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On S_θ-summability in neutrosophic-2-normed spaces

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Abstract. In the present paper, we aim to define S_θ-summability in neutrosophic 2-normed spaces and study some of its properties. We provide examples that shows our method of summability is stronger in these spaces. Finally we introduce S_θ-Cauchy and S_θ-completeness and prove that every neutrosophic-2-normed spaces is S_θ-complete.

Keywords: S_θ-convergence, S_θ-Cauchy, lacunary sequence, neutrosophic-2-normed spaces.

1. Introduction

Statistical convergence was initially introduced by Fast [9] and later connected to summability theory by Schoenberg [12]. The concept was subsequently advanced by researchers such as Maddox [11], Connor [13], Fridy [14], Mursaleen and Edely [21], Šalát [31], and Kumar and Mursaleen [33], among numerous others.

Lacunary statistical convergence was studied by Fridy and Orhan [16] and was defined as follows: "By a lacunary sequence we mean an increasing integer sequence $\theta = (k_s)$ with $k_0 = 0$ and $h_s = k_s - k_{s-1} \to \infty$ as $s \to \infty$. The intervals determined by θ will be denoted by $I_s = (k_{s-1}, k_s]$ and the ratio $\frac{k_s}{k_{s-1}}$ will be abbreviated as q_s. For $\Re \subseteq \mathbb{N}$, the number $\delta_\theta(\Re) = \lim_{s\to\infty} \dfrac{1}{h_s}|\{k \in I_s : k \in \Re\}|$ is called θ-density of $\Re$, provided the limit exists. A sequence $y = (y_k)$ is said to be lacunary statistically convergent (briefly S_θ-convergent) to y_0 if for each $\wp > 0$, $\lim_s \frac{1}{h_s}|\{k \in I_s : |y_k - y_0| \geq \wp\}| = 0$ or equivalently, the set $\Re(\wp)$ has θ-density zero, where $\Re(\wp) = \{k \in \mathbb{N} : |y_k - y_0| \geq \wp\}$. In this case, we write $S_\theta - \lim_{k\to\infty} y_k = y_0$." Additional noteworthy contributions to lacunary statistical convergence can be explored in references such as [7], [22], [26], and [35].

On the other hand, Zadeh [19] introduced the concept of fuzzy sets as a more suitable approach for addressing problems that cannot be adequately modeled using crisp set theory due to significant uncertainty in the data. Fuzzy set theory finds extensive applications in various scientific domains, including artificial intelligence, engineering, medicine, robotics, and numerous other fields, aiming to attain more effective solutions. Atanassov introduced intuitionistic fuzzy sets (IFS) in 1986 as an extension of fuzzy sets to better handle uncertainty. After introducing intuitionistic fuzzy sets, progressive developments were made in this field, as seen in [15], [27], etc.

Smarandache [35] proposed neutrosophic sets (NS) as another interesting generalization of fuzzy sets by introducing the indeterminacy function to intuitionistic fuzzy sets. Neutrosophic sets (NS) offer a more flexible and comprehensive way to represent uncertainty, imprecision, and indeterminacy in addressing the complexities of real-world situations. For ongoing development on neutrosophic sets (NS) and their applications, we refer to [1], [23], etc.

Kirişçi and Şimşek [20] established the concept of neutrosophic norm and investigated statistical convergence within the framework of neutrosophic normed spaces. For a comprehensive perspective in this direction, we recommend to the reader [2], [3], [4], [32], etc. Nowadays, the area of summability in these spaces is of much interest. Several summability approaches so far developed, including statistical convergence, ideal convergence, and lacunary statistical convergence in these spaces (see [5], [10], [18], [24], [25], [29], [34]). Recently in [30], the concept of neutrosophic-2-norm is introduced where the authors studied statistical convergence in neutrosophic-2-normed spaces. In the present work, we define a more general summability method, called S_θ-summability in $N-2-NS$ and develop some of its properties. We organize the paper as follows, the first and second sections are introductory and provide basic information needed in the sequel. In section 3, we define S_θ-summability in $N-2-NS$ and obtain interesting results. In section 4, we introduce S_θ-Cauchy and S_θ-completeness in $N-2-NS$. Finally, in the last section, we provide a brief conclusion regarding the whole work.

2. **Preliminaries**

This section commences with a concise overview of specific definitions and results needed in the sequel. In the course of this study, the notation $\mathbb{R}^+$ will be used to represent the open interval $(0, \infty)$, while $\mathbb{N}$ will represent the set of natural numbers.

Definition 2.1 [6] "Let $\Im = [0, 1]$. A function $\circ : \Im \times \Im \to \Im$ is said to be a t-norm for all $\mu_1, \mu_2, \mu_3, \mu_4 \in \Im$, we have

(i) $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$;

(ii)$\mu_1 \circ (\mu_2 \circ \mu_3) = (\mu_1 \circ \mu_2) \circ \mu_3$;

(iii) $\circ$ is continuous;

(iv) $\mu_1 \circ 1 = \mu_1$ for every $\mu_1 \in \Im$ and

(v) $\mu_1 \circ \mu_2 \leq \mu_3 \circ \mu_4$ whenever $\mu_1 \leq \mu_3$ and $\mu_2 \leq \mu_4$".

Definition 2.2 [6] "Let $\Im = [0,1]$. A function $\diamond : \Im \times \Im \to \Im$ is said to be a continuous triangular conorm or t-conorm for all $\mu_1, \mu_2, \mu_3, \mu_4 \in \Im$, we have

(i) $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$;

(ii)$\mu_1 \circ (\mu_2 \circ \mu_3) = (\mu_1 \circ \mu_2) \circ \mu_3$;

(iii) $\circ$ is continuous;

(iv) $\mu_1 \diamond 0 = \mu_1$ for every $\mu_1 \in \Im$ and

(v) $\mu_1 \circ \mu_2 \leq \mu_3 \circ \mu_4$ whenever $\mu_1 \leq \mu_3$ and $\mu_2 \leq \mu_4$".

We now recall the idea of two norm introduced in the paper [28].

Definition 2.3 [28]"Let X be a $d-$dimensional real vector space, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ fulfilling the below listed requirements: For all $\varrho_1, \varrho_2 \in X$, and scalar α, we have

(i) $||\varrho_1, \varrho_2|| = 0$ iff ϱ_1 and ϱ_2 are linearly dependent;

(ii) $||\varrho_1, \varrho_2|| = ||\varrho_2, \varrho_1||$;

(iii)$||\alpha\varrho_1, \varrho_2|| = |\alpha|||\varrho_1, \varrho_2||$ and

(iv) $||\varrho_1, \varrho_2 + \varrho_3|| \leq ||\varrho_1, \varrho_2|| + ||\varrho_1, \varrho_3||$.

The pair $(X, ||.,.||)$ is known as 2-normed space in this case.

Let $X = \mathbb{R}^2$ and for $\varrho_1 = (p_0, p_0')$ and $\varrho_2 = (q_0, q_0')$ we define $||\varrho_1, \varrho_2|| = |p_0 q_0' - p_0' q_0|$, then $||\varrho_1, \varrho_2||$ is a 2-norm on $X = \mathbb{R}^2$".

Recently, Murtaza et al. [30] defined neutrosophic 2-normed spaces as follows:

Definition 2.4 [30] "Let F is a vector space, $N_2 = (\{(\varrho_1, \varrho_2), G(\varrho_1, \varrho_2), B(\varrho_1, \varrho_2), Y(\varrho_1, \varrho_2)\} : (\varrho_1, \varrho_2) \in F \times F)$ be a 2-norm space s.t. $N_2 : F \times F \times \mathbb{R}^+ \to [0,1]$. If $\circ, \diamond$ respectively denotes t-norm and t-conorm, then the four-tuple $X = (F, N_2, \circ, \diamond)$ is known as neutrosophic 2-normed spaces (briefly $N-2-NS$) if for every $\varrho_1, \varrho_2, \omega \in X$, $\varsigma, \mu \geq 0$ and $\xi \neq 0$:

(i) $0 \leq G(\varrho_1, \varrho_2; \varsigma) \leq 1$, $0 \leq B(\varrho_1, \varrho_2; \varsigma) \leq 1$ and $0 \leq Y(\varrho_1, \varrho_2; \varsigma) \leq 1$ for every $\varsigma \in \mathbb{R}^+$;

(ii) $G(\varrho_1, \varrho_2; \varsigma) + B(\varrho_1, \varrho_2; \varsigma) + Y(\varrho_1, \varrho_2; \varsigma) \leq 3$;

(iii) $G(\varrho_1, \varrho_2; \varsigma) = 1$ iff ϱ_1, ϱ_2 are linearly dependent;

(iv) $G(\xi\varrho_1, \ \varrho_2; \ \varsigma) = G(\varrho_1, \varrho_2; \frac{\varsigma}{|\xi|})$ for each $\varsigma \neq 0$;

(v) $G(\varrho_1, \varrho_2; \varsigma) \circ G(\varrho_1, \omega; \mu) \leq G(\varrho_1, \varrho_2 + \omega; \varsigma + \mu)$;

(vi) $G(\varrho_1, \varrho_2; .) : (0, \infty) \to [0,1]$ is a non-decreasing function that runs continuously;

(vii) $\lim_{\varsigma \to \infty} G(\varrho_1, \varrho_2; \varsigma) = 1$;

(viii) $G(\varrho_1, \varrho_2; \varsigma) = G(\varrho_2, \varrho_1; \varsigma)$

(ix) $B(\varrho_1, \varrho_2; \varsigma) = 0$ iff ϱ_1, ϱ_2 are linearly dependent;

(x) $B(\xi\varrho_1, \varrho_2; \varsigma) = B(\varrho_1, \varrho_2; \frac{\varsigma}{|\xi|})$ for each $\varsigma \neq 0$;

(xi) $B(\varrho_1, \varrho_2; \varsigma) \diamond B(\varrho_1, \omega; \mu) \geq B(\varrho_1, \varrho_2 + \omega; \varsigma + \mu)$;

(xii) $B(\varrho_1, \varrho_2; .) : (0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;

(xiii) $\lim_{\varsigma\to\infty} B(\varrho_1, \varrho_2; \varsigma) = 0$;

(xiv) $B(\varrho_1, \varrho_2; \varsigma) = B(\varrho_2, \varrho_1; \varsigma)$;

(xv) $Y(\varrho_1, \varrho_2; \varsigma) = 0$ iff ϱ_1, ϱ_2 are linearly dependent;

(xvi)$Y(\xi\varrho_1,\ \varrho_2;\ \varsigma) = Y(\varrho_1,\ \varrho_2; \frac{\varsigma}{|\xi|})$ for each $\varsigma \neq 0$;

(xvii) $Y(\varrho_1, \varrho_2; \varsigma) \diamond Y(\varrho_1, \omega; \mu) \geq Y(\varrho_1, \varrho_2 + \omega; \varsigma + \mu)$;

(xviii) $Y(\varrho_1, \varrho_2; .) : (0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;

(xix) $\lim_{\lambda\to\infty} Y(\varrho_1, \varrho_2; \varsigma) = 0$;

(xx) $Y(\varrho_1, \varrho_2; \varsigma) = Y(\varrho_2, \varrho_1; \varsigma)$;

(xxi) if $\varsigma \leq 0$, then $G(\varrho_1, \varrho_2; \varsigma) = 0$, $B(\varrho_1, \varrho_2; \varsigma) = 1$, $Y(\varrho_1, \varrho_2; \varsigma) = 1$.

In this case, we call $N_2 = N_2(G, B, Y)$, a neutrosophic 2-norm on F . From now on wards, unless otherwise stated by X we shall denote the $N - 2 - NS$ $(F, N_2, \circ, \diamond)$.

A sequence (y_k) in X is said to be convergent to $y_0 \in X$ if for each $0 < \wp < 1$ and $\varsigma > 0$, $\exists$ a positive integer m s.t. $G(y_k - y_0, \omega; \varsigma) > 1 - \wp$, $B(y_k - y_0, \omega; \varsigma) < \wp$ and $Y(y_k - y_0, \omega; \varsigma) < \wp$ for all $k \geq m$ and $\omega \in X$ which is equivalently to say $\lim_{k\to\infty} G(y_k - y_0, \omega; \varsigma) = 1$, $\lim_{k\to\infty} B(y_k - y_0, \omega; \varsigma) = 0$ and $\lim_{k\to\infty} Y(y_k - y_0, \omega; \varsigma) = 0$. In this case, we write $N_2 - \lim_{k\to\infty} y_k = y_0$.

A sequence (y_k) in X is said to be Cauchy if for each $0 < \wp < 1$ and $\varsigma > 0$, $\exists$ a positive integer m s.t. $G(y_k - y_n, \omega; \varsigma) > 1 - \wp$, $B(y_k - y_n, \omega; \varsigma) < \wp$ and $Y(y_k - y_n, \omega; \varsigma) < \wp$ $\forall$ $k, n \geq m$ and $\forall$ $\omega \in X$."

3. Lacunary statistical Convergence in $N - 2 - NS$

Definition 3.1 A sequence $y = (y_k)$ in X is called lacunary statistical convergent (or S_θ-convergent) to y_0 w.r.t neutrosophic 2-norm N_2, if for each $\wp > 0$ and $\varsigma > 0$

$$\lim_{s\to\infty} \frac{1}{h_s} \Bigg| \Bigg\{ k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1 - \wp \text{ or } B(y_k - y_0, \omega; \varsigma) \geq \wp,\ Y(y_k - y_0, \omega; \varsigma) \geq \wp \Bigg\} \Bigg| = 0 \text{ for every } \omega \in X;$$

or, $\delta_\theta\left(\mathfrak{A}\left(\wp, \varsigma\right)\right) = 0$, where

$$\mathfrak{A}(\wp, \varsigma) = \{k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1 - \wp \text{ or } B(y_k - y_0, \omega; \varsigma) \geq \wp,\ Y(y_k - y_0, \omega; \varsigma) \geq \wp\}.$$

In present case, we denote $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$.

We now give the following Lemma and prove the uniqueness theorem.

Lemma 3.1 $y = (y_k)$ in X, the subsequent assertions are equivalent:

(i) $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$;

(ii) $\delta_\theta\{k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1 - \wp\} = \delta_\theta\{k \in I_s : B(y_k - y_0, \omega; \varsigma) \geq \wp\} = \delta_\theta\{k \in I_s : Y(y_k - y_0, \omega; \varsigma) \geq \wp\} = 0$;

(iii) $\delta_\theta\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \wp$ and $B(y_k - y_0, \omega; \varsigma) < \wp, Y(y_k - y_0, \omega; \varsigma) < \wp\} = 1$;

(iv) $\delta_\theta\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \wp\} = \delta_\theta\{k \in I_s : B(y_k - y_0, \omega; \varsigma) < \wp\} = \delta_\theta\{k \in I_s : Y(y_k - y_0, \omega; \varsigma) < \wp\} = 1$ and

(v) $S_\theta(N_2) - \lim_{k\to\infty} G(y_k - y_0, \omega; \varsigma) = 1$, $S_\theta(N_2) - \lim_{k\to\infty} B(y_k - y_0, \omega; \varsigma) = S_\theta(N_2) - \lim_{k\to\infty} Y(y_k - y_0, \omega; \varsigma) = 0$.

Theorem 3.1 For any sequence $y = (y_k)$ in X, if $S_\theta(N_2) - \lim_{k\to\infty} y_k$ exists, then it is unique.

Proof. Suppose that $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_1$ and $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_2$. For given $\wp > 0$, choose $\nu > 0$ s.t.

$$(1 - \nu) \circ (1 - \nu) > 1 - \wp \text{ and } \nu \diamond \nu < \wp. \tag{1}$$

For any $\varsigma > 0$ and any $w \in X$. Define the following sets:

$$K_{G,1}(\nu, \varsigma) = \{k \in I_s : G(y_k - y_1, \omega; \frac{\varsigma}{2}) \leq 1 - \nu\},$$
$$K_{G,2}(\nu, \varsigma) = \{k \in I_s : G(y_k - y_2, \omega; \frac{\varsigma}{2}) \leq 1 - \nu\};$$
$$K_{B,1}(\nu, \varsigma) = \{k \in I_s : B(y_k - y_1, \omega; \frac{\varsigma}{2}) \geq \nu\},$$
$$K_{B,2}(\nu, \varsigma) = \{k \in I_s : B(y_k - y_2, \omega; \frac{\varsigma}{2}) \geq \nu\};$$
$$K_{Y,1}(\nu, \varsigma) = \{k \in I_s : \mathcal{Y}(y_k - y_1, \omega; \frac{\varsigma}{2}) \geq \nu\};$$
$$K_{Y,2}(\nu, \varsigma) = \{k \in I_s : Y(y_k - y_2, \omega; \frac{\varsigma}{2}) \geq \nu\}.$$

Since $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_1$, so by lemma 3.1, we get $\delta_\theta\{K_{G,1}(\nu, \varsigma)\} = \delta_\theta\{K_{B,1}(\nu, \varsigma)\} = \delta_\theta\{K_{Y,1}(\nu, \varsigma)\} = 0$ and therefore $\delta_\theta\{K^C_{G,1}(\nu, \varsigma)\} = \delta_\theta\{K^C_{B,1}(\nu, \varsigma)\} = \delta_\theta\{K^C_{Y,1}(\nu, \varsigma)\} = 1$. Furthermore, using $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_2$, we get, $\delta_\theta\{K_{G,2}(\nu, \varsigma)\} = \delta_\theta\{K_{B,2}(\nu, \varsigma)\} = \delta_\theta\{K_{Y,2}(\nu, \varsigma)\} = 0$ and therefore $\delta_\theta\{K^C_{G,2}(\nu, \varsigma)\} = \delta_\theta\{K^C_{B,2}(\nu, \varsigma)\} = \delta_\theta\{K^C_{Y,2}(\nu, \varsigma)\} = 1$. Now define $K_{G,B,Y}(\wp, \varsigma) = \{K_{G,1}(\nu, \varsigma) \cup K_{G,2}(\nu, \varsigma)\} \cap \{K_{B,1}(\nu, \varsigma) \cup K_{B,2}(\nu, \varsigma)\} \cap \{K_{Y,1}(\nu, \varsigma) \cup K_{Y,2}(\nu, \varsigma)\}$. Then $\delta_\theta(\{K_{G,B,Y}(\wp, \varsigma)\}) = 0$ which implies $\delta(\{K^C_{G,B,Y}(\wp, \varsigma)\}) = 1$. Let $m \in K^C_{G,B,Y}(\wp, \varsigma)$, then we have

Case 1. $m \in \{K_{G,1}(\nu, \varsigma) \cup K_{G,2}(\nu, \varsigma)\}^C$,

Case 2. $m \in \{K_{B,1}(\nu,\varsigma) \cup K_{B,2}(\nu,\varsigma)\}^C$,

Case 3. $m \in \{K_{Y,1}(\nu,\varsigma) \cup K_{Y,2}(\nu,\varsigma)\}^C$.

Case 1: Let, $m \in \{K_{G,1}(\nu,\varsigma) \cup K_{G,2}(\nu,\varsigma)\}^C$, then $m \in K^C_{G,1}(\nu,\varsigma)$ and $m \in K^C_{G,2}(\nu,\varsigma)$. Therefore, for any $\omega \in X$ we have

$$G(y_m - y_1, \omega; \frac{\varsigma}{2}) > 1 - \nu \text{ and } G(y_m - y_2, \omega; \frac{\varsigma}{2}) > 1 - \nu. \tag{2}$$

Now

$$\begin{aligned} G(y_1 - y_2, \omega; \varsigma) &\geq G(y_m - y_1, \omega, ; \frac{\varsigma}{2}) \circ G(y_m - y_2, \omega; \frac{\varsigma}{2}) \\ &> (1-\nu) \circ (1-\nu) \text{ by}(2) \\ &> 1 - \wp. \text{ by } (1) \end{aligned}$$

Since $\wp > 0$ is arbitrary, so we have $G(y_1 - y_2, \omega; \varsigma) = 1 \ \forall \varsigma > 0$, and therefore $y_1 - y_2 = 0$. This shows that $y_1 = y_2$.

Case 2: Let $m \in \{K_{B,1}(\nu,\varsigma) \cup K_{B,2}(l,\varsigma)\}^C$, then $m \in K^C_{B,1}(\nu,\varsigma)$ and $m \in K^C_{B,2}(\nu,\varsigma)$. Therefore, for $\omega \in X$ we have

$$B(y_m - y_1, \omega; \frac{\varsigma}{2}) < \nu \text{ and } B(y_m - y_2, \omega; \frac{\varsigma}{2}) < \nu. \tag{3}$$

Now

$$\begin{aligned} B(y_1 - y_2, \omega; \varsigma) &\leq B(y_m - y_1, \omega, ; \frac{\varsigma}{2}) \diamond B(y_m - y_2, \omega; \frac{\varsigma}{2}) \\ &< \nu \diamond \nu \text{ by } (3) \\ &< \wp. \text{ by } (1) \end{aligned}$$

Since $\wp > 0$ is arbitrary, so we have $B(y_1 - y_2, \omega; \varsigma) = 0 \ \forall \varsigma > 0$, and therefore $y_1 - y_2 = 0$. This shows that $y_1 = y_2$.

Case 3: Let $m \in \{K_{Y,1}(\nu,\varsigma) \cup K_{Y,2}(\nu,\varsigma)\}^C$, then $m \in K^C_{Y,1}(\nu,\varsigma)$ and $m \in K^C_{Y,2}(\nu,\varsigma)$. Therefore, for $\omega \in X$ we have

$$Y(y_m - y_1, \omega; \frac{\varsigma}{2}) < \nu \text{ and } Y(y_m - y_2, \omega; \frac{\varsigma}{2}) < \nu. \tag{4}$$

Now

$$\begin{aligned} Y(y_1 - y_2, \omega; \varsigma) &\leq Y(y_m - y_1, \omega; \frac{\varsigma}{2}) \diamond Y(y_m - y_2, \omega; \frac{\varsigma}{2}) \\ &< \nu \diamond \nu \text{ by } (4) \\ &< \wp. \text{ by } (1) \end{aligned}$$

Since $\wp > 0$ is arbitrary, so we have $Y(y_1 - y_2, \omega; \varsigma) = 0 \ \forall \varsigma > 0$, and therefore $y_1 - y_2 = 0$. This shows that $y_1 = y_2$.

Hence in all cases, we get $y_1 = y_2$. □

Theorem 3.2 Let $y = (y_k)$ be any sequence in X. If $N_2 - \lim_{k\to\infty} y_k = y_0$, then $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$.

Proof Let $N_2 - \lim_{k\to\infty} y_k = y_0$. Then for every $\wp > 0$ and $\varsigma > 0, \exists$ an integer $k_0 \in \mathbb{N}$ s.t. $G(y_k - y_0, \omega; \varsigma) > 1 - \wp$ and $B(y_k - y_0, \omega; \varsigma) < \wp$, $Y(y_k - y_0, \omega; \varsigma) < \wp$ $\forall k \geq k_0$ and every $\omega \in X$. Hence, the set $\{k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1 - \wp$ or $B(y_k - y_0, \omega; \varsigma) \geq \wp$, $Y(y_k - y_0, \omega; \varsigma) \geq \wp\}$ has a finitely many terms whose θ-density is zero. Therefore, $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$. $\square$

But the converse of the above theorem is not true in general.

Example 3.1 Let $(\mathbb{R}^2, |.|)$ be 2-normed space. For $\tau_1, \tau_2 \in [0, 1]$. Let $\tau_1 \circ \tau_2 = \tau_1\tau_2$ and $\tau_1 \diamond \tau_2 = \min\{\tau_1 + \tau_2, 1\}$. Choose $(\varrho_1, \varrho_2) \in \mathbb{R}^2$ and $\varsigma > 0$ with $\varsigma > \|\varrho_1, \varrho_2\|$. Define $G(\varrho_1, \varrho_2; \varsigma) = \frac{\varsigma}{\varsigma + \|\varrho_1, \varrho_2\|}$, $B(\varrho_1, \varrho_2; \varsigma) = \frac{\|\varrho_1, \varrho_2\|}{\varsigma + \|\varrho_1, \varrho_2\|}$ and $Y(\varrho_1, \varrho_2; \varsigma) = \frac{\|\varrho_1, \varrho_2\|}{\varsigma}$, then it is easy to see that $X = (\mathbb{R}^2, N_2, \circ, \diamond)$ is a $N - 2 - NS$. Define a sequence $y = (y_k)$ by

$$y_k = \begin{cases} (k, 0) \text{ if } k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s, s \in \mathbb{N} \\ (0, 0) \text{ otherwise.} \end{cases}$$

Now, for each $\wp > 0$ and $\varsigma > 0$, let

$$\begin{aligned}
\mathfrak{A}(\wp, \varsigma) = \Big\{ & k \in I_s : G(y_k - 0, \omega; \varsigma) \leq 1 - \wp \text{ or} \\
& B(y_k - 0, \omega; \varsigma) \geq \wp,\ Y(y_k - 0, \omega; \varsigma) \geq \wp \Big\} \\
= & \left\{ k \in I_s : \frac{\varsigma}{\varsigma + \|y_k, \omega\|} \leq 1 - \wp \text{ or } \frac{\|y_k, \omega\|}{\varsigma + \|y_k, \omega\|} \geq \wp,\ \frac{\|y_k, \omega\|}{\varsigma} \geq \wp \right\} \\
= & \left\{ k \in I_s : \|y_k, \omega\| \geq \frac{\varsigma\wp}{1 - \wp} \text{ or } \|y_k, \omega\| \geq \varsigma\wp \right\} \\
= & \{ k \in I_s : k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s; s \in \mathbb{N} \}
\end{aligned}$$

and so we get

$$\frac{1}{h_s}|\mathfrak{A}(\wp, \varsigma)| \leq \frac{1}{h_s}|\{k \in I_s : k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s; s \in \mathbb{N}\}| \leq \frac{[\sqrt{h_s}]}{h_s}.$$

Taking $s \to \infty$,

$$\lim_{s\to\infty} \frac{1}{h_s}|\mathfrak{A}(\wp, \varsigma)| \leq \lim_{n\to\infty} \frac{[\sqrt{h_s}]}{h_s} = 0;$$

i.e., $\delta_\theta(\mathfrak{A}(\wp, \varsigma)) = 0$. Hence, $y = (y_k)$ is S_θ-convergent to 0. But the sequence $y = (y_k)$ is not N_2-convergent to 0.

Theorem 3.3 Let $y = (y_k)$ and $z = (z_k)$ be any two sequences in X s.t $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_1$

and $S_\theta(N_2) - \lim_{k\to\infty} z_k = z_1$, then
(i) $S_\theta(N_2) - \lim_{k\to\infty}(y_k + z_k) = y_1 + z_1$ and
(ii) $S_\theta(N_2) - \lim_{k\to\infty}(cy_k) = cy_1$, where $0 \neq c \in F$.

Proof. The proof of this theorem can be derived in a manner similar to the proof of theorem 3.1 and is therefore omitted. □

We now have the following interesting implication.

Theorem 3.4 A sequence $y = (y_k)$ in X is $S_\theta(N_2)$-convergent to y_0 iff $\exists$ a subset $\Re = \{k_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ with $\delta_\theta(\Re) = 1$ and $N_2 - \lim_{n\to\infty} y_{k_n} = y_0$.

Proof. Assume that $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$. For any $\varsigma > 0$, $l \in \mathbb{N}$ and $\omega \in X$, define the set

$$\Re_{N_2}(l,\varsigma) = \{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \frac{1}{l} \text{ and } B(y_k - y_0, \omega; \varsigma) < \frac{1}{l}, Y(y_k - y_0, \omega; \varsigma) < \frac{1}{l}\}.$$

Since $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$, it is clear that for $\varsigma > 0$ and $l \in \mathbb{N}$, $\Re_{N_2}(l+1,\varsigma) \subset \Re_{N_2}(l,\varsigma)$ and

$$\delta_\theta(\Re_{N_2}(l,\varsigma)) = 1. \tag{5}$$

Let r_1 be an arbitrary number in $\Re_{N_2}(1,\varsigma)$. Then, $\exists\ r_2 \in \Re_{N_2}(2,\varsigma), (r_2 > r_1)$, s.t $\forall\ n \geq r_2$, $\frac{1}{h_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \frac{1}{2}$ and $B(y_k - y_0, \omega; \varsigma) < \frac{1}{2}$, $Y(y_k - y_0, \omega; \varsigma) < \frac{1}{2}\}| > \frac{1}{2}$. Similarly, $\exists\ r_3 \in \Re_{N_2}(3,\varsigma), (r_3 > r_2)$, such that for all $n \geq r_3$, $\frac{1}{h_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \frac{1}{3}$ and $B(y_k - y_0, \omega; \varsigma) < \frac{1}{3}$, $Y(y_k - y_0, \omega; \varsigma) < \frac{1}{3}\}| > \frac{2}{3}$ and so on. So we can establish a sequence $\{r_l\}_{l\in\mathbb{N}}$ satisfying $r_l \in \Re_{N_2}(l,\varsigma)$. For all $n \geq r_l (l \in \mathbb{N})$, we have $\frac{1}{h_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \frac{1}{l}$ and $B(y_k - y_0, \omega; \varsigma) < \frac{1}{l}$, $Y(y_k - y_0, \omega; \varsigma) < \frac{1}{l}\}| > \frac{l-1}{l}$.

Define $\Re = \{n \in \mathbb{N} : 1 < n < r_1\} \cup \{\bigcup_{l\in\mathbb{N}}\{n \in \Re_{N_2}(l,\varsigma) : r_l \leq n < r_{l+1}\}\}$, Then for $r_l \leq n < r_l + 1$, we have $\frac{1}{h_s}|\{k \in I_s : k \in \Re\}| \geq \frac{1}{h_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \frac{1}{l}$ and $B(y_k - y_0, \omega; \varsigma) < \frac{1}{l}$, $Y(y_k - y_0, \omega; \varsigma) < \frac{1}{l}\}| > \frac{l-1}{l}$ and hence $\delta_\theta(\Re) = 1$ as $k \to \infty$. Now we have to demonstrate that $N_2 - \lim_{n\to\infty} \mathfrak{u}_{k_n} = \mathfrak{u}_0$. Let $\wp > 0$ and select $l \in \mathbb{N}$ with $\frac{1}{l} < \wp$. Furthermore, let $n \geq r_l$ and $n \in \Re$. Then, by definition of $\Re$, $\exists\ n_0 \geq l$ s.t, $r_{n_0} \leq n < r_{n_0+1}$ and $n \in \Re_{N_2}(l,\varsigma)$. Thus, for each $\wp > 0$, and for $\omega \in X$ we have $G(y_n - y_0, \omega; \varsigma) > 1 - \frac{1}{l} > 1 - \wp$ and $B(y_n - y_0, \omega; \varsigma) < \frac{1}{l} < \wp$, $Y(y_n - y_0, \omega; \varsigma) < \frac{1}{l} < \wp\ \forall\ n \geq r_l$ and $n \in \Re$. Hence $N_2 - \lim_{n\to\infty} y_{k_n} = y_0$.

Conversely, suppose that $\exists$ a subset $\Re = \{k_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ with $\delta_\theta\{\Re\} = 1$ and $N_2 - \lim_{n\in\Re} y_{k_n} = y_0$. Let $\wp > 0$ and $\varsigma > 0\ \exists\ k_{n_0} \in \mathbb{N}$ s.t $G(y_{k_n} - y_0, \omega; \varsigma) > 1 - \wp$ and $B(y_{k_n} - y_0, \omega; \varsigma) < \wp$, $Y(y_{k_n} - y_0, \omega; \varsigma) < \wp$ for each $k_n \geq k_{n_0}$ and $\omega \in X$. This implies $\mathfrak{T}_{N_2}(\wp,\varsigma) = \{k \in I_s : G(y_{k_n} - y_0, \omega; \varsigma) \leq 1 - \wp$ and $B(y_{k_n} - y_0, \omega; \varsigma) \geq \wp$, $Y(y_{k_n} - y_0, \omega; \varsigma) \geq \wp\}$

$\subseteq \mathbb{N} - \{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\}$ and therefore $\delta_\theta\{\mathfrak{T}_{N_2}(\wp,\varsigma)\} \leq \delta_\theta(\mathbb{N}) - \delta_\theta(\{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\})$. As $\delta_\theta\{\Re\} = 1$, so $\delta_\theta\{\mathfrak{T}_{N_2}(\wp,\varsigma)\} = 0$. This shows that $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$ and therefore the completes proof of the theorem. □

"For $\upsilon \in X, \varsigma > 0, \alpha \in (0,1)$ and $\omega \in X$, the ball centered at υ with radius α is denoted and defined by $H(\upsilon,\alpha,\varsigma) = \{u \in X : G(\upsilon - u,\omega,\varsigma) > 1-\alpha$ and $B(\upsilon - u,\omega,\varsigma) < \alpha,\ Y(\upsilon - u,\omega,\varsigma) < \alpha\}$."

Theorem 3.5 Let X be a $N-2-NS$. For any lacunary sequence $\theta = (k_s), S_\theta(N_2) \subseteq S(N_2)$ iff $\lim_s \sup q_s < \infty$.

Proof. If $\lim_s \sup q_s < \infty$, then $\exists\ M > 0$ s.t $q_s < M\ \forall\ s$. Suppose that $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$ and for $\varsigma > 0, \alpha \in (0,1), \omega \in X$, let

$$T_s = \Big|\Big\{k \in I_s : G(y_k - y_0,\omega;\varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0,\omega;\varsigma) \geq \alpha,\ Y(y_k - y_0,\omega;\varsigma) \geq \alpha\Big\}\Big|.$$

Let $\wp > 0$. Since $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$, then $\exists\ s_0 \in \mathbb{N}$ s.t

$$\frac{T_s}{h_s} < \wp\ \forall\ s > s_0. \tag{6}$$

Now, Let $C = \max\{T_s : 1 \leq s \leq s_0\}$ and r be an integer such that $k_{s-1} < r < k_s$. Then we write

$$\begin{aligned}
&\frac{1}{r}|\{k \leq r : G(y_k - y_0,\omega;\varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0,\omega;\varsigma) \geq \alpha,\ Y(y_k - y_0,\omega;\varsigma) \geq \alpha\}| \\
&\leq \frac{1}{k_{s-1}}|\{k \leq k_s : G(y_k - y_0,\omega;\varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0,\omega;\varsigma) \geq \alpha,\ Y(y_k - y_0,\omega;\varsigma) \geq \alpha\}| \\
&= \frac{1}{k_{s-1}}\{T_1 + T_2 + \ldots + T_{s_0} + T_{s_0+1} + \ldots + T_s\} \\
&\leq \frac{C}{k_{s-1}}s_0 + \frac{1}{k_{s-1}}\left\{h_{s_0+1}\frac{T_{s_0+1}}{h_{s_0+1}} + \ldots + h_s\frac{T_s}{h_s}\right\} \\
&\leq \frac{s_0C}{k_{s-1}} + \frac{1}{k_{s-1}}\left(\sup_{s>s_0}\frac{T_s}{h_s}\right)\{h_{s_0+1} + \ldots + h_s\} \\
&\leq \frac{s_0C}{k_{s-1}} + \wp\frac{k_s - k_{s_0}}{k_{s-1}} \quad \text{by (6)} \\
&\leq \frac{s_0C}{k_{s-1}} + \wp q_s \\
&\leq \frac{s_0C}{k_{s-1}} + \wp M.
\end{aligned}$$

To prove the converse, assume that $\lim_s \sup q_s = \infty$. Let $\beta(\neq 0) \in X$. By applying the concept from [5], we can obtain a subsequence $(k_{s(l)})$ of $\theta = (k_s)$ s.t $q_{s(l)} > l$. Define a sequence $y = (y_k)$ by

$$y_k = \begin{cases} \beta \text{ if } k_{s(l)-1} < k \leq 2k_{s(l)-1} \text{for some } l = 1, 2, 3, \ldots \\ 0 \text{ otherwise.} \end{cases}$$

Since $\beta(\neq 0)$, so we can select $\varsigma > 0, \alpha \in (0,1)$ and $\omega \in X$ s.t $\beta \notin H(0, \alpha, \varsigma)$. Now for $l > 1$,

$$\begin{aligned} \frac{1}{h_{s(l)}} |\{k \leq k_{s(l)} : G(y_k, \omega; \varsigma) \leq 1 - \alpha \text{ or} \\ B(y_k, \omega; \varsigma) \geq \alpha,\ Y(y_k, \omega; \varsigma) \geq \alpha\}| \\ \leq \frac{1}{h_{s(l)}} \left(k_{s(l)-1}\right) \\ = \frac{1}{k_{s(l)} - k_{s(l)-1}} \left(k_{s(l)-1}\right) \\ < \frac{1}{l-1}. \end{aligned}$$

Thus, we have $y \in S_\theta(N_2)$. But $y \notin S(N_2)$. For

$$\begin{aligned} \frac{1}{2k_{s(l)-1}} |\{k \leq 2k_{s(l)-1} : G(y_k, \omega; \varsigma) \leq 1 - \alpha \text{ or} \\ B(y_k, \omega; \varsigma) \geq \alpha,\ Y(y_k, \omega; \varsigma) \geq \alpha\}| \\ \geq \frac{1}{2k_{s(l)-1}} \{k_{s(1)-1} + k_{s(2)-1} + \ldots + k_{s(l)-1}\} \\ > \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{k_{s(l)}} |\{k \leq k_{s(l)} : G(y_k - \beta, \omega; \varsigma) \leq 1 - \alpha \text{ or} \\ B(y_k - \beta, \omega; \varsigma) \geq \alpha,\ Y(y_k - \beta, \omega; \varsigma) \geq \alpha\}| \\ \geq \frac{k_{s(l)} - 2k_{s(l)-1}}{k_{s(l)}} \\ \geq 1 - \frac{2}{l}. \end{aligned}$$

This shows that $y = (y_k)$ is not S-convergent w.r.t N_2. □

Theorem 3.6 Let X be a $N-2-NS$. For any lacunary sequence $\theta = (k_s), S(N_2) \subseteq S_\theta(N_2)$ iff $\lim_s \inf q_s > 1$.

Proof. Assume that $\lim_s \inf q_s > 1$, then $\exists\ \eta > 0$ s.t $q_s \geq 1 + \eta$ for sufficiently large s, which implies that

$$\frac{h_s}{k_s} \geq \frac{\eta}{1+\eta}.$$

If $y = (y_k)$ is S-convergent to y_0 w.r.t N_2, then for each $\varsigma > 0, \alpha \in (0,1), \omega \in X$ and sufficiently large s, we have

$$\frac{1}{k_s}|\{k \leq k_s : G(y_k - y_0, \omega; \varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0, \omega; \varsigma) \geq \alpha,\ Y(y_k - y_0, \omega; \varsigma) \geq \alpha\}|$$

$$\geq \frac{1}{k_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0, \omega; \varsigma) \geq \alpha,\ Y(y_k - y_0, \omega; \varsigma) \geq \alpha\}|$$
$$\geq \frac{\eta}{1+\eta}\frac{1}{h_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0, \omega; \varsigma) \geq \alpha,\ Y(y_k - y_0, \omega; \varsigma) \geq \alpha\}|.$$

Since $y = (y_k) \in S(N_2)$, it follows that $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$.
To prove the converse, assume that $\liminf_s q_s = 1$. Applying the concept from [5], we can obtain a subsequence $(k_{s(l)})$ of $\theta = (k_s)$ s.t

$$\frac{k_{s(l)}}{k_{s(l)-1}} < 1 + \frac{1}{l} \text{ and } \frac{k_{s(l)} - 1}{k_{s(l-1)}} > l \text{ where } s(l) \geq s(l-1) + 2.$$

Let $\beta(\neq 0) \in X$. Define a sequence $y = (y_k)$ by

$$y_k = \begin{cases} \beta \text{ if } k \in I_{s(l)} \text{for some } l = 1,2,3,\ldots \\ 0 \text{ otherwise.} \end{cases}$$

We now show that $y = (y_k)$ is S-convergent to 0 w.r.t N_2. Let $\varsigma > 0, \alpha \in (0,1)$ and $\omega \in X$. Choose $\varsigma_1 > 0$ and $\alpha_1 \in (0,1)$ such that for previously chosen $\omega \in X$, $H(0, \alpha_1, \varsigma_1) \subset H(0, \alpha, \varsigma)$ and $\beta \notin H(0, \alpha_1, \varsigma_1)$. Also for each $r \in \mathbb{N}$, we can find $l_r > 0$ s.t $k_{s(l_r)-1} < r \leq k_{s(l_r)}$. Then for each $r \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{1}{r}|\{k \leq r : G(y_k, \omega; \varsigma) \leq 1-\alpha \text{ or} \\
&\qquad\qquad B(y_k, \omega; \varsigma) \geq \alpha,\ Y(y_k, \omega; \varsigma) \geq \alpha\}| \\
&\leq \frac{1}{k_{s(l_r)-1}}|\{k \leq r : G(y_k, \omega; \varsigma_1) \leq 1-\alpha_1 \text{ or} \\
&\qquad\qquad B(y_k, \omega; \varsigma_1) \geq \alpha_1,\ Y(y_k, \omega; \varsigma_1) \geq \alpha_1\}| \\
&\leq \frac{1}{k_{s(l_r)-1}}\big\{|\{k \leq k_{s(l_r)} : G(y_k, \omega; \varsigma_1) \leq 1-\alpha_1 \text{ or} \\
&\qquad\qquad B(y_k, \omega; \varsigma_1) \geq \alpha_1,\ Y(y_k, \omega; \varsigma_1) \geq \alpha_1\}| \\
&\quad + |\{k_{s(l_r)-1} < k \leq r : G(y_k, \omega; \varsigma_1) \leq 1-\alpha_1 \text{ or} \\
&\qquad\qquad B(y_k, \omega; \varsigma_1) \geq \alpha_1,\ Y(y_k, \omega; \varsigma_1) \geq \alpha_1\}|\big\} \\
&\leq \frac{k_{s(l_r-1)}}{k_{s(l_r)-1}} + \frac{1}{k_{s(l_r)-1}}(k_{s(l_r)} - k_{s(l_r)-1}) \\
&< \frac{1}{l_r} + 1 + \frac{1}{l_r} - 1 \\
&= \frac{2}{l_r}.
\end{aligned}$$

It follows that $y = (y_k)$ is S-convergent to 0. Now we will prove that $y = (y_k)$ is not S_θ-convergent w.r.t N_2. Since $\beta \neq 0$, so we can select $\varsigma > 0, \alpha \in (0,1)$ and $\omega \in X$ s.t $\beta \notin H(0, \varsigma, \alpha)$. Thus

$$\begin{aligned}
&\lim_{l\to\infty} \frac{1}{h_{s(l)}}|\{k \in I_{s(l)} : G(y_k, \omega; \varsigma) \leq 1-\alpha \text{ or} \\
&\qquad\qquad B(y_k, \omega; \varsigma) \geq \alpha,\ Y(y_k, \omega; \varsigma) \geq \alpha\}| \\
&= \lim_{l\to\infty} \frac{1}{h_{s(l)}}(k_{s(l)} - k_{s(l)-1}) \\
&= \lim_{l\to\infty} \frac{1}{h_{s(l)}}(h_{s(l)}) \\
&= 1,
\end{aligned}$$

and for $s \neq s_l$,

$$\begin{aligned}
&\lim_{l\to\infty} \frac{1}{h_s}|\{k \in I_s : G(y_k - \beta, \omega; \varsigma) \leq 1-\alpha \text{ or} \\
&\qquad\qquad B(y_k - \beta, \omega; \varsigma) \geq \alpha,\ Y(y_k - \beta, \omega; \varsigma) \geq \alpha\}| = 1 \neq 0.
\end{aligned}$$

Hence neither β nor 0 can be the S_θ-limit of the sequence $y = (y_k)$ w.r.t N_2. Furthermore, there is no other element in X that can be the S_θ-limit of y. Therefore $y \notin S_\theta(N_2)$. □

Theorems 3.5 and 3.6 together give the following result.

Theorem 3.7 Let X be a $N-2-NS$. For any lacunary sequence $\theta = (k_s), S(N_2) = S_\theta(N_2)$ iff $1 < \lim_s \inf q_s \leq \lim_s \sup q_s < \infty$.

4. Lacunary statistical completeness in $N-2-NS$

Definition 4.1 A sequence $y = (y_k)$ in X is called lacunary statistically Cauchy (or S_θ-Cauchy) w.r.t neutrosophic 2-norm N_2 if for each $\wp > 0$ and $\varsigma > 0$, $\exists\ r \in \mathbb{N}$ s.t.

$$\lim_{s\to\infty} \frac{1}{h_s}\Bigg|\Bigg\{k \in I_s : G(y_k - y_r, \omega; \varsigma) \leq 1 - \wp \text{ or } B(y_k - y_r, \omega; \varsigma) \geq \wp,\ Y(y_k - y_r, \omega; \varsigma) \geq \wp\Bigg\}\Bigg| = 0 \ \forall\ \omega \in X$$

or $\delta(\mathfrak{A}(\wp, \varsigma)) = 0$ where

$$\mathfrak{A}(\wp, \varsigma) = \{k \in I_s :\ G(y_k - y_r, \omega; \varsigma) \leq 1 - \wp \text{ or } B(y_k - y_r, \omega; \varsigma) \geq \wp,\ Y\ (y_k - y_r, \omega; \varsigma) \geq \wp\}.$$

Theorem 4.1 Every $S_\theta(N_2)$-convergent sequence in X is $S_\theta(N_2)$-Cauchy.

Proof. Let $y = (y_k)$ be the S_θ-convergent sequence to y_0. Let $\wp > 0$ and $\varsigma > 0$. Select $\nu > 0$ s.t. (1) is satisfied. Define

$$\mathfrak{A}(\nu, \varsigma) = \{k \in I_s :\ G(y_k - y_0, \omega;\ \frac{\varsigma}{2}) \leq 1 - \nu \text{ or } B(y_k - y_0, \omega; \frac{\varsigma}{2}) \geq \nu\ \ Y(y_k - y_0, \omega; \frac{\varsigma}{2}) \geq \nu\},$$

then $\delta_\theta(\mathfrak{A}(\nu, \varsigma)) = 0$ and $\delta_\theta(\mathfrak{A}^C(\nu, \varsigma)) = 1$. Let $p \in \mathfrak{A}^C(\nu, \varsigma)$ then for $\omega \in X$, we have $G(y_p - y_0, \omega; \frac{\varsigma}{2}) > 1 - \nu$ and $B(y_p - y_0, \omega; \frac{\varsigma}{2}) < \nu, Y(y_p - y_0, \omega; \frac{\varsigma}{2}) < \nu$.

Now let $M(\wp, \varsigma) = \{k \in I_s : G(y_k - y_p, \omega; \varsigma) \leq 1 - \wp \text{ or } B(y_k - y_p, \omega; \varsigma) \geq \wp, Y(y_k - y_p, \omega; \varsigma) \geq \wp\}$.

We claim that $M(\wp, \varsigma) \subset \mathfrak{A}(l, \varsigma)$. Let $r \in M(\wp, \varsigma)$, then we have $G(y_r - y_p, \omega; \varsigma) \leq 1 - \wp$ or $B(y_r - y_p, \omega; \varsigma) \geq \wp, Y(y_r - y_p, \omega; \varsigma) \geq \wp$.

Case (i): Suppose $G(y_r - y_p, \omega; \varsigma) \leq 1 - \wp$, then $G(y_r - y_0, \omega; \frac{\varsigma}{2}) \leq 1 - \nu$ and therefore $r \in \mathfrak{A}(\nu, \varsigma)$. As otherwise, i.e, if $G(y_r - y_0, \omega; \frac{\varsigma}{2}) > 1 - \nu$, then

$$\begin{aligned} 1 - \wp \geq G(y_r - y_p, \omega; \varsigma) &\geq G(y_q - y_0, \omega; \frac{\varsigma}{2}) \circ G(y_p - y_0, \omega; \frac{\varsigma}{2}) \\ &> (1 - \nu) \circ (1 - \nu) \\ &> 1 - \wp (\text{not possible}) \quad . \end{aligned}$$

Thus, $M(\wp, \varsigma) \subset \mathfrak{A}(\nu, \varsigma)$.

Case (ii): Suppose $B(y_r - y_p, \omega; \varsigma) \geq \wp$, then $B(y_r - y_0, \omega; \frac{\varsigma}{2}) \geq \nu$ and therefore $r \in A(\nu, \varsigma)$.

As otherwise, i.e, if $B(y_r - y_0, \omega; \frac{\varsigma}{2}) < \nu$, then

$$\wp \leq B(y_r - y_p, \omega; \varsigma) \leq B(y_r - y_0, \omega; \frac{\varsigma}{2}) \diamond B(y_p - y_0, \omega; \frac{\varsigma}{2})$$
$$< \nu \diamond \nu$$
$$< \wp (\text{not possible}) \quad .$$

Also, suppose $Y(y_r - y_p, \omega; \varsigma) \geq \wp$, then $Y(y_r - y_0, \omega; \frac{\varsigma}{2}) \geq \nu$ and therefore $r \in \mathfrak{A}(\nu, \varsigma)$. As otherwise, i.e, if $B(y_r - y_0, \omega; \frac{\varsigma}{2}) < \nu$, then

$$\wp \leq Y(y_r - y_p, \omega; \varsigma) \leq Y(y_r - y_0, \omega; \frac{\varsigma}{2}) \diamond Y(y_p - y_0, \omega; \frac{\varsigma}{2})$$
$$< \nu \diamond \nu$$
$$< \wp (\text{not possible}) \quad .$$

Thus, $M(\wp, \varsigma) \subset \mathfrak{A}(\nu, \varsigma)$.
Hence in all cases, $M(\wp, \varsigma) \subset \mathfrak{A}(\nu, \varsigma)$. Since $\delta_\theta(\mathfrak{A}(\nu, \varsigma)) = 0$, so $\delta_\theta(M(\wp, \varsigma)) = 0$ and therefore $y = (y_k)$ is $S_\theta(N_2)$-Cauchy. □

Definition 4.2 A neutrosophic 2-normed space X is called $S_\theta(N_2)$-complete if every $S_\theta(N_2)$-Cauchy sequence in X is $S_\theta(N_2)$-convergent in X.

Theorem 4.2 Every $N-2-NS$ X is $S_\theta(N_2)$-complete.

Proof Let $y = (y_k)$ be $S_\theta(N_2)$-Cauchy sequence in X. Suppose on the contrary that $y = (y_k)$ is not $S_\theta(N_2)$-convergent. Let $\wp > 0$ and $\varsigma > 0$, then $\exists\ r \in \mathbb{N}$ such that $\omega \in X$ if we define

$$\mathfrak{A}(\wp, \varsigma) = \{k \in I_s : G(y_k - y_r, \omega; \varsigma) \leq 1 - \wp \text{ or}$$
$$B(y_k - y_r, \omega; \varsigma) \geq \wp, Y(y_k - y_r, \omega; \varsigma) \geq \wp\} \text{ and}$$

$$\mathfrak{T}(\wp, \varsigma) = \{k \in I_s : G(y_k - y_0, \omega; \frac{\varsigma}{2}) > 1 - \wp \text{ and}$$
$$B(y_k - y_0, \omega; \frac{\varsigma}{2}) < \wp, Y(y_k - y_0, \omega; \frac{\varsigma}{2}) < \wp\},$$

then $\delta_\theta(\mathfrak{A}(\wp, \varsigma)) = \delta_\theta(\mathfrak{T}(\wp, \varsigma)) = 0$ and therefore we have $\delta_\theta(\mathfrak{A}^C(\wp, \varsigma)) = \delta_\theta(\mathfrak{T}^C(\wp, \varsigma)) = 1$. Since $G(y_k - y_r, \omega; \varsigma) \geq 2G(y_k - y_0, \omega; \frac{\varsigma}{2}) > 1 - \wp$ and $B(y_k - y_r, \omega; \varsigma) \leq 2B(y_k - y_0, \omega; \frac{\varsigma}{2}) < \wp$, $Y(y_k - y_r, \omega; \varsigma) \leq 2Y(y_k - y_0, \omega; \frac{\varsigma}{2}) < \wp$, if $G(y_k - y_0, \omega; \frac{\varsigma}{2}) > \frac{1-\wp}{2}$ and $B(y_k - y_0, \omega; \frac{\varsigma}{2}) < \frac{\wp}{2}$, $Y(y_k - y_0, \omega; \frac{\varsigma}{2}) < \frac{\wp}{2}$. We have $\delta_\theta(\{k \in I_s : G(y_k - y_r, \omega; \varsigma) > 1 - \wp$ and $B(y_k - y_r, \omega; \varsigma) < \wp$, $Y(y_k - y_r, \omega; \varsigma) < \wp\}) = 0$. i.e., $\delta_\theta(\mathfrak{A}^C(\wp, \varsigma)) = 0$ which contradicts the fact that $\delta_\theta(\mathfrak{A}^C(\wp, \varsigma)) = 1$. Hence, $y = (y_k)$ is S_θ-convergent w.r.t. N_2. □

Theorem 4.3 For any sequence $y = (y_k)$ in X, the subsequent assertions are equivalent.

(i) $y = (y_k)$ is a $S_\theta(N_2)$-Cauchy sequence.

(ii) $\exists$ a subset $\Re = \{k_n\}$ of $\mathbb{N}$ with $\delta_\theta(\Re) = 1$ and subsequence $(y_{k_n})_{n\in\mathbb{N}}$ is a $S_\theta(N_2)$-Cauchy sequence over $\Re$.

Proof. The proof of this theorem can be derived in a similar manner to the proof of theorem 3.4.

5. Conclusion

The fuzzy norm is a very helpful tool to analyze many situations in the real world where the crisp norm is found difficult due to huge uncertainty. In the present work, we define and study S_θ-convergence, S_θ-Cauchy and S_θ-completeness in a more general setting, i.e., in neutrosophic 2-normed spaces. The results presented in this paper will be helpful for many problems of fuzzy functional analysis in which ordinary norm can not be predictable and therefore one looks forward towards a fuzzy norm or a generalized fuzzy norm.

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