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# An Introduction to some Methods for Solving A Large System Linear Neutrosophic Equations

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**Abstract:** This paper aims to extend the methods of steepest descent and conjugate directions to the neutrosophic field R(I). the generalizations were built similarly to the classic algorithms, starting by generalizing the quadratic forms to R(I). Geometric isometry (AH-Isometry) S was used as the tool, and many examples are presented in the main paragraphs. The simple extension method can be generalized to other linear and non-linear methods.

**Keywords:** Quadratic forms; neutrosophic field; Steepest Descent; Conjugate Directions; Neutrosophic Matrix.

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## 1. Introduction

Methods for the steepest descent [1], conjugate directions [2], and conjugate gradients [3] were constructed and derived from quadratic forms.

It is not hard to see the advantages of The Method of Steepest Descent; it is simple, easy, and popular. It uses a zig-zag path from an arbitrary point until it converges to the solution.

Conjugate direction methods have been developed to speed up the slow convergence of the steepest descent method; more details can be found in [4].

The nature of the consideration of how these two methods work in the neutrosophic field will not change.

Despite the great importance of optimization in modern mathematics, the importance of neutrosophic in prediction, and the existence of many studies that have developed many optimization concepts in the field of neutrosophic, as described in [5-13]. However, no study has established a method for solving large linear system neutrosophic equations.

This paper lays a foundation for specifically addressing this problem by extending the steepest descent and conjugate directions methods to the R(I).

To ensure more effective and general results, the matrix $A_N$ was treated such that its real section differs from its neutrosophic section.

AH-isometry $S$ was used to speed up the results because it is a simple and effective tool that saves the properties of the classic study in R(I), which is defined as followed:

$$S : R(I) \to R \times R$$
$$S(a+bI) = (a, a+b)$$

where $R(I)=\{a+bI\ ;a,b\in R\}$.

And its invert is

$$S^{-1}:R\times R\to R(I)$$

With some basic properties

$$S\left[(a+bI)+(c+dI)\right]=S(a+bI)+S(c+dI)$$

$$S\left[(a+bI).(c+dI)\right]=S(a+bI).S(c+dI).$$

Other details and applications of this tool can be found in [14-18]. Starting from derivate more generalized quadratic form

$$f(x_N)=\frac{1}{2}x_N^T A_N x_N - b_N^T x_N + c_N$$

The methods for steepest descent and conjugate directions were generalized to the neutrosophic field. It can be seen that the classical definitions used in this paper were simply extended to R(I).

## 2. Preliminaries

Steepest descent and conjugate directions are the most popular iterative methods for solving large systems of linear equations. Each method is effective for systems of the form

$$A\xi=b. \quad (1)$$

where $\xi$ is an unknown vector, $b$ is a known vector, and $A$ is a known, square, positive-definite (or positive-indefinite) matrix.

A matrix $A$ is positive-definite if, for every nonzero vector $\xi$,

$$\xi^T A\xi>0. \quad (2)$$

Where $\xi^T$ is the Transpose of $\xi$.

**The Quadratic Form**

A quadratic form is a scalar, quadratic function of a vector with the form

$$f(\xi)=\frac{1}{2}\xi^T A\xi-b^T\xi+c, \quad (3)$$

Where $A$ is a matrix, $\xi$ and $b$ are vectors, and $c$ is a scalar constant.

However, condition (2) is not a very intuitive idea, as it affects the shape of quadratic forms.

The gradient of a quadratic form is defined to be

$$f'(\xi)=\begin{bmatrix}\frac{\partial}{\partial\xi_1}f(\xi)\\ \frac{\partial}{\partial\xi_2}f(\xi)\\ \vdots\\ \frac{\partial}{\partial\xi_n}f(\xi)\end{bmatrix}. \quad (4)$$

One can apply Equation (4) to Equation (3), and derive, then, we obtain

$$f'(\xi)=\frac{1}{2}A^{T}\xi+\frac{1}{2}A\xi-b \tag{5}$$

If $A$ is symmetric, this equation reduces to

$$f'(\xi)=A\xi-b \tag{6}$$

Setting the gradient to zero, we obtain Equation (1).

**3. The Neutrosophic Quadratic Form**

**Definition (1):** The neutrosophic form of a matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ could be written as following:

$$A_N=\begin{bmatrix} a_{11}^N & a_{12}^N \\ a_{12}^N & a_{22}^N \end{bmatrix}$$

where $a_{ij}^N=a_{ij}+Ia_{ij}$, $i,j=1,2$.

In fact, $A_N:=A+IA=\begin{bmatrix} a_{11}^N & a_{12}^N \\ a_{21}^N & a_{22}^N \end{bmatrix}+I\begin{bmatrix} a_{11}^N & a_{12}^N \\ a_{21}^N & a_{22}^N \end{bmatrix}=\begin{bmatrix} a_{11}^N & a_{12}^N \\ a_{21}^N & a_{22}^N \end{bmatrix}+\begin{bmatrix} a_{11}^N I & a_{12}^N I \\ a_{21}^N I & a_{22}^N I \end{bmatrix}=\begin{bmatrix} a_{11}^N & a_{12}^N \\ a_{21}^N & a_{22}^N \end{bmatrix}.$

When $A_N$ is a symmetric matrix, then their componentes are symmetrics. One can considered a more generalized form of $A_N$ by taking non-coincide components.

Transpose of $A_N$ is defined as: $A_N^T:=[A+IA]^T=A^T+IA^T=\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}+I\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}=\begin{bmatrix} a_{11}^N & a_{21}^N \\ a_{12}^N & a_{22}^N \end{bmatrix}.$

For $m=2, n=1$, we have $A_N^T=\begin{bmatrix} a_{11}^N & a_{21}^N \end{bmatrix}.$, and $A_N=\begin{bmatrix} a_{11}^N \\ a_{21}^N \end{bmatrix}$

By looking to the equation $A\xi=b$, where $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\xi=\begin{bmatrix} \xi_{11} \\ \xi_{21} \end{bmatrix}$, $b=\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$,

and despite it is easy to find the neutrosophic form of it, we will consider more generalized form:

$$A_N x_N=b_N,$$

where

$$A_N=A+I\Lambda=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+I\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix},\ x_N=\xi+I\eta=\begin{bmatrix} \xi_{11} \\ \xi_{21} \end{bmatrix}+I\begin{bmatrix} \eta_{11} \\ \eta_{21} \end{bmatrix},\ b_N=b+IB=\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}+I\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix},$$

that we can rewrite it as

$$A\xi+I\left(A\eta+\Lambda\xi+\Lambda\eta\right)=b+IB. \tag{7}$$

Note that, if we make the neutrosophic part in (7) is equal to zero, then we will find the classical equation $A\xi=b$.

**The Neutrosophic Quadratic Form**

To find The Neutrosophic Quadratic Form, we begin Form the well-known relation

$$f(\xi)=\frac{1}{2}\xi^{T}A\xi-b^{T}\xi+c.$$

That which, according to the neutrosophic form, simply, we have

$$f(x_N)=\frac{1}{2}x_N^{T}A_Nx_N-b_N^{T}x_N+c_N,\ c_N=c+Id. \tag{8}$$

By taking $S$ for the both sides of (8):

$$\begin{aligned}S[f(x_N)]&=S(1/2)S(x_N^{T})S(A_N)S(x_N)-S(b_N^{T})S(x_N)+S(c_N)\\&=(1/2,1/2)(\xi,\xi+\eta)^{T}(A,A+\Lambda)(\xi,\xi+\eta)-(b,b+B)^{T}(\xi,\xi+\eta)+(c,c+d)\\&=(1/2,1/2)(\xi^{T},\xi^{T}+\eta^{T})(A,A+\Lambda)(\xi,\xi+\eta)-(b^{T},b^{T}+B^{T})(\xi,\xi+\eta)+(c,c+d)\\&=\left[\frac{1}{2}\xi^{T}A\xi,\frac{1}{2}(\xi^{T}+\eta^{T})(A+\Lambda)(\xi+\eta)\right]-\left[b^{T}\xi,(b^{T}+B^{T})(\xi+\eta)\right]+(c,c+d)\\&=\left[\frac{1}{2}\xi^{T}A\xi-b^{T}\xi+c,\frac{1}{2}(\xi^{T}+\eta^{T})(A+\Lambda)(\xi+\eta)-(b^{T}+B^{T})(\xi+\eta)+(c+d)\right].\end{aligned}$$

Taking $S^{-1}$, then we obtain

$$\begin{aligned}f(x_N)=&\frac{1}{2}\xi^{T}A\xi-b^{T}\xi+c+\\&+I\left\{\frac{1}{2}(\xi^{T}+\eta^{T})(A+\Lambda)(\xi+\eta)-(b^{T}+B^{T})(\xi+\eta)+(c+d)-\left[\frac{1}{2}\xi^{T}A\xi-b^{T}\xi+c\right]\right\}.\end{aligned}$$

Derivate the last function with respect to $\xi,\eta$ respectively:

$$\begin{aligned}f'^{N}_{\xi}&=\frac{\partial f(x_N)}{\partial\xi}=\frac{1}{2}(A+A^{T})\xi-b+I\left[\frac{1}{2}(A+\Lambda+A^{T}+\Lambda^{T})(\xi+\eta)-(b+B)-\left(\frac{1}{2}(A+A^{T})\xi-b\right)\right]\\&=\frac{1}{2}(A+A^{T})\xi-b+I\left[\frac{1}{2}(\Lambda+\Lambda^{T})(\xi+\eta)+\frac{1}{2}(A+A^{T})\eta-B\right].\\f'^{N}_{\eta}&=\frac{\partial f(x_N)}{\partial\eta}=I\left[\frac{1}{2}(A+\Lambda+A^{T}+\Lambda^{T})(\xi+\eta)-(b+B)\right].\end{aligned}$$

By making $f'(x_N)=0$, then it leads to solve the system:

$$(*)\quad\begin{cases}\frac{1}{2}(A+A^{T})\xi-b=0 & (9)\\ \frac{1}{2}(\Lambda+\Lambda^{T})(\xi+\eta)+\frac{1}{2}(A+A^{T})\eta-B=0 & (10)\\ \frac{1}{2}(A+\Lambda+A^{T}+\Lambda^{T})(\xi+\eta)-(b+B)=0 & (11)\end{cases}$$

If $A_N$ is symmetric, then, $A=A^{T},\Lambda=\Lambda^{T}$, and the system becomes:

$$(**)\quad\begin{cases}A\xi-b=0 & (12)\\ \Lambda(\xi+\eta)+A\eta-B=0 & (13)\\ (A+\Lambda)(\xi+\eta)-(b+B)=0 & (14)\end{cases}$$

In both systems, we note that the third equation is a summation of the first and the second equation.

**Examples:**

1. Let $A_N=\begin{bmatrix}1+I & 4+3I\\ -1+2I & -2-I\end{bmatrix}$, then $A=\begin{bmatrix}1 & 4\\ -1 & -2\end{bmatrix}$, $\Lambda=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$. $x_N=\begin{bmatrix}\xi_1+I\eta_1\\ \xi_2+I\eta_2\end{bmatrix}$, then
$\xi=\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}, \eta=\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}$. $b_N=\begin{bmatrix}2+I\\ -3-2I\end{bmatrix}$, then $b=\begin{bmatrix}2\\ -3\end{bmatrix}, B=\begin{bmatrix}1\\ -2\end{bmatrix}$.

By returning to (*) relations, we have for (9):

$$A^T=\begin{bmatrix}1 & -1\\ 4 & -2\end{bmatrix}, \text{ and } (1/2)\left(A+A^T\right)=\begin{bmatrix}1 & 3/2\\ 3/2 & -2\end{bmatrix},$$

$$\begin{bmatrix}1 & 3/2\\ 3/2 & -2\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}=\begin{bmatrix}2\\ -3\end{bmatrix}\xrightarrow{\text{yields}}\xi=\begin{bmatrix}-2/17\\ 24/17\end{bmatrix}.$$

And for (10), we have

$$\frac{1}{2}\left(\Lambda+\Lambda^T\right)(\xi+\eta)=\begin{bmatrix}1 & 5/2\\ 5/2 & -1\end{bmatrix}\begin{bmatrix}-2/17+\eta_1\\ 24/17+\eta_2\end{bmatrix}=\begin{bmatrix}58/17+\eta_1+(5/2)\eta_2\\ -29/17+(5/2)\eta_1-\eta_2\end{bmatrix},$$

$$\frac{1}{2}\left(A+A^T\right)\eta=\begin{bmatrix}1 & 3/2\\ 3/2 & -2\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}=\begin{bmatrix}\eta_1+(3/2)\eta_2\\ (3/2)\eta_1-2\eta_2\end{bmatrix},$$

and

$$\begin{bmatrix}58/17+\eta_1+(5/2)\eta_2\\ -29/17+(5/2)\eta_1-\eta_2\end{bmatrix}+\begin{bmatrix}\eta_1+(3/2)\eta_2\\ (3/2)\eta_1-2\eta_2\end{bmatrix}=\begin{bmatrix}1\\ -2\end{bmatrix}$$

By solving the system

$$\begin{cases}58/17+2\eta_1+4\eta_2=1,\\ -29/17+4\eta_1-3\eta_2=-2.\end{cases}$$

We obtain

$$\eta=\begin{bmatrix}-13/34\\ -7/17\end{bmatrix}, \text{ and } x_N=\begin{bmatrix}(-2/17)+I(-13/34)\\ (24/17)+I(-7/17)\end{bmatrix}.$$

This solution satisfied (11), since

$$\frac{1}{2}\left(A+\Lambda+A^T+\Lambda^T\right)(\xi+\eta)=\begin{bmatrix}2 & 4\\ 4 & -3\end{bmatrix}\begin{bmatrix}-1/2\\ 1\end{bmatrix}=\begin{bmatrix}3\\ -5\end{bmatrix}=b+B.$$

2. Let $A_N$ be a symmetric matrix, i.e. $A_N=\begin{bmatrix}2-I & 5+2I\\ 5+2I & -1+3I\end{bmatrix}$, $x_N=\begin{bmatrix}\xi_1+I\eta_1\\ \xi_2+I\eta_2\end{bmatrix}$, and $b_N=\begin{bmatrix}1+2I\\ 2+3I\end{bmatrix}$.

By returning to the (**), we have for (12):

$$\begin{bmatrix}2 & 5\\ 5 & -1\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix},$$

$$\begin{bmatrix}2\xi_1+5\xi_2\\ 5\xi_1-\xi_2\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix},$$

then

$$\xi=\begin{bmatrix}11/27\\ 1/27\end{bmatrix}.$$

and for (13):

$$\begin{bmatrix}-1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}(11/27)+\eta_1\\ (1/27)+\eta_2\end{bmatrix}+\begin{bmatrix}2 & 5\\ 5 & -1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}=\begin{bmatrix}2\\ 3\end{bmatrix},$$

then we find that

$$\eta=\begin{bmatrix}266/1269\\ 385/1269\end{bmatrix},$$

and

$$x_N=\begin{bmatrix}11/27\\ 1/27\end{bmatrix}+I\begin{bmatrix}266/1269\\ 385/1269\end{bmatrix}.$$

The solution satisfies (14):

$$(A+\Lambda)(\xi+\eta)-(b+B)=\begin{bmatrix}1 & 7\\ 7 & 2\end{bmatrix}\begin{bmatrix}29/47\\ 16/47\end{bmatrix}-\begin{bmatrix}3\\ 5\end{bmatrix}=0.$$

**A special case**

If $A=\Lambda,\ \xi=\eta,\ b=B$ then the neutrosophic Quadratic Form is:

$$f(x_N)=\frac{1}{2}\xi^T A\xi-b^T\xi+c+I\left(\frac{3}{2}\xi^T A\xi-3b^T\xi+c\right), \tag{15}$$

And

$$f'(x_N)=\frac{1}{2}A\xi+\frac{1}{2}A^T\xi-b+I\left(\frac{3}{2}A\xi+\frac{3}{2}A^T\xi-3b\right),$$

by making $f'(x_N)=0$, that leads to

$$\left(A+A^T\right)\xi=2b. \tag{16}$$

This means that the neutrosophic quadratic form of the classical form can be found directly by solving the well-known equation (16), which turns out to be (1) when $A$ is symmetric, which we will see later in an example.

## 4. The Method of neutrosophic Steepest Descent

Derivate the function (8) by using the relation (4), we find

$$f'(x_N)=\frac{\partial f(x_N)}{\partial\xi}+\frac{\partial f(x_N)}{\partial\eta}=$$

$$=\frac{1}{2}\left(A+A^T\right)\xi-b+I\left[\left(A+\Lambda+A^T+\Lambda^T\right)(\xi+\eta)-(2b+2B)-\left(\frac{1}{2}\left(A+A^T\right)\xi-b\right)\right]$$

Let us denote $x_{(i)}^N=\xi_{(i)}+\eta_{(i)}I$ to $x_N=\xi+\eta I$ in step $(i)$, then

$$-f'\left(x_{(i)}^N\right)=b-\frac{1}{2}\left(A+A^T\right)\xi_{(i)}+I\left[(2b+2B)-\left(A+\Lambda+A^T+\Lambda^T\right)\left(\xi_{(i)}+\eta_{(i)}\right)-\left(b-\frac{1}{2}\left(A+A^T\right)\xi_{(i)}\right)\right]$$

$$=b+(b+2B)I-\left\{\frac{1}{2}\left(A+A^T\right)+\left[\frac{1}{2}\left(A+A^T\right)+\left(\Lambda+\Lambda^T\right)\right]I\right\}\xi_{(i)}-$$

$$-\left\{\frac{1}{2}\left(A+A^T\right)+\left[\frac{1}{2}\left(A+A^T\right)+\left(\Lambda+\Lambda^T\right)\right]I\right\}\eta_{(i)}I.$$

Putting $k_N=b+(b+2B)I,\ t_N=\frac{1}{2}\left(A+A^T\right)+\left[\frac{1}{2}\left(A+A^T\right)+\left(\Lambda+\Lambda^T\right)\right]I$, and suppose that

$$r_{(i)}^{N}=-f'\left(x_{(i)}^{N}\right),$$

then

$$r_{(i)}^{N}=k_{N}-t_{N}x_{(i)}^{N}, \tag{17}$$

Let us define the neutrosophic error as

$$e_{(i)}^{N}=x_{(i)}^{N}-x_{N}=\left(\xi_{(i)}+\eta_{(i)}I\right)-(\xi+\eta I)=(\xi_{(i)}-\xi)+\left(\eta_{(i)}-\eta\right)I\ .$$

Note that $r_{(i)}^{N}=-t_{N}e_{(i)}^{N}$.

Let us start from a point $x_{(0)}^{N}$, and we will choose a point $x_{(1)}^{N}$ such that $x_{(1)}^{N}=x_{(0)}^{N}+\alpha_{N}.r_{(0)}^{N}$

where $\alpha_N$ is a neutrosophic number, which is minimizes $f$ when $\frac{df\left(x_{(1)}^{N}\right)}{d\alpha_{N}}$ is equal to zero.

To determine $\alpha_N$, we have $f'\left(x_{(i)}^{N}\right)=-r_{(i)}^{N}$ and

$$\frac{df\left(x_{(1)}^{N}\right)}{d\alpha_{N}}=\left[f'\left(x_{(1)}^{N}\right)\right]^{T}\frac{d\left(x_{(1)}^{N}\right)}{d\alpha_{N}}=\left[f'\left(x_{(1)}^{N}\right)\right]^{T}r_{(0)}^{N}$$

Setting the last expression to zero, then $\left[r_{(1)}^{N}\right]^{T}r_{(0)}^{N}=0$, by using (11), we have

$$\left[k_{N}-t_{N}x_{(1)}^{N}\right]^{T}r_{(0)}^{N}=0$$

$$\left[k_{N}-t_{N}\left(x_{(0)}^{N}+\alpha_{N}.r_{(0)}^{N}\right)\right]^{T}r_{(0)}^{N}=0$$

$$\left[k_{N}-t_{N}x_{(0)}^{N}\right]^{T}r_{(0)}^{N}-\alpha_{N}\left[t_{N}r_{(0)}^{N})\right]^{T}r_{(0)}^{N}=0$$

$$\left[k_{N}-t_{N}x_{(0)}^{N}\right]^{T}r_{(0)}^{N}=\alpha_{N}\left[t_{N}r_{(0)}^{N})\right]^{T}r_{(0)}^{N}$$

$$\left[r_{(0)}^{N}\right]^{T}r_{(0)}^{N}=\alpha_{N}\left[r_{(0)}^{N})\right]^{T}\left[t_{N}\right]^{T}r_{(0)}^{N}$$

$$\alpha_{N}=\frac{\left[r_{(0)}^{N}\right]^{T}r_{(0)}^{N}}{\left[r_{(0)}^{N}\right]^{T}\left[t_{N}\right]^{T}r_{(0)}^{N}}.$$

This formula can be rewritten with more specifically way as followed:
By returning to (5), it is easy to see that

$$-f'(\xi)=b-\frac{1}{2}\left(A+A^{T}\right)\xi,$$

and

$$\left[2f_{\eta}^{'N}-f'(\xi)\right]I=\left[\left(A+\Lambda+A^{T}+\Lambda^{T}\right)(\xi+\eta)-(2b+2B)-\left(\frac{1}{2}\left(A+A^{T}\right)\xi-b\right)\right]I.$$

Then

$$r_{(0)}^{N}=-f'\left(x_{(0)}^{N}\right)=-f'\left(\xi_{(0)}\right)+I\left[-2f_{\eta_{(0)}}^{'N}+f'\left(\xi_{(0)}\right)\right],$$

and one can write

$$\left[r_{(0)}^{N}\right]^{T}=\left[-f'\left(x_{(0)}^{N}\right)\right]^{T}=\left[-f'\left(\xi_{(0)}\right)\right]^{T}+I\left[\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}+\left[f'\left(\xi_{(0)}\right)\right]^{T}\right],$$

$$S\left\{\left[r_{(0)}^{N}\right]^{T}r_{(0)}^{N}\right\}=S\left\{\left[r_{(0)}^{N}\right]^{T}\right\}S\left\{r_{(0)}^{N}\right\}=\left(\left[-f'\left(\xi_{(0)}\right)\right]^{T},\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\right)\left(-f'\left(\xi_{(0)}\right),-2f'^{N}_{\eta_{(0)}}\right)=$$

$$=\left(\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right),\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\left(-2f'^{N}_{\eta_{(0)}}\right)\right).$$

and

$$S^{-1}\left(S\left\{\left[r_{(0)}^{N}\right]^{T}r_{(0)}^{N}\right\}\right)=\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)+I\left(\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\left(-2f'^{N}_{\eta_{(0)}}\right)-\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)\right).$$

Suppose that

$$\varphi=\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right),\ \psi=\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\left(-2f'^{N}_{\eta_{(0)}}\right)-\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right).$$

Then

$$\left[r_{(0)}^{N}\right]^{T}r_{(0)}^{N}=\varphi+I\psi .$$

And by using the same tool $S$, we find

$$S^{-1}\left(S\left\{\left[r_{(0)}^{N}\right]^{T}\left[t_{N}\right]^{T}r_{(0)}^{N}\right\}\right)=\left[r_{(0)}^{N}\right]^{T}\left[t_{N}\right]^{T}r_{(0)}^{N}=$$

$$=\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left[\frac{1}{2}\left(A+A^{T}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)+I\left[\left(\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\right)\left[\left(A+A^{T}\right)^{T}+\left(\Lambda+\Lambda^{T}\right)^{T}\right]\left(-2f'^{N}_{\eta_{(0)}}\right)-\right.$$

$$\left.-\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left[\frac{1}{2}\left(A+A^{T}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)\right].$$

Putting

$$\Phi=\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(\frac{1}{2}\left(A^{T}+A\right)\right)\left(-f'\left(\xi_{(0)}\right)\right),$$

$$\Psi=\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\left[(A^{T}+A)+\left(\left(\Lambda^{T}+\Lambda\right)\right)\right]\left(-2f'^{N}_{\eta_{(0)}}\right)-\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(\frac{1}{2}\left(A^{T}+A\right)\right)\left(-f'\left(\xi_{(0)}\right)\right).$$

Then $\left[r_{(0)}^{N}\right]^{T}\left[t_{N}\right]^{T}r_{(0)}^{N}=\Phi+I\Psi$ and $\alpha_N$ takes the form

$$\alpha_{N}=\frac{\varphi+I\psi}{\Phi+I\Psi}.$$

$$S\left(\alpha_{N}\right)=S\left(\frac{\varphi+I\psi}{\Phi+I\Psi}\right)=\frac{S\left(\varphi+I\psi\right)}{S\left(\Phi+I\Psi\right)}=\frac{\left(\varphi,\varphi+\psi\right)}{\left(\Phi,\Phi+\Psi\right)}=\left(\frac{\varphi}{\Phi},\frac{\varphi+\psi}{\Phi+\Psi}\right)$$

$$S^{-1}\left(S\left(\alpha_{N}\right)\right)=\alpha_{N}=\frac{\varphi}{\Phi}+I\left[\frac{\varphi+\psi}{\Phi+\Psi}-\frac{\varphi}{\Phi}\right],$$

or

$$\alpha_{N}=\frac{\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)}{\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(\frac{1}{2}\left(A^{T}+A\right)\right)\left(-f'\left(\xi_{(0)}\right)\right)}+$$

$$+I\left[\frac{\left[-2f'^{N}_{\eta(0)}\right]^{T}\left(-2f'^{N}_{\eta(0)}\right)}{\left[-2f'^{N}_{\eta(0)}\right]^{T}\left[(A^{T}+A)+\left((\Lambda^{T}+\Lambda)\right)\right]\left(-2f'^{N}_{\eta(0)}\right)}-\frac{\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)}{\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(\frac{1}{2}\left(A^{T}+A\right)\right)\left(-f'\left(\xi_{(0)}\right)\right)}\right].$$

Notice that the real part of $\alpha_N$ is the well-known classic form of $\alpha$ .
As a result, The Method of neutrosophic Steepest Descent is:

$$r^{N}_{(i)} = k_N - t_N x^{N}_{(i)},$$

$$\alpha^{N}_{(i)} = \frac{\left[r^{N}_{(i)}\right]^{T} r^{N}_{(i)}}{\left[r^{N}_{(i)}\right]^{T}\left[t_N\right]^{T} r^{N}_{(i)}},$$

$$x^{N}_{(i+1)} = x^{N}_{(i)} + \alpha^{N}_{(i)}.r^{N}_{(i)}.$$

Under conditions of multiplication of matrix, we can premultiplying both sides of the last equation by $-t_N$ and adding $k_N$ , then we have

$$r^{N}_{(i+1)} = r^{N}_{(i)} - \alpha^{N}_{(i)} t_N . r^{N}_{(i)} \tag{18}$$

**Example 1:**

Let us start with $A_N = \begin{bmatrix} 4-4I & 2+2I \\ 2+2I & 2-2I \end{bmatrix}$, $b_N = \begin{bmatrix} -1+I \\ 1-I \end{bmatrix}$, and $x^{N}_{(i)} = \begin{bmatrix} -\frac{1}{4}+\frac{1}{4}I \\ 0 \end{bmatrix}$.

$$k_N = b + (b+2B)I = \begin{bmatrix} -1+I \\ 1-I \end{bmatrix},$$

$$t_N = \frac{1}{2}\left(A+A^{T}\right) + \left[\frac{1}{2}\left(A+A^{T}\right)+\left(\Lambda+\Lambda^{T}\right)\right]I = \begin{bmatrix} 4-4I & 2+6I \\ 2+6I & 2-2I \end{bmatrix},$$

$$r^{N}_{(i)} = k_N - t_N x^{N}_{(i)} = \begin{bmatrix} -1+I \\ 1-I \end{bmatrix} - \begin{bmatrix} 4-4I & 2+6I \\ 2+6I & 2-2I \end{bmatrix}\begin{bmatrix} -\frac{1}{4}+\frac{1}{4}I \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2}-\frac{3}{2}I \end{bmatrix}.$$

$$\left[r^{N}_{(i)}\right]^{T} r^{N}_{(i)} = \frac{9}{4}-\frac{9}{4}I.$$

$$\left[r^{N}_{(i)}\right]^{T}\left[t_N\right]^{T} r^{N}_{(i)} = \frac{9}{2}-\frac{9}{2}I.$$

$$\alpha^{N}_{(i)} = \frac{\frac{9}{4}(1-I)}{\frac{9}{2}(1-I)} = \frac{1}{2}.$$

By using (18), we have

$$r^{N}_{(i+1)} = \begin{bmatrix} 0 \\ \frac{3}{2}-\frac{3}{2}I \end{bmatrix} - \left(\frac{1}{2}\right)\begin{bmatrix} 4-4I & 2+6I \\ 2+6I & 2-2I \end{bmatrix}\begin{bmatrix} 0 \\ \frac{3}{2}-\frac{3}{2}I \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}+\frac{3}{2}I \\ 0 \end{bmatrix}.$$

$$\left[r^{N}_{(i+1)}\right]^{T} r^{N}_{(i+1)} = \begin{bmatrix} -\frac{3}{2}+\frac{3}{2}I & 0 \end{bmatrix}\begin{bmatrix} -\frac{3}{2}+\frac{3}{2}I \\ 0 \end{bmatrix} = \frac{9}{4}-\frac{9}{4}I.$$

$$\left[r^{N}_{(i+1)}\right]^{T}\left[t_N\right]^{T} r^{N}_{(i+1)} = \begin{bmatrix} -\frac{3}{2}+\frac{3}{2}I & 0 \end{bmatrix}\begin{bmatrix} 4-4I & 2+6I \\ 2+6I & 2-2I \end{bmatrix}\begin{bmatrix} -\frac{3}{2}+\frac{3}{2}I \\ 0 \end{bmatrix} = 9-9I.$$

Hence

$$\alpha_{(i+1)}^{N}=\frac{\frac{9}{4}(1-I)}{9(1-I)}=\frac{1}{4}.$$

**Example 2:**

Let $A_N=\begin{bmatrix}4-\frac{1}{2}I & 2-\frac{1}{2}I\\ -2+\frac{1}{2}I & -1\end{bmatrix}$, $b_N=\begin{bmatrix}1-I\\ -1+2I\end{bmatrix}$, and $x_{(i)}^{N}=\begin{bmatrix}\frac{1}{4}-\frac{1}{4}I\\ 0\end{bmatrix}$.

Then

$$k_N=b+(b+2B)I=\begin{bmatrix}1-I\\ -1+3I\end{bmatrix},$$

$$t_N=\frac{1}{2}\left(A+A^T\right)+\left[\frac{1}{2}\left(A+A^T\right)+\left(\Lambda+\Lambda^T\right)\right]I=\begin{bmatrix}4+3I & 2I\\ 2I & -1+2I\end{bmatrix},$$

$$r_{(i)}^{N}=k_N-t_N x_{(i)}^{N}=\begin{bmatrix}1-I\\ -1+3I\end{bmatrix}-\begin{bmatrix}4+3I & 2I\\ 2I & -1+2I\end{bmatrix}\begin{bmatrix}\frac{1}{4}-\frac{1}{4}I\\ 0\end{bmatrix}=\begin{bmatrix}0\\ -1+3I\end{bmatrix}.$$

$$\left[r_{(i)}^{N}\right]^T r_{(i)}^{N}=1+3I.$$

$$\left[r_{(i)}^{N}\right]^T \left[t_N\right]^T r_{(i)}^{N}=-1+5I.$$

$$\alpha_{(i)}^{N}=\frac{1+3I}{-1+5I}=-1+2I.$$

By using (18), we have

$$r_{(i+1)}^{N}=\begin{bmatrix}0\\ -1+3I\end{bmatrix}-(-1+2I)\begin{bmatrix}4+3I & 2I\\ 2I & -1+2I\end{bmatrix}\begin{bmatrix}0\\ -1+3I\end{bmatrix}=\begin{bmatrix}-4I\\ 0\end{bmatrix}.$$

$$\left[r_{(i+1)}^{N}\right]^T r_{(i+1)}^{N}=\begin{bmatrix}-4I & 0\end{bmatrix}\begin{bmatrix}-4I\\ 0\end{bmatrix}=16I.$$

$$\left[r_{(i+1)}^{N}\right]^T \left[t_N\right]^T r_{(i+1)}^{N}=\begin{bmatrix}-4I & 0\end{bmatrix}\begin{bmatrix}4+3I & 2I\\ 2I & -1+2I\end{bmatrix}\begin{bmatrix}-4I\\ 0\end{bmatrix}=112I.$$

Hence

$$\alpha_{(i+1)}^{N}=\frac{16I}{112I}=\frac{1}{7}I.$$

**5. The method of Neutrosophic Conjugate Directions**

As in the classic case, we use coordinate axes as search directions. Every step consists of two paths, the first leads to the correct $x_{(1)}^{N}$ – coordinate, and the second hit the desired point, after n steps, it will be done.

**Definition (2):** Let $A_N$ be a symmetric matrix, two nonzero vectors $d_{(i)}^{N},d_{(j)}^{N}$ are said to be $A_N$ - orthogonal, if

$$\left[d_{(i)}^{N}\right]^T .A_N .d_{(j)}^{N}=0_N$$

where $i\neq j$.

In general, we take

$$x_{(i+1)}^{N}=x_{(i)}^{N}+\alpha_{(i)}^{N}d_{(i)}^{N} \tag{19}$$

By taking into consideration the fact that $e_{(i+1)}^{N}$ should be orthogonal to $d_{(i)}^{N}$, we can find the value of $\alpha_{(i)}^{N}$, which eliminates the need to step in the direction $d_{(i)}^{N}$ again. We have

$$\left[d_{(i)}^{N}\right]^{T} . e_{(i+1)}^{N} = 0_N$$

$$\left[d_{(i)}^{N}\right]^{T} . \left(e_{(i)}^{N} + \alpha_{(i)}^{N} d_{(i)}^{N}\right) = 0_N$$

Which leads to

$$\alpha_{(i)}^{N} = \frac{-\left[d_{(i)}^{N}\right]^{T} e_{(i)}^{N}}{\left[d_{(i)}^{N})\right]^{T} d_{(i)}^{N}}. \tag{20}$$

Knowing $e_{(i)}^{N}$ guarantees computing $\alpha_{(i)}^{N}$, As it has known in classical case, the solution is to be the two vectors $d_{(i)}^{N}$ and $d_{(j)}^{N}$ are $A_N$ -orthogonal. To make $e_{(i+1)}^{N}$ be $A_N$ -orthogonal to $d_{(i)}^{N}$, it is sufficient to find the minimum point along the direction $d_{(i)}^{N}$:

$$\frac{d}{d\alpha^N} f\left(x_{(i+1)}^{N}\right) = 0_N$$

$$\left[f'\left(x_{(i+1)}^{N}\right)\right]^{T} . \frac{d}{d\alpha^N} x_{(i+1)}^{N} = 0_N$$

$$-\left[r_{(i+1)}^{N}\right]^{T} d_{(i)}^{N} = 0_N$$

$$\left[d_{(i)}^{N}\right]^{T} . A_N . e_{(i+1)}^{N} = 0_N$$

Now, the equation (20) becomes with $A_N$ -orthogonal search directions, as following:

$$\alpha_{(i)}^{N} = \frac{-\left[d_{(i)}^{N}\right]^{T} . A_N . e_{(i)}^{N}}{\left[d_{(i)}^{N})\right]^{T} . A_N . d_{(i)}^{N}}$$

$$\alpha_{(i)}^{N} = \frac{\left[d_{(i)}^{N}\right]^{T} . r_{(i)}^{N}}{\left[d_{(i)}^{N})\right]^{T} . A_N . d_{(i)}^{N}}.$$

Which could be to calculate.

Let us write $\alpha_{(i)}^{N}$ as $\alpha_{(i)}^{N} = X + IY$. Suppose that $d_{(i)}^{N} = d_{(i)} + ID_{(i)}$, then $\left[d_{(i)}^{N})\right]^{T} = d_{(i)}^{T} + ID_{(i)}^{T}$.

$$S\left\{\left[d_{(i)}^{N}\right]^{T} . r_{(i)}^{N}\right\} = S\left(\left[d_{(i)}^{N}\right]^{T}\right) S\left(r_{(i)}^{N}\right) = \left(d_{(i)}{}^{T}, d_{(i)}{}^{T} + D_{(i)}{}^{T}\right)\left(-f'\left(\xi_{(i)}\right), -2f'^{N}_{\eta_{(i)}}\right)$$
$$= \left(d_{(i)}{}^{T}\left(-f'\left(\xi_{(i)}\right)\right), \left(d_{(i)}{}^{T} + D_{(i)}{}^{T}\right)\left[-2f'^{N}_{\eta_{(i)}}\right]\right).$$

Then

$$\left[d_{(i)}^{N}\right]^{T} . r_{(i)}^{N} = d_{(i)}{}^{T}\left(-f'\left(\xi_{(i)}\right)\right) + I\left[\left(d_{(i)}{}^{T} + D_{(i)}{}^{T}\right)\left[-2f'^{N}_{\eta_{(i)}}\right] - d_{(i)}{}^{T}\left(-f'\left(\xi_{(i)}\right)\right)\right].$$

$$S\left\{\left[d_{(i)}^{N})\right]^{T} . A_N . d_{(i)}^{N}\right\} = \left(\left(d_{(i)}{}^{T}, d_{(i)}{}^{T} + D_{(i)}{}^{T}\right)\right)\left(A, A + \Lambda\right)\left(d_{(i)}, d_{(i)} + D_{(i)}\right)$$
$$= \left(d_{(i)}{}^{T} A . d_{(i)}, \left(d_{(i)}{}^{T} + D_{(i)}{}^{T}\right)\left(A + \Lambda\right)\left(d_{(i)} + D_{(i)}\right)\right).$$

Hence

$$\left[d_{(i)}^{N})\right]^{T} . A_N . d_{(i)}^{N} = d_{(i)}{}^{T} A . d_{(i)} + I\left[\left(d_{(i)}{}^{T} + D_{(i)}{}^{T}\right)\left(A + \Lambda\right)\left(d_{(i)} + D_{(i)}\right) - d_{(i)}{}^{T} A . d_{(i)}\right].$$

Then we can write

$$S\left(\alpha_{(i)}^{N}\right)=S\left(\frac{\left[d_{(i)}^{N}\right]^{T}.r_{(i)}^{N}}{\left[d_{(i)}^{N})\right]^{T}.A_{N}.d_{(i)}^{N}}\right)=\frac{S\left(\left[d_{(i)}^{N}\right]^{T}.r_{(i)}^{N}\right)}{S\left(\left[d_{(i)}^{N})\right]^{T}.A_{N}.d_{(i)}^{N}\right)}$$

$$=\frac{\left(d_{(i)}^{T}\left(-f'\left(\xi_{(i)}\right)\right),\left(d_{(i)}^{T}+D_{(i)}^{T}\right)\left[-2f'^{N}_{\eta(i)}\right]\right)}{\left(d_{(i)}^{T}A.d_{(i)},\left(d_{(i)}^{T}+D_{(i)}^{T}\right)(A+\Lambda)\left(d_{(i)}+D_{(i)}\right)\right)}$$

$$=\left(\frac{d_{(i)}^{T}\left(-f'\left(\xi_{(i)}\right)\right)}{d_{(i)}^{T}A.d_{(i)}},\frac{\left(d_{(i)}^{T}+D_{(i)}^{T}\right)\left[-2f'^{N}_{\eta(i)}\right]}{\left(d_{(i)}^{T}+D_{(i)}^{T}\right)(A+\Lambda)\left(d_{(i)}+D_{(i)}\right)}\right)$$

Finally

$$\alpha_{(i)}^{N}=\frac{d_{(i)}^{T}\left(-f'\left(\xi_{(i)}\right)\right)}{d_{(i)}^{T}A.d_{(i)}}+I\left[\frac{\left(d_{(i)}^{T}+D_{(i)}^{T}\right)\left[-2f'^{N}_{\eta(i)}\right]}{\left(d_{(i)}^{T}+D_{(i)}^{T}\right)(A+\Lambda)\left(d_{(i)}+D_{(i)}\right)}-\frac{d_{(i)}^{T}\left(-f'\left(\xi_{(i)}\right)\right)}{d_{(i)}^{T}A.d_{(i)}}\right].$$

We can see that the classical $\alpha_{(i)}$ is the real part of $\alpha_{(i)}^{N}$.

**Example:**

To find the minimizer of

$$f\left(x_{N}\right)=\frac{1}{2}x_{N}^{T}\begin{bmatrix}4-4I & 2+2I\\ 2+2I & 2-2I\end{bmatrix}x_{N}-\begin{bmatrix}-1+I\\ 1-I\end{bmatrix}^{T}x_{N}.$$

Let $x_{(0)}^{N}=\begin{bmatrix}0\\0\end{bmatrix}$, and $A_N$ -conjugate directions $d_{(0)}^{N}=\begin{bmatrix}1-I\\0\end{bmatrix}$ and $d_{(1)}^{N}=\begin{bmatrix}-3/8-(3/8)I\\3/4+(3/4)I\end{bmatrix}$.

$$t_{N}=\begin{bmatrix}4-4I & 2+6I\\ 2+6I & 2-2I\end{bmatrix}.$$

$$r_{(0)}^{N}=k_{N}-t_{N}x_{(0)}^{N}=k_{N}=\begin{bmatrix}-1+I\\ 1-I\end{bmatrix}.$$

$$\alpha_{(0)}^{N}=\frac{\begin{bmatrix}1-I & 0\end{bmatrix}\begin{bmatrix}-1+I\\ 1-I\end{bmatrix}}{\begin{bmatrix}1-I & 0\end{bmatrix}\begin{bmatrix}4-4I & 2+2I\\ 2+2I & 2-2I\end{bmatrix}\begin{bmatrix}1-I\\ 0\end{bmatrix}}=\frac{-1+I}{4-4I}=\frac{-1}{4}.$$

Thus

$$x_{(1)}^{N}=x_{(0)}^{N}+\alpha_{(0)}^{N}d_{(0)}^{N}=\begin{bmatrix}-1/4+(1/4)I\\ 0\end{bmatrix}.$$

And

$$r_{(1)}^{N}=k_{N}-t_{N}x_{(1)}^{N}=\begin{bmatrix}0\\ \frac{3}{2}-\frac{3}{2}I\end{bmatrix}.$$

$$\alpha_{(1)}^{N}=\frac{\begin{bmatrix}-\frac{3}{8}-\frac{3}{8}I & \frac{3}{4}+\frac{3}{4}I\end{bmatrix}\begin{bmatrix}0\\ \frac{3}{2}-\frac{3}{2}I\end{bmatrix}}{\begin{bmatrix}-\frac{3}{8}-\frac{3}{8}I & \frac{3}{4}+\frac{3}{4}I\end{bmatrix}\begin{bmatrix}4-4I & 2+2I\\ 2+2I & 2-2I\end{bmatrix}\begin{bmatrix}-\frac{3}{8}-\frac{3}{8}I\\ \frac{3}{4}+\frac{3}{4}I\end{bmatrix}}=\frac{\frac{9}{8}-\frac{9}{8}I}{\frac{9}{16}-\frac{153}{16}I}=2-2I.$$

And

$$x_{(2)}^{N} = x_{(1)}^{N} + \alpha_{(1)}^{N} d_{(1)}^{N} = \begin{bmatrix} -1+I \\ \frac{3}{2} - \frac{3}{2} I \end{bmatrix}.$$

Let us prove that the algorithm of Neutrosophic Conjugate Directions can compute $x$ in $n$ steps: Using the error term as a linear neutrosophic combination of the neutrosophic search directions

$$e_{(0)}^{N} = \sum_{j=1}^{n-1} \delta_j^N d_{(j)}^{N}$$

where $\delta_j^N$ are neutrosophic numbers one can compute simply.

Since that the search directions are $A_N$ -orthogonal, then we have

$$\left[d_{(k)}^{N}\right]^T .A_N .e_{(0)}^{N} = \sum_j \delta_{(j)}^N \left[d_{(k)}^{N}\right]^T .A_N .d_{(j)}^{N}$$
$$= \delta_{(k)}^N \left[d_{(k)}^{N}\right]^T .A_N .d_{(k)}^{N}$$

hence

$$\delta_{(k)}^N = \frac{\left[d_{(k)}^{N}\right]^T .A_N .e_{(0)}^{N}}{\left[d_{(k)}^{N}\right]^T .A_N .d_{(k)}^{N}} = \frac{\left[d_{(k)}^{N}\right]^T .A_N .\left(e_{(0)}^{N} + \sum_{i=1}^{k-1} \alpha_{(i)}^N d_{(i)}^N\right)}{\left[d_{(k)}^{N}\right]^T .A_N .d_{(k)}^{N}} = \frac{\left[d_{(k)}^{N}\right]^T .A_N .e_{(k)}^{N}}{\left[d_{(k)}^{N}\right]^T .A_N .d_{(k)}^{N}}$$

We can see that $\alpha_{(k)}^N = -\delta_{(k)}^N$, hence we can write

$$e_{(i)}^{N} = e_{(0)}^{N} + \sum_{j=0}^{i-1} \alpha_{(j)}^N d_{(j)}^N = \sum_{j=0}^{n-1} \delta_{(j)}^N d_{(j)}^N - \sum_{j=0}^{i-1} \delta_{(j)}^N d_{(j)}^N$$
$$= \sum_{j=0}^{n-1} \delta_{(j)}^N d_{(j)}^N - \sum_{j=0}^{i-1} \delta_{(j)}^N d_{(j)}^N = \sum_{j=i}^{n-1} \delta_{(j)}^N d_{(j)}^N.$$

## 6. Conclusions

In this study, the neutrosophic quadratic form, method of neutrosophic steepest descent, and method of neutrosophic conjugate directions were introduced. Many examples have been discussed. The author did not address the topic of convergence study, so it did not adhere to the examples concerned with the fulfillment of the condition $x_N^T A_N x_N > 0$.

It is possible to work in many research directions, such as Markov chains. It is also possible to introduce a disturbance operator on the behavior of the moving point and then work on processing and correction to reach the optimal solution. One can also return to [14] and examine the application of the (AH) in generalizing stable distributions.

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