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Topological structures of fuzzy neutrosophic rough sets

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Abstract. In this paper, we examine the fuzzy neutrosophic relation having a special property that can be equivalently characterised by the essential properties of the lower and upper fuzzy neutrosophic rough approximation operators. Further, we prove that the set of all lower approximation sets based on fuzzy neutrosophic equivalence approximation space forms a fuzzy neutrosophic topology. Also, we discuss the necessary and sufficient conditions such that the FN interior (closure) equals FN lower (upper) approximation operator.

Keywords: Fuzzy neutrosophic rough set, Approximation operators, approximation spaces, rough sets, topological spaces.

1 Introduction

A rough set, first described by Pawlak, is a formal approximation of a crisp set in terms of a pair of sets which give the lower and the upper approximation of the original set. The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most successful approach is based on the fuzzy set notion proposed by L. Zadeh. Rough set theory proposed by Z. Pawlak in [10] presents still another attempt to this problem. Rough sets have been proposed for a very wide variety of applications. In particular, the rough set approach seems to be important for Artificial Intelligence and cognitive sciences, especially in machine learning, knowledge discovery, data mining, expert systems, approximate reasoning and pattern recognition.

Neutrosophic Logic has been proposed by Florentine Smarandache [11, 12] which is based on non-standard analysis that was given by Abraham Robinson in 1960s. Neutrosophic Logic was developed to represent mathematical model of uncertainty, vagueness, ambiguity, imprecision undefined, incompleteness, inconsistency, redundancy, contradiction. The neutrosophic logic is a formal frame to measure truth, indeterminacy and falsehood. In Neutrosophic set, indeterminacy is quantified explicitly whereas the truth membership, indeterminacy membership and falsity membership are independent. This assumption is very important in a lot of situations such as information fusion when we try to combine the data from different sensors.

In this paper we focus on the study of the relationship between fuzzy neutrosophic rough approximtion operators and fuzzy neutrosophic topological spaces.

2 Preliminaries

Definition2.1 [1]:
A fuzzy neutrosophic set A on the universe of discourse X is defined as
A= $\{\langle x, T_A(x), I_A(x), F_A(x)\rangle, x \in X\}$
where $T, I, F: X \to [0, 1]$ and
$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 2.2[1]:
A fuzzy neutrosophic relation U is a fuzzy neutrosophic subset R=$\{<x, y>, T_R(x,y), I_R(x,y), F_R(x,y) / x, y \in U\}$
$T_R: U \times U \to [0,1]$, $I_R: U \times U \to [0,1]$, $F_R: U \times U \to [0,1]$
Satisfies $0 \le T_R(x,y) + I_R(x,y) + F_R(x,y) \le 3$ for all (x,y) $\in U \times U$.

Definition 2.3[4]:
Let U be a non empty universe of discourse. For an arbitrary fuzzy neutrosophic relation R over $U \times U$ the pair (U, R) is called fuzzy neutrosophic approximation space. For any $A \in FN(U)$, we define the upper and lower approximation with respect to (U, R), denoted by $\overline{R}$ and $\underline{R}$ respectively.

$$\overline{R}(A) = \{< x, T_{\overline{R}(A)}(x), I_{\overline{R}(A)}(x), F_{\overline{R}(A)}(x) > / x \in U\}$$

$$\underline{R}(A) = \{< x, T_{\underline{R}(A)}(x), I_{\underline{R}(A)}(x), F_{\underline{R}(A)}(x) > x \in U\}$$

$$T_{\overline{R}(A)}(x) = \bigvee_{y \in U} [T_R(x,y) \wedge T_A(y)]$$

$$I_{\overline{R}(A)}(x) = \bigvee_{y \in U} [I_R(x,y) \wedge I_A(y)]$$

$$F_{\overline{R}(A)}(x) = \bigwedge_{y \in U} [F_R(x,y) \wedge T_A(y)]$$

$T_{\underline{R}(A)}(x) = \bigwedge_{y\in U} [F_R(x,y) \wedge T_A(y)]$

$I_{\underline{R}(A)}(x) = \bigwedge_{y\in U} [1 - I_R(x,y) \wedge I_A(y)]$

$F_{\underline{R}(A)}(x) = \bigvee_{y\in U} [T_R(x,y) \wedge F_A(y)]$

The pair $(\underline{R}, \overline{R})$ is fuzzy neutrosophic rough set of A with respect to (U,R) and $\overline{R}, \underline{R}$:FN(U)→FN(U) are refered to as upper and lower Fuzzy neutrosophic rough approximation operators respectively.

Theorem2.4[4]:
Let (U, R) be fuzzy neutrosophic approximation space. And $A \in FN(U)$, the upper FN approximation operator can be represented as follows $\forall\, x \in U$,

(1) $T_{\overline{R}(A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(A_\alpha)(x)]$

$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(A_{\alpha+})(x)]$

$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_{\alpha+}}(A_\alpha)(x)]$

$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_{\alpha+}}(A_{\alpha+})(x)]$

(2) $I_{\overline{R}(A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R\alpha}(A\alpha)(x)]$

$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R\alpha}(A\alpha_+)(x)]$

$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R\alpha}(A\alpha)(x)]$

$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R\alpha+}(A\alpha_+)(x)]$

(3) $F_{\overline{R}(A)}(u) = \bigwedge_{\alpha\in[0,1]} [\alpha \bigvee (1 - \overline{R^{\propto}}(A^\alpha)(x))]$

$= \bigwedge_{\alpha\in[0,1]} [\alpha \bigvee (1 - \overline{R^{\propto}}(A^{\alpha^+})(x))]$

$= \bigwedge_{\alpha\in[0,1]} [\alpha \bigvee (1 - \overline{R^{\propto+}}(A^\alpha)(x))]$

$= \bigwedge_{\alpha\in[0,1]} [\alpha \bigvee (1 - \overline{R^{\propto+}}(A^{\alpha^+})(x))]$

(3) $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R_{\alpha+}}(A_{\alpha+}) \subseteq \overline{R_{\alpha+}}(A_\alpha) \subseteq \overline{R_{(\alpha)}}(A_\alpha) \subseteq [\overline{R}(A)]_\alpha$

(4) $[\overline{R}(A)]\alpha + \subseteq \overline{R\propto +}(A\alpha +) \subseteq \overline{R\propto +}(A\alpha) \subseteq \overline{R\propto}(A\alpha) \subseteq [\overline{R}(A)]\alpha$

(6) $[\overline{R}(A)]^{\alpha^+} \subseteq \overline{R^{\propto+}}(A^{\alpha^+}) \subseteq \overline{R^{\propto+}}(A^\alpha) \subseteq \overline{R^{\propto}}(A^\alpha) \subseteq [\overline{R}(A)]_\alpha$

Theorem2.5[4]: Let (U, R) be fuzzy neutrosophic approximation space. And $A \in FN(U)$, the upper FN approximation operator can be represented as follows $\forall\, x \in U$,

(1) $T_{\underline{R}(A)}(u) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee 1 - \underline{R^{\propto}}(A_{\alpha+})(x)]$

$= \bigwedge_{\alpha\in[0,1]} [\alpha \vee 1 - \underline{R^{\propto}}(A_\alpha)(x)]$

$= \bigwedge_{\alpha\in[0,1]} [\alpha \vee 1 - \underline{R^{\propto+}}(A_{\alpha+})(x)]$

$= \bigwedge_{\alpha\in[0,1]} [\alpha \vee 1 - \underline{R_{\alpha+}}(A_{\alpha+})(x)]$

(2) $I_{\underline{R}(A)}(u) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee 1 - \underline{R(1-\propto)}(A\alpha +)(x)]$

$= \bigwedge_{\alpha\in[0,1]} [\alpha \vee 1 - \underline{R(1-\propto)A\alpha}(x)]$

$= \bigwedge_{\alpha\in[0,1]} [\alpha \vee 1 - \underline{R(1-\propto +)}(A\alpha +)(x)]$

$= \bigwedge_{\alpha\in[0,1]} [\alpha \vee 1 - \underline{R(1-\propto +)}(A\alpha +)(x)]$

(3) $F_{\underline{R}(A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha \bigvee (1 - \underline{R_{\propto}}(A^\alpha)(x))]$

$= \bigvee_{\alpha\in[0,1]} [\alpha \bigvee (1 - \underline{R_{\propto}}(A^{\alpha^+})(x))]$

$= \bigvee_{\alpha\in[0,1]} [\alpha \bigvee (1 - \underline{R_{\propto+}}(A^{\alpha^+})(x))]$

$= \bigvee_{\alpha\in[0,1]} [\alpha \bigvee (1 - \underline{R_{\propto+}}(A^\alpha)(x))]$

(3) $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R^{\propto}}(A_{\alpha+}) \subseteq \underline{R^{\propto+}}(A_{\alpha+}) \subseteq \underline{R^{\propto+}}(A_\alpha) \subseteq [\underline{R}(A)]_\alpha$

(4) $[\underline{R}(A)]\alpha + \subseteq \overline{R1-\propto}(A\alpha +) \subseteq \overline{R1-\propto +}(A\alpha) \subseteq \underline{R}1-\propto +(A\alpha) \subseteq [\overline{R}(A)]\alpha$

(6) $[\underline{R}(A)]^{\alpha^+} \subseteq \underline{R_{\propto}}(A^{\alpha^+}) \subseteq \underline{R_{\propto+}}(A^{\alpha^+}) \subseteq \underline{R_{\propto+}}(A^\alpha) \subseteq [\underline{R}(A)]_\alpha$

3. Equivalence relation on fuzzy neutrosophic rough sets

In this section we tend to prove the fuzzy neutrosophic relation having a special property such as reflexivity and transitivity, can be equivalentely characterised by the essential properties of lower and upper approximation operators.

Theorem 3.1:
Let R be a fuzzy neutrosophic relation on U and $\overline{R}$ and $\underline{R}$ the lower and upper approximation operators induced by (U, R). Then

(1) R is reflexive ⟺
 R1) $\underline{R}$ (A)⊆ A , ∀ A ∈ FN(U) ,
 R2) A ⊆ $\overline{R}$ (A), ∀ A ∈ FN(U).

(2) R is symmetric ⟺

S1) $T_{\overline{R}(1_x)}(y) = T_{\overline{R}(1_y)}(x)$, ∀ (x,y)∈ U ×U,
S2) $I_{\overline{R}(1_x)}(y) = I_{\overline{R}(1_y)}(x)$, ∀ (x,y) ∈ U ×U,
S3) $F_{\overline{R}(1_x)}(y) = F_{\overline{R}(1_x)}(y)$, ∀ (x,y)∈ U ×U,
S4) $T_{\underline{R}(1_{U-\{x\}})}(y) = T_{\underline{R}(1_{U-\{y\}})}(x)$, ∀ (x,y)∈ U ×U,
S5) $I_{\underline{R}(1_{U-\{x\}})}(y) = I_{\underline{R}(1_{U-\{y\}})}(x)$, ∀ (x,y)∈ U ×U,
S6) $F_{\underline{R}(1_{U-\{x\}})}(y) = F_{\underline{R}(1_{U-\{y\}})}(x)$, ∀ (x,y)∈ U ×U.

(3) R is transitive ⟺
T1) $\underline{R}$ (A) ⊆ $\underline{R}$ ($\underline{R}$ (A)) ∀ A ∈ FN(U)
T2) $\overline{R}$ ($\overline{R}$ (A)) ⊆ $\overline{R}$ (A), ∀ A ∈ FN(U)

Proof:
(1)R1 and R2 are equivalent because of the duality of the lower and upper fuzzy neutrosophic rough approximation operators. We need to prove that reflexivity of R is equivalent to R2.
Assume that R is reflexive. For any A ∈ FN(U) and x ∈ U, by the reflexivity of R we have $T_R(x,x) = 1$, $I_R(x,x) = 1$, $F_R(x,x) = 0$. Then

$$T_{\overline{R}(A)}(x) = \vee_{y\in U}[T_R(x,y)\wedge T_A(y)]$$
$$\geq T_R(x,x)\wedge T_A(x) = T_A(x)$$
$$I_{\overline{R}(A)}(x) = \vee_{y\in U}[I_R(x,y)\wedge I_A(y)]$$
$$\geq I_R(x,x)\wedge I_A(x) = I_A(x)$$
$$F_{\overline{R}(A)}(x) = \wedge_{y\in U}[F_R(x,y)\vee F_A(y)]$$
$$\geq F_R(x,x)\vee F_A(x) = F_A(x)$$

Thus A ⊆ $\overline{R}$ (A), ∀ A ∈ FN(U). R2 holds.

Conversely, assume that R2 holds.
For any $x \in U$, since $A \subseteq \overline{R}(A)$ for all $A \in FN(U)$.
Let A= 1_x , we have

$$1 = T_{1_x}(x) \leq T_{\overline{R}(1_x)}(x) = \vee_{y\in U}[T_R(x,y)\wedge T_{1x}(y)]$$
$$= T_R(x,x)$$
$$1 = I_{1_x}(x) \leq I_{\overline{R}(1_x)}(x) = \vee_{y\in U}[I_R(x,y)\wedge I_{1x}(y)]$$
$$= I_R(x,x)$$
$$0 = F_{1_x}(x) \geq F_{\overline{R}(1_x)}$$
$$\wedge_{y\in U}[F_R(x,y)\vee F_{1_x}(y)] = F_R(x,x)$$

Hence,
$T_R(x,x)$ =1, $I_R(x,x)$ =1, $F_R(x,x)$ =0.
Thus we can conclude, that FN relation R is reflexive.

(2) For any $(x,y)\in U\times U$, we have

$$T_{\overline{R}(1_y)}(x) = \vee_{y'\in U}[T_R(x,y')\wedge T_{1_y}(y')] = T_R(x,y)$$
$$I_{\overline{R}(1_y)}(x) = \vee_{y'\in U}[I_R(x,y')\wedge I_{1_y}(y')] = I_R(x,y)$$
$$F_{\overline{R}(1_y)}(x) = \wedge_{y'\in U}[F_R(x,y')\vee F_{1_y}(y') = F_R(x,y)$$

Also, we have

$$T_{\overline{R}(1_x)}(y) = \vee_{y'\in U}[T_R(y,y')\wedge T_{1x}(y') = T_R(y,x)$$
$$I_{\overline{R}(1_x)}(y) = \vee_{y'\in U}[I_R(y,y')\wedge I_{1x}(y') = I_R(y,x)$$
$$F_{\overline{R}(1_x)}(y) = \wedge_{y'\in U}[F_R(y,y')\vee F_{1x}(y') = F_R(y,x)$$

We know, R is symmetric if and only if
$T_R(x,y) = T_R(y,x)$, $I_R(x,y) = I_R(y,x)$, $F_R(x,y) = F_R(y,x)$ and S1, S2, S3 holds and similarly we can prove R is symmetric if and only if S4, S5, S6 holds.

(3) It can be easily verified that T1 and T2 are equivalent. We claim to prove that transitivity of R is equivalent to T2. Assume that R is transitive and $A \in FN(U)$. For any $x,y,z \in U$, we have

$$T_R(x,z) \geq \vee_{y\in U}[T_R(x,y)\wedge T_R(y,z)]$$
$$I_R(x,z) \geq \vee_{y\in U}[I_R(x,y)\wedge I_R(y,z)]$$
$$F_R(x,z) \leq \vee_{y\in U}[F_R(x,y)\wedge F_R(y,z)] .$$

We obtain,

$$T_{\overline{R}(\overline{R}(A))}(x) = \vee_{y\in U}[T_R(x,y)\wedge T_{\overline{R}(A)}(y)]$$
$$= \vee_{y\in U}[T_R(x,y)\wedge \vee_{z\in U}[T_R(y,z)\wedge T_A(z)]$$
$$= \vee_{y\in U}\vee_{z\in U}[T_R(x,y)\wedge T_R(y,z)\wedge T_A(z)]$$
$$= \vee_{y\in U}[\vee_{z\in U}(T_R(x,y)\wedge T_R(y,z))\wedge T_A(z)]$$
$$\leq \vee_{z\in U}[T_R(x,z)\wedge T_A(z) = T_{\overline{R}(A)}(x)$$
$$I_{\overline{R}(\overline{R}(A))}(x) = \vee_{y\in U}[I_R(x,y)\wedge I_{\overline{R}(A)}(y)]$$
$$= \vee_{y\in U}[I_R(x,y)\wedge \vee_{z\in U}[I_R(y,z)\wedge I_A(z)]$$
$$= \vee_{y\in U}\vee_{z\in U}[I_R(x,y)\wedge I_R(y,z)\wedge I_A(z)]$$

$= \underset{y\in U}{\vee}[\underset{z\in U}{\vee}(I_R(x,y)\wedge I_R(y,z))\wedge I_A(z)]$

$\leq \underset{z\in U}{\vee}[I_R(x,z)\wedge I_A(z) = I_{\overline{R}(A)}(x)$

$F_{\overline{R}(\overline{R}(A))}(x) = \underset{y\in U}{\wedge}[F_R(x,y)\vee F_{\overline{R}(A)}(y)]$

$= \underset{y\in U}{\wedge}[F_R(x,y)\vee \underset{z\in U}{\wedge}[F_R(y,z)\vee F_A(z)]$

$= \underset{y\in U}{\wedge}\underset{z\in U}{\wedge}[F_R(x,y)\vee F_R(y,z)\vee F_A(z)]$

$= \underset{y\in U}{\wedge}[\underset{z\in U}{\wedge}(F_R(x,y)\vee F_R(y,z))\vee F_A(z)]$

$\geq \underset{z\in U}{\wedge}[F_R(x,z)\vee F_A(z) = F_{\overline{R}(A)}(x)$

Thus, $\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$, $\forall$ A $\in$ FN(U), T2 holds.

Conversely, assume that T2 holds, For any $x, y, z \in U$

And $\lambda_1,\lambda_2,\lambda_3 \in [0,1]$, if $T_R(x,y) \geq \lambda_1$, $T_R(y,z) \geq \lambda_1$ $I_R(y,z) \geq \lambda_2$ $I_R(y,z) \geq \lambda_2$, $F_R(x,y) \leq \lambda_3$, $F_R(x,y) \leq \lambda_3$ then by T2, we have

$T_{\overline{R}(\overline{R}(1_z))}(x) \leq T_{\overline{R}(1_z)}(x)$

$= \underset{y\in U}{\vee}[T_R(x,y)\wedge T_{1_z}(y)] = T_R(x,z)$.

$I_{\overline{R}(\overline{R}(1_z))}(x) \leq I_{\overline{R}(1_z)}(x)$

$= \underset{y\in U}{\vee}[I_R(x,y)\wedge I_{1_z}(y)] = I_R(x,z)$.

$F_{\overline{R}(\overline{R}(1_z))}(x) \geq F_{\overline{R}(1_z)}(x)$

$= \underset{y\in U}{\wedge}[F_R(x,y)\vee F_{1_z}(y)] = F_R(x,z)$.

On otherhand,

$T_{\overline{R}(\overline{R}(1_z))}(x) = \underset{\alpha\in[0,1]}{\vee}[\alpha\wedge \overline{R}_\alpha(\overline{R}(1_z))_\alpha(x)]$

$= \sup\{\alpha\in[0,1]\bigcap x\in \overline{R}_\alpha(\overline{R}(1_z))_\alpha\}$

$= \sup\{\alpha\in[0,1]\bigcap \overline{R}_\alpha \cap(\overline{R}(1_z))_\alpha \neq \phi\}$

$= \sup\{\alpha\in[0,1]/\exists u\in U[T_R(x,u)\geq\alpha, T_{\overline{R}(1_z)}(\alpha)\geq\alpha]\}$

$= \sup\{\alpha\in[0,1]/\exists u\in U[T_R(x,u)\geq\alpha, T_R(u,z)\geq\alpha]\}$

$\geq T_R(x,y)\wedge T_R(y,z)\geq\lambda_1$

Thus we obtain $T_R(x,z)\geq\lambda_1$,and

$I_{\overline{R}(\overline{R}(1_z))}(x) = \underset{\alpha\in[0,1]}{\vee}[\alpha\wedge \overline{R}_\alpha(\overline{R}(1_z))_\alpha(x)]$

$= \sup\{\alpha\in[0,1]\bigcap x\in \overline{R}_\alpha(\overline{R}(1_z))_\alpha\}$

$= \sup\{\alpha\in[0,1]\bigcap \overline{R}_\alpha \cap(\overline{R}(1_z))_\alpha \neq \phi\}$

$= \sup\{\alpha\in[0,1]/\exists u\in U[I_R(x,u)\geq\alpha, I_{\overline{R}(1_z)}(\alpha)\geq\alpha]\}$

$= \sup\{\alpha\in[0,1]/\exists u\in U[I_R(x,u)\geq\alpha, I_R(u,z)\geq\alpha]\}$

$\geq I_R(x,y)\wedge I_R(y,z)\geq\lambda_2$

Thus we obtain $I_R(x,z)\geq\lambda_2$. also

$F_{\overline{R}(\overline{R}(1_z))}(x) = \underset{\alpha\in[0,1]}{\wedge}[\alpha\vee \overline{R}_\alpha(\overline{R}(1_z))_\alpha(x)]$

$= \inf\{\alpha\in[0,1]\bigcap x\in \overline{R}_\alpha(\overline{R}(1_z))_\alpha\}$

$= \inf\{\alpha\in[0,1]\bigcap \overline{R}_\alpha \cap(\overline{R}(1_z))_\alpha \neq \phi\}$

$= \inf\{\alpha\in[0,1]/\exists u\in U[F_R(x,u)\geq\alpha, F_{\overline{R}(1_z)}(\alpha)\geq\alpha]\}$

$= \inf\{\alpha\in[0,1]/\exists u\in U[F_R(x,u)\geq\alpha, F_R(u,z)\geq\alpha]\}$

$\geq F_R(x,y)\wedge F_R(y,z)\geq\lambda_3$

Thus $F_R(y,z)\leq\lambda_3$.

Hence, FN relation is transitive.

Corollary 3.2:
Let (U,R) be a fuzzy neutrosophic reflexive and transitive aproxiation space, i.e R is a fuzzy neutrosophic reflexive and transitive relation on U, and $\underline{R}$ and $\overline{R}$ the lower and upper FN rough approximation operator induced by (U,R). Then

(RT1) $\underline{R}(A) = \underline{R}(\underline{R}(A))\forall FN(U)$

(RT2) $\overline{R}(\overline{R}(A)) = \overline{R}(A)$

4. Relation between fuzzy neutrosophic approximation spaces and fuzzy neutrosophic topological spaces.

In this section, we generalise Fuzzy neutrosophic rough set theory in fuzzy neutrosophic topological spaces and investigate the relations between fuzzy neutrosophic rough set approximation and topologies.

4.1. From a fuzzy neutrosophic approximation space to fuzzy neutrosophic topological space

In this subsection, we assume that $U \neq \phi$ is a universe of discourse, R a fuzzy neutrosophic reflexive and transitive binary relation on U and $\underline{R}$ and $\overline{R}$ the lower and upper FN rough approximation operator induced by (U,R).

Theorem 4.1.1:

Let J be an index set, $A_j \in FN(U)$. Then

$\underline{R}(\underset{j\in J}{\cup}\underline{R}(A_j)) = \underset{j\in J}{\cup}\underline{R}(A_j)$.

Proof:
By reflexivity of R and Theorem (3.1), we have
$\underline{R}(\underset{j\in J}{\cup}\underline{R}(A_j)) \subseteq \underset{j\in J}{\cup}\underline{R}(A_j)$.

Since $\underset{j\in J}{\cup}\underline{R}(A_j) \supseteq \underline{R}(A_j)$, for all $j \in J$.

We have, $\underline{R}(\underset{j\in J}{\cup}\underline{R}(A_j)) \supseteq \underline{R}(\underline{R}(A_j))$

By transitivity of Rand theorem(3.1)
$\underline{R}(\underline{R}(A_j)) \supseteq \underline{R}(A_j)$.

Thus $\underline{R}(\underset{j\in J}{\cup}\underline{R}(A_j)) \subseteq \underline{R}(A_j)$, for all $j \in J$.

Consequently,
$\underline{R}(\underset{j\in J}{\cup}\underline{R}(A_j)) \supseteq \underset{j\in J}{\cup}\underline{R}(A_j)$

Hence we conclude
$\underline{R}(\underset{j\in J}{\cup}\underline{R}(A_j)) = \underset{j\in J}{\cup}\underline{R}(A_j)$.

Theorem 4.1.2:
$\tau_R = \{\underline{R}(A) / A \in FN(U)\}$ is a fuzzy neutrosophic toplogy on U.

Proof:
(I) In terms of Theorem (1) [4] we have $\underline{R}(1\sim) = 1\sim$, thus $1\sim \in \tau_R$ Since R is reflexive, by theorem 3.1, we have $\underline{R}(0\sim) = 0\sim$,therefore $0\sim \in \tau_R$,

(II) $\forall$ A, B $\in$ FN(U), since $\underline{R}(A), \underline{R}(B) \in \tau_R$ by theorem (1) [4] we have $\underline{R}(A) \cap \underline{R}(B) = \underline{R}(A \cap B) \in \tau_R$

(III) $\forall A_j \in FN(U), j \in J, J$ is an index set, by theorem 4.1.1 we have
$\underline{R}(\underset{j\in J}{\cup}\underline{R}(A_j)) = \underset{j\in J}{\cup}\underline{R}(A_j)$.

Thus $\underset{j\in J}{\cup}\underline{R}(A_j) \in \tau_R$.

Therefore, $\tau_R = \{\underline{R}(A) / A \in FN(U)\}$ is a fuzzy neutrosophic toplogy on U.

Therefore Theorem 4.1.2 states that a fuzzy neutrosophic reflexive and transitive approximation space can generate fuzzy neutrosophic topolgical space such that the family of all lower approximations of fuzzy neutrosophic sets with respect to fuzzy neutrosophic approximation space forms fuzzy neutrosophic topology.

Theorem 4.1.3:
Let (U, τ_R) be the fuzzy neutrosophic topological space induced from a fuzzy neutrsophic reflexive and transitive approximation space (U,R), i.e $\tau_R = \{\underline{R}(A) / A \in FN(U)\}$. Then, $\forall A \in FN(U)$.

$$1) \underline{R}(A) = \text{int}(A) = \cup\{\underline{R}(B) \cap R(B) \subseteq A, B \in FN(U)\}$$
$$2) \overline{R}(A) = cl(A) = \cap\{\sim \underline{R}(B) \cap \sim \underline{R}(B) \supseteq A, B \in FN(U)\}$$
$$= \cap\{\overline{R}(B) \cap \overline{R}(B) \supseteq A, B \in FN(U)\}$$

Proof:
(1) Since R is reflexive, by Theorem 3.1, we have

$\underline{R}(A) \subseteq A.$

Thus $\underline{R}(A) \subseteq \cup\{\underline{R}(B) \cap R(B) \subseteq A, B \in FN(U)\}$.

On other hand $\cup\{\underline{R}(B) \cap R(B)\} \subseteq A$,
then by Theorem 3.1
We obtain $\underline{R}(\cup\{\underline{R}(B) \cap R(B)\}) \subseteq \underline{R}(A)$ .In terms of Theorem 3.2 we concude $\cup\{\underline{R}(B) \cap R(B) \subseteq A\}$
$\cup\{\underline{R}(B) \cap R(B) \subseteq A\} = \underline{R}\,(\cup\{\underline{R}(B) \cap R(B) \subseteq A\})$

Hence, $\underline{R}(A) = \text{int}(A) = \cup\{\underline{R}(B) \cap R(B) \subseteq A\}$

(2) Follows from the duality of $\overline{R}$ and $\underline{R}$ and (1)

Theorem 4.1.4 :
Let (U, R) be a fuzzy neutrosophic reflexive and transitive approxiatin space and (U, τ) the fuzzy neutrosophic topological space induced by (U,R). Then
$T_R(x,y) = \bigwedge_{B \in y_\tau} T_B(x)$, $I_B(x,y) = \bigwedge_{B \in y_\tau} I_B(x)$,

$F_R(x,y) = \bigvee_{B \in y_\tau} F_B(x)$, $\forall x, y \in U$.

Where
$(y)_\tau = \{B \in FN(U) \cap \sim B \in \tau_R,$
$T_B(y) = 1, I_B(y) = 1, F_B(y) = 0\}$

Proof:
For any $x, y \in U$, by Thm 4.1.2 we have
$\overline{R}(1_y) = cl(1_y)$.

Also, $T_{\overline{R}(1_y)}(x) = \underset{u\in U}{\vee}[T_R(x,u) \wedge T_{1_y}(u)] = T_R(x,y)$

$I_{\overline{R}(1_y)}(x) = \underset{u\in U}{\vee}[I_R(x,u) \wedge I(u)] = I_R(x,y)$

$F_{\overline{R}(1_y)}(x) = \underset{u\in U}{\wedge}[F_R(x,u) \vee F_{1_y}(u)] = F_R(x,y)$

On other hand $cl(1_y)$
$= \cap\{B \in FN(U) \cap B\}$ is a FN closed set and $1_y \subseteq$ B}
$= \cap\{B \in FN(U) \cap \sim B \in \tau_R\}$ and $1_y \subseteq$ B}
Then

$T_{cl(1_y)}(x) = \wedge\{T_B(x) \cap \sim B \in \tau_R, B \supseteq 1_y\}$

$= \wedge\{T_B(x) \cap \sim B \in \tau_R,$

$T_B(y) = 1, I_B(y) = 1, F_B(y) = 0$

$\wedge_{B \in (y)_\tau} T_B(x)$

$I_{cl(1_y)}(x) = \wedge\{T_B(x) \cap \sim B \in \tau_R, B \supseteq 1_y\}$

$= \wedge\{I_B(x) \cap \sim B \in \tau_R,$

$T_B(y) = 1, I_B(y) = 1, F_B(y) = 0$

$\wedge_{B \in (y)_\tau} I_B(x)$

$F_{cl(1_y)}(x) = \vee\{F_B(x) \cap \sim B \in \tau_R, B \supseteq 1_y\}$

$= \vee\{F_B(x) \cap \sim B \in \tau_R,$

$T_B(y) = 1, I_B(y) = 1, F_B(y) = 0$

$= \vee_{B \in (y)_\tau} F_B(x)$

Hence,

$T_R(x, y) = \wedge_{B \in y_\tau} T_B(x),\ I_R(x, y) = \wedge_{B \in y_\tau} I_B(x),$

$F_R(x, y) = \vee_{B \in y_\tau} F_B(x),\ \forall x, y \in U$.

4.2. Fuzzy neutrsophic approximation space

In this section we discuss the suffiecient and necessary conditions under which a FN topological space be associated with a fuzzy neutrosophic approximation space and proved $cl(A) = \overline{R}(A)$ and $\text{int}(A) = \underline{R}(A)$.

Definition 4.2.1:

If P:FN(U)→FN(U) is an operator from FN(U) to FN(U), we can define three operators from F(U) to F(U), denoted by P_T, P_I, P_F , such that $P_T(T_A) = T_{P(A)}$ and $P_T(T_A) = I_{P(A)}$ and $P_T(T_A) = F_{P(A)}$.

That is $P(A) = P((T_A, I_A, F_A))$

$= (T_{P(A)}, I_{P(A)}, F_{P(A)})$

$= P_T(T_A), P_I(I_A), P_F(F_A)$

Theorem 4.2.2:

Let (U, τ) be fuzzy neutrosophic topological space and Cl, int:FN(U)→FN(U) the fuzzy neurtrosophic closure operator and fuzzy neutrosophic interior operator respectively. Then there exits a fuzzy neutrosophic reflexive and transitive relation R on U such that $\overline{R}(A) = cl(A)$ and $\underline{R}(A) = \text{int}(A)$ for all $A \in FN(U)$

Iff cl satisfies the fowing conditions (C1) and (C2), or equivalently, int satisfies the following conditions (I1) and (I2).

$(I1)\, cl(A \cap \overline{\overline{(\alpha, \beta, \gamma)}}) = cl(A) \cap \overline{\overline{(\alpha, \beta, \gamma)}}$

$\forall A \in FN(U), \forall \alpha, \beta, \gamma \in [0,1]$

With $\alpha + \beta + \gamma \leq 3$

(I2) cl($(\cup_{i \in J} A_i)$) = $\cup_{i \in J}$ cl(c), $A_i \in FN(U), i \in J$, J is any index set.

$(C1)\, \text{int}(A \cup \overline{\overline{(\alpha, \beta, \gamma)}}) = \text{int}(A) \cup \overline{\overline{(\alpha, \beta, \gamma)}}$

$\forall A \in FN(U), \forall \alpha, \beta, \gamma \in [0,1]$

(C2)int($(\cap_{i \in J} A_i)$)= $\cap_{i \in J}$ int(A_i), $A_i \in FN(U), i \in J$, J is any index set.

Proof:

Assume that there exists a fuzzy neutrosophic reflexive and transitive relation R on U such that $\overline{R}(A) = cl(A)$ and $\underline{R}(A) = \text{int}(A)$ for all $A \in FN(U)$, then by theorem 3.1, it can be easily seen that (C1), (C2), (I1), (I2) easily hold.

Converesly, Assume that closure operator cl:FN(U)→FN(U) satisfies conitions (C1) and (C2) and the interior operator int:FN(U)→FN(U) satifies the conditions (I1) and (I2).

For the closure operator we derive operators cl_T , cl_I and cl_F from FN(U) to FN(U) such that $cl_T(T_A) = T_{cl(A)}$, $cl_T(I_A) = I_{cl(A)}$, $cl_T(F_A) = F_{cl(A)}$. Likewise, from the interior operator int we have three operators $\text{int}_T, \text{int}_I, \text{int}_F$ from FN(U) to FN(U) such that $\text{int}_T(T_A) = T_{\text{int(A)}}$ $\text{int}_T(T_A) = I_{\text{int(A)}}$, $\text{int}_T(T_A) = F_{\text{int(A)}}$. We now define a FN relation R on U by cl as follows: for $(x, y) \in U \times U$.

$T_R(x, y) = cl_T(T_{1_y})(x)$, $I_R(x, y) = cl_I(I_{1_y})(x)$, $F_R(x, y) = cl_F(F_{1_y})(x)$

For $A \in FN(U)$

$T_A = \cup_{y \in U} [T_{1_y} \cap \overline{\overline{T_A(\text{y})}}],$

$I_A = \cup_{y \in U} [I_{1_y} \cap \overline{\overline{I_A(\text{y})}}],$

$F_A = \underset{y\in U}{\cup}[F_{1_y} \cap \overline{\overline{F_A(y)}}]$,

We also observe that (C1) implies (CT1), (CI1) and (CF1), and (C2) implies (CT2), (CI2) and (CF2)

(CT1) $(cl_T(T_{A\cap\overline{\overline{(\alpha,\beta,\gamma)}}}) = cl_T(T_{A\cap\overline{\overline{(\alpha)}}}) = cl_T(T_A)\cap\overline{\overline{\alpha}}$

$\forall A \in FN(U), \forall \alpha,\beta,\gamma \in [0,1]$ with $\alpha+\beta+\gamma \le 3$.

(CI1) $(cl_I(I_{A\cap\overline{\overline{(\alpha,\beta,\gamma)}}}) = cl_I(I_{A\cap\overline{\overline{(\beta)}}}) = cl_I(I_A)\cap\overline{\overline{\beta}}$

$\forall A \in FN(U), \forall \alpha,\beta,\gamma \in [0,1]$ with $\alpha+\beta+\gamma \le 3$.

(CF1) $(cl_F(F_{A\cap\overline{\overline{(\alpha,\beta,\gamma)}}}) = cl_F(F_{A\cap\overline{\overline{(\gamma)}}}) = cl_F(F_A)\cap\overline{\overline{\gamma}}$

$\forall A \in FN(U), \forall \alpha,\beta,\gamma \in [0,1]$ with $\alpha+\beta+\gamma \le 3$.

(CT2) $cl_T(T\underset{i\in J}{\cup}A_i) = cl_T(\underset{i\in J}{\cup}T_{A_i})$

$= \underset{i\in J}{\cup} cl_T(T_{A_i})$, $A_i \in FN(U)$, $i \in J$, J is any index set.

(CI2) $cl_I(I\underset{i\in J}{\cup}A_i) = cl_I(\underset{i\in J}{\cup}I_{A_i})$

$= \underset{i\in J}{\cup} cl_I(I_{A_i})$, $A_i \in FN(U)$, $i \in J$, J is any index set.

, J is any index set.

(CF2) $cl_F(F\underset{i\in J}{\cup}A_i) = cl_F(\underset{i\in J}{\cup}F_{A_i})$

$= \underset{i\in J}{\cup} cl_F(F_{A_i})$, $A_i \in FN(U)$, $i \in J$, J is any index set.

Then for any , $x \in U$ according to definition 4.2.1, and above properties, we have

$T_{\overline{R}(A)}(x) = \underset{y\in U}{\vee}[T_R(x,y)\wedge T_A(y))]$

$= \underset{y\in U}{\vee}[cl_T(T_{1_y})(y)\wedge T_A(y))]$

$= \underset{y\in U}{\vee}[(cl_T(T_{1_y})\cap\overline{\overline{T_A(y)}})]$

$= \underset{y\in U}{\vee}[(cl_T(T_{1_y}\cap\overline{\overline{T_A(y)}}))]$

$= [\underset{y\in U}{\cup}(cl_T(T_{1_y}\cap\overline{\overline{T_A(y)}}))](x)$

$= [(cl_T(\underset{y\in U}{\cup}(T_{1_y}\cap\overline{\overline{T_A(y)}}))](x)$

$= (cl_T(T_A)(x) = T_{cl(A)}(x)$

$I_{\overline{R}(A)}(x) = \underset{y\in U}{\vee}[I_R(x,y)\wedge I_A(y))]$

$= \underset{y\in U}{\vee}[cl_I(I_{1_y})(y)\wedge I_A(y))]$

$= \underset{y\in U}{\vee}[(cl_I(I_{1_y})\cap\overline{\overline{I_A(y)}})]$

$= \underset{y\in U}{\vee}[(cl_I(I_{1_y}\cap\overline{\overline{I_A(y)}}))]$

$= [\underset{y\in U}{\cup}(cl_I(I_{1_y}\cap\overline{\overline{I_A(y)}}))](x)$

$= [(cl_I(\underset{y\in U}{\cup}(I_{1_y}\cap\overline{\overline{I_A(y)}}))](x)$

$= (cl_I(I_A)(x) = I_{cl(A)}(x)$

$F_{\overline{R}(A)}(x) = \underset{y\in U}{\wedge}[F_R(x,y)\vee F_A(y))]$

$= \underset{y\in U}{\wedge}[cl_F(F_{1_y})(y)\vee F_A(y))]$

$= \underset{y\in U}{\wedge}[(cl_F(F_{1_y})\cup F\overline{\overline{I_A(y)}})]$

$= \underset{y\in U}{\wedge}[(cl_F(F_{1_y}\cup\overline{\overline{F_A(y)}}))]$

$= [\underset{y\in U}{\cap}(cl_F(F_{1_y}\cup\overline{\overline{F_A(y)}}))](x)$

$= [(cl_F(\underset{y\in U}{\cap}(F_{1_y}\cup\overline{\overline{F_A(y)}}))](x)$

$= (cl_F(F_A)(x) = F_{cl(A)}(x)$

Thus $cl(A) = \overline{R}(A)$.

Similary we can prove $\text{int}(A) = \underline{R}(A)$

Conclusion:

In this paper we defined the topological structures of fuzzy neutrosophic rough sets. We found that fuzzy neutrosophic topological space can be induced by fuzzy rough approximation operator if and only if fuzzy neutrosophic relation is reflexive and transitive. Also we have investigated the sufficient and necessary condition for which a fuzzy neutrosophic topological space can associate with fuzzy neutrosophic reflexive and transitive rough approximation space such that FN rough upper approximation equals closure and FN rough lower approximation equals interior operator.

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