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New algebraic structure for Diophantine neutrosophic subbisemirings of bisemirings

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Abstract. In this paper, we define the concept of Diophantine neutrosophic subbisemiring (DioNSBS) of bisemirings (BSs). The DioNSBS is the new approach for fuzzy subbisemiring (FSBS) over a BS. Let Ξ be the Diophantine neutrosophic subset (DioNSS) in $\mathcal{T}$, we show that $\Xi = \langle(\mho^{\mathcal{T}}_{\Xi}, \mho^{\mathcal{I}}_{\Xi}, \mho^{\mathcal{F}}_{\Xi}), (\Gamma_{\Xi}, \Lambda_{\Xi}, \Theta_{\Xi})\rangle$ is a DioNSBS of $\mathcal{T}$ if and only if all non-empty level set $\Xi^{(\beta,\gamma)}$ is a subbisemiring (SBS) of $\mathcal{T}$, $\forall \beta, \gamma \in [0,1]$. Let Ξ be the DioNSBS of a BS $\mathcal{T}$ and Z be the strongest Diophantine neutrosophic relation of $\mathcal{T}$. Then Ξ is a DioNSBS of $\mathcal{T}$ if and only if Z is a DioNSBS of $\mathcal{T} \times \mathcal{T}$. Let $\Xi_1, \Xi_2, \ldots, \Xi_n$ be the family of DioNSBSs of $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$, respectively. We show that $\Xi_1 \times \Xi_2 \times \ldots \times \Xi_n$ is a DioNSBS of $\mathcal{T}_1 \times \mathcal{T}_2 \times \ldots \times \mathcal{T}_n$. The homomorphic image of every DioNSBS is a DioNSBS. Let Ξ be any DioNSBS of $\mathcal{T}$, then pseudo Diophantine neutrosophic coset $(a\Xi)^p$ is a DioNSBS of $\mathcal{T}$, for every $a \in \mathcal{T}$. Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. The homomorphic preimage of every DioNSBS of $\mathcal{T}_2$ is a DioNSBS of $\mathcal{T}_1$. Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. Let Ξ and Δ be any two DioNSBSs of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, then $\Xi \times \Delta$ is a DioNSBS of $\mathcal{T}_1 \times \mathcal{T}_2$. If $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ is a homomorphism, then $\mathcal{L}(\Xi_{(\beta,\gamma)})$ is a level SBS of DioNSBS Z of $\mathcal{T}_2$. Examples are given to demonstrate our findings.

Keywords: BS; FSBS; NSBS; DioNSBS.

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1. **Introduction**

Most real-world problems are characterized by uncertainty. Numerous uncertain theories, such as the fuzzy set (FS) [31], intuitionistic fuzzy set (IFS) [5], Pythagorean fuzzy set (PFS)

[28] and neutrosophic set (NSS) [26] are proposed to deal with the uncertainties. An FS is one in which every element in the universe is a member, but only to a degree of belongingness that ranges from zero to one. In the set of elements, these grades are known as membership values. Clustering techniques [29] are used in applications of FSs like regression prediction for fuzzy time series [27]and fuzzy c-numbersIn applications that require precise data. Atanassov [5] introduced the idea of an IFS. An organization whose membership degree and non-membership degree values are less than or equal to one. Occasionally, we have difficulty making decisions when the combined value of the membership degree and the non-membership degree is greater than one. As part of a generalization of IFS, Yager [28] introduced the concept of PFS as defined by the sum of membership degrees with non-membership degrees having a value less than or equal to one. The numerous applications based on PFSs were addressed by Akram et al. [2–4]. The study of semirings resulted from Dedekind's engagement with commutative ring theories. Vandiver [30] introduces semirings as part of his generalization of rings. In the 1880s, the German mathematician Dedekind began to investigate semirings and commutative rings as ideals. Vandiver developed a fundamental algebraic structure in 1934 due to his later research on semirings. A distributive lattice was essentially a generalization of rings. On the other hand, semiring theory has advanced since 1950. Rings and distributive lattices were essentially generalized. The theory of semirings has nevertheless been developing since 1950. Iseki [8, 9] was introduced by the semiring concept that is not always commutative under either operation. Without zero, Iseki [10] demonstrated numerous significant results based on semirings by using this abstraction for semirings. Many authors and academics have described the various ideals based on semirings [7]. Semigroups, semirings, and hypersemigroups are a few examples of ordered algebraic structures that many writers have researched. Zadeh invented the concept of FS [31]. A function described by a membership value is what this definition refers to as an FS. In real unit intervals, degrees are taken. A combination of membership and non-membership has been considered, and an insufficient definition has been reached. NSS extend FS and IFS by delineating truth and indeterminacy memberships separately. To manage the uncertainty presented, Atanassov [5] described a set referred to as an IFS. Several application-related problems are present in this information set, and Smarandache [26] proposed neutrosophy to address these issues. Reference parameters were included in the discussion of the linear Diophantine fuzzy set (LDFS) by Riaz et al. [23]. Because reference parameters are used, the LDFS is more effective and adaptable than other methods. By modifying the reference parameter's physical sense, the LDFS classifies the data in MADM difficulties. A fundamental difference between FS and IFS can be found in neutrosophy, which focuses on neutral cognition. Smarandache [26] invented neurosophic logic. Each proposition is given an estimated degree of truth, degree of ambiguity and degree of falsity according to this logic. Every component of

the cosmos in NSS has a degree of truth, indeterminacy and falsity that ranges from $[0, 1]$. The FS, interval-valued FS and classical sets can be generalized to an NSS from a philosophical perspective.

A semiring $(S, +, \cdot)$ is a non-empty set in which $(S, +)$ and $(S, \cdot)$ are semigroups such that "$\cdot$" is distributive over "$+$" [7]. In 1993, Ahsan et al. [1] introduced the notion of fuzzy semirings. In 2001, Sen and Ghosh were introduced in BSs. A bisemiring (BS) $(\mathcal{T}, +, \circ, \times)$ is an algebraic structure in which $(\mathcal{T}, +, \circ)$ and $(\mathcal{T}, \circ, \times)$ are semirings in which $(\mathcal{T}, +), (\mathcal{T}, \circ)$ and $(\mathcal{T}, \times)$ are semigroups such that (i) $\tau_a \circ (\tau_b + \tau_c) = (\tau_a \circ \tau_b) + (\tau_a \circ \tau_c)$, (ii) $(\tau_b + \tau_c) \circ \tau_a = (\tau_b \circ \tau_a) + (\tau_c \circ \tau_a)$ (iii) $\tau_a \times (\tau_b \circ \tau_c) = (\tau_a \times \tau_b) \circ (\tau_a \times \tau_c)$ and (iv) $(\tau_b \circ \tau_c) \times \tau_a = (\tau_b \times \tau_a) \circ (\tau_c \times \tau_a), \forall \tau_a, \tau_c \in \mathcal{T}$ [25]. A non-empty subset Ξ of a BS $(\mathcal{T}, +, \circ, \times)$ is an SBS if and only if $\tau_a + \tau_b \in \Xi, \tau_a \circ \tau_b \in \Xi$ and $\tau_a \times \tau_b \in \Xi, \forall \tau_a, \tau_b, \tau_c \in \Xi$ [6]. Palanikumar et al. discussed the various ideal structures of SBS theory and its applications [11]- [20]. The concept of DioNSBSs is introduced in this study. This paper is focused on the following: The introduction is in Section 1. The preliminary definitions and results are found in Section 2. Section 3 introduces the notion of DioNSBS and its several illustrative examples.

2. **Basic Concepts**

Definition 2.1. [26] An NSS Ξ in the universe $\mathcal{U}$ is $\Xi = \{\epsilon, \mho_{\Xi}^{\mathcal{T}}(\epsilon), \mho_{\Xi}^{\mathcal{I}}(\epsilon), \mho_{\Xi}^{\mathcal{F}}(\epsilon) \mid \epsilon \in \mathcal{U}\}$, where $\mho_{\Xi}^{\mathcal{T}}(\epsilon)$, $\mho_{\Xi}^{\mathcal{I}}(\epsilon)$ $\mho_{\Xi}^{\mathcal{F}}(\epsilon)$ represents the degree of truth-membership, indeterminacy membership and falsity-membership of Ξ, respectively. The mapping $\mho_{\Xi}^{\mathcal{T}}, \mho_{\Xi}^{\mathcal{I}}, \mho_{\Xi}^{\mathcal{F}} : \mathcal{U} \to [0, 1]$ and $0 \leq \mho_{\Xi}^{\mathcal{T}}(\epsilon) + \mho_{\Xi}^{\mathcal{I}}(\epsilon) + \mho_{\Xi}^{\mathcal{F}}(\epsilon) \leq 3$.

Definition 2.2. [26] Let $\Xi_1 = \langle \mho_{\Xi_1}^{\mathcal{T}}, \mho_{\Xi_1}^{\mathcal{I}}, \mho_{\Xi_1}^{\mathcal{F}} \rangle$, $\Xi_2 = \langle \mho_{\Xi_2}^{\mathcal{T}}, \mho_{\Xi_2}^{\mathcal{I}}, \mho_{\Xi_2}^{\mathcal{F}} \rangle$ and $\Xi_3 = \langle \mho_{\Xi_3}^{\mathcal{T}}, \mho_{\Xi_3}^{\mathcal{I}}, \mho_{\Xi_3}^{\mathcal{F}} \rangle$ be the three neutrosophic numbers over $\mathcal{U}$. Then
(i) ${\Xi_1}^c = \langle \mho_{\Xi_1}^{\mathcal{F}}, \mho_{\Xi_1}^{\mathcal{I}}, \mho_{\Xi_1}^{\mathcal{T}} \rangle$
(ii) $\Xi_2 \veebar \Xi_3 = \left\langle \max(\mho_{\Xi_2}^{\mathcal{T}}, \mho_{\Xi_3}^{\mathcal{T}}), \min(\mho_{\Xi_2}^{\mathcal{I}}, \mho_{\Xi_3}^{\mathcal{I}}), \min(\mho_{\Xi_2}^{\mathcal{F}}, \mho_{\Xi_3}^{\mathcal{F}}) \right\rangle$
(iii) $\Xi_2 \barwedge \Xi_3 = \left\langle \min(\mho_{\Xi_2}^{\mathcal{T}}, \mho_{\Xi_3}^{\mathcal{T}}), \max(\mho_{\Xi_2}^{\mathcal{I}}, \mho_{\Xi_3}^{\mathcal{I}}), \max(\mho_{\Xi_2}^{\mathcal{F}}, \mho_{\Xi_3}^{\mathcal{F}}) \right\rangle$
(iv) $\Xi_2 \geq \Xi_3$ iff $\mho_{\Xi_2}^{\mathcal{T}} \geq \mho_{\Xi_3}^{\mathcal{T}}$ and $\mho_{\Xi_2}^{\mathcal{I}} \leq \mho_{\Xi_3}^{\mathcal{I}}$ and $\mho_{\Xi_2}^{\mathcal{F}} \leq \mho_{\Xi_3}^{\mathcal{F}}$
(v) $\Xi_2 = \Xi_3$ iff $\mho_{\Xi_2}^{\mathcal{T}} = \mho_{\Xi_3}^{\mathcal{T}}$ and $\mho_{\Xi_2}^{\mathcal{I}} = \mho_{\Xi_3}^{\mathcal{I}}$ and $\mho_{\Xi_2}^{\mathcal{F}} = \mho_{\Xi_3}^{\mathcal{F}}$.

Definition 2.3. [26] For any NSS $\Xi = \{\xi_a, \mho_{\Xi}^{\mathcal{T}}(\xi_a), \mho_{\Xi}^{\mathcal{I}}(\xi_a), \mho_{\Xi}^{\mathcal{F}}(\xi_a)\}$ of $\mathcal{U}$, we defined a (τ, σ)-cut of as the crisp subset $\{\xi_a \in U \mid \mho_{\Xi}^{\mathcal{T}}(\xi_a) \geq \tau, \mho_{\Xi}^{\mathcal{I}}(\xi_a) \geq \tau, \mho_{\Xi}^{\mathcal{F}}(\xi_a) \leq \sigma\}$.

Definition 2.4. [26] Let Ξ and Δ be be two NSSs of $\mathcal{T}$. The Cartesian product of Ξ and Δ is defined as $\Xi \times \Delta = \{\mho_{\Xi\times\Delta}^{\mathcal{T}}(\xi_a, \xi_b), \mho_{\Xi\times\Delta}^{\mathcal{I}}(\xi_a, \xi_b), \mho_{\Xi\times\Delta}^{\mathcal{F}}(\xi_a, \xi_b) \mid \ for\ all\ \xi_a, \xi_b \in \mathcal{T}\}$, where $\mho_{\Xi\times\Delta}^{\mathcal{T}}(\xi_a, \xi_b) = \min\{\mho_{\Xi}^{\mathcal{T}}(\xi_a), \mho_{\Delta}^{\mathcal{T}}(\xi_b)\}, \mho_{\Xi\times\Delta}^{\mathcal{I}}(\xi_a, \xi_b) = \frac{\mho_{\Xi}^{\mathcal{I}}(\xi_a) + \mho_{\Delta}^{\mathcal{I}}(\xi_b)}{2}, \mho_{\Xi\times\Delta}^{\mathcal{F}}(\xi_a, \xi_b) = \max\{\mho_{\Xi}^{\mathcal{F}}(\xi_a), \mho_{\Delta}^{\mathcal{F}}(\xi_b)\}$.

Definition 2.5. [?] An FS Ξ of a BS $(\mathcal{T}, \oslash_1, \oslash_2, \oslash_3)$ is said to be a fuzzy subbisemiring (FSBS) of $\mathcal{T}$ if $\mho_\Xi(\xi_a \oslash_1 \xi_b) \geq \min\{\mho_\Xi(\xi_a), \mho_\Xi(\xi_b)\}, \mho_\Xi(\xi_a \oslash_2 \xi_b) \geq \min\{\mho_\Xi(\xi_a), \mho_\Xi(\xi_b)\}, \mho_\Xi(\xi_a \oslash_3 \xi_b) \geq \min\{\mho_\Xi(\xi_a), \mho_\Xi(\xi_b)\}$, $\forall \xi_a, \xi_b \in \mathcal{T}$.

Definition 2.6. [?] An FS Ξ of a BS $(\mathcal{T}, \oslash_1, \oslash_2, \oslash_3)$ is said to be a fuzzy normal subbisemiring (FNSBS) of $\mathcal{T}$ if $\mho_\Xi(\xi_a \oslash_1 \xi_b) = \mho_\Xi(\xi_b \oslash_1 \xi_a), \mho_\Xi(\xi_a \oslash_2 \xi_b) = \mho_\Xi(\xi_b \oslash_2 \xi_a), \mho_\Xi(\xi_a \oslash_3 \xi_b) = \mho_\Xi(\xi_b \oslash_3 \xi_a)$, $\forall \xi_a, \xi_b \in \mathcal{T}$.

Definition 2.7. [6] Let $(\mathcal{T}, +, \cdot, \times)$ and $(\mathcal{T}_1, \circledast, \circ, \otimes)$ be two BSs. A mapping $\kappa : \mathcal{T} \to \mathcal{T}_1$ is said to be a homomorphism if $\kappa(\xi_a + \xi_b) = \kappa(\xi_a) \circledast \kappa(\xi_b), \kappa(\xi_a \cdot \xi_b) = \kappa(\xi_a) \circ \kappa(\xi_b), \kappa(\xi_a \times \xi_b) = \kappa(\xi_a) \otimes \kappa(\xi_b)$, $\forall \xi_a, \xi_b \in \mathcal{T}$.

3. Diophantine Neutrosophic Subbisemirings

In the following, let $\mathcal{T}$ denote a BS unless otherwise stated.

Definition 3.1. A DioNSS Ξ in $\mathcal{U}$ is $\Xi = \Big\{\epsilon, \Big(\mho^{\mathcal{T}}_\Xi(\epsilon), \mho^{\mathcal{I}}_\Xi(\epsilon), \mho^{\mathcal{F}}_\Xi(\epsilon)\Big), \Big(\Gamma_\Xi(\epsilon), \Lambda_\Xi(\epsilon), \Theta_\Xi(\epsilon)\Big) \mid \epsilon \in \mathcal{U}\Big\}$, where $\mho^{\mathcal{T}}_\Xi(\epsilon)$, $\mho^{\mathcal{I}}_\Xi(\epsilon)$ $\mho^{\mathcal{F}}_\Xi(\epsilon)$ represents the degree of truth-membership, degree of indeterminacy membership and degree of falsity-membership of Ξ, respectively, and $\Gamma_\Xi(\epsilon) + \Lambda_\Xi(\epsilon) + \Theta_\Xi(\epsilon) \leq 1$. The mapping $\mho^{\mathcal{T}}_\Xi, \mho^{\mathcal{I}}_\Xi, \mho^{\mathcal{F}}_\Xi : \mathcal{U} \to [0,1]$ and $0 \leq (\Gamma_\Xi(\epsilon) \cdot \mho^{\mathcal{T}}_\Xi(\epsilon)) + (\Lambda_\Xi(\epsilon) \cdot \mho^{\mathcal{I}}_\Xi(\epsilon)) + (\Theta_\Xi(\epsilon) \cdot \mho^{\mathcal{F}}_\Xi(\epsilon)) \leq 2$.

Definition 3.2. A DioNSS Ξ of $\mathcal{T}$ is said to be a DioNSBS of $\mathcal{T}$ if $(\forall \zeta, \eta \in \mathcal{T})$

$$\begin{Bmatrix} \mho^{\mathcal{T}}_\Xi(\zeta \oslash_1 \eta) \geq \min\{\mho^{\mathcal{T}}_\Xi(\zeta), \mho^{\mathcal{T}}_\Xi(\eta)\} \\ \mho^{\mathcal{T}}_\Xi(\zeta \oslash_2 \eta) \geq \min\{\mho^{\mathcal{T}}_\Xi(\zeta), \mho^{\mathcal{T}}_\Xi(\eta)\} \\ \mho^{\mathcal{T}}_\Xi(\zeta \oslash_3 \eta) \geq \min\{\mho^{\mathcal{T}}_\Xi(\zeta), \mho^{\mathcal{T}}_\Xi(\eta)\} \end{Bmatrix} \quad \begin{Bmatrix} \mho^{\mathcal{I}}_\Xi(\zeta \oslash_1 \eta) \geq \frac{\mho^{\mathcal{I}}_\Xi(\zeta) + \mho^{\mathcal{I}}_\Xi(\eta)}{2} \\ \text{OR} \\ \mho^{\mathcal{I}}_\Xi(\zeta \oslash_2 \eta) \geq \frac{\mho^{\mathcal{I}}_\Xi(\zeta) + \mho^{\mathcal{I}}_\Xi(\eta)}{2} \\ \text{OR} \\ \mho^{\mathcal{I}}_\Xi(\zeta \oslash_3 \eta) \geq \frac{\mho^{\mathcal{I}}_\Xi(\zeta) + \mho^{\mathcal{I}}_\Xi(\eta)}{2} \end{Bmatrix}$$

$$\begin{Bmatrix} \mho^{\mathcal{F}}_\Xi(\zeta \oslash_1 \eta) \leq \max\{\mho^{\mathcal{F}}_\Xi(\zeta), \mho^{\mathcal{F}}_\Xi(\eta)\} \\ \mho^{\mathcal{F}}_\Xi(\zeta \oslash_2 \eta) \leq \max\{\mho^{\mathcal{F}}_\Xi(\zeta), \mho^{\mathcal{F}}_\Xi(\eta)\} \\ \mho^{\mathcal{F}}_\Xi(\zeta \oslash_3 \eta) \leq \max\{\mho^{\mathcal{F}}_\Xi(\zeta), \mho^{\mathcal{F}}_\Xi(\eta)\} \end{Bmatrix}$$

$$\begin{Bmatrix} \Gamma_\Xi(\zeta \oslash_1 \eta) \geq \min\{\Gamma_\Xi(\zeta), \Gamma_\Xi(\eta)\} \\ \Gamma_\Xi(\zeta \oslash_2 \eta) \geq \min\{\Gamma_\Xi(\zeta), \Gamma_\Xi(\eta)\} \\ \Gamma_\Xi(\zeta \oslash_3 \eta) \geq \min\{\Gamma_\Xi(\zeta), \Gamma_\Xi(\eta)\} \end{Bmatrix} \quad \begin{Bmatrix} \Lambda_\Xi(\zeta \oslash_1 \eta) \geq \frac{\Lambda_\Xi(\zeta) + \Lambda_\Xi(\eta)}{2} \\ \text{OR} \\ \Lambda_\Xi(\zeta \oslash_2 \eta) \geq \frac{\Lambda_\Xi(\zeta) + \Lambda_\Xi(\eta)}{2} \\ \text{OR} \\ \Lambda_\Xi(\zeta \oslash_3 \eta) \geq \frac{\Lambda_\Xi(\zeta) + \Lambda_\Xi(\eta)}{2} \end{Bmatrix}$$

$$\begin{Bmatrix} \Theta_\Xi(\zeta \oslash_1 \eta) \leq \max\{\Theta_\Xi(\zeta), \Theta_\Xi(\eta)\} \\ \Theta_\Xi(\zeta \oslash_2 \eta) \leq \max\{\Theta_\Xi(\zeta), \Theta_\Xi(\eta)\} \\ \Theta_\Xi(\zeta \oslash_3 \eta) \leq \max\{\Theta_\Xi(\zeta), \Theta_\Xi(\eta)\} \end{Bmatrix}.$$

Example 3.3. Let $\mathcal{T} = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ be the BS with the tables:

$\oslash_1$	θ_a	θ_b	θ_c	θ_d
θ_a	θ_a	θ_a	θ_a	θ_a
θ_b	θ_a	θ_b	θ_a	θ_b
θ_c	θ_a	θ_a	θ_c	θ_c
θ_d	θ_a	θ_b	θ_c	θ_d

$\oslash_2$	θ_a	θ_b	θ_c	θ_d
θ_a	θ_a	θ_b	θ_c	θ_d
θ_b	θ_b	θ_b	θ_d	θ_d
θ_c	θ_c	θ_d	θ_c	θ_d
θ_d	θ_d	θ_d	θ_d	θ_d

$\oslash_3$	θ_a	θ_b	θ_c	θ_d
θ_a	θ_a	θ_a	θ_a	θ_a
θ_b	θ_a	θ_b	θ_c	θ_d
θ_c	θ_d	θ_d	θ_d	θ_d
θ_d	θ_d	θ_d	θ_d	θ_d

	$\theta = \theta_a$	$\theta = \theta_b$	$\theta = \theta_c$	$\theta = \theta_d$
$\left(\mho_\Xi^{\mathcal{T}}(\theta), \Gamma_\Xi(\theta)\right)$	$(0.97, 0.40)$	$(0.95, 0.35)$	$(0.92, 0.25)$	$(0.94, 0.30)$
$\left(\mho_\Xi^{\mathcal{I}}(\theta), \Lambda_\Xi(\theta)\right)$	$(0.80, 0.25)$	$(0.78, 0.20)$	$(0.73, 0.10)$	$(0.75, 0.15)$
$\left(\mho_\Xi^{\mathcal{F}}(\theta), \Theta_\Xi(\theta)\right)$	$(0.85, 0.30)$	$(0.89, 0.35)$	$(0.91, 0.45)$	$(0.90, 0.40)$

Clearly, Ξ is a DioNSBS of $\mathcal{T}$.

Theorem 3.4. *The intersection of a family of DioNSBSs of $\mathcal{T}$ is a DioNSBS of $\mathcal{T}$.*

Proof. Let $\{Z_i : i \in \mathcal{I}\}$ be a family of DioNSBSs of $\mathcal{T}$ and $\Xi = \bigcap_{i \in \mathcal{I}} Z_i$.
Let ζ and η in $\mathcal{T}$. Then

$$\begin{aligned} \mho_\Xi^{\mathcal{T}}(\zeta \oslash_1 \eta) &= \inf_{i \in \mathcal{I}} \mho_{Z_i}^{\mathcal{T}}(\zeta \oslash_1 \eta) \\ &\geq \inf_{i \in \mathcal{I}} \min\{\mho_{Z_i}^{\mathcal{T}}(\zeta), \mho_{Z_i}^{\mathcal{T}}(\eta)\} \\ &= \min\left\{\inf_{i \in \mathcal{I}} \mho_{Z_i}^{\mathcal{T}}(\zeta), \inf_{i \in \mathcal{I}} \mho_{Z_i}^{\mathcal{T}}(\eta)\right\} \\ &= \min\{\mho_\Xi^{\mathcal{T}}(\zeta), \mho_\Xi^{\mathcal{T}}(\eta)\}. \end{aligned}$$

Similarly, $\mho_\Xi^{\mathcal{T}}(\zeta \oslash_2 \eta) \geq \min\{\mho_\Xi^{\mathcal{T}}(\zeta), \mho_\Xi^{\mathcal{T}}(\eta)\}$, $\mho_\Xi^{\mathcal{T}}(\zeta \oslash_3 \eta) \geq \min\{\mho_\Xi^{\mathcal{T}}(\zeta), \mho_\Xi^{\mathcal{T}}(\eta)\}$. Now,

$$\begin{aligned} \mho_\Xi^{\mathcal{I}}(\zeta \oslash_1 \eta) &= \inf_{i \in \mathcal{I}} \mho_{Z_i}^{\mathcal{I}}(\zeta \oslash_1 \eta) \\ &\geq \inf_{i \in \mathcal{I}} \frac{\mho_{Z_i}^{\mathcal{I}}(\zeta) + \mho_{Z_i}^{\mathcal{I}}(\eta)}{2} \\ &= \frac{\inf_{i \in \mathcal{I}} \mho_{Z_i}^{\mathcal{I}}(\zeta) + \inf_{i \in \mathcal{I}} \mho_{Z_i}^{\mathcal{I}}(\eta)}{2} \\ &= \frac{\mho_\Xi^{\mathcal{I}}(\zeta) + \mho_\Xi^{\mathcal{I}}(\eta)}{2}. \end{aligned}$$

Similarly, $\mho^{\mathcal{I}}_{\Xi}(\zeta \oslash_2 \eta) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta)+\mho^{\mathcal{I}}_{\Xi}(\eta)}{2}$ and $\mho^{\mathcal{I}}_{\Xi}(\zeta \oslash_3 \eta) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta)+\mho^{\mathcal{I}}_{\Xi}(\eta)}{2}$. Now,

$$\begin{aligned}
\mho^{\mathcal{F}}_{\Xi}(\zeta \oslash_1 \eta) &= \sup_{i\in\mathcal{I}} \mho_{Z_i}(\zeta \oslash_1 \eta) \\
&\leq \sup_{i\in\mathcal{I}} \max\{\mho_{Z_i}(\zeta), \mho_{Z_i}(\eta)\} \\
&= \max\left\{\sup_{i\in\mathcal{I}} \mho_{Z_i}(\zeta), \sup_{i\in\mathcal{I}} \mho_{Z_i}(\eta)\right\} \\
&= \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\}.
\end{aligned}$$

Similarly, $\mho^{\mathcal{F}}_{\Xi}(\zeta \oslash_2 \eta) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\}$, $\mho^{\mathcal{F}}_{\Xi}(\zeta \oslash_3 \eta) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\}$.

$$\begin{aligned}
\Gamma_{\Xi}(\zeta \oslash_1 \eta) &= \inf_{i\in\mathcal{I}} \Gamma_{Z_i}(\zeta \oslash_1 \eta) \\
&\geq \inf_{i\in\mathcal{I}} \min\{\Gamma_{Z_i}(\zeta), \Gamma_{Z_i}(\eta)\} \\
&= \min\left\{\inf_{i\in\mathcal{I}} \Gamma_{Z_i}(\zeta), \inf_{i\in\mathcal{I}} \Gamma_{Z_i}(\eta)\right\} \\
&= \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\}.
\end{aligned}$$

Similarly, $\Gamma_{\Xi}(\zeta \oslash_2 \eta) \geq \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\}$, $\Gamma_{\Xi}(\zeta \oslash_3 \eta) \geq \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\}$. Now,

$$\begin{aligned}
\Lambda_{\Xi}(\zeta \oslash_1 \eta) &= \inf_{i\in\mathcal{I}} \Lambda_{Z_i}(\zeta \oslash_1 \eta) \\
&\geq \inf_{i\in\mathcal{I}} \frac{\Lambda_{Z_i}(\zeta) + \Lambda_{Z_i}(\eta)}{2} \\
&= \frac{\inf_{i\in\mathcal{I}} \Lambda_{Z_i}(\zeta) + \inf_{i\in\mathcal{I}} \Lambda_{Z_i}(\eta)}{2} \\
&= \frac{\Lambda_{\Xi}(\zeta) + \Lambda_{\Xi}(\eta)}{2}.
\end{aligned}$$

Similarly, $\Lambda_{\Xi}(\zeta \oslash_2 \eta) \geq \frac{\Lambda_{\Xi}(\zeta)+\Lambda_{\Xi}(\eta)}{2}$ and $\Lambda_{\Xi}(\zeta \oslash_3 \eta) \geq \frac{\Lambda_{\Xi}(\zeta)+\Lambda_{\Xi}(\eta)}{2}$. Now,

$$\begin{aligned}
\Theta_{\Xi}(\zeta \oslash_1 \eta) &= \sup_{i\in\mathcal{I}} \Theta_{Z_i}(\zeta \oslash_1 \eta) \\
&\leq \sup_{i\in\mathcal{I}} \max\{\Theta_{Z_i}(\zeta), \Theta_{Z_i}(\eta)\} \\
&= \max\left\{\sup_{i\in\mathcal{I}} \Theta_{Z_i}(\zeta), \sup_{i\in\mathcal{I}} \Theta_{Z_i}(\eta)\right\} \\
&= \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\}.
\end{aligned}$$

Similarly, $\Theta_{\Xi}(\zeta \oslash_2 \eta) \leq \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\}$, $\Theta_{\Xi}(\zeta \oslash_3 \eta) \leq \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\}$. Hence Ξ is a DioNSBS of $\mathcal{T}$.

Theorem 3.5. *If Ξ and Δ are two DioNSBSs of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, then $\Xi \times \Delta$ is a DioNSBS of $\mathcal{T}_1 \times \mathcal{T}_2$.*

Proof. Let Ξ and Δ be two DioNSBSs of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let $\zeta_1, \zeta_2 \in \mathcal{T}_1$ and $\eta_1, \eta_2 \in \mathcal{T}_2$. Then (ζ_1, η_1) and (ζ_2, η_2) are in $\mathcal{T}_1 \times \mathcal{T}_2$. Now,

$$\begin{aligned}
\mho^{\mathcal{T}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_1(\zeta_2,\eta_2)] &= \mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_1\oslash_1\zeta_2, \eta_1\oslash_1\eta_2)\\
&= \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1\oslash_1\zeta_2), \mho^{\mathcal{T}}_{\Delta}(\eta_1\oslash_1\eta_2)\}\\
&\geq \min\{\min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1), \mho^{\mathcal{T}}_{\Xi}(\zeta_2)\}, \min\{\mho^{\mathcal{T}}_{\Delta}(\eta_1), \mho^{\mathcal{T}}_{\Delta}(\eta_2)\}\}\\
&= \min\{\min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1), \mho^{\mathcal{T}}_{\Delta}(\eta_1)\}, \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_2), \mho^{\mathcal{T}}_{\Delta}(\eta_2)\}\}\\
&= \min\{\mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}.
\end{aligned}$$

Also, $\mho^{\mathcal{T}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_2(\zeta_2,\eta_2)] \geq \min\{\mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$ and $\mho^{\mathcal{T}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_3(\zeta_2,\eta_2)] \geq \min\{\mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$. Now,

$$\begin{aligned}
\mho^{\mathcal{I}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_1(\zeta_2,\eta_2)] &= \mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_1\oslash_1\zeta_2, \eta_1\oslash_1\eta_2)\\
&= \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1\oslash_1\zeta_2) + \mho^{\mathcal{I}}_{\Delta}(\eta_1\oslash_1\eta_2)}{2}\\
&\geq \frac{1}{2}\left[\frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1) + \mho^{\mathcal{I}}_{\Xi}(\zeta_2)}{2} + \frac{\mho^{\mathcal{I}}_{\Delta}(\eta_1) + \mho^{\mathcal{I}}_{\Delta}(\eta_2)}{2}\right]\\
&= \frac{1}{2}\left[\frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1) + \mho^{\mathcal{I}}_{\Delta}(\eta_1)}{2} + \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_2) + \mho^{\mathcal{I}}_{\Delta}(\eta_2)}{2}\right]\\
&= \frac{1}{2}\Big[\mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_1,\eta_1) + \mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big].
\end{aligned}$$

Also, $\mho^{\mathcal{I}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_2(\zeta_2,\eta_2)] \geq \frac{1}{2}\Big[\mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_1,\eta_1) + \mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big]$ and $\mho^{\mathcal{I}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_3(\zeta_2,\eta_2)] \geq \frac{1}{2}\Big[\mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_1,\eta_1) + \mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big]$. Now,

$$\begin{aligned}
\mho^{\mathcal{F}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_1(\zeta_2,\eta_2)] &= \mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_1\oslash_1\zeta_2, \eta_1\oslash_1\eta_2)\\
&= \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1\oslash_1\zeta_2), \mho^{\mathcal{F}}_{\Delta}(\eta_1\oslash_1\eta_2)\}\\
&\leq \max\{\max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2)\}, \max\{\mho^{\mathcal{F}}_{\Delta}(\eta_1), \mho^{\mathcal{F}}_{\Delta}(\eta_2)\}\}\\
&= \max\{\max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Delta}(\eta_1)\}, \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_2), \mho^{\mathcal{F}}_{\Delta}(\eta_2)\}\}\\
&= \max\{\mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}.
\end{aligned}$$

Also, $\mho^{\mathcal{F}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_2(\zeta_2,\eta_2)] \leq \max\{\mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$ and $\mho^{\mathcal{F}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_3(\zeta_2,\eta_2)] \leq \max\{\mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$.

$$\begin{aligned}
\Gamma_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_1(\zeta_2,\eta_2)] &= \Gamma_{\Xi\times\Delta}(\zeta_1\oslash_1\zeta_2, \eta_1\oslash_1\eta_2)\\
&= \min\{\Gamma_{\Xi}(\zeta_1\oslash_1\zeta_2), \Gamma_{\Delta}(\eta_1\oslash_1\eta_2)\}\\
&\geq \min\{\min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}, \min\{\Gamma_{\Delta}(\eta_1), \Gamma_{\Delta}(\eta_2)\}\}\\
&= \min\{\min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Delta}(\eta_1)\}, \min\{\Gamma_{\Xi}(\zeta_2), \Gamma_{\Delta}(\eta_2)\}\}\\
&= \min\{\Gamma_{\Xi\times\Delta}(\zeta_1,\eta_1), \Gamma_{\Xi\times\Delta}(\zeta_2,\eta_2)\}.
\end{aligned}$$

Also, $\Gamma_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_2 (\zeta_2,\eta_2)] \geq \min\{\Gamma_{\Xi\times\Delta}(\zeta_1,\eta_1),\Gamma_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$ and $\Gamma_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_3 (\zeta_2,\eta_2)] \geq \min\{\Gamma_{\Xi\times\Delta}(\zeta_1,\eta_1),\Gamma_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$. Now,

$$\begin{aligned}\Lambda_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_1(\zeta_2,\eta_2)] &= \Lambda_{\Xi\times\Delta}(\zeta_1\oslash_1\zeta_2,\eta_1\oslash_1\eta_2)\\ &= \frac{\Lambda_\Xi(\zeta_1\oslash_1\zeta_2)+\Lambda_\Delta(\eta_1\oslash_1\eta_2)}{2}\\ &\geq \frac{1}{2}\Big[\frac{\Lambda_\Xi(\zeta_1)+\Lambda_\Xi(\zeta_2)}{2}+\frac{\Lambda_\Delta(\eta_1)+\Lambda_\Delta(\eta_2)}{2}\Big]\\ &= \frac{1}{2}\Big[\frac{\Lambda_\Xi(\zeta_1)+\Lambda_\Delta(\eta_1)}{2}+\frac{\Lambda_\Xi(\zeta_2)+\Lambda_\Delta(\eta_2)}{2}\Big]\\ &= \frac{1}{2}\Big[\Lambda_{\Xi\times\Delta}(\zeta_1,\eta_1)+\Lambda_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big].\end{aligned}$$

Also, $\Lambda_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_2 (\zeta_2,\eta_2)] \geq \frac{1}{2}\Big[\Lambda_{\Xi\times\Delta}(\zeta_1,\eta_1)+\Lambda_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big]$ and $\Lambda_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_3 (\zeta_2,\eta_2)] \geq \frac{1}{2}\Big[\Lambda_{\Xi\times\Delta}(\zeta_1,\eta_1)+\Lambda_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big]$. Now,

$$\begin{aligned}\Theta_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_1(\zeta_2,\eta_2)] &= \Theta_{\Xi\times\Delta}(\zeta_1\oslash_1\zeta_2,\eta_1\oslash_1\eta_2)\\ &= \max\{\Theta_\Xi(\zeta_1\oslash_1\zeta_2),\Theta_\Delta(\eta_1\oslash_1\eta_2)\}\\ &\leq \max\{\max\{\Theta_\Xi(\zeta_1),\Theta_\Xi(\zeta_2)\},\max\{\Theta_\Delta(\eta_1),\Theta_\Delta(\eta_2)\}\}\\ &= \max\{\max\{\Theta_\Xi(\zeta_1),\Theta_\Delta(\eta_1)\},\max\{\Theta_\Xi(\zeta_2),\Theta_\Delta(\eta_2)\}\}\\ &= \max\{\Theta_{\Xi\times\Delta}(\zeta_1,\eta_1),\Theta_{\Xi\times\Delta}(\zeta_2,\eta_2)\}.\end{aligned}$$

Also, $\Theta_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_2 (\zeta_2,\eta_2)] \leq \max\{\Theta_{\Xi\times\Delta}(\zeta_1,\eta_1),\Theta_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$ and $\Theta_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_3 (\zeta_2,\eta_2)] \leq \max\{\Theta_{\Xi\times\Delta}(\zeta_1,\eta_1),\Theta_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$. Hence $\Xi\times\Delta$ is a DioNSBS of $\mathcal{T}$.

Corollary 3.6. *If $\Xi_1,\Xi_2,\ldots,\Xi_n$ are DioNSBSs of $\mathcal{T}_1,\mathcal{T}_2,\ldots,\mathcal{T}_n$, respectively, then $\Xi_1\times\Xi_2\times\ldots\times\Xi_n$ is a DioNSBS of $\mathcal{T}_1\times\mathcal{T}_2\times\ldots\times\mathcal{T}_n$.*

Definition 3.7. Let Ξ be a DioNSS in $\mathcal{T}$, the strongest Diophantine neutrosophic relation on $\mathcal{T}$. That is a Diophantine neutrosophic relation on Ξ is Z given by

$$\left\{\begin{array}{l}\mho_Z^{\mathcal{T}}(\zeta,\eta)=\min\{\mho_Z^{\mathcal{T}}(\zeta),\mho_Z^{\mathcal{T}}(\eta)\}\\ \mho_Z^{\mathcal{I}}(\zeta,\eta)=\frac{\mho_Z^{\mathcal{I}}(\zeta)+\mho_Z^{\mathcal{I}}(\eta)}{2}\\ \mho_Z^{\mathcal{F}}(\zeta,\eta)=\max\{\mho_Z^{\mathcal{F}}(\zeta),\mho_Z^{\mathcal{F}}(\eta)\}\end{array}\right\}\quad\left\{\begin{array}{l}\Gamma_Z(\zeta,\eta)=\min\{\Gamma_Z(\zeta),\Gamma_Z(\eta)\}\\ \Lambda_Z(\zeta,\eta)=\frac{\Lambda_Z(\zeta)+\Lambda_Z(\eta)}{2}\\ \Theta_Z(\zeta,\eta)=\max\{\Theta_Z(\zeta),\Theta_Z(\eta)\}\end{array}\right\}.$$

Theorem 3.8. *Let Ξ be the DioNSBS of $\mathcal{T}$ and Z be the strongest Diophantine neutrosophic relation of $\mathcal{T}$. Then Ξ is a DioNSBS of $\mathcal{T}$ if and only if Z is a DioNSBS of $\mathcal{T}\times\mathcal{T}$.*

Proof. Let Ξ be the DioNSBS of $\mathcal{T}$ and Z be the strongest Diophantine neutrosophic relation of $\mathcal{T}$. Then for any $\zeta = (\zeta_1, \zeta_2)$ and $\eta = (\eta_1, \eta_2)$ are in $\mathcal{T} \times \mathcal{T}$. We have

$$\begin{aligned}
\mho_Z^{\mathcal{T}}(\zeta \oslash_1 \eta) &= \mho_Z^{\mathcal{T}}[((\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \mho_Z^{\mathcal{T}}(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \min\{\mho_\Xi^{\mathcal{T}}(\zeta_1 \oslash_1 \eta_1), \mho_\Xi^{\mathcal{T}}(\zeta_2 \oslash_1 \eta_2)\} \\
&\geq \min\{\min\{\mho_\Xi^{\mathcal{T}}(\zeta_1), \mho_\Xi^{\mathcal{T}}(\eta_1)\}, \min\{\mho_\Xi^{\mathcal{T}}(\zeta_2), \mho_\Xi^{\mathcal{T}}(\eta_2)\}\} \\
&= \min\{\min\{\mho_\Xi^{\mathcal{T}}(\zeta_1), \mho_\Xi^{\mathcal{T}}(\zeta_2)\}, \min\{\mho_\Xi^{\mathcal{T}}(\eta_1), \mho_\Xi^{\mathcal{T}}(\eta_2)\}\} \\
&= \min\{\mho_Z^{\mathcal{T}}(\zeta_1, \zeta_2), \mho_Z^{\mathcal{T}}(\eta_1, \eta_2)\} \\
&= \min\{\mho_Z^{\mathcal{T}}(\zeta), \mho_Z^{\mathcal{T}}(\eta)\}.
\end{aligned}$$

Also, $\mho_Z^{\mathcal{T}}(\zeta \oslash_2 \eta) \geq \min\{\mho_Z^{\mathcal{T}}(\zeta), \mho_Z^{\mathcal{T}}(\eta)\}, \mho_Z^{\mathcal{T}}(\zeta \oslash_3 \eta) \geq \min\{\mho_Z^{\mathcal{T}}(\zeta), \mho_Z^{\mathcal{T}}(\eta)\}$. Now,

$$\begin{aligned}
\mho_Z^{\mathcal{I}}(\zeta \oslash_1 \eta) &= \mho_Z^{\mathcal{I}}[((\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \mho_Z^{\mathcal{I}}(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \frac{\mho_\Xi^{\mathcal{I}}(\zeta_1 \oslash_1 \eta_1) + \mho_\Xi^{\mathcal{I}}(\zeta_2 \oslash_1 \eta_2)}{2} \\
&\geq \frac{1}{2}\left[\frac{\mho_\Xi^{\mathcal{I}}(\zeta_1) + \mho_\Xi^{\mathcal{I}}(\eta_1)}{2} + \frac{\mho_\Xi^{\mathcal{I}}(\zeta_2) + \mho_\Xi^{\mathcal{I}}(\eta_2)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\mho_\Xi^{\mathcal{I}}(\zeta_1) + \mho_\Xi^{\mathcal{I}}(\zeta_2)}{2} + \frac{\mho_\Xi^{\mathcal{I}}(\eta_1) + \mho_\Xi^{\mathcal{I}}(\eta_2)}{2}\right] \\
&= \frac{\mho_Z^{\mathcal{I}}(\zeta_1, \zeta_2) + \mho_Z^{\mathcal{I}}(\eta_1, \eta_2)}{2} \\
&= \frac{\mho_Z^{\mathcal{I}}(\zeta) + \mho_Z^{\mathcal{I}}(\eta)}{2}.
\end{aligned}$$

Also, $\mho_Z^{\mathcal{I}}(\zeta \oslash_2 \eta) \geq \frac{\mho_Z^{\mathcal{I}}(\zeta)+\mho_Z^{\mathcal{I}}(\eta)}{2}$ and $\mho_Z^{\mathcal{I}}(\zeta \oslash_3 \eta) \geq \frac{\mho_Z^{\mathcal{I}}(\zeta)+\mho_Z^{\mathcal{I}}(\eta)}{2}$. Similarly, $\mho_Z^{\mathcal{F}}(\zeta \oslash_1 \eta) \leq \max\{\mho_Z^{\mathcal{F}}(\zeta), \mho_Z^{\mathcal{F}}(\eta)\}, \mho_Z^{\mathcal{F}}(\zeta \oslash_2 \eta) \leq \max\{\mho_Z^{\mathcal{F}}(\zeta), \mho_Z^{\mathcal{F}}(\eta)\}$ and $\mho_Z^{\mathcal{F}}(\zeta \oslash_3 \eta) \leq \max\{\mho_Z^{\mathcal{F}}(\zeta), \mho_Z^{\mathcal{F}}(\eta)\}$. Now,

$$\begin{aligned}
\Gamma_Z(\zeta \oslash_1 \eta) &= \Gamma_{\Xi Z}[((\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \Gamma_Z(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \min\{\Gamma_\Xi(\zeta_1 \oslash_1 \eta_1), \Gamma_\Xi(\zeta_2 \oslash_1 \eta_2)\} \\
&\geq \min\{\min\{\Gamma_\Xi(\zeta_1), \Gamma_\Xi(\eta_1)\}, \min\{\Gamma_\Xi(\zeta_2), \Gamma_\Xi(\eta_2)\}\} \\
&= \min\{\min\{\Gamma_\Xi(\zeta_1), \Gamma_\Xi(\zeta_2)\}, \min\{\Gamma_\Xi(\eta_1), \Gamma_\Xi(\eta_2)\}\} \\
&= \min\{\Gamma_Z(\zeta_1, \zeta_2), \Gamma_Z(\eta_1, \eta_2)\} \\
&= \min\{\Gamma_Z(\zeta), \Gamma_Z(\eta)\}.
\end{aligned}$$

Also, $\Gamma_Z(\zeta \oslash_2 \eta) \geq \min\{\Gamma_Z(\zeta), \Gamma_Z(\eta)\}$ and $\Gamma_Z(\zeta \oslash_3 \eta) \geq \min\{\Gamma_Z(\zeta), \Gamma_Z(\eta)\}$. Now,

$$\begin{aligned}
\Lambda_Z(\zeta \oslash_1 \eta) &= \Lambda_Z[((\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \Lambda_Z(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \frac{\Lambda_\Xi(\zeta_1 \oslash_1 \eta_1) + \Lambda_\Xi(\zeta_2 \oslash_1 \eta_2)}{2} \\
&\geq \frac{1}{2}\left[\frac{\Lambda_\Xi(\zeta_1) + \Lambda_\Xi(\eta_1)}{2} + \frac{\Lambda_\Xi(\zeta_2) + \Lambda_\Xi(\eta_2)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\Lambda_\Xi(\zeta_1) + \Lambda_\Xi(\zeta_2)}{2} + \frac{\Lambda_\Xi(\eta_1) + \Lambda_\Xi(\eta_2)}{2}\right] \\
&= \frac{\Lambda_Z(\zeta_1, \zeta_2) + \Lambda_Z(\eta_1, \eta_2)}{2} \\
&= \frac{\Lambda_Z(\zeta) + \Lambda_Z(\eta)}{2}.
\end{aligned}$$

Also, $\Lambda_Z(\zeta \oslash_2 \eta) \geq \frac{\Lambda_Z(\zeta)+\Lambda_Z(\eta)}{2}$ and $\Lambda_Z(\zeta \oslash_3 \eta) \geq \frac{\Lambda_Z(\zeta)+\Lambda_Z(\eta)}{2}$. Similarly, $\Theta_Z(\zeta \oslash_1 \eta) \leq \max\{\Theta_Z(\zeta), \Theta_Z(\eta)\}, \Theta_Z(\zeta \oslash_2 \eta) \leq \max\{\Theta_Z(\zeta), \Theta_Z(\eta)\}$ and $\Theta_Z(\zeta \oslash_3 \eta) \leq \max\{\Theta_Z(\zeta), \Theta_Z(\eta)\}$. Hence Z is a DioNSBS of $\mathcal{T} \times \mathcal{T}$.

Conversely, assume that Z is a DioNSBS of $\mathcal{T} \times \mathcal{T}$, then for any $\zeta = (\zeta_1, \zeta_2)$ and $\eta = (\eta_1, \eta_2)$ are in $\mathcal{T} \times \mathcal{T}$. We have

$$\begin{aligned}
\min\{\mho_\Xi^\mathcal{T}(\zeta_1 \oslash_1 \eta_1), \mho_\Xi^\mathcal{T}(\zeta_2 \oslash_1 \eta_2)\} &= \mho_Z^\mathcal{T}(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \mho_Z^\mathcal{T}[(\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \mho_Z^\mathcal{T}(\zeta \oslash_1 \eta) \\
&\geq \min\{\mho_Z^\mathcal{T}(\zeta), \mho_Z^\mathcal{T}(\eta)\} \\
&= \min\{\mho_Z^\mathcal{T}(\zeta_1, \zeta_2)\}, \mho_Z^\mathcal{T}(\eta_1, \eta_2)\} \\
&= \min\{\min\{\mho_\Xi^\mathcal{T}(\zeta_1), \mho_\Xi^\mathcal{T}(\zeta_2)\}, \min\{\mho_\Xi^\mathcal{T}(\eta_1), \mho_\Xi^\mathcal{T}(\eta_2)\}\}.
\end{aligned}$$

If $\mho_\Xi^\mathcal{T}(\zeta_1 \oslash_1 \eta_1) \leq \mho_\Xi^\mathcal{T}(\zeta_2 \oslash_1 \eta_2)$, then $\mho_\Xi^\mathcal{T}(\zeta_1) \leq \mho_\Xi^\mathcal{T}(\zeta_2)$ and $\mho_\Xi^\mathcal{T}(\eta_1) \leq \mho_\Xi^\mathcal{T}(\eta_2)$. We get $\mho_\Xi^\mathcal{T}(\zeta_1 \oslash_1 \eta_1) \geq \min\{\mho_\Xi^\mathcal{T}(\zeta_1), \mho_\Xi^\mathcal{T}(\eta_1)\}$ $\forall \zeta_1, \eta_1 \in \mathcal{T}$ and $\min\{\mho_\Xi^\mathcal{T}(\zeta_1 \oslash_2 \eta_1), \mho_\Xi^\mathcal{T}(\zeta_2 \oslash_2 \eta_2)\} \geq \min\{\min\{\mho_\Xi^\mathcal{T}(\zeta_1), \mho_\Xi^\mathcal{T}(\zeta_2)\}, \min\{\mho_\Xi^\mathcal{T}(\eta_1), \mho_\Xi^\mathcal{T}(\eta_2)\}\}$. If $\mho_\Xi^\mathcal{T}(\zeta_1 \oslash_2 \eta_1) \leq \mho_\Xi^\mathcal{T}(\zeta_2 \oslash_2 \eta_2)$, then $\mho_\Xi^\mathcal{T}(\zeta_1 \oslash_2 \eta_1) \geq \min\{\mho_\Xi^\mathcal{T}(\zeta_1), \mho_\Xi^\mathcal{T}(\eta_1)\}$. So, $\min\{\mho_\Xi^\mathcal{T}(\zeta_1 \oslash_3 \eta_1), \mho_\Xi^\mathcal{T}(\zeta_2 \oslash_3 \eta_2)\} \geq \min\{\min\{\mho_\Xi^\mathcal{T}(\zeta_1), \mho_\Xi^\mathcal{T}(\zeta_2)\}, \min\{\mho_\Xi^\mathcal{T}(\eta_1), \mho_\Xi^\mathcal{T}(\eta_2)\}\}$. If $\mho_\Xi^\mathcal{T}(\zeta_1 \oslash_3 \eta_1) \leq \mho_\Xi^\mathcal{T}(\zeta_2 \oslash_3 \eta_2)$, then

$\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_3 \eta_1) \geq \min\{\mho^{\mathcal{I}}_{\Xi}(\zeta_1), \mho^{\mathcal{I}}_{\Xi}(\eta_1)\}$. Now,

$$\begin{aligned}
\frac{1}{2}\Big[\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_1 \eta_1) + \mho^{\mathcal{I}}_{\Xi}(\zeta_2 \oslash_1 \eta_2)\Big] &= \mho^{\mathcal{I}}_{Z}(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2)\\
&= \mho^{\mathcal{I}}_{Z}[(\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)]\\
&= \mho^{\mathcal{I}}_{Z}(\zeta \oslash_1 \eta)\\
&\geq \frac{\mho^{\mathcal{I}}_{Z}(\zeta) + \mho^{\mathcal{I}}_{Z}(\eta)}{2}\\
&= \frac{\mho^{\mathcal{I}}_{Z}(\zeta_1, \zeta_2) + \mho^{\mathcal{I}}_{Z}(\eta_1, \eta_2)}{2}\\
&= \frac{1}{2}\left[\frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1) + \mho^{\mathcal{I}}_{\Xi}(\zeta_2)}{2} + \frac{\mho^{\mathcal{I}}_{\Xi}(\eta_1) + \mho^{\mathcal{I}}_{\Xi}(\eta_2)}{2}\right].
\end{aligned}$$

If $\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_1 \eta_1) \leq \mho^{\mathcal{I}}_{\Xi}(\zeta_2 \oslash_1 \eta_2)$, then $\mho^{\mathcal{I}}_{\Xi}(\zeta_1) \leq \mho^{\mathcal{I}}_{\Xi}(\zeta_2)$ and $\mho^{\mathcal{I}}_{\Xi}(\eta_1) \leq \mho^{\mathcal{I}}_{\Xi}(\eta_2)$. We get, $\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_1 \eta_1) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1)+\mho^{\mathcal{I}}_{\Xi}(\eta_1)}{2}$. Similarly, $\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_2 \eta_1) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1)+\mho^{\mathcal{I}}_{\Xi}(\eta_1)}{2}$ and $\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_3 \eta_1) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1)+\mho^{\mathcal{I}}_{\Xi}(\eta_1)}{2}$. Similarly to prove that $\max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_1 \eta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2 \oslash_1 \eta_2)\} \leq \max\{\max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2)\}, \max\{\mho^{\mathcal{F}}_{\Xi}(\eta_1), \mho^{\mathcal{F}}_{\Xi}(\eta_2)\}\}$. If $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_1 \eta_1) \geq \mho^{\mathcal{F}}_{\Xi}(\zeta_2 \oslash_1 \eta_2)$, then $\mho^{\mathcal{F}}_{\Xi}(\zeta_1) \geq \mho^{\mathcal{F}}_{\Xi}(\zeta_2)$ and $\mho^{\mathcal{F}}_{\Xi}(\eta_1) \geq \mho^{\mathcal{F}}_{\Xi}(\eta_2)$. We get, $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_1 \eta_1) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\eta_1)\}$. So, $\max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_2 \eta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2 \oslash_2 \eta_2)\} \leq \max\{\max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2)\}, \max\{\mho^{\mathcal{F}}_{\Xi}(\eta_1), \mho^{\mathcal{F}}_{\Xi}(\eta_2)\}\}$. If $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_2 \eta_1) \geq \mho^{\mathcal{F}}_{\Xi}(\zeta_2 \oslash_2 \eta_2)$, then $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_2 \eta_1) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\eta_1)\}$. So, $\max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_3 \eta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2 \oslash_3 \eta_2)\} \leq \max\{\max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2)\}, \max\{\mho^{\mathcal{F}}_{\Xi}(\eta_1), \mho^{\mathcal{F}}_{\Xi}(\eta_2)\}\}$. If $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_3 \eta_1) \geq \mho^{\mathcal{F}}_{\Xi}(\zeta_2 \oslash_3 \eta_2)$, then $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_3 \eta_1) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\eta_1)\}$. Now,

$$\begin{aligned}
\min\{\Gamma_{\Xi}(\zeta_1 \oslash_1 \eta_1), \Gamma_{\Xi}(\zeta_2 \oslash_1 \eta_2)\} &= \Gamma_Z(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2)\\
&= \Gamma_Z[(\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)]\\
&= \Gamma_Z(\zeta \oslash_1 \eta)\\
&\geq \min\{\Gamma_Z(\zeta), \Gamma_Z(\eta)\}\\
&= \min\{\Gamma_Z(\zeta_1, \zeta_2)\}, \Gamma_Z(\eta_1, \eta_2)\}\\
&= \min\{\min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}, \min\{\Gamma_{\Xi}(\eta_1), \Gamma_{\Xi}(\eta_2)\}\}.
\end{aligned}$$

If $\Gamma_{\Xi}(\zeta_1 \oslash_1 \eta_1) \leq \Gamma_{\Xi}(\zeta_2 \oslash_1 \eta_2)$, then $\Gamma_{\Xi}(\zeta_1) \leq \Gamma_{\Xi}(\zeta_2)$ and $\Gamma_{\Xi}(\eta_1) \leq \Gamma_{\Xi}(\eta_2)$. We get $\Gamma_{\Xi}(\zeta_1 \oslash_1 \eta_1) \geq \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\eta_1)\}$ $\forall \zeta_1, \eta_1 \in \mathcal{T}$ and $\min\{\Gamma_{\Xi}(\zeta_1 \oslash_2 \eta_1), \Gamma_{\Xi}(\zeta_2 \oslash_2 \eta_2)\} \geq \min\{\min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}, \min\{\Gamma_{\Xi}(\eta_1), \Gamma_{\Xi}(\eta_2)\}\}$. If $\Gamma_{\Xi}(\zeta_1 \oslash_2 \eta_1) \leq \Gamma_{\Xi}(\zeta_2 \oslash_2 \eta_2)$, then $\Gamma_{\Xi}(\zeta_1 \oslash_2 \eta_1) \geq \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\eta_1)\}$. So, $\min\{\Gamma_{\Xi}(\zeta_1 \oslash_3 \eta_1), \Gamma_{\Xi}(\zeta_2 \oslash_3 \eta_2)\} \geq \min\{\min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}, \min\{\Gamma_{\Xi}(\eta_1), \Gamma_{\Xi}(\eta_2)\}\}$. If $\Gamma_{\Xi}(\zeta_1 \oslash_3 \eta_1) \leq \Gamma_{\Xi}(\zeta_2 \oslash_3 \eta_2)$, then

$\Gamma_\Xi(\zeta_1 \oslash_3 \eta_1) \geq \min\{\Gamma_\Xi(\zeta_1), \Gamma_\Xi(\eta_1)\}$. Now,

$$\begin{aligned}
\frac{1}{2}\Big[\Lambda_\Xi(\zeta_1 \oslash_1 \eta_1) + \Lambda_\Xi(\zeta_2 \oslash_1 \eta_2)\Big] &= \Lambda_Z(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \Lambda_Z[(\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \Lambda_Z(\zeta \oslash_1 \eta) \\
&\geq \frac{\Lambda_Z(\zeta) + \Lambda_Z(\eta)}{2} \\
&= \frac{\Lambda_Z(\zeta_1, \zeta_2) + \Lambda_Z(\eta_1, \eta_2)}{2} \\
&= \frac{1}{2}\Bigg[\frac{\Lambda_\Xi(\zeta_1) + \Lambda_\Xi(\zeta_2)}{2} + \frac{\Lambda_\Xi(\eta_1) + \Lambda_\Xi(\eta_2)}{2}\Bigg].
\end{aligned}$$

If $\Lambda_\Xi(\zeta_1 \oslash_1 \eta_1) \leq \Lambda_\Xi(\zeta_2 \oslash_1 \eta_2)$, then $\Lambda_\Xi(\zeta_1) \leq \Lambda_\Xi(\zeta_2)$ and $\Lambda_\Xi(\eta_1) \leq \Lambda_\Xi(\eta_2)$. We get, $\Lambda_\Xi(\zeta_1 \oslash_1 \eta_1) \geq \frac{\Lambda_\Xi(\zeta_1)+\Lambda_\Xi(\eta_1)}{2}$. Similarly, $\Lambda_\Xi(\zeta_1 \oslash_2 \eta_1) \geq \frac{\Lambda_\Xi(\zeta_1)+\Lambda_\Xi(\eta_1)}{2}$ and $\Lambda_\Xi(\zeta_1 \oslash_3 \eta_1) \geq \frac{\Lambda_\Xi(\zeta_1)+\Lambda_\Xi(\eta_1)}{2}$. Similarly to prove that $\max\{\Theta_\Xi(\zeta_1 \oslash_1 \eta_1), \Theta_\Xi(\zeta_2 \oslash_1 \eta_2)\} \leq \max\{\max\{\Theta_\Xi(\zeta_1), \Theta_\Xi(\zeta_2)\}, \max\{\Theta_\Xi(\eta_1), \Theta_\Xi(\eta_2)\}\}$. If $\Theta_\Xi(\zeta_1 \oslash_1 \eta_1) \geq \Theta_\Xi(\zeta_2 \oslash_1 \eta_2)$, then $\Theta_\Xi(\zeta_1) \geq \Theta_\Xi(\zeta_2)$ and $\Theta_\Xi(\eta_1) \geq \Theta_\Xi(\eta_2)$. We get, $\Theta_\Xi(\zeta_1 \oslash_1 \eta_1) \leq \max\{\Theta_\Xi(\zeta_1), \Theta_\Xi(\eta_1)\}$. So, $\max\{\Theta_\Xi(\zeta_1 \oslash_2 \eta_1), \Theta_\Xi(\zeta_2 \oslash_2 \eta_2)\} \leq \max\{\max\{\Theta_\Xi(\zeta_1), \Theta_\Xi(\zeta_2)\}, \max\{\Theta_\Xi(\eta_1), \Theta_\Xi(\eta_2)\}\}$. If $\Theta_\Xi(\zeta_1 \oslash_2 \eta_1) \geq \Theta_\Xi(\zeta_2 \oslash_2 \eta_2)$, then $\Theta_\Xi(\zeta_1 \oslash_2 \eta_1) \leq \max\{\Theta_\Xi(\zeta_1), \Theta_\Xi(\eta_1)\}$. So, $\max\{\Theta_\Xi(\zeta_1 \oslash_3 \eta_1), \Theta_\Xi(\zeta_2 \oslash_3 \eta_2)\} \leq \max\{\max\{\Theta_\Xi(\zeta_1), \Theta_\Xi(\zeta_2)\}, \max\{\Theta_\Xi(\eta_1), \Theta_\Xi(\eta_2)\}\}$. If $\Theta_\Xi(\zeta_1 \oslash_3 \eta_1) \geq \Theta_\Xi(\zeta_2 \oslash_3 \eta_2)$, then $\Theta_\Xi(\zeta_1 \oslash_3 \eta_1) \leq \max\{\Theta_\Xi(\zeta_1), \Theta_\Xi(\eta_1)\}$. Hence Ξ is a DioNSBS of $\mathcal{T}$.

Theorem 3.9. *Let Ξ be a DioNSS in $\mathcal{T}$. Then $\Xi = \langle(\mho_\Xi^{\mathcal{T}}, \mho_\Xi^{\mathcal{I}}, \mho_\Xi^{\mathcal{F}}), (\Gamma_\Xi, \Lambda_\Xi, \Theta_\Xi)\rangle$ is a DioNSBS of $\mathcal{T}$ if and only if all non-empty level set $\Xi^{(\beta,\gamma)}$ is an SBS of $\mathcal{T}$ for $\beta, \gamma \in [0,1]$.*

Proof. Assume that Ξ is a DioNSBS of $\mathcal{T}$. For each $\beta, \gamma \in [0,1]$ and $\zeta_1, \zeta_2 \in \Xi^{(\beta,\gamma)}$. We have $\mho_\Xi^{\mathcal{T}}(\zeta_1) \geq \beta, \mho_\Xi^{\mathcal{T}}(\zeta_2) \geq \beta$, $\mho_\Xi^{\mathcal{I}}(\zeta_1) \geq \beta, \mho_\Xi^{\mathcal{I}}(\zeta_2) \geq \beta$, $\mho_\Xi^{\mathcal{F}}(\zeta_1) \leq \gamma, \mho_\Xi^{\mathcal{F}}(\zeta_2) \leq \gamma$ and $\Gamma_\Xi(\zeta_1) \geq \beta, \Gamma_\Xi(\zeta_2) \geq \beta$, $\Lambda_\Xi(\zeta_1) \geq \beta, \Lambda_\Xi(\zeta_2) \geq \beta$ and $\Theta_\Xi(\zeta_1) \leq \gamma, \Theta_\Xi(\zeta_2) \leq \gamma$. Now, $\mho_\Xi^{\mathcal{T}}(\zeta_1 \oslash_1 \zeta_2) \geq \min\{\mho_\Xi^{\mathcal{T}}(\zeta_1), \mho_\Xi^{\mathcal{T}}(\zeta_2)\} \geq \beta$ and $\mho_\Xi^{\mathcal{I}}(\zeta_1 \oslash_1 \zeta_2) \geq \frac{\mho_\Xi^{\mathcal{I}}(\zeta_1)+\mho_\Xi^{\mathcal{I}}(\zeta_2)}{2} \geq \frac{t+t}{2} = t$ and $\mho_\Xi^{\mathcal{F}}(\zeta_1 \oslash_1 \zeta_2) \leq \max\{\mho_\Xi^{\mathcal{F}}(\zeta_1), \mho_\Xi^{\mathcal{F}}(\zeta_2)\} \leq \gamma$. Similarly, $\Gamma_\Xi(\zeta_1 \oslash_1 \zeta_2) \geq \min\{\Gamma_\Xi(\zeta_1), \Gamma_\Xi(\zeta_2)\} \geq \beta$ and $\Lambda_\Xi(\zeta_1 \oslash_1 \zeta_2) \geq \frac{\Lambda_\Xi(\zeta_1)+\Lambda_\Xi(\zeta_2)}{2} \geq \frac{t+t}{2} = t$ and $\Theta_\Xi(\zeta_1 \oslash_1 \zeta_2) \leq \max\{\Theta_\Xi(\zeta_1), \Theta_\Xi(\zeta_2)\} \leq \gamma$. This implies that $\zeta_1 \oslash_1 \zeta_2 \in \Xi^{(\beta,\gamma)}$. Similarly, $\zeta_1 \oslash_2 \zeta_2 \in \Xi^{(\beta,\gamma)}$ and $\zeta_1 \oslash_3 \zeta_2 \in \Xi^{(\beta,\gamma)}$. Therefore $\Xi^{(\beta,\gamma)}$ is a SBS of $\mathcal{T}$ for each $\beta, \gamma \in [0,1]$.

Conversely, assume that $\Xi^{(\beta,\gamma)}$ is an SBS of $\mathcal{T}$ for each $\beta, \gamma \in [0,1]$. Suppose if there exist $\zeta_1, \zeta_2 \in \mathcal{T}$ such that $\mho_\Xi^{\mathcal{T}}(\zeta_1 \oslash_1 \zeta_2) < \min\{\mho_\Xi^{\mathcal{T}}(\zeta_1), \mho_\Xi^{\mathcal{T}}(\zeta_2)\}$, $\mho_\Xi^{\mathcal{I}}(\zeta_1 \oslash_1 \zeta_2) < \frac{\mho_\Xi^{\mathcal{I}}(\zeta_1)+\mho_\Xi^{\mathcal{I}}(\zeta_2)}{2}$, $\mho_\Xi^{\mathcal{F}}(\zeta_1 \oslash_1 \zeta_2) > \max\{\mho_\Xi^{\mathcal{F}}(\zeta_1), \mho_\Xi^{\mathcal{F}}(\zeta_2)\}$ and $\Gamma_\Xi(\zeta_1 \oslash_1 \zeta_2) < \min\{\Gamma_\Xi(\zeta_1), \Gamma_\Xi(\zeta_2)\}$, $\Lambda_\Xi(\zeta_1 \oslash_1 \zeta_2) < \frac{\Lambda_\Xi(\zeta_1)+\Lambda_\Xi(\zeta_2)}{2}$ and $\Theta_\Xi(\zeta_1 \oslash_1 \zeta_2) > \max\{\Theta_\Xi(\zeta_1), \Theta_\Xi(\zeta_2)\}$. Select $\beta, \gamma \in [0,1]$ such that $\mho_\Xi^{\mathcal{T}}(\zeta_1 \oslash_1 \zeta_2) < \beta \leq \min\{\mho_\Xi^{\mathcal{T}}(\zeta_1), \mho_\Xi^{\mathcal{T}}(\zeta_2)\}$ and $\mho_\Xi^{\mathcal{I}}(\zeta_1 \oslash_1 \zeta_2) < \beta \leq \frac{\mho_\Xi^{\mathcal{I}}(\zeta_1)+\mho_\Xi^{\mathcal{I}}(\zeta_2)}{2}$ and $\mho_\Xi^{\mathcal{F}}(\zeta_1 \oslash_1 \zeta_2) > \gamma \geq \max\{\mho_\Xi^{\mathcal{F}}(\zeta_1), \mho_\Xi^{\mathcal{F}}(\zeta_2)\}$. Then $\zeta_1, \zeta_2 \in \Xi^{(\beta,\gamma)}$, but $\zeta_1 \oslash_1 \zeta_2 \notin \Xi^{(\beta,\gamma)}$. This contradicts to that $\Xi^{(\beta,\gamma)}$ is an SBS of $\mathcal{T}$. Hence $\mho^{\mathcal{T}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1), \mho^{\mathcal{T}}_{\Xi}(\zeta_2)\}$, $\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1)+\mho^{\mathcal{I}}_{\Xi}(\zeta_2)}{2}$ and $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2)\}$. Select $\beta, \gamma \in [0,1]$ such that $\Gamma_{\Xi}(\zeta_1 \oslash_1 \zeta_2) < \beta \leq \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}$ and $\Lambda_{\Xi}(\zeta_1 \oslash_1 \zeta_2) < \beta \leq \frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Xi}(\zeta_2)}{2}$ and $\Theta_{\Xi}(\zeta_1 \oslash_1 \zeta_2) > \gamma \geq \max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\zeta_2)\}$. Then $\zeta_1, \zeta_2 \in \Xi^{(\beta,\gamma)}$, but $\zeta_1 \oslash_1 \zeta_2 \notin \Xi^{(\beta,\gamma)}$. This contradicts to that $\Xi^{(\beta,\gamma)}$ is an SBS of $\mathcal{T}$. Hence $\Gamma_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}$, $\Lambda_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Xi}(\zeta_2)}{2}$ and $\Theta_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \leq \max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\zeta_2)\}$. Similarly, $\oslash_2$ and $\oslash_3$ cases. Hence $\Xi = \langle(\mho^{\mathcal{T}}_{\Xi}, \mho^{\mathcal{I}}_{\Xi}, \mho^{\mathcal{F}}_{\Xi}), (\Gamma_{\Xi}, \Lambda_{\Xi}, \Theta_{\Xi})\rangle$ is a DioNSBS of $\mathcal{T}$.

Definition 3.10. Let Ξ be any DioNSBS of $\mathcal{T}$, $a \in \mathcal{T}$ and P is any set. Then the pseudo Diophantine neutrosophic coset $(a\Xi)^p$ is defined by

$$\left\{\begin{matrix} ((a\mho^{\mathcal{T}}_{\Xi})^p)(\zeta) = p(a)\mho^{\mathcal{T}}_{\Xi}(\zeta) \\ ((a\mho^{\mathcal{I}}_{\Xi})^p)(\zeta) = p(a)\mho^{\mathcal{I}}_{\Xi}(\zeta) \\ ((a\mho^{\mathcal{F}}_{\Xi})^p)(\zeta) = p(a)\mho^{\mathcal{F}}_{\Xi}(\zeta) \end{matrix}\right\} \quad \left\{\begin{matrix} ((a\Gamma_{\Xi})^p)(\zeta) = p(a)\Gamma_{\Xi}(\zeta) \\ ((a\Lambda_{\Xi})^p)(\zeta) = p(a)\Lambda_{\Xi}(\zeta) \\ ((a\Theta_{\Xi})^p)(\zeta) = p(a)\Theta_{\Xi}(\zeta) \end{matrix}\right\}$$

for every $\zeta \in \mathcal{T}$ and for some $p \in P$.

Theorem 3.11. *Let Ξ be any DioNSBS of $\mathcal{T}$, then the pseudo Diophantine neutrosophic coset $(a\Xi)^p$ is a DioNSBS of $\mathcal{T}$, for every $a \in \mathcal{T}$.*

Proof. Let Ξ be any DioNSBS of $\mathcal{T}$ and for every $\zeta, \eta \in \mathcal{T}$. Now, $((a\mho^{\mathcal{T}}_{\Xi})^p)(\zeta \oslash_1 \eta) = p(a)\ \mho^{\mathcal{T}}_{\Xi}(\zeta \oslash_1 \eta) \geq p(a)\ \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta), \mho^{\mathcal{T}}_{\Xi}(\eta)\} = \min\{p(a)\ \mho^{\mathcal{T}}_{\Xi}(\zeta), p(a)\ \mho^{\mathcal{T}}_{\Xi}(\eta)\} = \min\{((a\mho^{\mathcal{T}}_{\Xi})^p)(\zeta), ((a\mho^{\mathcal{T}}_{\Xi})^p)(\eta)\}$. Thus, $((a\mho^{\mathcal{T}}_{\Xi})^p)(\zeta \oslash_1 \eta) \geq \min\{((a\mho^{\mathcal{T}}_{\Xi})^p)(\zeta), ((a\mho^{\mathcal{T}}_{\Xi})^p)(\eta)\}$. Now, $((a\mho^{\mathcal{I}}_{\Xi})^p)(\zeta \oslash_1 \eta) = p(a)\ \mho^{\mathcal{I}}_{\Xi}(\zeta \oslash_1 \eta) \geq p(a)\left[\frac{\mho^{\mathcal{I}}_{\Xi}(\zeta)+\mho^{\mathcal{I}}_{\Xi}(\eta)}{2}\right] = \frac{p(a)\ \mho^{\mathcal{I}}_{\Xi}(\zeta)+p(a)\ \mho^{\mathcal{I}}_{\Xi}(\eta)}{2} = \frac{((a\mho^{\mathcal{I}}_{\Xi})^p)(\zeta)+((a\mho^{\mathcal{I}}_{\Xi})^p)(\eta)}{2}$. Thus, $((a\mho^{\mathcal{I}}_{\Xi})^p)(\zeta \oslash_1 \eta) \geq \frac{((a\mho^{\mathcal{I}}_{\Xi})^p)(\zeta)+((a\mho^{\mathcal{I}}_{\Xi})^p)(\eta)}{2}$. Now, $((a\mho^{\mathcal{F}}_{\Xi})^p)(\zeta \oslash_1 \eta) = p(a)\ \mho^{\mathcal{F}}_{\Xi}(\zeta \oslash_1 \eta) \leq p(a)\ \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\} = \max\{p(a)\ \mho^{\mathcal{F}}_{\Xi}(\zeta), p(a)\ \mho^{\mathcal{F}}_{\Xi}(\eta)\} = \max\{((a\mho^{\mathcal{F}}_{\Xi})^p)(\zeta), ((a\mho^{\mathcal{F}}_{\Xi})^p)(\eta)\}$. Thus, $((a\mho^{\mathcal{F}}_{\Xi})^p)(\zeta \oslash_1 \eta) \leq \max\{((a\mho^{\mathcal{F}}_{\Xi})^p)(\zeta), ((a\mho^{\mathcal{F}}_{\Xi})^p)(\eta)\}$. Now,

$$\begin{aligned} ((a\Gamma_{\Xi})^p)(\zeta \oslash_1 \eta) &= p(a)\ \Gamma_{\Xi}(\zeta \oslash_1 \eta) \\ &\geq p(a)\ \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\} \\ &= \min\{p(a)\ \Gamma_{\Xi}(\zeta), p(a)\ \Gamma_{\Xi}(\eta)\} \\ &= \min\{((a\Gamma_{\Xi})^p)(\zeta), ((a\Gamma_{\Xi})^p)(\eta)\}. \end{aligned}$$

Thus, $((a\Gamma_\Xi)^p)(\zeta \oslash_1 \eta) \geq \min\{((a\Gamma_\Xi)^p)(\zeta), ((a\Gamma_\Xi)^p)(\eta)\}$. Now,

$$\begin{aligned} ((a\Lambda_\Xi)^p)(\zeta \oslash_1 \eta) &= p(a)\ \Lambda_\Xi(\zeta \oslash_1 \eta) \\ &\geq p(a)\left[\frac{\Lambda_\Xi(\zeta)+\Lambda_\Xi(\eta)}{2}\right] \\ &= \frac{p(a)\ \Lambda_\Xi(\zeta)+p(a)\ \Lambda_\Xi(\eta)}{2} \\ &= \frac{((a\Lambda_\Xi)^p)(\zeta)+((a\Lambda_\Xi)^p)(\eta)}{2}. \end{aligned}$$

Thus, $((a\Lambda_\Xi)^p)(\zeta \oslash_1 \eta) \geq \frac{((a\Lambda_\Xi)^p)(\zeta)+((a\Lambda_\Xi)^p)(\eta)}{2}$. Now,

$$\begin{aligned} ((a\Theta_\Xi)^p)(\zeta \oslash_1 \eta) &= p(a)\ \Theta_\Xi(\zeta \oslash_1 \eta) \\ &\leq p(a)\ \max\{\Theta_\Xi(\zeta), \Theta_\Xi(\eta)\} \\ &= \max\{p(a)\ \Theta_\Xi(\zeta), p(a)\ \Theta_\Xi(\eta)\} \\ &= \max\{((a\Theta_\Xi)^p)(\zeta), ((a\Theta_\Xi)^p)(\eta)\}. \end{aligned}$$

Thus, $((a\Theta_\Xi)^p)(\zeta \oslash_1 \eta) \leq \max\{((a\Theta_\Xi)^p)(\zeta), ((a\Theta_\Xi)^p)(\eta)\}$. Similarly, $\oslash_2$ and $\oslash_3$ cases. Hence $(a\Xi)^p$ is a DioNSBS of $\mathcal{T}$.

Definition 3.12. Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. Let $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ be any function and Ξ be any DioNSBS in $\mathcal{T}_1$, Z be any DioNSBS in $\mathcal{L}(\mathcal{T}_1) = \mathcal{T}_2$. If $\mho_\Xi = \langle(\mho_\Xi^\mathcal{T}, \mho_\Xi^\mathcal{I}, \mho_\Xi^\mathcal{F}), (\Gamma_\Xi, \Lambda_\Xi, \Theta_\Xi)\rangle$ is a DioNSS in $\mathcal{T}_1$, then $\mho_Z$ is a DioNSS in $\mathcal{T}_2$, defined by $\forall \zeta \in \mathcal{T}_1$ and $\eta \in \mathcal{T}_2$,

$$\mho_Z^\mathcal{T}(\eta) = \begin{cases} \sup \mho_\Xi^\mathcal{T}(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 0 & \text{otherwise} \end{cases} \qquad \mho_Z^\mathcal{I}(\eta) = \begin{cases} \sup \mho_\Xi^\mathcal{I}(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 0 & \text{otherwise} \end{cases}$$

$$\mho_Z^\mathcal{F}(\eta) = \begin{cases} \inf \mho_\Xi^\mathcal{F}(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 1 & \text{otherwise} \end{cases}$$

$$\Gamma_Z(\eta) = \begin{cases} \sup \Gamma_\Xi(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 0 & \text{otherwise} \end{cases} \qquad \Lambda_Z(\eta) = \begin{cases} \sup \Lambda_\Xi(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 0 & \text{otherwise} \end{cases}$$

$$\Theta_Z(\eta) = \begin{cases} \inf \Theta_\Xi(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 1 & \text{otherwise} \end{cases}$$

which is called the image of $\mho_\Xi$ under $\mathcal{L}$.

Similarly, If $\mho_Z = \langle(\mho_Z^\mathcal{T}, \mho_Z^\mathcal{I}, \mho_Z^\mathcal{F}), (\Gamma_Z, \Lambda_Z, \Theta_Z)\rangle$ is a DioNSS in $\mathcal{T}_2$, then DioNSS $\mho_\Xi = \mathcal{L} \circ \mho_Z$ in $\mathcal{T}_1$ [i.e., the DioNSS defined by $\mho_\Xi(\zeta) = \mho_Z(\mathcal{L}(\zeta))$] is called the preimage of $\mho_Z$ under $\mathcal{L}$.

Theorem 3.13. *Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. The homomorphic image of every DioNSBS of $\mathcal{T}_1$ is a DioNSBS of $\mathcal{T}_2$.*

Proof. Let $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ be any homomorphism. Then $\mathcal{L}(\zeta \circledast_1 \eta) = \mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta), \mathcal{L}(\zeta \circledast_2 \eta) = \mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) = \mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)$ $\forall \zeta, \eta \in \mathcal{T}_1$. Let $Z = \mathcal{L}(\Xi)$, Ξ is any DioNSBS of $\mathcal{T}_1$. Let $\mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{T}_2$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$ and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\mho^{\mathcal{T}}_{\Xi}(\zeta) = \sup\limits_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \mho^{\mathcal{T}}_{\Xi}(\zeta')$ and $\mho^{\mathcal{T}}_{\Xi}(\eta) = \sup\limits_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \mho^{\mathcal{T}}_{\Xi}(\zeta')$. Now,

$$\begin{aligned}
\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta))} \mho^{\mathcal{T}}_{\Xi}(\zeta'') \\
&= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \mho^{\mathcal{T}}_{\Xi}(\zeta'') \\
&= \mho^{\mathcal{T}}_{\Xi}(\zeta \circledast_1 \eta) \\
&\geq \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta), \mho^{\mathcal{T}}_{\Xi}(\eta)\} \\
&= \min\{\mho^{\mathcal{T}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{T}}_{Z}\mathcal{L}(\eta)\}.
\end{aligned}$$

Thus, $\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \min\{\mho^{\mathcal{T}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{T}}_{Z}\mathcal{L}(\eta)\}$. Similarly, $\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \geq \min\{\mho^{\mathcal{T}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{T}}_{Z}\mathcal{L}(\eta)\}$ and $\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)) \geq \min\{\mho^{\mathcal{T}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{T}}_{Z}\mathcal{L}(\eta)\}$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$ and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\mho^{\mathcal{I}}_{\Xi}(\zeta) = \sup\limits_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \mho^{\mathcal{I}}_{\Xi}(\zeta')$ and $\mho^{\mathcal{I}}_{\Xi}(\eta) = \sup\limits_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \mho^{\mathcal{I}}_{\Xi}(\zeta')$. Now,

$$\begin{aligned}
\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta))} \mho^{\mathcal{I}}_{\Xi}(\zeta'') \\
&= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \mho^{\mathcal{I}}_{\Xi}(\zeta'') \\
&= \mho^{\mathcal{I}}_{\Xi}(\zeta \circledast_1 \eta) \\
&\geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta) + \mho^{\mathcal{I}}_{\Xi}(\eta)}{2} \\
&= \frac{\mho^{\mathcal{I}}_{Z}\mathcal{L}(\zeta) + \mho^{\mathcal{I}}_{Z}\mathcal{L}(\eta)}{2}.
\end{aligned}$$

Thus, $\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \frac{\mho^{\mathcal{I}}_{Z}\mathcal{L}(\zeta) + \mho^{\mathcal{I}}_{Z}\mathcal{L}(\eta)}{2}$. Similarly, $\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \geq \frac{\mho^{\mathcal{I}}_{Z}\mathcal{L}(\zeta) + \mho^{\mathcal{I}}_{Z}\mathcal{L}(\eta)}{2}$ and $\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)) \geq \frac{\mho^{\mathcal{I}}_{Z}\mathcal{L}(\zeta) + \mho^{\mathcal{I}}_{Z}\mathcal{L}(\eta)}{2}$. Let $\mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{T}_2$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$ and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\mho^{\mathcal{F}}_{\Xi}(\zeta) = \inf\limits_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \mho^{\mathcal{F}}_{\Xi}(\zeta')$ and $\mho^{\mathcal{F}}_{\Xi}(\eta) = \inf\limits_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \mho^{\mathcal{F}}_{\Xi}(\zeta')$. Now,

$$\begin{aligned}
\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \inf_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta))} \mho^{\mathcal{F}}_{\Xi}(\zeta'') \\
&= \inf_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \mho^{\mathcal{F}}_{\Xi}(\zeta'') \\
&= \mho^{\mathcal{F}}_{\Xi}(\zeta \circledast_1 \eta) \\
&\leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\} \\
&= \max\{\mho^{\mathcal{F}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{F}}_{Z}\mathcal{L}(\eta)\}.
\end{aligned}$$

Thus, $\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\mho^{\mathcal{F}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{F}}_{Z}\mathcal{L}(\eta)\}$. Similarly, $\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \leq \max\{\mho^{\mathcal{F}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{F}}_{Z}\mathcal{L}(\eta)\}$ and $\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)) \leq \max\{\mho^{\mathcal{F}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{F}}_{Z}\mathcal{L}(\eta)\}$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$

and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\Gamma_\Xi(\zeta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \Gamma_\Xi(\zeta')$ and $\Gamma_\Xi(\eta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \Gamma_\Xi(\zeta')$. Now,

$$\begin{aligned}
\Gamma_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta)\otimes_1 \mathcal{L}(\eta))} \Gamma_\Xi(\zeta'') \\
&= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \Gamma_\Xi(\zeta'') \\
&= \Gamma_\Xi(\zeta \circledast_1 \eta) \\
&\geq \min\{\Gamma_\Xi(\zeta), \Gamma_\Xi(\eta)\} \\
&= \min\{\Gamma_Z\mathcal{L}(\zeta), \Gamma_Z\mathcal{L}(\eta)\}.
\end{aligned}$$

Thus, $\Gamma_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \min\{\Gamma_Z\mathcal{L}(\zeta), \Gamma_Z\mathcal{L}(\eta)\}$. Similarly, $\Gamma_Z(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \geq \min\{\Gamma_Z\mathcal{L}(\zeta), \Gamma_Z\mathcal{L}(\eta)\}$ and $\Gamma_Z(\mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)) \geq \min\{\Gamma_Z\mathcal{L}(\zeta), \Gamma_Z\mathcal{L}(\eta)\}$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$ and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\Lambda_\Xi(\zeta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \Lambda_\Xi(\zeta')$ and $\Lambda_\Xi(\eta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \Lambda_\Xi(\zeta')$. Now,

$$\begin{aligned}
\Lambda_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta)\otimes_1 \mathcal{L}(\eta))} \Lambda_\Xi(\zeta'') \\
&= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \Lambda_\Xi(\zeta'') \\
&= \Lambda_\Xi(\zeta \circledast_1 \eta) \\
&\geq \frac{\Lambda_\Xi(\zeta) + \Lambda_\Xi(\eta)}{2} \\
&= \frac{\Lambda_Z\mathcal{L}(\zeta) + \Lambda_Z\mathcal{L}(\eta)}{2}.
\end{aligned}$$

Thus, $\Lambda_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \frac{\Lambda_Z\mathcal{L}(\zeta)+\Lambda_Z\mathcal{L}(\eta)}{2}$. Similarly, $\Lambda_Z(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \geq \frac{\Lambda_Z\mathcal{L}(\zeta)+\Lambda_Z\mathcal{L}(\eta)}{2}$ and $\Lambda_Z(\mathcal{L}(\zeta)\otimes_3\mathcal{L}(\eta)) \geq \frac{\Lambda_Z\mathcal{L}(\zeta)+\Lambda_Z\mathcal{L}(\eta)}{2}$. Let $\mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{T}_2$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$ and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\Theta_\Xi(\zeta) = \inf_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \Theta_\Xi(\zeta')$ and $\Theta_\Xi(\eta) = \inf_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \Theta_\Xi(\zeta')$. Now,

$$\begin{aligned}
\Theta_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \inf_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta)\otimes_1 \mathcal{L}(\eta))} \Theta_\Xi(\zeta'') \\
&= \inf_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \Theta_\Xi(\zeta'') \\
&= \Theta_\Xi(\zeta \circledast_1 \eta) \\
&\leq \max\{\Theta_\Xi(\zeta), \Theta_\Xi(\eta)\} \\
&= \max\{\Theta_Z\mathcal{L}(\zeta), \Theta_Z\mathcal{L}(\eta)\}.
\end{aligned}$$

Thus, $\Theta_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\Theta_Z\mathcal{L}(\zeta), \Theta_Z\mathcal{L}(\eta)\}$. Similarly, $\Theta_Z(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \leq \max\{\Theta_Z\mathcal{L}(\zeta), \Theta_Z\mathcal{L}(\eta)\}$ and $\Theta_Z(\mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)) \leq \max\{\Theta_Z\mathcal{L}(\zeta), \Theta_Z\mathcal{L}(\eta)\}$. Hence Z is a DioNSBS of $\mathcal{T}_2$.

Theorem 3.14. *Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. The homomorphic preimage of DioNSBS of $\mathcal{T}_2$ is a DioNSBS of $\mathcal{T}_1$.*

Proof. Let $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ be any homomorphism. Then $\mathcal{L}(\zeta \circledast_1 \eta) = \mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta), \mathcal{L}(\zeta \circledast_2 \eta) = \mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) = \mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)\ \forall \zeta, \eta \in \mathcal{T}_1$. Let $Z = \mathcal{L}(\Xi)$, where Z is any DioNSBS of $\mathcal{T}_2$. Let $\zeta, \eta \in \mathcal{T}_1$. Now, $\mho^{\mathcal{T}}_{\Xi}(\zeta \circledast_1 \eta) = \mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta \circledast_1 \eta)) = \mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \min\{\mho^{\mathcal{T}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{T}}_{Z}\mathcal{L}(\eta)\} = \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta), \mho^{\mathcal{T}}_{\Xi}(\eta)\}$. Thus, $\mho^{\mathcal{T}}_{\Xi}(\zeta \circledast_1 \eta) \geq \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta), \mho^{\mathcal{T}}_{\Xi}(\eta)\}$. Now, $\mho^{\mathcal{I}}_{\Xi}(\zeta \circledast_1 \eta) = \mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta \circledast_1 \eta)) = \mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \frac{\mho^{\mathcal{I}}_{Z}\mathcal{L}(\zeta) + \mho^{\mathcal{I}}_{Z}\mathcal{L}(\eta)}{2} = \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta) + \mho^{\mathcal{I}}_{\Xi}(\eta)}{2}$. Thus, $\mho^{\mathcal{I}}_{\Xi}(\zeta \circledast_1 \eta) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta) + \mho^{\mathcal{I}}_{\Xi}(\eta)}{2}$. Now, $\mho^{\mathcal{F}}_{\Xi}(\zeta \circledast_1 \eta) = \mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta \circledast_1 \eta)) = \mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\mho^{\mathcal{F}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{F}}_{Z}\mathcal{L}(\eta)\} = \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\}$. Thus, $\mho^{\mathcal{F}}_{\Xi}(\zeta \circledast_1 \eta) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\}$. Now, $\Gamma_{\Xi}(\zeta \circledast_1 \eta) = \Gamma_Z(\mathcal{L}(\zeta \circledast_1 \eta)) = \Gamma_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \min\{\Gamma_Z\mathcal{L}(\zeta), \Gamma_Z\mathcal{L}(\eta)\} = \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\}$. Thus, $\Gamma_{\Xi}(\zeta \circledast_1 \eta) \geq \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\}$. Now, $\Lambda_{\Xi}(\zeta \circledast_1 \eta) = \Lambda_Z(\mathcal{L}(\zeta \circledast_1 \eta)) = \Lambda_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \frac{\Lambda_Z\mathcal{L}(\zeta) + \Lambda_Z\mathcal{L}(\eta)}{2} = \frac{\Lambda_{\Xi}(\zeta) + \Lambda_{\Xi}(\eta)}{2}$. Thus, $\Lambda_{\Xi}(\zeta \circledast_1 \eta) \geq \frac{\Lambda_{\Xi}(\zeta) + \Lambda_{\Xi}(\eta)}{2}$. Now, $\Theta_{\Xi}(\zeta \circledast_1 \eta) = \Theta_Z(\mathcal{L}(\zeta \circledast_1 \eta)) = \Theta_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\Theta_Z\mathcal{L}(\zeta), \Theta_Z\mathcal{L}(\eta)\} = \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\}$. Thus, $\Theta_{\Xi}(\zeta \circledast_1 \eta) \leq \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\}$. Similarly to prove two other operations, Ξ is a DioNSBS of $\mathcal{T}_1$.

Theorem 3.15. *Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. If $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ is a homomorphism, then $\mathcal{L}(\Xi_{(\beta,\gamma)})$ is a level SBS of DioNSBS Z of $\mathcal{T}_2$.*

Proof. Let $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ be any homomorphism. Then $\mathcal{L}(\zeta \circledast_1 \eta) = \mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta), \mathcal{L}(\zeta \circledast_2 \eta) = \mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) = \mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)\ \forall \zeta, \eta \in \mathcal{T}_1$. Let $Z = \mathcal{L}(\Xi)$, Ξ is a DioNSBS of $\mathcal{T}_1$. By Theorem 3.13, Z is a DioNSBS of $\mathcal{T}_2$. Let $\Xi_{(\beta,\gamma)}$ be any level SBS of Ξ. Suppose that $\zeta, \eta \in \Xi_{(\beta,\gamma)}$. Then $\mathcal{L}(\zeta \circledast_1 \eta), \mathcal{L}(\zeta \circledast_2 \eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) \in \Xi_{(\beta,\gamma)}$. Now, $\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta)) = \mho^{\mathcal{T}}_{\Xi}(\zeta) \geq \beta, \mho^{\mathcal{T}}_{Z}(\mathcal{L}(\eta)) = \mho^{\mathcal{T}}_{\Xi}(\eta) \geq \beta$. Thus, $\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \mho^{\mathcal{T}}_{\Xi}(\zeta \circledast_1 \eta) \geq \beta$. Now, $\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta)) = \mho^{\mathcal{I}}_{\Xi}(\zeta) \geq \beta, \mho^{\mathcal{I}}_{Z}(\mathcal{L}(\eta)) = \mho^{\mathcal{I}}_{\Xi}(\eta) \geq \beta$. Thus, $\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \mho^{\mathcal{I}}_{\Xi}(\zeta \circledast_1 \eta) \geq \beta$. Now, $\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta)) = \mho^{\mathcal{F}}_{\Xi}(\zeta) \leq \gamma, \mho^{\mathcal{F}}_{Z}(\mathcal{L}(\eta)) = \mho^{\mathcal{F}}_{\Xi}(\eta) \leq \gamma$. Thus, $\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \mho^{\mathcal{F}}_{\Xi}(\zeta \circledast_1 \eta) \leq \gamma$, $\forall \mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{T}_2$. Now, $\Gamma_{\Xi Z}(\mathcal{L}(\zeta)) = \Gamma_{\Xi}(\zeta) \geq \beta, \Gamma_Z(\mathcal{L}(\eta)) = \Gamma(\eta) \geq \beta$. Thus, $\Gamma_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \Gamma_{\Xi}(\zeta \circledast_1 \eta) \geq \beta$. Now, $\Lambda_Z(\mathcal{L}(\zeta)) = \Lambda_{\Xi}(\zeta) \geq \beta, \Lambda_Z(\mathcal{L}(\eta)) = \Lambda_{\Xi}(\eta) \geq \beta$. Thus, $\Lambda_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \Lambda_{\Xi}(\zeta \circledast_1 \eta) \geq \beta$. Now, $\Theta_Z(\mathcal{L}(\zeta)) = \Theta_{\Xi}(\zeta) \leq \gamma, \Theta_Z(\mathcal{L}(\eta)) = \Theta_{\Xi}(\eta) \leq \gamma$. Thus, $\Theta_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \Theta_{\Xi}(\zeta \circledast_1 \eta) \leq \gamma$, $\forall \mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{T}_2$. Similarly to prove other operations, hence $\mathcal{L}(\Xi_{(\beta,\gamma)})$ is a level SBS of DioNSBS Z of $\mathcal{T}_2$.

Theorem 3.16. *Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. If $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ is any homomorphism, then $\Xi_{(\beta,\gamma)}$ is a level SBS of DioNSBS Ξ of $\mathcal{T}_1$.*

Proof. Let $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ be any homomorphism. Then $\mathcal{L}(\zeta \circledast_1 \eta) = \mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta), \mathcal{L}(\zeta \circledast_2 \eta) = \mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) = \mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)\ \forall \zeta, \eta \in \mathcal{T}_1$. Let $Z = \mathcal{L}(\Xi)$, Z is a DioNSBS of $\mathcal{T}_2$. By Theorem 3.14, Ξ is a DioNSBS of $\mathcal{T}_1$. Let $\mathcal{L}(\Xi_{(\beta,\gamma)})$ be a level SBS of Z. Suppose that $\mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{L}(\Xi_{(\beta,\gamma)})$. Then $\mathcal{L}(\zeta \circledast_1 \eta), \mathcal{L}(\zeta \circledast_2 \eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) \in \mathcal{L}(\Xi_{(\beta,\gamma)})$. Now, $\mho_{\Xi}^{\mathcal{T}}(\zeta) = \mho_{Z}^{\mathcal{T}}(\mathcal{L}(\zeta)) \geq t, \mho_{\Xi}^{\mathcal{T}}(\eta) = \mho_{Z}^{\mathcal{T}}(\mathcal{L}(\eta)) \geq \beta$. Thus, $\mho_{\Xi}^{\mathcal{T}}(\zeta \circledast_1 \eta) \geq \min\{\mho_{\Xi}^{\mathcal{T}}(\zeta), \mho_{\Xi}^{\mathcal{T}}(\eta)\} \geq \beta$. Now, $\mho_{\Xi}^{\mathcal{I}}(\zeta) = \mho_{Z}^{\mathcal{I}}(\mathcal{L}(\zeta)) \geq t, \mho_{\Xi}^{\mathcal{I}}(\eta) = \mho_{Z}^{\mathcal{I}}(\mathcal{L}(\eta)) \geq \beta$. Thus, $\mho_{\Xi}^{\mathcal{I}}(\zeta \circledast_1 \eta) \geq \frac{\mho_{\Xi}^{\mathcal{I}}(\zeta)+\mho_{\Xi}^{\mathcal{I}}(\eta)}{2} \geq \beta$. Now, $\mho_{\Xi}^{\mathcal{F}}(\zeta) = \mho_{Z}^{\mathcal{F}}(\mathcal{L}(\zeta)) \leq \gamma, \mho_{\Xi}^{\mathcal{F}}(\eta) = \mho_{Z}^{\mathcal{F}}(\mathcal{L}(\eta)) \leq \gamma$. Thus, $\mho_{\Xi}^{\mathcal{F}}(\zeta \circledast_1 \eta) = \mho_{Z}^{\mathcal{F}}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\mho_{\Xi}^{\mathcal{F}}(\zeta), \mho_{\Xi}^{\mathcal{F}}(\eta)\} \leq \gamma, \forall \zeta, \eta \in \mathcal{T}_1$. Now, $\Gamma_{\Xi}(\zeta) = \Gamma_{\Xi Z}(\mathcal{L}(\zeta)) \geq t, \Gamma_{\Xi}(\eta) = \Gamma_{Z}(\mathcal{L}(\eta)) \geq \beta$. Thus, $\Gamma_{\Xi}(\zeta \circledast_1 \eta) \geq \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\} \geq \beta$. Now, $\Lambda_{\Xi}(\zeta) = \Lambda_{Z}(\mathcal{L}(\zeta)) \geq t, \Lambda_{\Xi}(\eta) = \Lambda_{Z}(\mathcal{L}(\eta)) \geq \beta$. Thus, $\Lambda_{\Xi}(\zeta \circledast_1 \eta) \geq \frac{\Lambda_{\Xi}(\zeta)+\Lambda_{\Xi}(\eta)}{2} \geq \beta$. Now, $\Theta_{\Xi}(\zeta) = \Theta_{Z}(\mathcal{L}(\zeta)) \leq \gamma, \Theta_{\Xi}(\eta) = \Theta_{Z}(\mathcal{L}(\eta)) \leq \gamma$. Thus, $\Theta_{\Xi}(\zeta \circledast_1 \eta) = \Theta_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\} \leq \gamma, \forall \zeta, \eta \in \mathcal{T}_1$. Similarly to prove other two operations, hence $\Xi_{(\beta,\gamma)}$ is a level SBS of DioNSBS Ξ of $\mathcal{T}_1$.

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