NSS

Neutrosophic Sets and Systems, Vol. 63, 2024

University of New Mexico

On λ-statistical and V_λ-statistical summability in neutrosophic-2-normed spaces

Sumaira Aslam[1,*], Vijay Kumar[1] and Archana Sharma[1]

[1]Department Of Mathematics, Chandigarh University, Mohali-140413, Punjab, India;

bhatsumair64@gmail.com, kaushikvjy@gmail.com and dr.archanasharma1022@gmail.com

[*]Correspondence: bhatsumair64@gmail.com.

Abstract. In the present paper, we aim to define λ-statistical summability, V_λ-statistical summability in neutrosophic-2-normed spaces (briefly called $N-2-NS$) and study some relationships among these notions. We give an example that shows λ-statistical summability is stronger method in neutrosophic-2-normed spaces. Finally, we define λ-statistically Cauchy sequence and λ-statistically completeness in neutrosophic-2-normed spaces and obtain the Cauchy convergence criteria in these spaces.

Keywords: λ-statistical convergence, V_λ-summable, λ-statistical Cauchy, neutrosophic-2-normed spaces.

---

1. Introduction

The idea of λ-statistical convergence was explored by Mursaleen [21] as a generalization of statistical convergence, initially introduced in [4] and [33] independently.

"Let $\lambda = (\lambda_n)\ ; \lambda_n \in \mathbb{R}^+ = (0, \infty)$ be a non decreasing sequence satisfying $\lim_{n\to\infty} \lambda_n = \infty$, $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$ and $I_n = [n - \lambda_n + 1, n]$. For $\mathfrak{R} \subseteq \mathbb{N}$, the λ-density of $\mathfrak{R}$ is defined by $\delta_\lambda(\mathfrak{R}) = \lim_n \frac{1}{\lambda_n} |\{k \in I_n : k \in \mathfrak{R}\}|$. A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ of numbers is said to be λ-statistical convergent to $\mathfrak{u}_0$ if for each $\mathfrak{y} > 0$, $\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |\mathfrak{u}_k - \mathfrak{u}_0| \geq \mathfrak{y}\}| = 0$, i.e., $\delta_\lambda(\mathfrak{R}_\mathfrak{y}) = 0$, where $\mathfrak{R}_\mathfrak{y} = \{k \in I_n : |\mathfrak{u}_k - \mathfrak{u}_0| \geq \mathfrak{y}\}$. We write, in this case $\mathcal{S}_\lambda - \lim_k \mathfrak{u}_k = \mathfrak{u}_0$."Subsequently, statistical convergence and its generalizations have been developed by numerous authors including Connor [3], Fridy [6], Hazarika et al. [8, 9], Kumar et al. [15], Maddox [20], Šalát [36] and many others.

On the other hand, many problems of real life can't be modeled via crispness due to huge uncertainty in data. In view of this Zadeh [41] defined fuzzy sets as generalization of crisp sets to deal with such problems.

One of interesting generalizations of fuzzy sets is due to Atanassov [1], called intuitionistic fuzzy sets by adding the non-membership function along with the membership function to the fuzzy sets. These sets have been applied to introduce new norms (see [5], [22]), topology [31], and metric [27] and found very useful where the crisp norms are not sufficient to work due to huge uncertainty. Intuitionistic fuzzy sets are naturally used to define intuitionistic fuzzy normed spaces [34]. Recently, statistical convergence and its generalizations have been extended and developed in these spaces (see [2], [15], [25], [26] and [38]).

Another, interesting generalization of fuzzy sets is given by Smarandache [35] by introducing the indeterminacy function to the intuitionistic fuzzy sets. For ongoing development on neutrosophic set (NS) and its applications, we refer to the reader [10], [18], [23], [29-31], etc. Kirişçi and Şimşek [13] used neutrosophic sets to define neutrosophic norm and studied statistical convergence in neutrosophic normed spaces(NNS). Nowadays, the area of summability in these spaces is of much interest. For a broad view in this direction, we recommend [28], [37-39], etc. Several summability approaches have been created, including statistical convergence, lacunary statistical convergence, and ideal convergence in these spaces (see [11], [12], [14], [16], [17], and [32]). Recently, in [24] the concept of neutrosophic-2-norm is introduced where the authors studied statistical convergence in neutrosophic-2-normed spaces. In the present study, we continue to define a more general summability method, called $\mathcal{S}_\lambda$-summability in $N-2-NS$ and develop some of its properties.

2. Prelimanaries

This section starts with a brief review on certain definitions and results needed in the sequel.

"For $\lambda = (\lambda_n)$ as defined above, the generalized de la Vallée-Poussin mean of $\mathfrak{u} = (\mathfrak{u}_k)$ is defined by $t_n(\mathfrak{u}) = \frac{1}{\lambda_n}\sum\limits_{k\in I_n} \mathfrak{u}_k$. Further, $\mathfrak{u} = (\mathfrak{u}_k)$ is called V_λ-summable to $\mathfrak{u}_0$(see[19]) if $\lim\limits_{n\to\infty} t_n(\mathfrak{u}) = \mathfrak{u}_0$. Let

$$[V_\lambda] = \left\{\mathfrak{u} = (\mathfrak{u}_n) : \exists\ \mathfrak{u}_0 \in \mathbb{R}, \lim_{n\to\infty} \frac{1}{\lambda}\sum_{k\in I_n} |\mathfrak{u}_k - \mathfrak{u}_0| = 0\right\}"$$

Definition 2.1 [40] "Let $\mathfrak{T} = [0,1]$. A binary operation $\circ : \mathfrak{T}\times\mathfrak{T} \to \mathfrak{T}$ is t-norm if $\forall\ \mathfrak{c},\ \mathfrak{e},\ \mathfrak{g},\ \mathfrak{h} \in \mathfrak{T}$ we have

1) $\circ$ is continuous, commutative and associative,

2) $\mathfrak{e} = \mathfrak{e}\circ 1$,

3) $\mathfrak{c}\circ\mathfrak{e} \le \mathfrak{g}\circ\mathfrak{h}$ whenever $\mathfrak{c} \le \mathfrak{g}$ and $\mathfrak{e} \le \mathfrak{h}$."

Definition 2.2 [40]"Let $\mathfrak{T} = [0,1]$. A binary operation $\diamond : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ is t-conorm if $\forall$ $\mathfrak{c},\ \mathfrak{e},\ \mathfrak{g},\ \mathfrak{h} \in \mathfrak{T}$ we have

1) $\diamond$ is continuous, commutative and associative,

2) $\mathfrak{e} = \mathfrak{e} \diamond 0$,

3) $\mathfrak{c} \diamond \mathfrak{e} \leq \mathfrak{g} \diamond \mathfrak{h}$ whenever $\mathfrak{c} \leq \mathfrak{g}$ and $\mathfrak{e} \leq \mathfrak{h}$."

Kirişci and Şimşek introduced the notion NNS in the following manner [13].

Definition 2.3 [13]"Consider $\mathfrak{F}$ to be a vector space, $N = \{\langle \mathfrak{V}, G(\mathfrak{V}), B(\mathfrak{V}), Y(\mathfrak{V})\rangle : \mathfrak{V} \in \mathfrak{F}\}$ be a normed space s.t $N : \mathfrak{F} \times \mathbb{R}^+ \to [0,\ 1]$. Let $\circ, \diamond$ be t-norm and t-conorm, respectively. If the followings conditions hold, then the four tuple $\mathfrak{U} = (\mathfrak{F}, N,\ \circ, \diamond\ )$ is called NNS, for $\mathfrak{r}, \mathfrak{s} \in \mathfrak{F}$, $\varrho,\ \omega > 0$ and for each $\varsigma \neq 0$,

(i) $0 \leq G(\mathfrak{r},\ \varrho) \leq 1$, $0 \leq B(\mathfrak{r},\ \varrho) \leq 1$, $0 \leq Y(\mathfrak{r},\ \varrho) \leq 1$ for $\varrho \in \mathbb{R}^+$;

(ii) $G(\mathfrak{r},\ \varrho) +\ B(\mathfrak{r},\ \varrho) + Y(\mathfrak{r},\ \varrho) \leq 3$ for $\varrho \in \mathbb{R}^+$;

(iii) $G(\mathfrak{r},\ \varrho) = 1$ (for $\varrho > 0$) iff $\mathfrak{r} = 0$;

(iv) $G(\varsigma\mathfrak{r}, \varrho) = G\left(\mathfrak{r},\ \frac{\varrho}{|\varsigma|}\right)$;

(v) $G(\mathfrak{r},\ \omega) \circ G(\mathfrak{s},\ \varrho) \leq G(\mathfrak{r} + \mathfrak{s},\ \omega + \varrho)$;

(vi) $G(\mathfrak{r},\ .)$ is non-decreasing and continuous function ;

(vii) $\lim_{\varrho\to\infty} G(\mathfrak{r},\ \varrho) = 1$;

(viii) $B(\mathfrak{r}, \varrho) = 0$ (for$\varrho > 0$) iff $\mathfrak{r} = 0$;

(ix) $B(\varsigma\mathfrak{r}, \varrho) = B\left(\mathfrak{r},\ \frac{\varrho}{|\varsigma|}\right)$;

(x) $B(\mathfrak{r},\ \omega) \diamond B(\mathfrak{s},\ \varrho) \geq B(\mathfrak{r} + \mathfrak{s}, \varrho + \omega)$;

(xi) $B(\mathfrak{r},\ .)$ is non-increasing and continuous function;

(xii) $\lim_{\lambda\to\infty} B(\mathfrak{r}, \varrho) = 0$;

(xiii) $Y(\mathfrak{r}, \varrho) = 0$ (for $\varrho > 0$) iff $\mathfrak{r} = 0$;

(xiv) $Y(\varsigma\mathfrak{r}, \varrho) = Y\left(\mathfrak{r},\ \frac{\varrho}{|\varsigma|}\right)$;

(xv) $Y(\mathfrak{r},\ \omega) \diamond Y(\mathfrak{s},\ \varrho) \geq Y(\mathfrak{r} + \mathfrak{s}, \varrho + \omega)$;

(xvi) $Y(\mathfrak{r},\ .)$ is non-increasing and continuous function;

(xvii) $\lim_{\lambda\to\infty} Y(\mathfrak{r}, \varrho) = 0$;

(xviii) If $\varrho \leq 0$, then $G(\mathfrak{r}, \varrho) = 0$, $B(\mathfrak{r}, \varrho) = 1$ and $Y(\mathfrak{r}, \varrho) = 1$.

Then $N = (G, B, Y)$ is called the neutrosophic norm."

We now recall the idea of two norm introduced in the paper [7].

Definition 2.4 [7]"Let $\mathfrak{U}$ be a linear space of dimension $d > 1$. A function $\|.,.\| : \mathfrak{U} \times \mathfrak{U} \to \mathbb{R}$ satisfying the prerequisites specified below: For all $\mathfrak{s}, \mathfrak{t}, \mathfrak{l} \in \mathfrak{U}$, and scalar $\mathfrak{c}$, we have

(i) $||\mathfrak{s}, \mathfrak{t}|| = 0$ iff $\mathfrak{s}$ and $\mathfrak{t}$ are linearly dependent;

(ii) $||\mathfrak{s}, \mathfrak{t}|| = ||\mathfrak{t}, \mathfrak{s}||$;

(iii)$||\mathfrak{c}\mathfrak{s}, \mathfrak{t}|| = |\mathfrak{c}|||\mathfrak{s}, \mathfrak{t}||$ and

(iv) $||\mathfrak{s}, \mathfrak{t} + \mathfrak{l}|| \leq ||\mathfrak{s}, \mathfrak{t}|| + ||\mathfrak{s}, \mathfrak{l}||$.

The pair $(\mathfrak{U}, ||.,.||)$ is then called an 2-normed space.

Let $\mathfrak{U} = \mathbb{R}^2$ and for $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2)$ and $\mathfrak{t} = (\mathfrak{t}_1, \mathfrak{t}_2)$ we define $||\mathfrak{s}, \mathfrak{t}|| = |\mathfrak{s}_1\mathfrak{t}_2 - \mathfrak{s}_2\mathfrak{t}_1|$, then $||\mathfrak{s}, \mathfrak{t}||$ is a 2-norm on $\mathfrak{U} = \mathbb{R}^2$".

Recently, Murtaza et al [24] defined neutrosophic-2- normed spaces as follows.

Definition 2.5 [24] "Consider $\mathfrak{F}$ to be a vector space, $N_2 = (\{(\mathfrak{r}, \mathfrak{s}), G(\mathfrak{r}, \mathfrak{s}), B(\mathfrak{r}, \mathfrak{s}), Y(\mathfrak{r}, \mathfrak{s})\} : (\mathfrak{r}, \mathfrak{s}) \in \mathfrak{F} \times \mathfrak{F})$ be a $2-$normed space s.t $N_2 : \mathfrak{F} \times \mathfrak{F} \times \mathbb{R}^+ \to [0, 1]$. Let $\circ$, $\diamond$ be t-norm and t-conorm respectively. If the following conditions hold, then the four-tuple $\mathfrak{U} = (\mathfrak{F}, N_2, \circ, \diamond)$ is called a neutrosophic $2-$normed spaces (briefly $N-2-NS$) if for each $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in \mathfrak{U}$, $\varrho, \omega \geq 0$ and $\varsigma \neq 0$:

(i) $0 \leq G(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 1$, $0 \leq B(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 1$ and $0 \leq Y(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 1$ for $\varrho \in \mathbb{R}^+$;

(ii) $G(\mathfrak{r}, \mathfrak{s}; \varrho) + B(\mathfrak{r}, \mathfrak{s}; \varrho) + Y(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 3$;

(iii) $G(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$ iff $\mathfrak{r}, \mathfrak{s}$ are linearly dependent;

(iv) $G(\varsigma\mathfrak{r},\ \mathfrak{s};\ \varrho) = G(\mathfrak{r}, \mathfrak{s}; \frac{\varrho}{|\varsigma|})$ for each $\varsigma \neq 0$;

(v) $G(\mathfrak{r}, \mathfrak{s}; \varrho) \circ G(\mathfrak{r}, \mathfrak{t}; \omega) \leq G(\mathfrak{r}, \mathfrak{s} + \mathfrak{t}; \varrho + \omega)$;

(vi) $G(\mathfrak{r}, \mathfrak{s}; .) : (0, \infty) \to [0, 1]$ is non-decreasing and continuous function;

(vii) $\lim_{\varrho \to \infty} G(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$;

(viii) $G(\mathfrak{r}, \mathfrak{s}; \varrho) = G(\mathfrak{s}, \mathfrak{r}; \varrho)$

(ix) $B(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$ iff $\mathfrak{r}, \mathfrak{s}$ are linearly dependent;

(x) $B(\varsigma\mathfrak{r}, \mathfrak{s}; \varrho) = B(\mathfrak{r}, \mathfrak{s}; \frac{\varrho}{|\varsigma|})$ for each $\varsigma \neq 0$;

(xi) $B(\mathfrak{r}, \mathfrak{s}; \varrho) \diamond B(\mathfrak{r}, \mathfrak{t}; \omega) \geq B(\mathfrak{r}, \mathfrak{s} + \mathfrak{t}; \varrho + \omega)$;

(xii) $B(\mathfrak{r}, \mathfrak{s}; .) : (0, \infty) \to [0, 1]$ is non-increasing and continuous function;

(xiii) $\lim_{\varrho \to \infty} B(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$;

(xiv) $B(\mathfrak{r}, \mathfrak{s}; \varrho) = B(\mathfrak{s}, \mathfrak{r}; \varrho)$;

(xvi) $Y(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$ iff $\mathfrak{r}, \mathfrak{s}$ are linearly dependent;

(xv)$Y(\varsigma\mathfrak{r},\ \mathfrak{s};\ \varrho) = Y(\mathfrak{r},\ \mathfrak{s}; \frac{\varrho}{|\varsigma|})$ for each $\varsigma \neq 0$;

(xvi) $Y(\mathfrak{r}, \mathfrak{s}; \varrho) \diamond Y(\mathfrak{r}, \mathfrak{t}; \omega) \geq Y(\mathfrak{r}, \mathfrak{s} + \mathfrak{t}; \varrho + \omega)$;

(xvii) $Y(\mathfrak{r}, \mathfrak{s}; .) : (0, \infty) \to [0, 1]$ is non-increasing and continuous function;

(xviii) $\lim_{\lambda \to \infty} Y(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$;

(xix) $Y(\mathfrak{r}, \mathfrak{s}; \varrho) = Y(\mathfrak{s}, \mathfrak{r}; \varrho)$;

(xx) if $\varrho \leq 0$, then $G(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$, $B(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$, $Y(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$.

In this case, $N_2 = (G, B, Y)_2$ is called a neutrosophic 2-norm. From now on wards, unless otherwise stated by $\mathfrak{U}$ we shall denote the $N-2-NS$ $(\mathfrak{F}, N_2, \circ, \diamond)$

A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $\mathfrak{U}$ is called convergent to $\mathfrak{u}_0$ if for each $\mathfrak{y} > 0$ and $\varrho > 0$, $\exists\ k_0 \in \mathbb{N}$ s.t $G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$, $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}$ and $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\ \forall\ k \geq k_0$ and $\mathfrak{w} \in \mathfrak{U}$ which is equivalently to say $\lim_{k\to\infty} G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = 1$, $\lim_{k\to\infty} B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = 0$ and $\lim_{k\to\infty} Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = 0$. In present case, we denote $N_2 - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$.

A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $\mathfrak{U}$ is called Cauchy if for each $\mathfrak{y} > 0$ and $\varrho > 0$, $\exists\ k_0 \in \mathbb{N}$ s.t $G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$, $B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) < \mathfrak{y}$ and $Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) < \mathfrak{y}\ \forall\ k, p \geq k_0$ and $\forall$ $\mathfrak{w} \in \mathfrak{U}$."

3. λ-Statistical Convergence in N-2-NS

In this section, we define and study λ-Statistical Convergence in $N-2-NS$ and develop some of its properties.

Definition 3.1 A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $N-2-NS$ $\mathfrak{U}$ is called λ-statistical convergent (or $\mathcal{S}_\lambda$-convergent) to $\mathfrak{u}_0$ if for each $\mathfrak{y} > 0$ and $\varrho > 0$

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or } B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y},\ Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y} \right\} \right| = 0\ \forall\ \mathfrak{w} \in \mathfrak{U};$$

or equivalently, $\delta_\lambda(\mathcal{A}(\mathfrak{y}, \varrho)) = 0$, where

$$\mathcal{A}(\mathfrak{y}, \varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or } B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y},\ Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\}.$$

In present case, we denote $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$.

We now give the following Lemma:

Lemma 3.1 For any sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $\mathfrak{U}$, the subsequent assertions are equivalent:

(i) $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$;

(ii) $\delta_\lambda\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}\} = \delta_\lambda\{k \in I_n : B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\} = \delta_\lambda\{k \in I_n : Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\} = 0$;

(iii) $\delta_\lambda\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\} = 1$;

(iv) $\delta_\lambda\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}\} = \delta_\lambda\{k \in I_n : B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\} = \delta_\lambda\{k \in I_n : Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\} = 1$ and

(v) $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = 1$, $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = \mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = 0$.

We now have the following interesting implication.

Theorem 3.1 Let $\mathfrak{u} = (\mathfrak{u}_k)$ be any sequence in $N-2-NS$ $\mathfrak{U}$. If $N_2 - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$, then $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$.

Proof. Let $N_2 - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$. Then for each $\mathfrak{y} > 0$ and $\varrho > 0, \exists$ an integer $k_0 \in \mathbb{N}$ s.t. $G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\ \forall k \geq k_0$ and every $\mathfrak{w} \in \mathfrak{U}$. Hence the set $\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}$ or $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\}$ has a finitely many terms whose λ-density is zero. Therefore, $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$. □

The converse of the above theorem is not true in general.

Example 3.1 Let $(\mathbb{R}^2, |.|)$ be a 2-normed space. For $\mathfrak{e}, \mathfrak{g} \in [0, 1]$. Let $\mathfrak{e} \circ \mathfrak{g} = \mathfrak{e}\mathfrak{g}$ and $\mathfrak{e} \diamond \mathfrak{g} = \min\{\mathfrak{e} + \mathfrak{g}, 1\}$. Choose $\mathfrak{s}, \mathfrak{t} \in \mathfrak{F}$, $\varrho > 0$ and $\varrho > \|\mathfrak{s}, \mathfrak{t}\|$. Define $G(\mathfrak{s}, \mathfrak{t}; \varrho) = \frac{\varrho}{\varrho + \|\mathfrak{s},\mathfrak{t}\|}$, $B(\mathfrak{s}, \mathfrak{t}; \varrho) = \frac{\|\mathfrak{s},\mathfrak{t}\|}{\varrho + \|\mathfrak{s},\mathfrak{t}\|}$ and $Y(\mathfrak{s}, \mathfrak{t}; \varrho) = \frac{\|\mathfrak{s},\mathfrak{t}\|}{\varrho}$, then it is clear that $\mathfrak{U} = (\mathbb{R}^2, N_2, \circ, \diamond)$ is a $N - 2 - NS$. Define $\mathfrak{u} = (\mathfrak{u}_k)$ by

$$\mathfrak{u}_k = \begin{cases} (k, 0) \text{ if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ (0, 0) \text{ otherwise.} \end{cases}$$

Now, for each $\mathfrak{y} > 0$ and $\varrho > 0$, let

$$\begin{aligned} \mathcal{A}(\mathfrak{y}, \varrho) &= \Big\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or} \\ &\qquad B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y},\ Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\Big\} \\ &= \left\{k \in I_n : \frac{\varrho}{\varrho + \|\mathfrak{u}_k, \mathfrak{w}\|} \leq 1 - \mathfrak{y} \text{ or} \frac{\|\mathfrak{u}_k, \mathfrak{w}\|}{\varrho + \|\mathfrak{u}_k, \mathfrak{w}\|} \geq \mathfrak{y},\ \frac{\|\mathfrak{u}_k, \mathfrak{w}\|}{\varrho} \geq \mathfrak{y}\right\} \\ &= \left\{k \in I_n : \|\mathfrak{u}_k, \mathfrak{w}\| \geq \frac{\varrho\mathfrak{y}}{1 - \mathfrak{y}} \text{or} \|\mathfrak{u}_k, \mathfrak{w}\| \geq \varrho\mathfrak{y}\right\} \\ &= \{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n\} \end{aligned}$$

and so we get

$$\frac{1}{\lambda_n}|\mathcal{A}(\mathfrak{y}, \varrho)| = \frac{1}{\lambda_n}|\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n\}| \leq \frac{[\sqrt{\lambda_n}]}{\lambda_n}.$$

Taking $n \to \infty$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\mathcal{A}(\mathfrak{y}, \varrho)| \leq \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]}{\lambda_n} = 0;$$

i.e., $\delta_\lambda(\mathcal{A}(\mathfrak{y}, \varrho)) = 0$. This shows that, $\mathfrak{u}_k \to 0\big(\mathcal{S}_\lambda(N_2)\big)$ But the sequence, $\mathfrak{u} = (\mathfrak{u}_k)$ is not N_2-convergent to 0.

Theorem 3.2 Let $\mathfrak{u} = (\mathfrak{u}_k)$ be any sequence in $N - 2 - NS\ \mathfrak{U}$, if $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k$ exists, then it is unique.

Proof. Let $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_1$ and $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_2$. Let $\mathfrak{y} > 0$, select $l > 0$ s.t.

$$(1 - l) \circ (1 - l) > 1 - \mathfrak{y} \text{ and } l \diamond l < \mathfrak{y}. \tag{1}$$

For $\varrho > 0$ and $\mathfrak{w} \in \mathfrak{U}$. Define the sets:

$$K_{G,1}(l,\varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_1, \mathfrak{w}; \frac{\varrho}{2}) \leq 1 - l\},$$
$$K_{G,2}(l,\varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \leq 1 - l\};$$
$$K_{B,1}(l,\varrho) = \{k \in I_n : B(\mathfrak{u}_k - \mathfrak{u}_1, \mathfrak{w}; \frac{\varrho}{2}) \geq l\},$$
$$K_{B,2}(l,\varrho) = \{k \in I_n : B(\mathfrak{u}_k - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \geq l\};$$
$$K_{Y,1}(l,\varrho) = \{k \in I_n : Y(\mathfrak{u}_k - \mathfrak{u}_1, \mathfrak{w}; \frac{\varrho}{2}) \geq l\};$$
$$K_{Y,2}(l,\varrho) = \{k \in I_n : Y(\mathfrak{u}_k - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \geq l\}.$$

Since $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_1$. By Lemma 3.1, we have $\delta_\lambda\{K_{G,1}(l,\varrho)\} = \delta_\lambda\{K_{B,1}(l,\varrho)\} = \delta_\lambda\{K_{Y,1}(l,\varrho)\} = 0$ and therefore $\delta_\lambda\{K^C_{G,1}(l,\varrho)\} = \delta_\lambda\{K^C_{B,1}(l,\varrho)\} = \delta_\lambda\{K^C_{Y,1}(l,\varrho)\} = 1$. Also, using $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_2$, we get, $\delta_\lambda\{K_{G,2}(l,\varrho)\} = \delta_\lambda\{K_{B,2}(l,\varrho)\} = \delta_\lambda\{K_{Y,2}(l,\varrho)\} = 0$ and therefore $\delta_\lambda\{K^C_{G,2}(l,\varrho)\} = \delta_\lambda\{K^C_{B,2}(l,\varrho)\} = \delta_\lambda\{K^C_{Y,2}(l,\varrho)\} = 1$. Now define $K_{G,B,Y}(\mathfrak{y},\varrho) = \{K_{G,1}(l,\varrho) \cup K_{G,2}(l,\varrho)\} \cap \{K_{B,1}(l,\varrho) \cup K_{B,2}(l,\varrho)\} \cap \{K_{Y,1}(l,\varrho) \cup K_{Y,2}(l,\varrho)\}$. Then observe that $\delta_\lambda(\{K_{G,B,Y}(\mathfrak{y},\varrho)\}) = 0$ which implies $\delta(\{K^C_{G,B,Y}(\mathfrak{y},\varrho)\}) = 1$. Let $m \in K^C_{G,B,Y}(\mathfrak{y},\varrho)$, then we have

Case 1. $m \in \{K_{G,1}(l,\varrho) \cup K_{G,2}(l,\varrho)\}^C$,
Case 2. $m \in \{K_{B,1}(l,\varrho) \cup K_{B,2}(l,\varrho)\}^C$,
Case 3. $m \in \{K_{Y,1}(l,\varrho) \cup K_{Y,2}(l,\varrho)\}^C$.

Case 1: Let, $m \in \{K_{G,1}(l,\varrho) \cup K_{G,2}(l,\varrho)\}^C$, then $m \in K^C_{G,1}(l,\varrho)$ and $m \in K^C_{G,2}(l,\varrho)$. Therefore, for any $\mathfrak{w} \in \mathfrak{U}$ we have

$$G(\mathfrak{u}_m - \mathfrak{u}_1, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l \text{ and } G(\mathfrak{u}_m - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l. \tag{2}$$

Now

$$\begin{aligned} G(\mathfrak{u}_1 - \mathfrak{u}_2, \mathfrak{w}; \varrho) &\geq G(\mathfrak{u}_m - \mathfrak{u}_1, \mathfrak{w}, ; \frac{\varrho}{2}) \circ G(\mathfrak{u}_m - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \\ &> (1-l) \circ (1-l) \text{ by}(2) \\ &> 1 - \mathfrak{y}. \text{ by (1)} \end{aligned}$$

Since $\mathfrak{y} > 0$ is arbitrary, so we have $G(\mathfrak{u}_1 - \mathfrak{u}_2, \mathfrak{w}; \varrho) = 1 \ \forall \varrho > 0$, and therefore $\mathfrak{u}_1 - \mathfrak{u}_2 = 0$. This shows that $\mathfrak{u}_1 = \mathfrak{u}_2$.

Case 2: Let, $m \in \{K_{B,1}(l,\varrho) \cup K_{B,2}(l,\varrho)\}^C$, then $m \in K^C_{B,1}(l,\varrho)$ and $m \in K^C_{B,2}(l,\varrho)$. Therefore, for $\mathfrak{w} \in \mathfrak{U}$ we have

$$B(\mathfrak{u}_m - \mathfrak{u}_1, \mathfrak{w}; \frac{\varrho}{2}) < l \text{ and } B(\mathfrak{u}_m - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) < l. \tag{3}$$

Now

$$B(\mathfrak{u}_1-\mathfrak{u}_2,\mathfrak{w};\varrho)\leq B(\mathfrak{u}_m-\mathfrak{u}_1,\mathfrak{w};\frac{\varrho}{2})\circ B(\mathfrak{u}_m-\mathfrak{u}_2,\mathfrak{w};\frac{\varrho}{2})$$
$$< l\diamond l \text{ by } (3)$$
$$< \mathfrak{y}. \text{ by } (1)$$

Since $\mathfrak{y}>0$ is arbitrary, so we have $B(\mathfrak{u}_1-\mathfrak{u}_2,\mathfrak{w};\varrho)=0\ \forall\varrho>0$, and therefore $\mathfrak{u}_1-\mathfrak{u}_2=0$. This shows that $\mathfrak{u}_1=\mathfrak{u}_2$.

Case 3: Let, $m\in\{K_{Y,1}(l,\varrho)\cup K_{Y,2}(l,\varrho)\}^C$, then $m\in K^C_{Y,1}(l,\varrho)$ and $m\in K^C_{Y,2}(l,\varrho)$. Therefore, for $\mathfrak{w}\in\mathfrak{U}$ we have

$$Y(\mathfrak{u}_m-\mathfrak{u}_1,\mathfrak{w};\frac{\varrho}{2})<l \text{ and } Y(\mathfrak{u}_m-\mathfrak{u}_2,\mathfrak{w};\frac{\varrho}{2})<l. \tag{4}$$

Now

$$Y(\mathfrak{u}_1-\mathfrak{u}_2,\mathfrak{w};\varrho)\leq Y(\mathfrak{u}_m-\mathfrak{u}_1,\mathfrak{w};\frac{\varrho}{2})\circ Y(\mathfrak{u}_m-\mathfrak{u}_2,\mathfrak{w};\frac{\varrho}{2})$$
$$< l\diamond l \text{ by } (4)$$
$$< \mathfrak{y}. \text{ by } (1)$$

Since $\mathfrak{y}>0$ is arbitrary, so we have $Y(\mathfrak{u}_1-\mathfrak{u}_2,\mathfrak{w};\varrho)=0\ \forall\varrho>0$, and therefore $\mathfrak{u}_1-\mathfrak{u}_2=0$. This shows that $\mathfrak{u}_1=\mathfrak{u}_2$.

Hence, in all three cases, we have $\mathfrak{u}_1=\mathfrak{u}_2$, i.e., λ-statistical limit of $\mathfrak{u}=(\mathfrak{u}_k)$ is unique. □

Theorem 3.3 Let $\mathfrak{q}=(\mathfrak{q}_k)$ and $\mathfrak{u}=(\mathfrak{u}_k)$ be two sequences in $N-2-NS$ $\mathfrak{U}$ s.t. $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}\mathfrak{q}_k=\mathfrak{q}_0$ and $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}\mathfrak{u}_k=\mathfrak{u}_0$. Then
(i) $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}(\mathfrak{q}_k+\mathfrak{u}_k)=\mathfrak{q}_0+\mathfrak{u}_0$.
(ii) $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}(\mathfrak{a}\mathfrak{q}_k)=\mathfrak{a}\mathfrak{q}_0$, where $\mathfrak{a}$ is any scalar.
Proof. The proof can be obtained analogously as the proof of theorem 3.2.□

Theorem 3.4 A sequence $\mathfrak{u}=(\mathfrak{u}_k)$ in $N-2-NS$ $\mathfrak{U}$ is $\mathcal{S}_\lambda$-convergent to $\mathfrak{u}_0$, iff $\exists$ a subset $\mathfrak{R}=\{k_n:n\in\mathbb{N}\}$ of $\mathbb{N}$ with $\delta_\lambda\{\mathfrak{R}\}=1$ and $N_2-\lim_{n\to\infty}\mathfrak{u}_{k_n}=\mathfrak{u}_0$.
Proof. First assume that $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}\mathfrak{u}_k=\mathfrak{u}_0$. For $\varrho>0$, $\mathfrak{j}\in\mathbb{N}$ and $\mathfrak{w}\in\mathfrak{U}$, define the set

$$\mathfrak{R}_{N_2}(\mathfrak{j},\varrho)=\{k\in I_n: G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)>1-\frac{1}{\mathfrak{j}} \text{ and}$$
$$B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\frac{1}{\mathfrak{j}}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\frac{1}{\mathfrak{j}}\}.$$

Since $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}\mathfrak{u}_k=\mathfrak{u}_0$, it is clear that for $\varrho>0$ and $\mathfrak{j}\in\mathbb{N}$, $\mathfrak{R}_{N_2}(\mathfrak{j}+1,\varrho)\subset\mathfrak{R}_{N_2}(\mathfrak{j},\varrho)$ and

$$\delta_\lambda(\mathfrak{R}_{N_2}(\mathfrak{j},\varrho))=1. \tag{5}$$

Let $m_1 \in \mathfrak{R}_{N_2}(1,\varrho)$. Then, $\exists\ m_2 \in \mathfrak{R}_{N_2}(2,\varrho),(m_2 > m_1)$, such that for all $n \geq m_2$, $\frac{1}{\lambda_n}|\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{2}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{2}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{2}\}| > \frac{1}{2}$. Similarly, $\exists\ m_3 \in \mathfrak{R}_{N_2}(3,\varrho),(m_3 > m_2)$, such that for all $n \geq m_3$, $\frac{1}{\lambda_n}|\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{3}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{3}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{3}\}| > \frac{2}{3}$ and so on. Thus, we can set a sequence $(m_\mathfrak{j})_{\mathfrak{j}\in\mathbb{N}}$ s.t $m_\mathfrak{j} \in \mathfrak{R}_{N_2}(\mathfrak{j},\varrho)$ and $\forall\ n \geq m_\mathfrak{j}(\mathfrak{j} \in \mathbb{N})$, $\frac{1}{\lambda_n}|\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{\mathfrak{j}}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}}\}| > \frac{\mathfrak{j}-1}{\mathfrak{j}}$. Now define $\mathfrak{R} = \{n \in \mathbb{N} : 1 < n < m_1\} \cup \{\bigcup_{\mathfrak{j}\in\mathbb{N}}\{n \in \mathfrak{R}_{N_2}(\mathfrak{j},\varrho) : m_\mathfrak{j} \leq n < m_{\mathfrak{j}+1}\}\}$. Then for $m_\mathfrak{j} \leq n < m_\mathfrak{j} + 1$, we have $\frac{1}{\lambda_n}|\{k \in I_n : k \in \mathfrak{R}\}| \geq \frac{1}{\lambda_n}|\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{\mathfrak{j}}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}}\}| > \frac{\mathfrak{j}-1}{\mathfrak{j}}$ and hence, $\delta_\lambda(\mathfrak{R}) = 1$ as $k \to \infty$. Now we have to demonstrate that $N_2 - \lim_{n\to\infty} \mathfrak{u}_{k_n} = \mathfrak{u}_0$. Let $\mathfrak{y} > 0$ and select $\mathfrak{j} \in \mathbb{N}$ s.t $\frac{1}{\mathfrak{j}} < \mathfrak{y}$. Furthermore, let $n \geq m_\mathfrak{j}$ and $n \in \mathfrak{R}$. Then, by definition of $\mathfrak{R}$, $\exists\ \mathfrak{k} \geq \mathfrak{j}$ s.t, $m_\mathfrak{k} \leq n < m_{\mathfrak{k}+1}$ and $n \in \mathfrak{R}_{N_2}(\mathfrak{j},\varrho)$. Thus, for each $\mathfrak{y} > 0$, and for $\mathfrak{w} \in \mathfrak{U}$, we have $G(\mathfrak{u}_n - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{\mathfrak{j}} > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_n - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}} < \mathfrak{y}$, $Y(\mathfrak{u}_n - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}} < \mathfrak{y}\ \forall\ n \geq m_\mathfrak{j}$ and $n \in \mathfrak{R}$. Hence $N_2 - \lim_{n\to\infty} \mathfrak{u}_{k_n} = \mathfrak{u}_0$.

Conversely, suppose $\exists$ a subset $\mathfrak{R} = \{k_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ with $\delta_\lambda\{\mathfrak{R}\} = 1$ and $N_2 - \lim_{n\to\infty} \mathfrak{u}_{k_n} = \mathfrak{u}_0$. Let $\mathfrak{y} > 0$ or $\varrho > 0$, $\exists\ k_{n_0} \in \mathbb{N}$ s.t $G(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}$, $Y(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\ \forall\ k_n \geq k_{n_0}$ and every $\mathfrak{w} \in \mathfrak{U}$. This implies $\mathfrak{D}_{N_2}(\mathfrak{y},\varrho) = \{k \in I_n : G(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}$ or $B(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}$, $Y(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\} \subseteq \mathbb{N} - \{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\}$. and therefore $\delta_\lambda\{\mathfrak{D}_{N_2}(\mathfrak{y},\varrho)\} \leq\ \delta_\lambda\{\mathbb{N}\} - \delta_\lambda\{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\} = 1 - 1 = 0$. This shows that $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$. □

4. V_λ-summability in N-2-NS

Definition 4.1 A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $N - 2 - NS\ \mathfrak{U}$ is called V_λ-summable to $\mathfrak{u}_0$ w.r.t the neutrosophic 2-norm N_2 if for each $\mathfrak{y} > 0, 0 < \varrho < 1$ and $\mathfrak{w} \in \mathfrak{U}$

$$G\left((\frac{1}{\lambda_n}\sum_{k\in I_n}\mathfrak{u}_k) - \mathfrak{u}_0, \mathfrak{w}; \varrho\right) > 1 - \mathfrak{y} \text{ and}$$

$$B\left((\frac{1}{\lambda_n}\sum_{k\in I_n}\mathfrak{u}_k) - \mathfrak{u}_0, \mathfrak{w}; \varrho\right) < \mathfrak{y}; Y\left((\frac{1}{\lambda_n}\sum_{k\in I_n}\mathfrak{u}_k) - \mathfrak{u}_0, \mathfrak{w}; \varrho\right) < \mathfrak{y}.$$

In present case, we denote $V_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$ or $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2))$.

Theorem 4.1 Let $\lambda = (\lambda_n)$ as defined above and $\mathfrak{u} = (\mathfrak{u}_k)$ be a sequence in $N - 2 - NS\ \mathfrak{U}$ then

(I) $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2)) \Rightarrow \mathfrak{u}_k \to \mathfrak{u}_0\big(\mathcal{S}_\lambda(N_2)\big)$ and the inclusion $V_\lambda(N_2) \subseteq \mathcal{S}_\lambda(N_2)$ is proper.

(II) If $\mathfrak{u} \in l_\infty(\mathfrak{U})$ and $\mathfrak{u}_k \to \mathfrak{u}_0\big(\mathcal{S}_\lambda(N_2)\big)$, then $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2))$.

(III) $\mathcal{S}_\lambda(N_2) \cap l_\infty(\mathfrak{U}) = V_\lambda(N_2) \cap l_\infty(\mathfrak{U})$, where $l_\infty(\mathfrak{U})$ is the space of all bounded sequences in $\mathfrak{U}$.

Proof. (I) Let $\mathfrak{y} > 0$ and $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2))$. We have,

$$\begin{aligned}
&\sum_{k\in I_n}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)\\
=&\sum_{\substack{k\in I_n\\ G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or}\\ B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}}}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)\\
+&\sum_{\substack{k\in I_n\\ G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)> 1-\mathfrak{y} \;\&\\ B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\mathfrak{y}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\mathfrak{y}}}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ and } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)\\
\ge&\sum_{\substack{k\in I_n\\ G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or}\\ B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}}}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)\\
&\ge \mathfrak{y}.\Big|\big\{k\in I_n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}\big\}\Big|.
\end{aligned}$$

Since $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2))$. Therefore, it follows that $\mathfrak{u}_k \to \mathfrak{u}_0\big(\mathcal{S}_\lambda(N_2)\big)$.

In order to show that the containment $V_\lambda(N_2) \subseteq \mathcal{S}_\lambda(N_2)$ is proper. We define a sequence $\mathfrak{u} = (\mathfrak{u}_k)$ by

$$\mathfrak{u}_k = \begin{cases} (k,0), & \text{for } n-[\sqrt{\lambda_n}]+1 \le k \le n,\\ (0,0), & \text{otherwise}\end{cases}$$

It is obvious that the sequence $(\mathfrak{u}_k)$ is unbounded. Then for each $\mathfrak{y} \in (0,1)$ and $\varrho > 0$ we have

$$\begin{aligned}
\frac{1}{\lambda_n}\Bigg|\Bigg\{k\in I_n : G(\mathfrak{u}_k-0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or}&\\
B(\mathfrak{u}_k-0,\mathfrak{w};\varrho)\ge\mathfrak{y}, Y(\mathfrak{u}_k-0,\mathfrak{w};\varrho)\ge\mathfrak{y}\Bigg\}\Bigg|&\\
=\frac{[\sqrt{\lambda_n}]}{\lambda_n}\to 0 \text{ as } n\to\infty.&
\end{aligned}$$

i.e., $\mathfrak{u}_k \to 0\big(\mathcal{S}_\lambda(N_2)\big)$.

Further,

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\Bigg(G(\mathfrak{u}_k-0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-0,\mathfrak{w};\varrho)\Bigg)\to\infty \text{ as } n\to\infty.$$

This implies that $\mathfrak{u}_k \nrightarrow 0(V_\lambda(N_2))$.

(II) Let $\mathfrak{u} = (\mathfrak{u}_k) \in l_\infty(\mathfrak{U})$ and $\mathfrak{u}_k \to \mathfrak{u}_0\big(\mathcal{S}_\lambda(N_2)\big)$. Then $\exists\, M > 0$ s.t $G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \ge 1-M$ or $B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \le M, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \le M\ \forall\, k$. Let $\mathfrak{y} > 0$ be arbitrary selected, now as in case (I) we can write

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)$$

$$=\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\leq 1-\mathfrak{y} \text{ or}\\ B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y}}}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)$$

$$+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)>1-\mathfrak{y} \ \&\\ B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\mathfrak{y}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\mathfrak{y}}}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ and } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)$$

$$\leq \frac{M}{\lambda_n}\left|\left\{k\in I_n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\leq 1-\mathfrak{y} \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y}\right\}\right|+\mathfrak{y},$$

which shows that $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2))$.

(III) Follows easily from part (I) and part (II). □

Theorem 4.2 Let $\mathfrak{u} = (\mathfrak{u}_k)$ be any sequence in $N-2-NS$ $\mathfrak{U}$. Then $\mathcal{S}(N_2) \subset \mathcal{S}_\lambda(N_2)$ if

$$\liminf_{n\to\infty}\frac{\lambda_n}{n} > 0. \tag{6}$$

Proof. Given $\mathfrak{y} > 0$ and $\varrho > 0$, we have

$$\begin{aligned}&\big\{k\leq n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\leq 1-\mathfrak{y} \text{ or}\\ &\qquad B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y},\ Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y}\big\}\\ &\supseteq \big\{k\in I_n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\leq 1-\mathfrak{y} \text{ or}\\ &\qquad B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y},\ Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y}\big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{n}\big|\big\{k\leq n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\leq 1-\mathfrak{y} \text{ or}\\ &\qquad B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y},\ Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y}\big\}\big|\\ &\geq \frac{1}{n}\big|\big\{k\in I_n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\leq 1-\mathfrak{y} \text{ or}\\ &\qquad B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y},\ Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y}\big\}\big|\\ &\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}\big|\big\{k\in I_n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\leq 1-\mathfrak{y} \text{ or}\\ &\qquad B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y},\ Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\geq\mathfrak{y}\big\}\big|.\end{aligned}$$

Taking $n\to\infty$ and using (6), we get $\mathfrak{u}_k \to \mathfrak{u}_0\big(\mathcal{S}(N_2)\big) \Rightarrow \mathfrak{u}_k \to \mathfrak{u}_0\big(\mathcal{S}_\lambda(N_2)\big)$.

5. **λ-Statistical completeness in N-2-NS**

Definition 5.1 A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $N-2-NS$ $\mathfrak{U}$ is called λ-statistically Cauchy(or $\mathcal{S}_\lambda$-Cauchy) if for each $\mathfrak{y} > 0$ and $\varrho > 0$, $\exists$ $p \in$ $\mathbb{N}$ s.t.

$$\lim_n \frac{1}{\lambda_n}|\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or } B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y},\ Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}\}| = 0\ \forall\ \mathfrak{w} \in \mathfrak{U}$$

or equivalently, $\delta_\lambda(\mathcal{A}(\mathfrak{y}, \varrho)) = 0$, where

$$\mathcal{A}(\mathfrak{y}, \varrho) = \{k \in I_n :\ G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or } B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y},\ Y\ (\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}\}$$

Theorem 5.1 Every $\mathcal{S}_\lambda$-convergent sequence in $N-2-NS$ $\mathfrak{U}$ is $\mathcal{S}_\lambda$-Cauchy.

Proof. Let $\mathfrak{u} = (\mathfrak{u}_k)$ be a $\mathcal{S}_\lambda$-convergent sequence with $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$. Let $\mathfrak{y} > 0$ and $\varrho > 0$. Choose $l > 0$ s.t. (1) is satisfied. Define a set,

$$\mathcal{A}(l,\ \varrho) = \{k \in I_n :\ G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w};\ \frac{\varrho}{2}) \leq 1 - l \text{ or } B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l\ \ Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l\},$$

then $\delta_\lambda(\mathcal{A}(l, \varrho)) = 0$ and therefore $\delta_\lambda(\mathcal{A}^C(l, \varrho)) = 1$. Let $p \in \mathcal{A}^C(l, \varrho)$ then for $\mathfrak{w} \in \mathfrak{U}$, we have $G(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l$ and $B(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < l, Y(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < l$.

Now, let $\mathcal{T}(\mathfrak{y}, \varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}$ or $B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}, Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}\}$. We claim that $\mathcal{T}(\mathfrak{y}, \varrho) \subset \mathcal{A}(l, \varrho)$. Let $m \in \mathcal{T}(\mathfrak{y}, \varrho)$, then we have $G(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}$ or $B(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}, Y(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}$.

Case (i): Suppose $G(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}$, then $G(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \leq 1 - l$ and therefore $m \in \mathcal{A}(l, \varrho)$. As otherwise, i.e, if $G(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l$, then

$$\begin{aligned} 1 - \mathfrak{y} \geq G(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) &\geq G(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \circ G(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \\ &> (1 - l) \circ (1 - l) \\ &> 1 - \mathfrak{y}.\text{(not possible)} \end{aligned}$$

Thus, $\mathcal{T}(\mathfrak{y}, \varrho) \subset \mathcal{A}(l, \varrho)$.

Case (ii): Suppose $B(\mathfrak{u}_m - v_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}$, then $B(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l$ and therefore $m \in \mathcal{A}(l, \varrho)$. As otherwise, i.e, if $B(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < l$, then

$$\begin{aligned} \mathfrak{y} \leq B(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) &\leq B(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \diamond B(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \\ &< l \diamond l \\ &< \mathfrak{y}.\text{(not possible)} \end{aligned}$$

Also, suppose $Y(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}$, then $Y(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l$ and therefore $m \in \mathcal{A}(l, \varrho)$. As otherwise, i.e, if $B(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < l$, then

$$\begin{aligned} \mathfrak{y} \leq Y(\mathfrak{u}_m - v_p, \mathfrak{w}; \varrho) \leq Y(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \diamond Y(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \\ < l \diamond l \\ < \mathfrak{y}.\text{(not possible)} \qquad . \end{aligned}$$

Thus, $\mathcal{T}(\mathfrak{y}, \varrho) \subset \mathcal{A}(l, \varrho)$.
Hence in all cases, $\mathcal{T}(\mathfrak{y}, \varrho) \subset \mathcal{A}(l, \varrho)$. Since $\delta_\lambda(\mathcal{A}(l, \varrho)) = 0$, so $\delta_\lambda(\mathcal{T}(\mathfrak{y}, \varrho)) = 0$ and therefore $\mathfrak{u} = (\mathfrak{u}_k)$ is λ-statistically Cauchy. □

Definition 5.2 A neutrosophic 2-normed space $\mathfrak{U}$ is called $\mathcal{S}_\lambda$-complete if every $\mathcal{S}_\lambda$-Cauchy sequence in $\mathfrak{U}$ is $\mathcal{S}_\lambda$-convergent in $\mathfrak{U}$.

Theorem 5.2 Every $N-2-NS$ $\mathfrak{U}$ is $\mathcal{S}_\lambda$-complete.
Proof. Let $\mathfrak{u} = (\mathfrak{u}_k)$ be $\mathcal{S}_\lambda$-Cauchy sequence in $\mathfrak{U}$. Suppose on the contrary that $\mathfrak{u} = (\mathfrak{u}_k)$ is not $\mathcal{S}_\lambda$-convergent. Let $\mathfrak{y} > 0$ and $\varrho > 0$, then $\exists\, p \in \mathbb{N}$ s.t $\mathfrak{w} \in \mathfrak{U}$ if we define

$$\begin{aligned} \mathcal{A}(\mathfrak{y}, \varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or} \\ B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}, Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}\} \text{ and} \end{aligned}$$

$$\begin{aligned} \mathcal{T}(\mathfrak{y}, \varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - \mathfrak{y} \text{ and} \\ B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \mathfrak{y}, Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \mathfrak{y}\}, \end{aligned}$$

then $\delta_\lambda(\mathcal{A}(\mathfrak{y}, \varrho)) = \delta_\lambda(\mathcal{T}(\mathfrak{y}, \varrho)) = 0$ and therefore we have $\delta_\lambda(\mathcal{A}^C(\mathfrak{y}, \varrho)) = \delta_\lambda(\mathcal{T}^C(\mathfrak{y}, \varrho)) = 1$. Since $G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq 2G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 2B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \mathfrak{y}$, $Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 2Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \mathfrak{y}$, if $G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) > \frac{1-\mathfrak{y}}{2}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \frac{\mathfrak{y}}{2}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \frac{\mathfrak{y}}{2}$. We have $\delta_\lambda(\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) < \mathfrak{y}$, $Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) < \mathfrak{y}\}) = 0$. i.e., $\delta_\lambda(\mathcal{A}^C(\mathfrak{y}, \varrho)) = 0$ which contradicts the fact that $\delta_\lambda(\mathcal{A}^C(\mathfrak{y}, \varrho)) = 1$. Therefore, $\mathfrak{u} = (\mathfrak{u}_k)$ is $\mathcal{S}_\lambda$-convergent and Hence $\mathcal{S}_\lambda$-complete. □

Theorem 5.3 For any sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $N-2-NS$ $\mathfrak{U}$, the subsequent assertions are equivalent:
(i) $\mathfrak{u} = (\mathfrak{u}_k)$ is $\mathcal{S}_\lambda$-Cauchy.
(ii) $\exists$ a subset $\mathfrak{R} = \{k_n\}$ of $\mathbb{N}$ with $\delta_\lambda\{\mathfrak{R}\} = 1$ and $\{\mathfrak{u}_{k_n}\}_{n\in\mathbb{N}}$ is $\mathcal{S}_\lambda$-Cauchy sequence over $\mathfrak{R}$.
Proof. The proof of the theorem can be obtained analogously as the proof of the theorem 3.4.

Aknowledgement : The authors express their gratitude to the reviewers for their valuable comments and suggestions towards the quality improvement of the paper.

References

[1] Atanassov, K.T., *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1), (1986), 87–96.

[2] Antal R., Chawla M., Kumar V., *Statistical* Λ*-Convergence in Intuitionistic fuzzy normed spaces*, Buletinul Academiei de Stiinte a Moldovei Mathematica, **3**(91), (2019), 22-33.

[3] Connor, J., *The statistical and strong p-Cesaro convergence of sequences.* Analysis,**8**(1-2), (1988), 47-64.

[4] Fast, H., *Sur la convergence statistique.* In Colloquium mathematicae, **2**(3-4), (1951), 241-244.

[5] Felbin, C., *Finite dimensional fuzzy normed linear spaces.* Fuzzy Sets System, **48**, (1992), 239–248.

[6] Fridy, J. A., *On statistical convergence.* Analysis, **5**(4), (1985) 301-314.

[7] Gahler, S., *2-metrische Räume und ihre topologeische Struktur*, Math. Nachr., **26** (1963), 115–148.

[8] Hazarika, B.; Savaş, E., λ*-statistical convergence in n-normed spaces.* Analele ştiinţifice ale Universităţii" Ovidius" Constanţa. Seria Matematică, **21**(2), (2013), 141-153.

[9] Hazarika, B., Kumar, V., Lafuerza-Guillén, B., *Generalized ideal convergence in intuitionistic fuzzy normed linear spaces.* Filomat, **27**, (2013), 811-820.

[10] Harnpornchai, N., Wonggattaleekam, W., *An Application of Neutrosophic Set to Relative Importance Assignment in AHP.* Mathematics, **9**(20), (2021), 2636.

[11] Karakus, S., Demirci, K., Duman, O., *Statistical convergence on intuitionistic fuzzy normed spaces.* Chaos, Solitons & Fractals, **35**, (2008), 763–769.

[12] Khan, V. A., Khan, M. D., Ahmad, M., *Some new type of lacunary statistically convergent sequences in neutrosophic normed space.* Neutrosophic Sets and Systems, **42**(1), (2021), 15.

[13] Kirişci, M., Şimşek, N., *Neutrosophic normed spaces and statistical convergence.* The Journal of Analysis, **28**, (2020), 1059-1073.

[14] Kumar, K., Kumar, V., *On the* I *and* I^*-*convergence of sequences in fuzzy normed spaces.* Advances in Fuzzy Sets, **3**, (2008), 341-365.

[15] Kumar. V., Mursaleen, M., *On* (λ, μ)*-Statistical convergence of double sequences on intuitionistic fuzzy normed spaces*, Filomat, **25**(2), (2011), 109–120.

[16] Kumar, V., *On* Λ*- statistical limit points in probabilistic normed spaces*, Atti della Fondazione Giorgio Ronchi, **65**(2),(2010), 1-10.

[17] Kumar, P., Kumar, V., Batia, S. S., *On* Λ*- statistical limit inferior and limit superior of order* β *for sequences of fuzzy numbers*, Gen. **25**(2), (2014), 6-18.

[18] Koundal, D., Gupta, S., Singh, S., *Applications of neutrosophic sets in medical image denoising and segmentation.* Infinite Study. (2016).

[19] Leindler, L., *Über die de la Vallée-Pousinsche Summierbarkeit allgen- meiner Othogonalreihen*, Acta Math. Acad. Sci. Hungar, **16**, (1965), 375- 387.

[20] Maddox, I. J., *Statistical convergence in a locally convex space.* In Mathematical Proceedings of the Cambridge Philosophical Society, **104**(1), (1988), 141-145.

[21] Mursaleen, M., λ*-statistical convergence.* Math Slovaca., **50**(1), (2000), 111–115.

[22] Mursaleen, M., Danish-Lohani, Q.M. *Intuitionistic fuzzy 2-normed space and some related concepts.*Chaos, Solitons & Fractals, **42**, (2009), 224–234.

[23] Majumdar, P., *Neutrosophic sets and its applications to decision making.* In Computational intelligence for big data analysis (2015), (pp. 97-115). Springer, Cham.

[24] Murtaza, S., Sharma, A., and Kumar, V., *Neutrosophic 2-normed space and generalized summability.* Neutrosophic sets and system. **55**(1), (2023), 414-426

[25] Meenakshi, C., Kumar, V., Saroa, M. S., *Some remarks on statistical summability of order* $\tilde{\alpha}$ *defined by generalized De la Vallée Poussin mean*, Boletim da Sociedade Paranaense de Mathematica, **33**(1),(2015), 147-156.

[26] Meenakshi, C., Saroa, M. S., Kumar, V., *On* Λ*-statistical convergence of order* α *in random 2-normed space*, Miskolc Mathematical Notes, **16**(2), (2015), 1003-1015.

[27] Park, J. H., *Intuitionistic fuzzy metric spaces.* Chaos, Solitons & Fractals, **22**, (2004), 1039-46.

[28] Praveena, U., Jeyaraman, M., *On Generalized Cesaro Summability Method In Neutrosophic Normed Spaces Using Two-Sided Taubarian Conditions.* Journal of algebraic statistics. **13**(3), (2022), 1313-1323.

[29] Parimala, M., Smarandache, F., Jafari, S., Udhayakumar, R., *On Neutrosophic* $\alpha\psi$*-closed sets.* information, **9**(5),(2018), 103.

[30] Parimala, M., Karthik, M., Jafari, S., Smarandache, F., Udhayakumar, R., *Decision-Making via Neutrosophic support soft Topological spaces*, Symmetry, **10**(6),(2018),217.

[31] Parimala, M., Karthik. M., Jafari, S., Smarandache, F., Udhayakumar, R., *Neutrosophic Nano ideal topological structure*, Neutrosophic sets and systems **24**, (2019),pp 70-76.

[32] Reena, A., Meenakshi, C., Vijay, K., *Some Remarks On Rough Statistical* Λ*-convergence of order* α.Ural Mathematical Journal, **7**(1), (2021),16-25.

[33] Schoenberg, I. J., *The integrability of certain functions and related summability methods.* The American mathematical monthly. **66**(5), (1959), 361-775.

[34] Saadati, R., Park, J. H., *On the Intuitionistic fuzzy topological spaces.* Chaos, Solitons & Fractals, **27**, (2006) 331-44.

[35] Smarandache, F., *Neutrosophic set, a generalisation of intuitionistic fuzzy sets*, Int J Pure Appl Math **24** (2005), 287–297.

[36] Šalát, T., *On statistically convergent sequences of real numbers.* Mathematica slovaca. **30**(2), (1980), 139-150.

[37] Sharma, A., Kumar, V., *Some remarks on generalized summability using difference operators on neutrosophic normed spaces*, J. of Ramanujan Society of Mathematics and Mathematical Sciences. **9**(2) (2022) 53-164.

[38] Sharma A. and Kumar V., *On generalized asymptotically equivalent double sequences through (V,*λ*,*μ*)-summability*, series Mathematics,(2013),3.

[39] Sharma A., Kumar, V., Ganaie, I. R., *Some remarks on* $I(S_\theta)$*-summability via neutrosophic norm*, Filomat., **37**(20), (2023), 6699-6707.

[40] Schweizer, B., Sklar, A., *Statistical metric spaces*, Pacific J. Math. **10**(1), (1960) 313-334.

[41] Zadeh, L. A., *Fuzzy sets.* Information Control, **8**, (1965) 338-53.

Received: Oct 9, 2023. Accepted: Jan 6, 2024