NSS

Neutrosophic Sets and Systems, Vol. 63, 2024

University of New Mexico

Neutrosophic Automata and Reverse Neutrosophic Automata

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Abstract. This research is dedicated to exploring the relationships among neutrosophic automata, reverse neutrosophic automata, and double neutrosophic automata. Through the utilization of these three automata, we establish definitions for a neutrosophic subsystem, a reverse neutrosophic subsystem, and a double neutrosophic subsystem, delving into various properties associated with them. Additionally, we aim to introduce the notion of categorical aspects concerning neutrosophic automata and reverse neutrosophic automata, along with their functorial relationship.

Keywords: neutrosophic automata; reverse neutrosophic automata; neutrosophic subsystem; reverse neutrosophic subsystem; category.

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1. **Introduction**

The field of automata theory has proven instrumental in addressing computational complexity issues, finding applications across computer science and discrete mathematics. Following Zadeh's [75] introduction of fuzzy set theory, scholars such as Wee [72] and Santos [52] initiated the exploration of fuzzy automata and languages to bridge the gap between the precision of computer languages and inherent vagueness. Malik and collaborators [32, 38] introduced a simpler notion of a fuzzy finite state machine, laying the groundwork for the algebraic study of fuzzy automata and languages. Numerous researchers (cf., e.g., [5–7,14–19,25,27,30,35–37,46–48,56–63,66,68,69,76]) have contributed to the development of fuzzy automata theory, with diverse focuses. Among these works, Jin and colleagues [17] delved into the algebraic study of fuzzy automata based on po-monoids, while Kim, Kim, and Cho [25] concentrated on the algebraic aspects of fuzzy automata theory. Moćkor [35–37] explored categorical concepts in fuzzy automata theory, and Abolpour and Zahedi [5–7] applied categorical concepts to automata with membership values in various lattice structures. The work of Qiu [46–48], Tiwari and their co-authors [62, 63, 66, 68, 69] pursued algebraic, topological, and categorical studies of fuzzy automata theory based on different lattice structures. Ignjatovic and collaborators [14] investigated the notion of determinism in fuzzy automata, while Anupam and co-authors [55–61, 64, 65] explored the topological, algebraic, and categorical aspects of more generalized fuzzy automata and fuzzy languages. These collective contributions reflect the rich and diverse landscape of research in fuzzy automata theory.

Recent advancements in fuzzy automata theory are highlighted in various works, including [7, 42, 61, 67]. Fuzzy automata find practical applications in engineering contexts, particularly in areas such as information representation, pattern recognition, and machine learning systems, as discussed in [38, 43, 44, 73]. Notably, [73] proposes a non-supervised learning scheme for automatic control and pattern recognition, emphasizing the simplicity in design and computation offered by fuzzy automata as a machine learning model.

In addressing computational uncertainty, alternative mathematical tools have emerged, such as bipolar-valued fuzzy sets [31], vague sets [12], and cubic sets [20]. The generalization trend of fuzzy sets has led to the development of neutrosophy, a philosophical branch introduced and studied by Florientin Samrandache [53, 54]. Neutrosophy serves as a method for handling the computational uncertainty inherent in real-life and scientific problems. Unlike fuzzy sets, neutrosophic sets introduced by Samrandache have three independent components: the degree of membership, the degree of non-membership, and the degree of indeterminacy. Although neutrosophic sets may pose challenges in practical engineering and scientific applications, Wang et al. [70, 71] have introduced the concepts of single-valued neutrosophic sets and interval neutrosophic sets as a more manageable instance of neutrosophic sets. From a practical perspective, neutrosophic set theory has demonstrated substantial success in various fields, including topology [13,41], control theory [39,40], decision-making problems [1,3,26,50], medical diagnosis [1, 51, 74], financial management [2], and smart product-service systems [4]. Neutrosophic automata, a more recent model stemming from fuzzy automata theory, has garnered attention from numerous researchers who have extensively explored automata theory within a neutrosophic framework [21–24, 33, 34]. Neutrosophic automata offer a valuable environment for handling ambiguous computations and have demonstrated their significance in addressing substantial challenges in learning management systems [49], topology [13, 41], and algebraic structures [21–24, 33], among other applications. The concept of category theory, initially introduced by Eilenberg and Mac Lane [10], is widely recognized. Subsequent development by various researchers [11, 28, 29] has showcased its utility in advancing theoretical computer science aspects, such as the design of functional and imperative programming languages, semantic models of programming languages, algorithm development, and polymorphism [45].

1.1. *Motivation*

Various researchers have integrated neutrosophic set theory into automata theory in different ways. However, there is a notable gap in exploring the algebraic properties of automata and reverse automata within a neutrosophic environment, particularly considering t-norm and implication operators. Additionally, the application of category theory and functors between neutrosophic automata and reverse neutrosophic automata remains unexplored. This paper aims to fill these gaps by investigating and introducing the algebraic properties of neutrosophic automata, incorporating a t-norm and implication operator. Furthermore, we present fundamental properties of category theory and explore functors connecting neutrosophic automata with reverse neutrosophic automata.

The paper's structure is outlined as follows:
Section 2: Provides an introduction to the paper's content.
Section 3: Introduces and explores the concepts of neutrosophic automata, reverse neutrosophic automata, as well as subsystems (including reverse and double subsystems) for neutrosophic automata within a neutrosophic environment. This section also delves into presenting various algebraic properties associated with neutrosophic automata.
Section 4: Focuses on the introduction and examination of homomorphism and strong homomorphism between neutrosophic automata, considering specific properties as their basis. Also, proposes categorical and functorial properties of both neutrosophic automata and reverse neutrosophic automata.
Section 5: The article ends with conclusion.

2. **Preliminaries**

Within this section, we revisit fundamental notations and concepts associated with neutrosophic sets, including neutrosophic t-norms, implication operators, and category theory. The foundation for understanding neutrosophic sets is drawn from the works of [53, 54], while the principles of categories and functors are referenced from [8,9]. The discussion commences with the following points.

Definition 2.1. A **neutrosophic set (NS, in short)** A on a non-empty set X is an object having the form $A = \{< b_1, F_A(b_1), G_A(b_1), H_A(b_1) >: b_1 \in X\}$, where the functions $F_A, G_A, H_A : X \to]0^-, 1^+[$ define respectively the degree of membership (or truth), the degree of indeterminacy and the degree of non-membership (or false) of each element $b_1 \in X$ to the set A . As, the sum of $F_A(b_1), G_A(b_1), H_A(b_1)$, have no restriction. So for each $b_1 \in X, 0^- \leq F_A(b_1) + G_A(b_1) + H_A(b_1) \leq 3^+$.

Remark 2.2. A Neutrosophic Set $A = < b_1, F_A(b_1), G_A(b_1), H_A(b_1) >: b_1 \in X$ is typically denoted as an ordered triple $< F_A, G_A, H_A >$ in the non-standard unit interval $]0^-, 1^+[$ on X. The neutrosophic sets (**NSs, in short)** $\mathbf{0_N}$ and $\mathbf{1_N}$ represent constant NSs in X and are defined as $0_N =< 0, 1, 1 >$ and $1_N =< 1, 0, 0 >$, where $0, 1 : X \to]0^-, 1^+[$ are defined respectively by $\mathbf{0}(\mathbf{b_1}) = \mathbf{0}$ and $\mathbf{1}(\mathbf{b_1}) = \mathbf{1}$. The NS $\eta = (\sigma, \beta, \gamma)$ such that $\widehat{\eta} = \widehat{(\sigma, \beta, \gamma)}$ is expressed as $\widehat{\eta}(b_1) = \eta$ for all $b_1 \in X$, where σ, β, and γ are the σ-valued, β-valued, and γ-valued constant neutrosophic sets in X respectively, with the condition $0^- \leq \sigma + \beta + \gamma \leq 3^+$.

This paper opts for the interval $[0, 1]$ instead of the notation $]0^-, 1^+[$ in consideration of practical applications, as the latter might pose challenges in real-world scenarios. Also, $\mathbf{NS(X)}$ will denote the family of all neutrosophic sets in X and I^* denotes the set $\{(b_1, b_2, b_3) : ((b_1, b_2, b_3) \in [0, 1] \times [0, 1] \times [0, 1], 0 \leq b_1 + b_2 + b_3 \leq 3\}$. A neutrosophic set $A =< F_A, G_A, H_A >$ in X will frequently be viewed as a function $A : X \to I^*$, given by $A(b_1) = \{F_A(b_1), G_A(b_1), H_A(b_1) : b_1 \in X\}$.

Firstly, we recall some basic properties of NS in X.

Definition 2.3. For NSs $A =< F_A, G_A, H_A >, B =< F_B, G_B, H_B >$ and $A_i =< F_{A_i}, G_{A_i}, H_{A_i} >, i \in J$ in $b_1 \in X$. We have

(1) $A \leq B$ if $F_A(b_1) \leq F_B(b_1), G_A(b_1) \geq G_B(b_1)$ and $H_A(b_1) \geq H_B(b_1)$;
(2) $\vee_{i\in J} A_i(b_1) = (\vee_{i\in J} F_{A_i}(b_1), \wedge_{i\in J} G_{A_i}(b_1), \wedge_{i\in J} H_{A_i}(b_1))$;
(3) $\wedge_{i\in J} A_i(b_1) = (\wedge_{i\in J} F_{A_i}(b_1), \vee_{i\in J} G_{A_i}(b_1), \vee_{i\in J} H_{A_i}(b_1))$;
(4) $A^c = (1 - F_A, 1 - G_A, 1 - H_A)$;
(5) $0_N \subseteq A \subseteq 1_N; 0_N^c = 1_N$ and $1_N^c = 0_N$;
(6) $A \cup 0_N = A, A \cup 1_N = 1_N$ and $A \cap 0_N = 0_N, A \cap 1_N = A$.

Example 2.4. Let $X = \{b_1, b_2\}, A = \{< b_1, 0.2, 1, 0.3 >, < b_2, 0.4, 0.5, 0.6 >\}$ and $B = \{< b_1, 0.1, 0.3, 0.8 >, < b_2, 0, 0, 0.9 >\}$ are two NSs on X. Then $A \cup B = \{< b_1, 0.2, 0.3, 0.3 >, < b_2, 0.4, 0, 0.6 >\}, A \cap B = \{< b_1, 0.1, 1, 0.8 >, < b_2, 0, 0.5, 0.9 >\}, A^c = \{< b_1, 0.8, 0, 0.7 >, < b_2, 0.6, 0.5, 0.4 >\}, A \cup 1_N = (1, 0, 0) = 1_N, A \cap 0_N = (0, 1, 1) = 0_N$ and $B \cap 1_N = (0.1, 0.3, 0.8) = B$.

Definition 2.5. (1) A neutrosophic t-norm $\otimes : I^* \times I^* \longrightarrow I^*$ be a mapping such for all $\sigma_N = (\sigma_1, \sigma_2, \sigma_3), \beta_N = (\beta_1, \beta_2, \beta_3), \gamma_N = (\gamma_1, \gamma_2, \gamma_3), \delta_N = (\delta_1, \delta_2, \delta_3) \in I^*$ which satisfies

(i) $\sigma_N \otimes 1_N = \sigma_N$ (border condition);

(ii) $\sigma_N \otimes \beta_N = \beta_N \otimes \sigma_N$, (commutativity);

(iii) $\sigma_N \otimes (\beta_N \otimes \gamma_N) = (\sigma_N \otimes \beta_N) \otimes \gamma_N$, (associativity);

(iv) $\sigma_N \leq \beta_N$ and $\gamma_N \leq \delta_N \Rightarrow \sigma_N \otimes \gamma_N \leq \beta_N \otimes \delta_N$, (monotonicity).

(2) The neutrosophic precomplement on I^* is the mapping $\neg : I^* \longrightarrow I^*$ such that $\neg(b_1, b_2, b_3) = (b_1, b_2, b_3) \to 0_N = (b_1, b_2, b_3) \to (0, 1, 1) = (b_1 \to 0, b_2 \leftarrow 1, b_3 \leftarrow 1), \forall b_1, b_2, b_3 \in X$.

(3) The implication operator $\longrightarrow: I^* \longrightarrow I^*$ is defined as;

$$\sigma_N \to \beta_N = \vee\{\gamma_N = (\gamma_1, \gamma_2, \gamma_3) \in I^* : \sigma_N \otimes \gamma_N \leq \beta_N\}, \forall \sigma_N = (\sigma_1, \sigma_2, \sigma_3), \beta_N = (\beta_1, \beta_2, \beta_3) \in I^* \text{ with respect to } \otimes.$$

For $\sigma_N = (\sigma_1, \sigma_2, \sigma_3) \in I^*$ and $A = (F_A, G_A, H_A) \in NS(X)$, the NS $\sigma_N \to A = (\sigma_1 \to F_A, \sigma_2 \leftarrow G_A, \sigma_3 \leftarrow H_A)$ in X is defined as

$$(\sigma_1 \to F_A)(b_1) = \begin{cases} 1 & \text{if } \sigma_1(b_1) \leq F_A(b_1) \\ F_A(b_1) & \text{if } \sigma_1(b_1) > F_A(b_1) \end{cases}$$

$$(\sigma_2 \leftarrow G_A)(b_1) = \begin{cases} 0 & \text{if } \sigma_2(b_1) \geq G_A(b_1) \\ G_A(b_1) & \text{if } \sigma_2(b_1) < G_A(b_1) \end{cases}$$

and

$$(\sigma_3 \leftarrow H_A)(b_1) = \begin{cases} 0 & \text{if } \sigma_3(b_1) \geq H_A(b_1) \\ H_A(b_1) & \text{if } \sigma_3(b_1) < H_A(b_1) \end{cases}$$

$\forall b_1 \in X$.

Proposition 2.6. *Let* $A = (F_A, G_A, H_A) \in NS(X)$ *and* $\sigma_N = (\sigma_1, \sigma_2, \sigma_3), \beta_N = (\beta_1, \beta_2, \beta_3), \gamma_N = (\gamma_1, \gamma_2, \gamma_3) \in I^*$. *Then*

(i) $1_N \to A = (1, 0, 0) \to (F_A, G_A, H_A) = (F_A, G_A, H_A) = A$;

(ii) $\sigma_N \otimes \beta_N \leq \gamma_N \Leftrightarrow \sigma_N \leq \beta_N \to \gamma_N$;

(iii) $(\sigma_N \otimes \beta_N) \to \gamma_N = \sigma_N \to (\beta_N \to \gamma_N)$;

(iv) $(\sigma_N \to \beta_N) \otimes (\beta_N \to \gamma_N) \leq \sigma_N \to \gamma_N$;

(v) $\sigma_N \otimes (\vee_{i \in I} \beta_{N_i}) = \vee_{i \in I}(\sigma_N \otimes \beta_{N_i})$;

(vi) $(\sigma_N \to \beta_N) \otimes \sigma_N \leq \beta_N$;

(vii) $(\sigma_N \otimes \beta_N) \to \gamma_N = (\beta_N \otimes \sigma_N) \to \gamma_N$;

(viii) *if* $\sigma_N \leq \beta_N \Rightarrow \neg\beta_N \leq \neg\sigma_N$.

Definition 2.7. The key component of a **category thoery** T contains:

(i) a **T**- objects;

(ii) For any pair of objects X and Y within the category **T**, there exists a set denoted as $\mathbf{T(X, Y)}$. The members of this set are referred to as **morphisms** (or **T**- morphisms), where each morphism ψ in $\mathbf{T(X, Y)}$ is represented as $\psi : X \to Y$. These morphisms have a specified domain X and codomain Y;

(iii) For every object X within the category **T**, a morphism denoted as $id_X : X \to X$ is termed the **identity morphism** on X; and

(iv) There exists a "composition law" linked to each pair of **T**-morphisms $\psi : X \to Y$ and $\chi : Y \to Z$, a **T**-morphism denoted as $\chi \circ \psi : X \to Z$ is termed the **composition** of ψ and χ, adhering to the following properties:

(a) for any **T**-morphisms $\psi : X \to Y, \chi : Y \to Z$, and $\Phi : Z \to W$, the composition follows the associativity property: $\Phi \circ (\chi \circ \psi) = (\Phi \circ \chi) \circ \psi$.

(b) for any **T**-morphism $\psi : X \to Y$, the identity morphism id_Y satisfies the properties: $id_Y \circ \psi = \psi$ and $\psi \circ id_X = \psi$.

For simplicity, we represent the object-class of the category **T** by **T** itself.

Definition 2.8. A **functor K** : **T** $\to$ **E** is a mapping that assigns each **T**-object X to a **E**-object $K(X)$ and every **T**-morphism $\psi : X \to Y$ to a **E**-morphism $K(\psi) : K(X) \to K(Y)$ follows the conditions that:

(a) For all **T**-morphisms $\psi : X \to Y$ and $\chi : Y \to Z$, $K(\chi \circ \psi) = K(\chi) \circ K(\psi)$, and

(b) For all $X \in \mathbf{T}$, $K(id_X) = id_{K(X)}$.

3. Neutrosophic automata

In this section, we present the concept of neutrosophic automata and reverse neutrosophic automata. The introduction of neutrosophic automata naturally leads to the development of neutrosophic subsystems, including reverse neutrosophic subsystems and double neutrosophic subsystems. Throughout this exploration, we delve into various properties, such as order-preserving maps, involution and some more, associated with these neutrosophic automata and subsystems. The discussion commences with the following points.

Definition 3.1. A **neutrosophic automaton, (NA, in short)** is a triple $L = (Q, X, \delta)$, where Q and X are non-empty sets referred to as the set of states and the set of inputs (with the identity denoted as e), respectively. The neutrosophic transition function is denoted as $\delta = (F_\delta, G_\delta, H_\delta)$ and is a neutrosophic subset of $Q \times X \times Q$. In other words, δ is a mapping $\delta : Q \times X \times Q \to I^*$.

Remark 3.2. (i) Let X^* as the free monoid generated by the set X, with e being its identity. The extension of δ is denoted as $\delta^* = (F_{\delta^*}, G_{\delta^*}, H_{\delta^*}) : Q \times X^* \times Q \to I^*$. This extension is characterized by the property that for any $q_1, q_2 \in Q, u \in X^*$, and $b_1 \in X$, the following holds:

$$F_{\delta^*}(q_1, e, q_2) = \begin{cases} 1 & \text{if } q_1 = q_2 \\ 0 & \text{if } q_1 \neq q_2, \end{cases} \quad G_{\delta^*}(q_1, e, q_2) = H_{\delta^*}(q_1, e, q_2) = \begin{cases} 0 & \text{if } q_1 = q_2 \\ 1 & \text{if } q_1 \neq q_2 \end{cases}$$

$F_{\delta^*}(q_1, ub_1, q_2) = \vee\{F_{\delta^*}(q_1, u, q_3) \otimes F_\delta(q_3, b_1, q_2) : q_3 \in Q\}, G_{\delta^*}(q_1, ub_1, q_2) = \wedge\{G_{\delta^*}(q_1, u, q_3) \otimes G_\delta(q_3, b_1, q_2) : q_3 \in Q\}$, and $H_{\delta^*}(q_1, ub_1, q_2) = \wedge\{H_{\delta^*}(q_1, u, q_3) \otimes H_\delta(q_3, b_1, q_2) : q_3 \in Q\}$.

(ii) For $u \in X^*$, we can establish a mapping $\delta_u = (F_{\delta_u}, G_{\delta_u}, H_{\delta_u}) : Q \times Q \to I^*$ such that $\forall q_1, q_2 \in Q, F_{\delta_u}(q_1, q_2) = F_{\delta^*}(q_1, u, q_2), G_{\delta_u}(q_1, q_2) = G_{\delta^*}(q_1, u, q_2)$ and $H_{\delta_u}(q_1, q_2) = H_{\delta^*}(q_1, u, q_2)$.

Definition 3.3. A **reverse neutrosophic automaton (RNA, in short)** of a NA $L = (Q, X, \delta)$ is a NA $\overline{L} = (Q, X, \overline{\delta})$, where $\overline{\delta} : Q \times X \times Q \to I^*$ is a mapping such that $\overline{\delta}(q_1, b_1, q_2) = \delta(q_2, b_1, q_1), \forall q_1, q_2 \in Q$ and $\forall b_1 \in X$.

Definition 3.4. Let $L = (Q, X, \delta)$ be a NA. Then $A = (F_A, G_A, H_A) \in NS(S)$ is called

(i) **neutrosophic subsystem, (NSS, in short)** of L if $F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_2), \forall q_1, q_2 \in Q$ and $\forall b_1 \in X$.

(ii) **reverse neutrosophic subsystem, (RNSS, in short)** of L if $F_A(q_2) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_1), G_A(q_2) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_1)$ and $H_A(q_2) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_1), \forall q_1, q_2 \in Q$ and $\forall b_1 \in X$.

(iii) **double neutrosophic subsystem, (DNSS, in short)** of L if it is both NSS and RNSS of L.

Proposition 3.5. *If A is a NSS in a NA $L = (Q, X, \delta)$, then A is a RNSS in a RNA $\overline{L} = (Q, X, \overline{\delta})$.*

Proof: Let $A = (F_A, G_A, H_A)$ be a NSS in L. Then $\forall q_1, q_2 \in Q$ and $\forall b_1 \in X, F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_2) \Rightarrow F_A(q_1) \otimes F_{\overline{\delta}}(q_2, b_1, q_1) \leq F_A(q_2), G_A(q_1) \otimes G_{\overline{\delta}}((q_2, b_1, q_1) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_{\overline{\delta}}((q_2, b_1, q_1) \geq H_A(q_2)$. Hence A is a RNSS in a RNA $\overline{L}$.

Proposition 3.6. *Let $L = (Q, X, \delta)$ be a NA and $A \in NS(S)$. Then*

(i) *$A = (F_A, G_A, H_A)$ is a NSS of L if and only if $A : (Q, X, \delta) \longrightarrow (I^*, \rightarrow)$ is an order preserving map.*

(ii) *$A = (F_A, G_A, H_A)$ is a RNSS of L if and only if $A : (Q, X, \overline{\delta}) \longrightarrow (I^*, \rightarrow)$ is an order preserving map.*

(iii) *$A = (F_A, G_A, H_A)$ is a DNSS of L if and only if $A : (Q, X, \delta) \longrightarrow (I^*, \rightarrow)$ is an order preserving map.*

Proof: (i) Let $A = (F_A, G_A, H_A) \in NS(S)$ be a NSS of L. Then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_2)$, then $F_\delta(q_1, b_1, q_2) \leq F_A(q_1) \rightarrow F_A(q_2), G_\delta(q_1, b_1, q_2) \geq G_A(q_1) \leftarrow G_A(q_2)$ and $H_\delta(q_1, b_1, q_2) \geq H_A(q_1) \leftarrow H_A(q_2)$ (cf., Proposition 2.6). Hence $A : (Q, X, \delta) \longrightarrow (I^*, \rightarrow)$ preserve order. Converse follows similarly.
(ii) Similar to (i).
(iii) Derives from (i) and (ii).

Proposition 3.7. *Let $L = (Q, X, \delta)$ be a NA and $q_1, q_3 \in Q, b_1 \in X$. Then*

(i) *$[q_3]^{\delta_{b_1}} = (F_{[q_3]^{\delta_{b_1}}}, G_{[q_3]^{\delta_{b_1}}}, H_{[q_3]^{\delta_{b_1}}}) \in NS(Q)$ such that $F_{[q_3]^{\delta_{b_1}}}(q_1) = F_{\delta_{b_1}}(q_3, q_1), G_{[q_3]^{\delta_{b_1}}}(q_1) = G_{\delta_{b_1}}(q_3, q_1)$ and $H_{[q_3]^{\delta_{b_1}}}(q_1) = H_{\delta_{b_1}}(q_3, q_1)$ is a NSS of L,*

(ii) *$[q_3]_{\delta_{b_1}} = (F_{[q_3]_{\delta_{b_1}}}, G_{[q_3]_{\delta_{b_1}}}, H_{[q_3]_{\delta_{b_1}}}) \in NS(Q)$ such that $F_{[q_3]_{\delta_{b_1}}}(q_1) = F_{\delta_{b_1}}(q_1, q_3), G_{[q_3]_{\delta_{b_1}}}(q_1) = G_{\delta_{b_1}}(q_1, q_3)$ and $H_{[q_3]_{\delta_{b_1}}}(q_1) = H_{\delta_{b_1}}(q_1, q_3)$ is a RNSS of L* , and

(iii) *$[q_3]^{\delta_{b_1}}$ and $[q_3]_{\delta_{b_1}}$ is a DNSS of L.*

Proof: (i) Let $F_{[q_3]^{\delta_{b_1}}}(q_1) = F_{\delta_{b_1}}(q_3, q_1), G_{[q_3]^{\delta_{b_1}}}(q_1) = G_{\delta_{b_1}}(q_3, q_1)$ and $H_{[q_3]^{\delta_{b_1}}}(q_1) = H_{\delta_{b_1}}(q_3, q_1)$. Then $F_{[q_3]^{\delta_{b_1}}}(q_1) \otimes F_\delta(q_1, b_1, q_2) = F_{\delta_{b_1}}(q_3, q_1) \otimes F_\delta(q_1, b_1, q_2) = F_\delta(q_3, b_1, q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_\delta(q_3, b_1, q_2) = F_{\delta_{b_1}}(q_3, q_2) = F_{[q_3]^{\delta_{b_1}}}(q_2), G_{[q_3]^{\delta_{b_1}}}(q_1) \otimes G_\delta(q_1, b_1, q_2) = G_{\delta_{b_1}}(q_3, q_1) \otimes G_\delta(q_1, b_1, q_2) = G_\delta(q_3, b_1, q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_\delta(q_3, b_1, q_2) = G_{\delta_{b_1}}(q_3, q_2) = G_{[q_3]^{\delta_{b_1}}}(q_2)$ and $H_{[q_3]^{\delta_{b_1}}}(q_1) \otimes H_\delta(q_1, b_1, q_2) = H_{\delta_{b_1}}(q_3, q_1) \otimes H_\delta(q_1, b_1, q_2) = H_\delta(q_3, a, q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_\delta(q_3, b_1, q_2) = H_{\delta_{b_1}}(q_3, q_2) = H_{[q_3]^{\delta_{b_1}}}(q_2)$, as δ_{b_1} is transitive. Hence $F_{[q_3]^{\delta_{b_1}}}(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_{[q_3]^{\delta_{b_1}}}(q_2), G_{[q_3]^{\delta_{b_1}}}(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_{[q_3]^{\delta_{b_1}}}(q_2)$ and $H_{[q_3]^{\delta_{b_1}}}(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_{[q_3]^{\delta_{b_1}}}(q_2)$. Thus $[q_3]^{\delta_{b_1}}$ is a NSS of L.
(ii) Derives from (i) and the transitivity of δ_{b_1}.
(iii) Derives from (i) and (ii).

Proposition 3.8. *Let $L = (Q, X, \delta)$ be a NA and $A \in NS(Q)$. Then*

(i) *if $A = (F_A, G_A, H_A)$ is a NSS of a NA L, then for each $\eta \in I^*, A \rightarrow \widehat{\eta}$ is a RNSS of L.*

(ii) *if $A = (F_A, G_A, H_A)$ is a RNSS of a NA L, then for each $\eta \in I^*, A \rightarrow \widehat{\eta}$ is a NSS of L.*

Proof: Let $A = (F_A, G_A, H_A)$ is a NSS of a NA L , i.e., $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq$

$H_A(q_2)$. Then, we have to show that $A \to \widehat{\eta}$ is a RNSS of L, or that $\forall q_1, q_2 \in Q$ and $b_1 \in X, (F_A(q_2) \to \sigma) \otimes F_\delta(q_1, b_1, q_2) \leq (F_A(q_1) \to \sigma), (G_A(q_2) \leftarrow \beta) \otimes G_\delta(q_1, b_1, q_2) \geq (G_A(q_1) \leftarrow \beta)$ and $(H_A(q_2) \leftarrow \gamma) \otimes H_\delta(q_1, b_1, q_2) \geq (H_A(q_1) \leftarrow \gamma)$ which implies that $(F_A(q_2) \to \sigma) \otimes F_\delta(q_1, b_1, q_2) \otimes F_A(q_1) \leq \sigma, (G_A(q_2) \leftarrow \beta) \otimes G_\delta(q_1, b_1, q_2) \otimes G_A(q_1) \geq \beta$ and $(H_A(q_2) \leftarrow \gamma) \otimes H_\delta(q_1, b_1, q_2) \otimes (H_A(q_1) \geq \gamma$. So $(F_A(q_2) \to \sigma) \otimes F_\delta(q_1, b_1, q_2) \otimes F_A(q_1) \leq (F_A(q_2) \to \sigma) \otimes F_A(q_2) \leq \sigma, (G_A(q_2) \leftarrow \beta) \otimes G_\delta(q_1, b_1, q_2) \otimes G_A(q_1) \geq (G_A(q_2) \leftarrow \beta) \otimes G_A(q_2) \geq \beta$ and $(H_A(q_2) \leftarrow \gamma) \otimes H_\delta(q_1, b_1, q_2) \otimes (H_A(q_1) \geq (H_A(q_2) \leftarrow \gamma) \otimes (H_A(q_2) \geq \gamma$ (cf., Proposition 2.6). Hence $A \to \widehat{\eta}$ is a RNSS of L.

(ii) In a similar manner, it can be prove that if $A = (F_A, G_A, H_A)$ is a RNSS of a NA L, then for each $\eta \in I^*, A \to \widehat{\eta}$ is a NSS of L.

Proposition 3.9. *Let $L = (Q, X, \delta)$ be a NA and $A \in NS(Q)$. Then*

(i) *if $A = (F_A, G_A, H_A)$ is a NSS of a NA L, then for each $\eta \in I^*, \widehat{\eta} \otimes A$ is a NSS of L.*

(ii) *if $A = (F_A, G_A, H_A)$ is a RNSS of a NA L, then for each $\eta \in I^*, \widehat{\eta} \otimes A$ is a RNSS of L.*

Proof: (i) Let $A = (F_A, G_A, H_A)$ is a NSS of a NA L and $\eta \in I^*$. Then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_2)$ which implies that $\forall q_1, q_2 \in Q$ and $b_1 \in X, (\sigma \otimes F_A(q_1)) \otimes F_\delta(q_1, b_1, q_2) \leq (\sigma \otimes F_A(q_2)), (\beta \otimes G_A(q_1)) \otimes G_\delta(q_1, b_1, q_2) \geq (\beta \otimes G_A(q_2))$ and $(\gamma \otimes H_A(q_1)) \otimes H_\delta(q_1, b_1, q_2) \geq (\gamma \otimes H_A(q_2))$. Hence $\widehat{\eta} \otimes A$ is a NSS of L.

(ii) In a similar manner, one can demonstrate that if $A = (F_A, G_A, H_A)$ is a RNSS of a NA L, then for each $\eta \in I^*, \widehat{\eta} \otimes A$ is a RNSS of L.

The following provides a characterization of the neutrosophic transition function of a NA based on its NSS.

Proposition 3.10. *For given a NA $L = (Q, X, \delta)$. We have*

(1) *let **E** be the family of all NSS. Then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_{\delta_{b_1}}(q_1, q_2) = \wedge\{F_A(q_1) \to F_A(q_2) : F_A \in \boldsymbol{E}\}; G_{\delta_{b_1}}(q_1, q_2) = \vee\{G_A(q_1) \leftarrow G_A(q_2) : G_A \in \boldsymbol{E}\}; H_{\delta_{b_1}}(q_1, q_2) = \vee\{H_A(q_1) \leftarrow H_A(q_2) : H_A \in \boldsymbol{E}\}$.*

(2) *let **E'** be the family of all RNSS. Then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_{\delta_{b_1}}(q_1, q_2) = \wedge\{F_A(q_2) \to F_A(q_1) : F_A \in \boldsymbol{E'}\}; G_{\delta_{b_1}}(q_1, q_2) = \vee\{G_A(q_2) \leftarrow G_A(q_1) : G_A \in \boldsymbol{E'}\}; H_{\delta_{b_1}}(q_1, q_2) = \vee\{H_A(q_2) \leftarrow H_A(q_1) : H_A \in \boldsymbol{E'}\}$.*

Proof: We only prove here for NSS of L. The RNSS of L can be proved in a similar way.

(i) Let A be a NSS of a NA L. Then $\forall q_1, q_2 \in Q, b_1 \in X, F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq$

$F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_2)$, i.e. $F_A(q_1) \otimes F_{\delta_{b_1}}(q_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_{\delta_{b_1}}(q_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_{\delta_{b_1}}(q_1, q_2) \geq H_A(q_2)$, or that $F_{\delta_{b_1}}(q_1, q_2) \leq F_A(q_1) \to F_A(q_2), G_{\delta_{b_1}}(q_1, q_2) \geq G_A(q_1) \leftarrow G_A(q_2)$ and $H_{\delta_{b_1}}(q_1, q_2) \geq H_A(q_1) \leftarrow H_A(q_2) \Rightarrow F_{\delta_{b_1}}(q_1, q_2) \leq \wedge\{F_A(q_1) \to F_A(q_2) : F_A \in \mathbf{E}\}, G_{\delta_{b_1}}(q_1, q_2) \geq \vee\{G_A(q_1) \leftarrow G_A(q_2) : G_A \in \mathbf{E}\}$ and $H_{\delta_{b_1}}(q_1, q_2) \geq \vee\{H_A(q_1) \leftarrow H_A(q_2) : H_A \in \mathbf{E}\}$. Next for $q_3 \in Q, b_1 \in X$, as $[q_3]^{\delta_{b_1}}(q_1) = (F_{[q_3]^{\delta_{b_1}}}(q_1), G_{[q_3]^{\delta_{b_1}}}(q_1), H_{[q_3]^{\delta_{b_1}}}(q_1))$ is a NSS of M. Then $\wedge\{F_{[q_3]^{\delta_{b_1}}}(q_1) \to F_{[q_3]^{\delta_{b_1}}}(q_2) : q_3 \in Q\} \leq \{F_{\delta_e}(q_1, q_1) \to F_{\delta_{b_1}}(q_1, q_2)\} = 1 \to F_{\delta_{b_1}}(q_1, q_2) = F_{\delta_{b_1}}(q_1, q_2), \vee\{G_{[q_3]^{\delta_{b_1}}}(q_1) \leftarrow G_{[q_3]^{\delta_{b_1}}}(q_2) : q_3 \in Q\} \geq \{G_{\delta_e}(q_1, q_1) \leftarrow G_{\delta_{b_1}}(q_1, q_2)\} = 0 \leftarrow G_{\delta_{b_1}}(q_1, q_2) = G_{\delta_{b_1}}(q_1, q_2)$ and $\vee\{H_{[q_3]^{\delta_{b_1}}}(q_1) \leftarrow H_{[q_3]^{\delta_{b_1}}}(q_2) : q_3 \in Q\} \geq \{H_{\delta_e}(q_1, q_1) \leftarrow H_{\delta_{b_1}}(q_1, q_2)\} = 0 \leftarrow H_{\delta_{b_1}}(q_1, q_2) = H_{\delta_{b_1}}(q_1, q_2)$ (cf., Proposition 2.6). Thus $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_{\delta_{b_1}}(q_1, q_2) = \wedge\{F_A(q_1) \to F_A(q_2) : F_A \in \mathbf{E}\}; G_{\delta_{b_1}}(q_1, q_2) = \vee\{G_A(q_1) \leftarrow G_A(q_2) : G_A \in \mathbf{E}\}; H_{\delta_{b_1}}(q_1, q_2) = \vee\{H_A(q_1) \leftarrow H_A(q_2) : H_A \in \mathbf{E}\}$.

Proposition 3.11. *Let $L = (Q, X, \delta)$ be a NA and $A \in NS(Q)$. Then*

(1) *if $A = (F_A, G_A, H_A)$ is a NSS of L, so for each $\eta \in I^*, \widehat{\eta} \to A$ is a NSS of L.*

(2) *if $A = (F_A, G_A, H_A)$ is a RNSS of L, so for each $\eta \in I^*, \widehat{\eta} \to A$ is a RNSS of L.*

Proof: We only prove here for NSS of L. The RNSS of L can be proved in a similar way. (i) Let $A = (F_A, G_A, H_A)$ be a NSS of a NA L and $\eta \in I^*$. Then $\forall q_1, q_2 \in Q$ and $b_1 \in X, (\sigma \to F_A(q_1)) \otimes (F_A(q_1) \to F_A(q_2)) \leq (\sigma \to F_A(q_2)), (\beta \leftarrow G_A(q_1)) \otimes (G_A(q_1) \leftarrow G_A(q_2)) \geq (\beta \leftarrow G_A(q_2))$ and $(\gamma \leftarrow H_A(q_1)) \otimes (H_A(q_1) \leftarrow H_A(q_2)) \geq (\gamma \leftarrow H_A(q_2))$ (cf., Proposition 2.6). So that $(F_A(q_1) \to F_A(q_2)) \leq (\sigma \to F_A(q_1)) \to (\sigma \to F_A(q_2)), (G_A(q_1) \leftarrow G_A(q_2)) \geq (\beta \leftarrow G_A(q_1)) \leftarrow (\beta \leftarrow G_A(q_2))$ and $(H_A(q_1) \leftarrow H_A(q_2)) \geq (\gamma \leftarrow H_A(q_1)) \leftarrow (\gamma \leftarrow H_A(q_2))$, or that $F_\delta(q_1, q_2) \leq (\sigma \to F_A(q_1)) \to (\sigma \to F_A(q_2)), G_\delta(q_1, q_2) \geq (\beta \leftarrow G_A(q_1)) \leftarrow (\beta \leftarrow G_A(q_2))$ and $H_\delta(q_1, q_2) \geq (\gamma \leftarrow H_A(q_1)) \leftarrow (\gamma \leftarrow H_A(q_2))$ (cf., Proposition 2.6), which implies that $(\sigma \to F_A(q_1)) \otimes F_\delta(q_1, q_2) \leq (\sigma \to F_A(q_2)), (\beta \leftarrow G_A(q_1)) \otimes G_\delta(q_1, q_2) \geq (\beta \leftarrow G_A(q_2))$ and $(\gamma \leftarrow H_A(q_1)) \otimes H_\delta(q_1, q_2) \geq (\gamma \leftarrow H_A(q_2))$. Thus $\widehat{\eta} \to A$ is a NSS of L.

Proposition 3.12. *Let $L = (Q, X, \delta)$ be a NA and $A \in NS(Q)$ is a RNSS of L if and only if it is a NSS of the RNA $\overline{L} = (Q, X, \overline{\delta})$.*

Proof: Let A is a NSS of the RNA $\overline{L} = (Q, X, \overline{\delta})$, then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_A(q_1) \otimes F_{\overline{\delta}}(q_1, b_1, q_2) \leq F_A(q_2); G_A(q_1) \otimes G_{\overline{\delta}}(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_{\overline{\delta}}(q_1, b_1, q_2) \geq H_A(q_2)$ if and only if $F_A(q_1) \otimes F_\delta(q_2, b_1, q_1) \leq F_A(q_2); G_A(q_1) \otimes G_\delta(q_2, b_1, q_1) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_2, b_1, q_1) \geq H_A(q_2)$. Thus A is a RNSS of L. Converse is trivial.

Proposition 3.13. *Let $L = (Q, X, \delta)$ be a NA with $A \in NS(Q)$ and let $\neg$ be involutive. Then*

(i) *If A is a NSS, then $\neg A = (\neg F_A, \neg G_A, \neg H_A)$ is a RNSS,* and

(ii) *if A is a RNSS, then $\neg A = (\neg F_A, \neg G_A, \neg H_A)$ is a NSS.*

(iii) *if A is a DNSS, then $\neg A = (\neg F_A, \neg G_A, \neg H_A)$ is also a DNSS.*

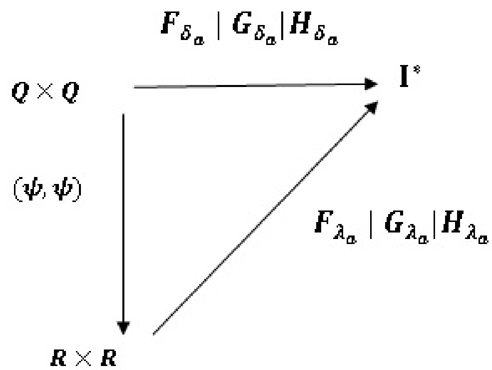
Proof: (i) Let $A = (F_A, G_A, H_A)$ is a NSS of L, then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_2)$, or that $\neg(F_A(q_1) \otimes F_\delta(q_1, b_1, q_2)) \geq \neg F_A(q_2); \neg(G_A(q_1) \otimes G_\delta(q_1, b_1, q_2)) \leq \neg G_A(q_2)$ and $\neg(H_A(q_1) \otimes H_\delta(q_1, b_1, q_2)) \leq \neg H_A(q_2)$ which implies that $(F_A(q_1) \otimes F_\delta(q_1, b_1, q_2)) \rightarrow 0 \geq \neg F_A(q_2); (G_A(q_1) \otimes G_\delta(q_1, b_1, q_2)) \leftarrow 1 \leq \neg G_A(q_2)$ and $(H_A(q_1) \otimes H_\delta(q_1, b_1, q_2)) \leftarrow 1 \leq \neg H_A(q_2) \Rightarrow (F_\delta(q_1, b_1, q_2) \otimes F_A(q_1)) \rightarrow 0 \geq \neg F_A(q_2); (G_\delta(q_1, b_1, q_2) \otimes G_A(q_1)) \leftarrow 1 \leq \neg G_A(q_2)$ and $(H_\delta(q_1, b_1, q_2) \otimes H_A(q_1)) \leftarrow 1 \leq \neg H_A(q_2) \Rightarrow F_\delta(q_1, b_1, q_2) \rightarrow (F_A(q_1) \rightarrow 0) \geq \neg F_A(q_2); G_\delta(q_1, b_1, q_2) \leftarrow (G_A(1_1) \leftarrow 1) \leq \neg G_A(q_2)$ and $H_\delta(q_1, b_1, q_2) \leftarrow (H_A(q_1) \leftarrow 1) \leq \neg H_A(q_2) \Rightarrow F_\delta(q_1, b_1, q_2) \rightarrow \neg F_A(q_1) \geq \neg F_A(q_2); G_\delta(q_1, b_1, q_2) \leftarrow \neg G_A(q_1) \leq \neg G_A(q_2)$ and $H_\delta(q_1, b_1, q_2) \leftarrow \neg H_A(q_1) \leq \neg H_A(q_2) \Rightarrow \neg F_A(q_2) \otimes F_\delta(q_1, b_1, q_2) \leq \neg F_A(q_1); \neg G_A(q_2) \otimes G_\delta(q_1, b_1, q_2) \geq \neg G_A(q_1)$ and $\neg H_A(q_2) \otimes H_\delta(q_1, b_1, q_2) \geq \neg H_A(q_1)$ (cf., Proposition 2.6). Hence $\neg A$ is a RNSS of L.

(ii) Let $A = (F_A, G_A, H_A)$ is a RNSS of L, then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_A(q_2) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_1); G_A(q_2) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_1)$ and $H_A(q_2) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_1)$, or that $\neg(F_A(q_2) \otimes F_\delta(q_1, b_1, q_2)) \geq \neg F_A(q_1); \neg(G_A(q_2) \otimes G_\delta(q_1, b_1, q_2)) \leq \neg G_A(q_1)$ and $\neg(H_A(q_2) \otimes H_\delta(q_1, b_1, q_2)) \leq \neg H_A(q_1)$ which implies that $(F_A(q_2) \otimes F_\delta(q_1, b_1, q_2)) \rightarrow 0 \geq \neg F_A(q_1); (G_A(q_2) \otimes G_\delta(q_1, b_1, q_2)) \leftarrow 1 \leq \neg G_A(q_1)$ and $(H_A(q_2) \otimes H_\delta(q_1, b_1, q_2)) \leftarrow 1 \leq \neg H_A(q_1) \Rightarrow (F_\delta(q_1, b_1, q_2) \otimes F_A(q_2)) \rightarrow 0 \geq \neg F_A(q_1); (G_\delta(q_1, b_1, q_2) \otimes G_A(q_2)) \leftarrow 1 \leq \neg G_A(q_1)$ and $(H_\delta(q_1, b_1, q_2) \otimes H_A(q_2)) \leftarrow 1 \leq \neg H_A(q_1) \Rightarrow F_\delta(q_1, b_1, q_2) \rightarrow (F_A(q_2) \rightarrow 0) \geq \neg F_A(q_1); G_\delta(q_1, b_1, q_2) \leftarrow (G_A(q_2) \leftarrow 1) \leq \neg G_A(q_1)$ and $H_\delta(q_1, b_1, q_2) \leftarrow (H_A(q_2) \leftarrow 1) \leq \neg H_A(q_1) \Rightarrow F_\delta(q_1, b_1, q_2) \rightarrow \neg F_A(q_2) \geq \neg F_A(q_1); G_\delta(q_1, b_1, q_2) \leftarrow \neg G_A(q_2) \leq \neg G_A(q_1)$ and $H_\delta(q_1, b_1, q_2) \leftarrow \neg H_A(q_2) \leq \neg H_A(q_1) \Rightarrow \neg F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq \neg F_A(q_2); \neg G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq \neg G_A(q_2)$ and $\neg H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq \neg H_A(q_2)$ (cf., Proposition 2.6). Hence $\neg A$ is a NSS of L.

(iii) Derives from (i) and (ii).

4. Neutrosophic automata and reverse neutrosophic automata: a categorical approach

In this section, we initially demonstrate that an isomorphism among neutrosophic automata (NA) establishes an equivalence relation. Additionally, we present the categorical characteristics of both neutrosophic automata and reverse neutrosophic automata. Furthermore, we identify the functorial relationship that exists between the categories of neutrosophic automata and reverse neutrosophic automata. The discussion begins with the following points.



(Fig.1)Homomorphism between L and N

Definition 4.1. Let $L = (Q, \delta)$ and $N = (R, \lambda)$ are two NA over X. A homomorphism from L to N is a function $\psi : Q \to R$ such that, for each element $b_1 \in X$, the diagram depicted in Figure 1 remains consistent.

Remark 4.2. (i) In Figure 1, the commutativity of a diagram signifies $(F_{\lambda_{b_1}} o(\psi, \psi))(q_1, q_2) = F_{\delta_{b_1}}(q_1, q_2); (G_{\lambda_{b_1}} o(\psi, \psi))(q_1, q_2) = G_{\delta_{b_1}}(q_1, q_2)$ and $(H_{\lambda_{b_1}} o(\psi, \psi))(q_1, q_2) = H_{\delta_{b_1}}(q_1, q_2), \forall q_1, q_2 \in Q$.

(ii) Throughout, we will use the notation $F_A|G_A|H_A$ diagrams to denote a neutrosophic set A. Furthermore, the commutativity of these diagrams remains consistent with the discussion in part (i).

Remark 4.3. (i). The pair (ψ_1, ψ_2) is known as a strong homomorphism if, $\forall (q_1, b_1, q_2) \in Q \times X \times Q, F_\lambda(\psi_1(q_1), \psi_2(b_1), \psi_1(q_2)) = \vee\{F_\delta(q_1, b_1, q_3) : q_3 \in Q, \psi_1(q_3) = \psi_1(q_2)\}, G_\lambda(\psi_1(q_3), \psi_2(b_1), \psi_1(q_2)) = \wedge\{G_\delta(q_1, b_1, q_3) : q_3 \in Q, \psi_1(q_3) = \psi_1(q_2)\}$ and $H_\lambda(\psi_1(q_1), \psi_2(b_1), \psi_1(q_2)) = \wedge\{H_\delta(q_1, b_1, q_3) : q_3 \in Q, \psi_1(q_3) = \psi_1(q_2)\}$.

(ii). A bijective homomorphism (strong homomorphism) with the property $\lambda(\psi_1(q_1), \psi_2(b_1), \psi_1(q_2)) = \delta(q_1, b_1, q_2)$ is called an isomorphism (strong isomorphism).

Definition 4.4. Let $L = (Q, X, \delta)$ and $N = (R, X, \lambda)$ be two NA and $\psi : L \longrightarrow N$ be a homomorphism. Then for $A \in NS(Q)$, the neutrosophic subset $\psi(A) \in NS(R)$ can be defined as

$$F_{\psi(A)}(q_3) = \begin{cases} \vee(F_A(q_1) : q_1 \in Q, \psi(q_1) = q_3) & \text{if } \psi^{-1}(q_3) \neq \phi \\ 0 & \text{if } \psi^{-1}(q_3) = \phi \end{cases}$$

$$G_{\psi(A)}(q_3) = \begin{cases} \wedge(G_A(q_1) : q_1 \in Q, \psi(q_1) = q_3) & \text{if } \psi^{-1}(q_3) \neq \phi \\ 1 & \text{if } \psi^{-1}(q_3) = \phi \; and \end{cases}$$

$$H_{\psi(A)}(q_3) = \begin{cases} \wedge(H_A(q_1) : q_1 \in Q, \psi(q_1) = q_3) & \text{if } \psi^{-1}(q_3) \neq \phi \\ 1 & \text{if } \psi^{-1}(q_3) = \phi, \end{cases}$$

In this context, we explore the properties of NSS under strong homomorphism.

Proposition 4.5. *Let $L = (Q, X, \delta)$ and $N = (R, X, \lambda)$ be two NA and $\psi : L \longrightarrow N$ be an onto strong homomorphism. Then for a NSS A of $L, \psi(A)$ is a NSS of N.*

Proof: Let $q_1, q_2 \in Q$ and $r_1, r_2 \in R$ such that $f(q_1) = r_1$ and $f(q_2) = r_2$. If A is a NSS of L, then $\forall r_1, r_2 \in R$ and $b_1 \in X$, we have $F_{\psi(A)}(r_1) \otimes F_\lambda(r_1, b_1, r_2) = F_A(r_1) \otimes F_\lambda(r_1, b_1, r_2) = F_A(q_1) \otimes F_\lambda(f(q_1), b_1, f(q_2))$ (where $f(q_1) = r_1, \forall q_1 \in Q$) $= F_A(q_1) \otimes \vee\{F_\delta(q_1, b_1, q_3) : q_3 \in Q, \psi(q_3) = \psi(q_2) = r_2\} = \vee\{F_A(q_1) \otimes F_\delta(q_1, b_1, q_3) : q_3 \in Q, \psi(q_3) = \psi(q_2) = r_2\} \leq \vee\{F_A(q_3) : q_3 \in Q, \psi(q_3) = \psi(q_2) = r_2\} = F_{\psi(A)}(r_2)$. Similarly, we can show that $G_{\psi(A)}(r_1) \otimes G_\lambda(r_1, b_1, r_2) \geq G_{\psi(A)}(r_2)$ and $H_{\psi(A)}(r_1) \otimes H_\lambda(r_1, b_1, r_2) \geq H_{\psi(A)}(r_2)$. Hence $\psi(A)$ is a NSS of N.

The proposition mentioned above holds true solely for NSS and does not apply to RNSS.

Proposition 4.6. *An isomorphism among NA establishes an equivalence relation.*

Proof:-The reflexivity and symmetry are evident. To establish transitivity, we let $(\psi_1, \psi_2) : L_1 \longrightarrow L_2$ and $(\chi_1, \chi_2) : L_2 \longrightarrow L_3$ where $\psi_1 : Q_1 \longrightarrow Q_2$, $\chi_1 : Q_2 \longrightarrow Q_3$ and $\psi_2, \chi_2 : X \longrightarrow X$ be the isomorphism of L_1 onto L_2 and L_2 onto L_3 respectively. Then $(\chi_1, \chi_2)o(\psi_1, \psi_2) : L_1 \longrightarrow L_3$ is bijective map from L_1 to L_3, where$((\chi_1, \chi_2)o(\psi_1, \psi_2))(q_1, b_1, q_1') = (\chi_1, \chi_2)((\psi_1, \psi_2)(q_1, b_1, q_1')), \forall (q_1, b_1, q_1') \in Q_1 \times X \times Q_1$.

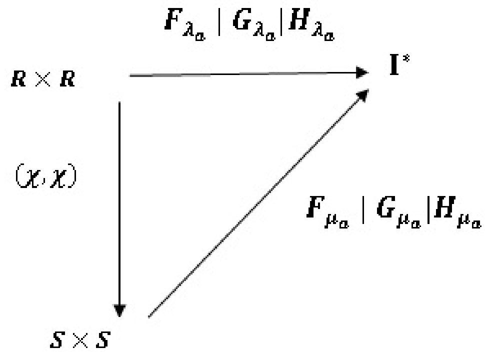
Since a map $(\psi_1, \psi_2) : L_1 \to L_2$ defined as $\psi_1(q_1) = q_2, \psi_1(q_1') = q_2', \psi_2(b_1) = b_1$ is an isomorphism. So, we have $F_{\delta_1}(q_1, b_1, q_1') = F_{\delta_2}(\psi_1(q_1), \psi_2(b_1), \psi_1(q_1')) = F_{\delta_2}(q_2, b_1, q_2')$. Similarly,$G_{\delta_1}(q_1, b_1, q_1') = G_{\delta_2}(q_2, b_1, q_2')$ and $H_{\delta_1}(q_1, b_1, q_1') = H_{\delta_2}(q_2, b_1, q_2'), \forall (q_1, b_1, q_1') \in Q_1 \times X \times Q_1$ and $\forall (q_2, b_1, q_2') \in Q_2 \times X \times Q_2$.

.....(1)

Next, since a map $(\chi_1, \chi_2) : L_2 \to L_3$ defined as $\chi_1(q_2) = q_3, \chi_1(q_2') = q_3'$ and $\chi_2(b_1) = b_1$ is an isomorphism. So, we have $F_{\delta_2}(q_2, b_1, q_2') = F_{\delta_3}(\chi_1(q_2), \chi_2(b_1), \chi_1(q_2')) = F_{\delta_3}(q_3, b_1, q_3')$. Similarly $G_{\delta_2}(q_2, b_1, q_2') = G_{\delta_3}(q_3, b_1, q_3')$ and $H_{\delta_2}(q_2, b_1, q_2') = H_{\delta_3}(q_3, b_1, q_3'), \forall (q_2, b_1, q_2') \in Q_2 \times X \times Q_2$ and $(q_3, b_1, q_3') \in Q_3 \times X \times Q_3$.

.........(2)

Thus from expressions $(1), (2)$ and$\psi_1(q_1) = q_2, \psi_1(q_1') = q_2', \psi_2(b_1) = b_1, \forall (q_1, b_1, q_1') \in Q_1 \times X \times Q_1$, we have $F_{\delta_1}(q_1, b_1, q_1') = F_{\delta_2}(\psi_1(q_1), \psi_2(b_1), \psi_1(q_1')) = F_{\delta_2}(q_2, b_1, q_2') = F_{\delta_3}(\chi_1(q_2), \chi_2(b_1), \chi_1(q_2')) = F_{\delta_3}((\chi_1, \chi_2)(q_2, b_1, q_2')) = F_{\delta_3}((\chi_1, \chi_2)((\psi_1, \psi_2)(q_1, b_1, q_1')) =$



(Fig.2)Homomorphism between N and P

$F_{\delta_3}((\chi_1, \chi_2)o(\psi_1, \psi_2))(q_1, b_1, q_1')$. Similarly $G_{\delta_1}(q_1, b_1, q_1') = G_{\delta_3}((\chi_1, \chi_2)o(\psi_1, \psi_2))(q_1, b_1, q_1')$ and $H_{\delta_1}(q_1, b_1, q_1') = H_{\delta_3}((\chi_1, \chi_2)o(\psi_1, \psi_2))(q_1, b_1, q_1'), \forall (q_1, b_1, q_1') \in Q_1 \times X \times Q_1$. Hence $(\chi_1, \chi_2)o(\psi_1, \psi_2)$ is an isomorphism between L_1 and L_3.

Proposition 4.7. *An isomorphism among RNA establishes an equivalence relation.*

Proof:- A direct consequence of the proposition 4.6.

Proposition 4.8. *An isomorphism among DNA establishes an equivalence relation.*

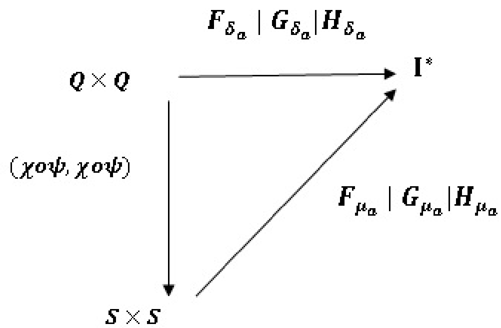
Proof:- This is a direct consequence of the propositions 4.6 and 4.7.

We will represent the category of NA over X as $\mathbf{NeA}(X)$ and the category of NA over X^* as $\mathbf{NeA}(X^*)$. Additionally, the object-class of the categories NeA(X) and NeA(X^*) will be denoted as NeA(X) and NeA(X^*), respectively. Now, we proceed with the following.

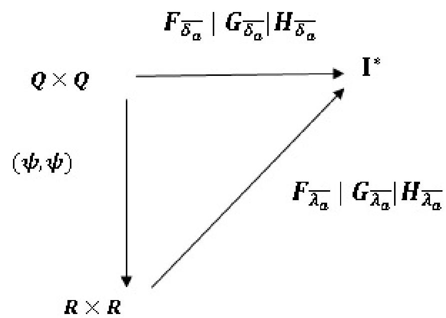
Proposition 4.9. *The class of NA over X and their homomorphisms constitute a category.*

Proof: We demonstrate solely that the composition of two homomorphisms is again a homomorphism, as follows, let $L = (Q, \delta), N = (R, \lambda)$ and $P = (S, \mu)$ be NA over X and $\psi : L \to N, \chi : N \to P$ be homomorphisms, i.e., $\psi : Q \to R, \chi : R \to S$ are the maps such that for all $b_1 \in X$, the diagrams in Fig.1 and Fig. 2 holds. Then the following shows that for all $b_1 \in X$, the diagram in Fig. 3 also hold. So, let $q_1, q_2 \in Q$. Then $(F_{\mu_{b_1}} o(\chi o\psi, \chi o\psi))(q_1, q_2) = F_{\mu_{b_1}}(\chi(\psi(q_1)), \chi(\psi(q_2))) = (F_{\mu_{b_1}} o(\chi, \chi))(\psi(q_1), \psi(q_2)) = F_{\lambda_{b_1}}(\psi(q_1), \psi(q_2)) = (F_{\lambda_{b_1}} o(\psi, \psi))(q_1, q_2) = F_{\delta_{b_1}}(q_1, q_2)$. Hence $F_{\delta_{b_1}} = F_{\mu_{b_1}} o(\chi o\psi, \chi o\psi)$. Similarly, we can show that $G_{\delta_{b_1}} = G_{\mu_{b_1}} o(\chi o\psi, \chi o\psi)$ and $H_{\delta_{b_1}} = H_{\mu_{b_1}} o(\chi o\psi, \chi o\psi)$. Thus $\chi o\psi : L \to P$ is a homomorphism.

We will represent the category of RNA over X as $\mathbf{RNeA}(X)$ and the category of RNA over



(Fig.3) Homomorphism between L and P



(Fig.4) Homomorphism between $\overline{L}$ and $\overline{N}$

X^* as **RNeA**(X^*). Additionally, the object-class of the categories $RNeA(X)$ and $RNeA(X^*)$ will be denoted as $RNeA(X)$ and $RNeA(X^*)$, respectively.

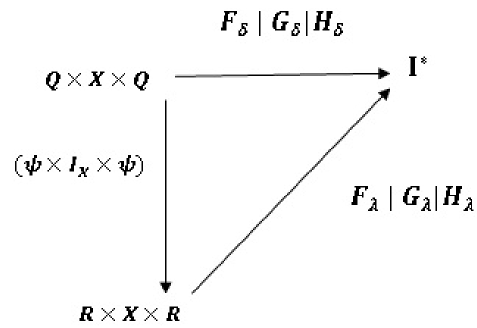
Definition 4.10. Let $\overline{L}$=(Q, $\overline{\delta}$) and $\overline{N}$=(R, $\overline{\lambda}$) be RNA over X. A homomorphism from $\overline{L}$ to $\overline{N}$ is a map $\psi : Q \to R$ such that for all $b_1 \in X$, the diagram in Fig.4 hold.

Now, we present the introduction of functors between the categories of NA as described earlier.

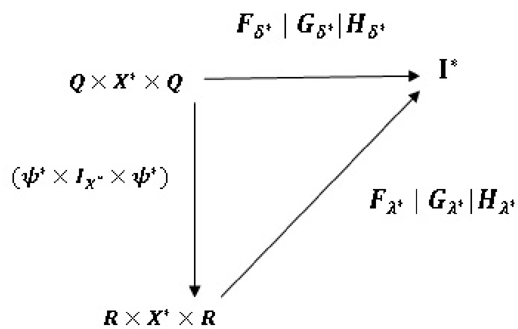
Proposition 4.11. *From $NeA(X)$ to $NeA(X^*)$, there exists a functor.*

Proof:- Let $L = (Q, X, \delta) \in NeA(X)$. We establish a mapping $K : NeA(X) \to NeA(X^*)$ such that $K(L) = (Q, X^*, \delta^*)$, then $K(L) \in NeA(X^*)$. Also, for a $NeA(X)$-morphism $\psi : L = (Q, X, \delta) \to N = (R, X, \lambda)$, let $K(\psi) : K(L) \to K(N)$,i.e., $K(\psi) = \psi^*$. Subsequently, it can be demonstrated that ψ^* is a $NeA(X^*)$-morphism from $K(L)$ to $K(N)$, i.e., the depicted diagram in Figure 5 is valid, indicating that the diagram in Figure 6 also holds. Consequently, based on Figure 5, we obtain

$F_\delta = F_\lambda o(\psi \times I_X \times \psi), G_\delta = G_\lambda o(\psi \times I_X \times \psi)$ and $H_\delta = H_\lambda o(\psi \times I_X \times \psi)$



(Fig.5) Morphism between L and N



(Fig.6) Morphism between $K(L)$and $K(N)$

Now, $K(F_\delta) = F_{\delta^*} = K[F_\lambda o(\psi \times I_X \times \psi)] = F_{\lambda^*} o(\psi^* \times I_{X^*} \times \psi^*)$. In a similar manner $K(G_\delta) = G_{\delta^*} = K[G_\lambda o(\psi \times I_X \times \psi)] = G_{\lambda^*} o(\psi^* \times I_{X^*} \times \psi^*)$ and $K(H_\delta) = H_{\delta^*} = K[H_\lambda o(\psi \times I_X \times \psi)] = H_{\lambda^*} o(\psi^* \times I_{X^*} \times \psi^*)$. This implies the validity of Figure 6. Additionally, the identity and composition properties of maps K are evident. Therefore, the mapping $K : NeA(X) \rightarrow NeA(X^*)$ is a functor.

Proposition 4.12. *From $NeA(X^*)$ to $NeA(X)$, there exists a functor.*

Proof:- Define a mapping $\beta : NeA(X^*) \rightarrow NeA(X)$ such that $\beta(L) = (Q, X, \delta), \forall L \in NeA(X^*)$. Then $\beta(L) \in NeA(X)$. Therefore, based on proposition 4.11, we demonstrate that β operates as a functor.

In this context, we present the functor between the category of RNA, as defined earlier.

Proposition 4.13. *From $RNeA(X)$ to $RNeA(X^*)$, there exists a functor.*

Proof:- This is a direct consequence of the proposition 4.11.

Proposition 4.14. *From $RNeA(X^*)$ to $RNeA(X)$, there exists a functor.*

Proof:-This is a direct consequence of the proposition 4.12.

5. Conclusions

This paper has introduced the novel concepts of neutrosophic automata and reverse neutrosophic automata, extending the groundwork laid by fuzzy automata. The exploration includes the introduction of neutrosophic subsystems, reverse neutrosophic subsystems, and double neutrosophic subsystems linked to these automata, with an investigation into algebraic results derived from these concepts. Additionally, the categorical properties of neutrosophic automata and their functorial relationships have been examined. In future work, the focus will extend to exploring the topological properties of neutrosophic automata based on the aforementioned concepts.

Conflicts of Interest: The authors assert that there are no conflicts of interest..

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Received: Oct 10, 2023. Accepted: Jan 9, 2024