{\mu}\rfloor<n\leq k} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\}$$

By definition of $\liminf$ in f_1, we have a subsequence (n_r) with

$$f_1(\mu) = \lim_{r\to\infty} \left\{ \min_{n_r<k\leq\lfloor\mu n_r\rfloor} \mathcal{H}(\Delta^m u_k - \Delta^m u_{n_r}, \mathfrak{y}) \right\}.$$

This gives rise another subsequence (k_r) satisfying $n_r < k_r \leq \lfloor\mu n_r\rfloor$ with

$$\min_{n_r<k\leq\lfloor\mu n_r\rfloor} \mathcal{H}(\Delta^m u_k - \Delta^m u_{n_r}, \mathfrak{y}) = \mathcal{H}(\Delta^m u_{k_r} - \Delta^m u_{n_r}, \mathfrak{y}).$$

Since, $n_r < k_r \leq \lfloor \mu n_r \rfloor$, so by Remark 3 [16], $n_r \in \left(\lfloor (\frac{k_r}{\mu} \rfloor, k_r \right)$, and therefore we have

$$
\begin{aligned}
f_2\left(\frac{1}{\mu}\right) = \liminf_{k\to\infty}\left\{\min_{\lfloor \frac{k}{\mu}\rfloor < n \leq k} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y})\right\} &\leq \lim_{r\to\infty}\left\{\min_{\lfloor \frac{k_r}{\mu}\rfloor < n \leq k_r} \mathcal{H}(\Delta^m u_{k_r} - \Delta^m u_n, \mathfrak{y})\right\} \\
&\leq \lim_{r\to\infty} \mathcal{H}(\Delta^m u_{k_r} - \Delta^m u_{n_r}, \mathfrak{y}) \\
&= \lim_{r\to\infty}\left\{\min_{n_r < k \leq \lfloor \mu n_r \rfloor} \mathcal{H}(\Delta^m u_k - \Delta^m u_{n_r}, \mathfrak{y})\right\} \\
&= f_1\left(\frac{1}{\mu}\right).
\end{aligned}
$$

Similarly, we can have $f_2\left(\frac{1}{\mu}\right) \geq f_1\left(\frac{1}{\mu}\right)$ by changing their roles and therefore we have $f_1\left(\frac{1}{\mu}\right) = f_2\left(\frac{1}{\mu}\right)$. This shows that both expressions

$$
\begin{aligned}
\sup_{\mu>1}\left[\liminf_{n\to\infty}\left\{\min_{n<k\leq\ \mu_n} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y})\right\}\right] = 1, \\
\sup_{0<\mu<1}\left[\liminf_{n\to\infty}\left\{\min_{\mu_n<k\leq\ n} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y})\right\}\right] = 1.
\end{aligned}
$$

are equivalent.

Following the same line of proof, one can easily obtain the equivalence of other pairs of expressions. □

Example 4.2 Consider the neutrosophic normed space $((\mathbb{R}, \circ, \diamond, \mathcal{H}, \mathcal{I}, \mathcal{J})$ as defined in Example 3.1.

Define a sequence (u_n) by $u_n = \sum_{i=1}^{n}\left(\frac{1}{i}\right)$ and take $\mathfrak{y} > 0$.

Let $0 < \epsilon < 1$ be given and select $\mu = \frac{\mathfrak{y}\epsilon}{1-\epsilon} + 1$.

Now, for all n satisfying $1 < n < k < \mu_n$, we have

$$
\begin{aligned}
\|u_k - u_n\| &= \left\|\sum_{i=1}^{k}\left(\frac{1}{i}\right) - \sum_{i=1}^{n}\left(\frac{1}{i}\right)\right\| \\
&= \left\|\sum_{i=n+1}^{k}\left(\frac{1}{i}\right)\right\| \leq \sum_{i=n+1}^{k}\left(\frac{1}{i}\right) \\
&< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} \\
&= \frac{k-n}{n} = \frac{k}{n} - 1 < \mu - 1 = \frac{\mathfrak{y}\epsilon}{1-\epsilon} \text{(by selection of } n,\ k \text{ and } \mu);
\end{aligned}
$$

and therefore

$$\mathcal{H}\left(u_k-u_n,\mathfrak{y}\right)=\frac{\mathfrak{y}}{\mathfrak{y}+\|u_k-u_n\|}>\frac{\mathfrak{y}}{\mathfrak{y}+\frac{\mathfrak{y}\epsilon}{1-\epsilon}}=1-\epsilon \quad \text{and}$$

$$\mathcal{I}\left(u_k-u_n,\mathfrak{y}\right)=\frac{\|u_k-u_n\|}{\mathfrak{y}+\|u_k-u_n\|}<\frac{\left(\frac{\mathfrak{y}\epsilon}{1-\epsilon}\right)}{\mathfrak{y}+\left(\frac{\mathfrak{y}\epsilon}{1-\epsilon}\right)}=\epsilon.$$

Similarly, one can have $\mathcal{J}\left(u_k-u_n,\mathfrak{y}\right)<\epsilon$.

This shows that (u_n) is slowly oscillating in $(\mathbb{R},\circ,\diamond,\mathcal{H},\mathcal{I},\mathcal{J})$. $\square$

Theorem 4.2 Let V be a normed space with norm $\|.\|$ and $(\mathbb{R},\circ,\diamond,\mathcal{H},\mathcal{I},\mathcal{J})$ be the neutrosophic normed space as in Example 3.1. Then, a sequence $u=(u_n)$ is Δ^m-slowly oscillating in V if and only if it is so in $(\mathbb{R},\circ,\diamond,\mathcal{H},\mathcal{I},\mathcal{J})$.

Proof. We first assume that $u=(u_n)$ is Δ^m-slowly oscillating in V. Let, $\mathfrak{y}>0$ and $0<\epsilon<1$. Select $\epsilon'=\frac{\mathfrak{y}\epsilon}{1-\epsilon}$, then by Remark 4.2 there exists $\mu>1$ and $n_0(\epsilon,\mathfrak{y})\in\mathbb{N}$ such that

$$\mathcal{H}(\Delta^m u_k-\Delta^m u_n,\mathfrak{y})>1-\epsilon \text{ and } \mathcal{I}(\Delta^m u_k-\Delta^m u_n,\mathfrak{y})<\epsilon, \mathcal{J}(\Delta^m u_k-\Delta^m u_n,\mathfrak{y})<\epsilon.$$

holds for every $n_0\le n<k\le\mu_n$.

This proves that $u=(u_n)$ is Δ^m-slowly oscillating in $(\mathbb{R},\circ,\diamond,\mathcal{H},\mathcal{I},\mathcal{J})$.

Conversely, assume that $u=(u_n)$ is Δ^m-slowly oscillating in $(\mathbb{R},\circ,\diamond,\mathcal{H},\mathcal{I},\mathcal{J})$. Then for $0<\epsilon<\frac{1}{2}$ and $\mathfrak{y}=1>0$, then there exists $\mu>1$ and $n_0(\epsilon,1)\in\mathbb{N}$ such that

$$\mathcal{H}(\Delta^m u_k-\Delta^m u_n,1)>1-\epsilon \text{ and } \mathcal{I}(\Delta^m u_k-\Delta^m u_n,1)<\epsilon, \mathcal{J}(\Delta^m u_k-\Delta^m u_n,1)<\epsilon.$$

holds for every $n_0\le n<k\le\mu_n$.

Now, for $n_0\le n<k\le\mu_n$, $\mathcal{H}(\Delta^m u_k-\Delta^m u_n,1)>1-\epsilon$ will immediately gives

$$1-\epsilon<\frac{1}{1+\|\Delta^m u_k-\Delta^m u_n\|} \quad \text{or} \quad \|\Delta^m u_k-\Delta^m u_n\|<\frac{\epsilon}{1+\epsilon}<2\epsilon=\epsilon',$$

and therefore $u=(u_n)$ is Δ^m-slowly oscillating in V. $\square$

Theorem 4.3 If $u=(u_n)$ is any Δ^m-slowly oscillating sequence in V, then

$$(i) \sup_{\mu>1}\left[\liminf_{n\to\infty}\mathcal{H}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k-\Delta^m u_n),\mathfrak{y}\right)\right]=1;$$

$$(ii) \inf_{\mu>1}\left[\limsup_{n\to\infty}\mathcal{I}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k-\Delta^m u_n),\mathfrak{y}\right)\right]=0;$$

$$(iii) \inf_{\mu>1}\left[\limsup_{n\to\infty}\mathcal{J}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k-\Delta^m u_n),\mathfrak{y}\right)\right]=0.$$

Proof. Suppose that $u = (u_n)$ is any Δ^m-slowly oscillating sequence in V. Then for $\mathfrak{y} > 0$ and $0 < \epsilon < 1$ there exists $\mu > 1$ and $n_0(\epsilon, \mathfrak{y}) \in \mathbb{N}$ such that

$$\mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) > 1 - \epsilon \text{ and } \mathcal{I}(\Delta^m u_k - \Delta^m x_n, \mathfrak{y}) < \epsilon, \mathcal{J}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) < \epsilon.$$

holds for every $n_0 \leq n < k \leq \mu_n$. Now,

$$\begin{aligned}\mathcal{H}\left(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right) &= \mathcal{H}\left(\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), (\mu_n - n)\mathfrak{y}\right)\\ &\geq \min\{\mathcal{H}(\Delta^m u_{n+1} - \Delta^m u_n), \mathcal{H}(\Delta^m u_{n+2} - \Delta^m u_n), \cdots \mathcal{H}(\Delta^m u_{\mu_n} - \Delta^m u_n)\}\\ &> 1 - \epsilon\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}\left(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right) &= \mathcal{I}\left(\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), (\mu_n - n)\mathfrak{y}\right)\\ &\leq \max\{\mathcal{I}(\Delta^m u_{n+1} - \Delta^m u_n), \mathcal{I}(\Delta^m u_{n+2} - \Delta^m u_n), \cdots \mathcal{I}(\Delta^m u_{\mu_n} - \Delta^m u_n)\}\\ &< \epsilon,\\ \mathcal{J}\left(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right) &= \mathcal{J}\left(\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), (\mu_n - n)\mathfrak{y}\right)\\ &\leq \max\{\mathcal{J}(\Delta^m u_{n+1} - \Delta^m u_n), Y(\Delta^m u_{n+2} - \Delta^m u_n), \cdots \mathcal{J}(\Delta^m u_{\mu_n} - \Delta^m u_n)\}\\ &< \epsilon.\end{aligned}$$

This proves the Theorem. □

Theorem 4.4 If $u = (u_n)$ is any Δ^m-slowly oscillating sequence in V which is $N(C, \Delta^m, 1)-$summable with $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$, then $N - \lim_{n\to\infty} \Delta^m u_n = u_0$.
Proof. The proof is omitted as it can be obtain with the help of Theorem 3.2 and Theorem 4.3. □.

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