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A Study on Neutrosophic Frontier and Neutrosophic Semi-frontier in Neutrosophic Topological Spaces

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ABSTRACT. In this paper neutrosophic frontier and neutrosophic semi-frontier in neutrosophic topology are introduced and several of their properties, characterizations and examples are established.

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I. INTRODUCTION

Theory of Fuzzy sets [21], Theory of Intuitionistic fuzzy sets [2], Theory of Neutrosophic sets [10] and the theory of Interval Neutrosophic sets [13] can be considered as tools for dealing with uncertainties. However, all of these theories have their own difficulties which are pointed out in [10]. In 1965, Zadeh [21] introduced fuzzy set theory as a mathematical tool for dealing with uncertainties where each element had a degree of membership. The Intuitionistic fuzzy set was introduced by Atanassov [2] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. The neutrosophic set was introduced by Smarandache [10] and explained, neutrosophic set is a generalization of Intuitionistic fuzzy set. In 2012, Salama, Alblowi [18], introduced the concept of Neutrosophic topological spaces. They introduced neutrosophic topological space as a generalization of Intuitionistic fuzzy topological space and a Neutrosophic set besides the degree of membership, the degree of indeterminacy and the degree of non-membership of each element.

The concepts of neutrosophic semi-open sets, neutrosophic semi-closed sets, neutrosophic semi-interior and neutrosophic semi-closure in neutrosophic topological spaces were introduced by P. Iswarya and Dr. K. Bageerathi [12] in 2016. Frontier and semifrontier in intuitionistic fuzzy topological spaces were introduced by Athar Kharal [4] in 2014. In this paper, we are extending the above concepts to neutrosophic topological spaces. We study some of the basic properties of neutrosophic frontier and neutrosophic semi-frontier in neutrosophic topological spaces with examples. Properties of neutrosophic semi-interior, neutrosophic semi-closure, neutrosophic frontier and neutrosophic semi-frontier have been obtained in neutrosophic product related spaces.

II. NEUTROSOPHIC FRONTIER

In this section, the concepts of the neutrosophic frontier in neutrosophic topological space are introduced and also discussed their characterizations with some related examples.

Definition 2.1 Let $\alpha, \beta, \lambda \in [0, 1]$ and $\alpha + \beta + \lambda \leq 1$. A neutrosophic point [*NP* for short] $x_{(\alpha,\beta,\lambda)}$ of X is a *NS* of X which is defined by

$$x_{(\alpha,\beta,\lambda)} = \begin{cases} (\alpha,\beta,\lambda), & y = x, \\ (0,0,1), & y \neq x. \end{cases}$$

In this case, x is called the support of $x_{(\alpha,\beta,\lambda)}$ and α, β and λ are called the value, intermediate value and the non-value of $x_{(\alpha,\beta,\lambda)}$, respectively. A *NP* $x_{(\alpha,\beta,\lambda)}$ is said to belong to a *NS* $A = \langle \mu_A, \sigma_A, \gamma_A \rangle$ in X, denoted by $x_{(\alpha,\beta,\lambda)} \in A$ if $\alpha \leq \mu_A(x)$, $\beta \leq \sigma_A(x)$ and $\lambda \geq \gamma_A(x)$. Clearly a neutrosophic point can be represented by an ordered triple of neutrosophic points as follows : $x_{(\alpha,\beta,\lambda)} = (x_\alpha , x_\beta , C (x_{C(\lambda)}))$. A class of all *NPs* in X is denoted as *NP* (X).

Definition 2.2 Let X be a *NTS* and let $A \in NS$ (X). Then $x_{(\alpha,\beta,\lambda)} \in NP$ (X) is called a neutrosophic frontier point [*NFP* for short] of A if $x_{(\alpha,\beta,\lambda)} \in NCl(A) \cap NCl(C(A))$. The intersection of all the *NFPs* of A is called a neutrosophic frontier of A and is denoted by $NFr(A)$. That is,
$NFr(A) = NCl(A) \cap NCl(C(A))$.

Proposition 2.3 For each $A \in NS(X)$, $A \cup NFr(A) \subseteq NCl(A)$.

Proof : Let A be the NS in the neutrosophic topological space X. Then by Definition 2.2,

$A \cup NFr(A) = A \cup (NCl(A) \cap NCl(C(A)))$
$= (A \cup NCl(A)) \cap (A \cup NCl(C(A)))$
$\subseteq NCl(A) \cap NCl(C(A))$
$\subseteq NCl(A)$

Hence $A \cup NFr(A) \subseteq NCl(A)$.

From the above proposition, the inclusion cannot be replaced by an equality as shown by the following example.

Example 2.4 Let X = { a, b } and τ = { 0_N, A, B, C, D, 1_N }. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are C (τ) = { 1_N, E, F, G, H, 0_N } where

A = ⟨ (0.5, 1, 0.1), (0.9, 0.2, 0.5) ⟩,
B = ⟨ (0.2, 0.5, 0.9), (0, 0.5, 1) ⟩,
C = ⟨ (0.5, 1, 0.1), (0.9, 0.5, 0.5) ⟩,
D = ⟨ (0.2, 0.5, 0.9), (0, 0.2, 1) ⟩,
E = ⟨ (0.1, 0, 0.5), (0.5, 0.8, 0.9) ⟩,
F = ⟨ (0.9, 0.5, 0.2), (1, 0.5, 0) ⟩,
G = ⟨ (0.1, 0, 0.5), (0.5, 0.5, 0.9) ⟩ and
H = ⟨ (0.9, 0.5, 0.2), (1, 0.8, 0) ⟩.

Here NCl (A) = 1_N and NCl (C (A)) = NCl (E) = E. Then by Definition 2.2, NFr (A) = E.

Also $A \cup NFr(A)$ = ⟨ (0.5, 1, 0.1), (0.9, 0.8, 0.5) ⟩ $\subseteq$ 1_N. Therefore $NCl(A) = 1_N \nsubseteq$ ⟨ (0.5, 1, 0.1), (0.9, 0.8, 0.5) ⟩.

Theorem 2.5 For a NS A in the NTS X, $NFr(A) = NFr(C(A))$.

Proof : Let A be the NS in the neutrosophic topological space X. Then by Definition 2.2,

$NFr(A) = NCl(A) \cap NCl(C(A))$
$= NCl(C(A)) \cap NCl(A)$
$= NCl(C(A)) \cap NCl(C(C(A)))$

Again by Definition 2.2,
$= NFr(C(A))$

Hence $NFr(A) = NFr(C(A))$.

Theorem 2.6 If a NS A is a NCS, then $NFr(A) \subseteq A$.

Proof : Let A be the NS in the neutrosophic topological space X. Then by Definition 2.2,

$NFr(A) = NCl(A) \cap NCl(C(A))$
$\subseteq NCl(A)$

By Definition 4.4 (a) [18] ,
$= A$

Hence $NFr(A) \subseteq A$, if A is NCS in X.

The converse of the above theorem needs not be true as shown by the following example.

Example 2.7 From Example 2.4, NFr (C) = G $\subseteq$ C. But C $\notin$ C (τ).

Theorem 2.8 If a NS A is NOS, then $NFr(A) \subseteq C(A)$.

Proof : Let A be the NS in the neutrosophic topological space X. Then by Definition 4.3 [18] , A is NOS implies $C(A)$ is NCS in X. By Theorem 2.6, $NFr(C(A)) \subseteq C(A)$ and by Theorem 2.5, we get $NFr(A) \subseteq C(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 2.9 From Example 2.4, NFr (G) = G $\subseteq$ C (G) = C. But G $\notin$ τ.

Theorem 2.10 For a NS A in the NTS X, $C(NFr(A)) = NInt(A) \cup NInt(C(A))$.

Proof : Let A be the NS in the neutrosophic topological space X. Then by Definition 2.2,

$C(NFr(A)) = C(NCl(A) \cap NCl(C(A)))$

By Proposition 3.2 (1) [18] ,
$= C(NCl(A)) \cup C(NCl(C(A)))$

By Proposition 4.2 (b) [18] ,
$= NInt(C(A)) \cup NInt(A)$

Hence $C(NFr(A)) = NInt(A) \cup NInt(C(A))$.

Theorem 2.11 Let $A \subseteq B$ and $B \in NC(X)$ (resp., $B \in NO(X)$). Then $NFr(A) \subseteq B$ (resp., $NFr(A) \subseteq C(B)$), where $NC(X)$ (resp., $NO(X)$) denotes the class of neutrosophic closed (resp., neutrosophic open) sets in X.

Proof : By Proposition 1.18 (d) [12] , $A \subseteq B$, $NCl(A) \subseteq NCl(B)$ -------------------- (1).

By Definition 2.2,

$NFr(A) = NCl(A) \cap NCl(C(A))$
$\subseteq NCl(B) \cap NCl(C(A))$ by (1)
$\subseteq NCl(B)$

By Definition 4.4 (b) [18] ,
$= B$

Hence $NFr(A) \subseteq B$.

Theorem 2.12 Let A be the NS in the NTS X. Then $NFr(A) = NCl(A) - NInt(A)$.

Proof : Let A be the NS in the neutrosophic topological space X. By Proposition 4.2 (b) [18] , $C(NCl(C(A))) = NInt(A)$ and by Definition 2.2,

$NFr(A) = NCl(A) \cap NCl(C(A))$
$= NCl(A) - C(NCl(C(A)))$
by using $A - B = A \cap C(B)$

By Proposition 4.2 (b) [18] ,
$= NCl(A) - NInt(A)$

Hence $NFr(A) = NCl(A) - NInt(A)$.

Theorem 2.13 For a *NS* A in the *NTS* X, $NFr\,(NInt\,(A)) \subseteq NFr\,(A)$.

Proof : Let A be the *NS* in the neutrosophic topological space X. Then by Definition 2.2,
$NFr\,(NInt\,(A)) = NCl\,(NInt\,(A)) \cap NCl\,(C\,(NInt\,(A)))$
By Proposition 4.2 (a) [18] ,
$= NCl\,(NInt\,(A)) \cap NCl\,(NCl\,(C\,(A)))$
By Definition 4.4 (b) [18] ,
$= NCl\,(NInt\,(A)) \cap NCl\,(C\,(A))$
By Definition 4.4 (a) [18] ,
$\subseteq NCl\,(A) \cap NCl\,(C\,(A))$
Again by Definition 2.2,
$= NFr\,(A)$
Hence $NFr\,(NInt\,(A)) \subseteq NFr\,(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 2.14 Let X = { a, b } and τ = { 0_N, A, B, C, D, 1_N }. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are C (τ) = { 1_N, E, F, G, H, 0_N } where
A = ⟨ (0.5, 0.6, 0.7), (0.1, 0.9, 0.4) ⟩,
B = ⟨ (0.3, 0.9, 0.2), (0.4, 0.1, 0.6) ⟩,
C = ⟨ (0.5, 0.9, 0.2), (0.4, 0.9, 0.4) ⟩,
D = ⟨ (0.3, 0.6, 0.7), (0.1, 0.1, 0.6) ⟩,
E = ⟨ (0.7, 0.4, 0.5), (0.4, 0.1, 0.1) ⟩,
F = ⟨ (0.2, 0.1, 0.3), (0.6, 0.9, 0.4) ⟩,
G = ⟨ (0.2, 0.1, 0.5), (0.4, 0.1, 0.4) ⟩ and
H = ⟨ (0.7, 0.4, 0.3), (0.6, 0.9, 0.1) ⟩.
Define A_1 = ⟨ (0.4, 0.2, 0.8), (0.4, 0.5, 0.1) ⟩. Then C (A_1) = ⟨ (0.8, 0.8, 0.4), (0.1, 0.5, 0.4) ⟩.
Therefore by Definition 2.2, $NFr\,(A_1) = H \nsubseteq 0_N = NFr\,(NInt\,(A_1))$.

Theorem 2.15 For a *NS* A in the *NTS* X, $NFr\,(NCl\,(A)) \subseteq NFr\,(A)$.

Proof : Let A be the *NS* in the neutrosophic topological space X. Then by Definition 2.2,
$NFr\,(NCl\,(A)) = NCl\,(NCl\,(A)) \cap NCl\,(C\,(NCl\,(A)))$
By Proposition 1.18 (f) [12] and 4.2 (b) [18] ,
$= NCl\,(A) \cap NCl\,(NInt\,(C\,(A)))$
By Proposition 1.18 (a) [12] ,
$\subseteq NCl\,(A) \cap NCl\,(C\,(A))$
Again by Definition 2.2,
$= NFr\,(A)$
Hence $NFr\,(NCl\,(A)) \subseteq NFr\,(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 2.16 From Example 2.14, let A_2 = ⟨ (0.7, 0.9, 0.2), (0.5, 0.9, 0.3) ⟩.
Then C (A_2) = ⟨ (0.2, 0.1, 0.7), (0.3, 0.1, 0.5) ⟩. Then by Definition 2.2, $NFr\,(A_2) = G$.
Therefore $NFr\,(A_2) = G \nsubseteq 0_N = NFr\,(NCl\,(A_2))$.

Theorem 2.17 Let A be the *NS* in the *NTS* X. Then $NInt\,(A) \subseteq A - NFr\,(A)$.

Proof : Let A be the *NS* in the neutrosophic topological space X. Now by Definition 2.2,
$A - NFr\,(A) = A - (NCl\,(A) \cap NCl\,(C\,(A)))$
$= (A - NCl\,(A)) \cup (A - NCl\,(C\,(A)))$
$= A - NCl\,(C\,(A))$
$\supseteq NInt\,(A)$.
Hence $NInt\,(A) \subseteq A - NFr\,(A)$.

Example 2.18 From Example 2.14, $A_1 - NFr\,(A_1)$ = ⟨ (0.3, 0.2, 0.8), (0.1, 0.1, 0.6) ⟩.
Therefore $A_1 - NFr\,(A_1)$ = ⟨ (0.3, 0.2, 0.8), (0.1, 0.1, 0.6) ⟩ $\nsubseteq 0_N = NInt\,(A_1)$.

Remark 2.19 In general topology, the following conditions are hold :
$NFr\,(A) \cap NInt\,(A) = 0_N$,
$NInt\,(A) \cup NFr\,(A) = NCl\,(A)$,
$NInt\,(A) \cup NInt\,(C\,(A)) \cup NFr\,(A) = 1_N$.
But the neutrosophic topology, we give counter-examples to show that the conditions of the above remark may not be hold in general.

Example 2.20 From Example 2.14,
$NFr\,(A_2) \cap NInt\,(A_2) = G \cap C = G \neq 0_N$.

$NInt\,(A_2) \cup NFr\,(A_2) = C \cup G = C \neq 1_N = NCl\,(A_2)$.

$NInt\,(A_2) \cup NInt\,(C\,(A_2)) \cup NFr\,(A_2) = C \cup 0_N \cup G = C \neq 1_N$.

Theorem 2.21 Let A and B be the two *NSs* in the *NTS* X. Then $NFr\,(A \cup B) \subseteq NFr\,(A) \cup NFr\,(B)$.

Proof : Let A and B be the two *NSs* in the *NTS* X. Then by Definition 2.2,
$NFr\,(A \cup B) = NCl\,(A \cup B) \cap NCl\,(C\,(A \cup B))$
By Proposition 3.2 (2) [18] ,
$= NCl\,(A \cup B) \cap NCl\,(\,C\,(A) \cap C\,(B)\,)$
by Proposition 1.18 (h) and (o) [12] ,
$\subseteq (NCl\,(A) \cup NCl\,(B)) \cap (NCl\,(C\,(A)) \cap NCl(C\,(B)))$
$= [(NCl\,(A) \cup NCl\,(B)\,) \cap NCl\,(C\,(A))\,]$
$\cap [\,(\,NCl\,(A) \cup NCl\,(B)\,) \cap NCl\,(C\,(B))\,]$
$= [(NCl\,(A) \cap NCl\,(C\,(A))) \cup (NCl\,(B) \cap NCl(C\,(A)))]$
$\cap [(NCl\,(A) \cap NCl\,(C\,(B))) \cup (NCl(B) \cap NCl(C\,(B)))]$
Again by Definition 2.2,
$= [NFr\,(A) \cup (\,NCl\,(B) \cap NCl\,(C\,(A)))\,]$
$\cap [\,(\,NCl\,(A) \cap NCl\,(C\,(B))\,) \cup NFr(B)\,]$
$= (\,NFr\,(A) \cup NFr\,(B)) \cap [\,(\,NCl\,(B) \cap NCl\,(C\,(A))\,)$
$\cup (\,NCl\,(A) \cap NCl\,(C\,(B))\,)\,]$

$\subseteq$ *NFr* (*A*) $\cup$ *NFr* (*B*).
Hence *NFr* ($A \cup B$) $\subseteq$ *NFr* (*A*) $\cup$ *NFr* (*B*).

The converse of the above theorem needs not be true as shown by the following example.

Example 2.22 By Example 2.14, we define
$A_1 = \langle$ (0.2, 0, 0.5), (0.4, 0.1, 0.1) $\rangle$,
$A_2 = \langle$ (0.7, 0.9, 0.2), (0.5, 0.9, 0.3) $\rangle$,
$A_1 \cup A_2 = A_3 = \langle$ (0.7, 0.9, 0.2), (0.5, 0.9, 0.1) $\rangle$ and
$A_1 \cap A_2 = A_4 = \langle$ (0.2, 0, 0.5), (0.4, 0.1, 0.3) $\rangle$. Then
C (A_1) = $\langle$ (0.5, 1, 0.2), (0.1, 0.9, 0.4) $\rangle$,
C (A_2) = $\langle$ (0.2, 0.1, 0.7), (0.3, 0.1, 0.5) $\rangle$,
C (A_3) = $\langle$ (0.2, 0.1, 0.7), (0.1, 0.1, 0.5) $\rangle$ and
C (A_4) = $\langle$ (0.5, 1, 0.2), (0.3, 0.9, 0.4) $\rangle$.
Therefore *NFr* (A_1) $\cup$ *NFr* (A_2) = E $\cup$ G = E $\nsubseteq$ G = *NFr* (A_3) = *NFr* ($A_1 \cup A_2$).

Note 2.23 The following example shows that *NFr* ($A \cap B$) $\nsubseteq$ *NFr* (*A*) $\cap$ *NFr* (*B*) and
NFr (*A*) $\cap$ *NFr* (*B*) $\nsubseteq$ *NFr* ($A \cap B$).

Example 2.24 From Example 2.22, *NFr* ($A_1 \cap A_2$) = *NFr* (A_4) = E $\nsubseteq$ G = *NFr* (A_1) $\cap$ *NFr* (A_2).
From Example 2.14, We define $B_1 = \langle$ (0.4, 0.5, 0.1), (0.2, 0.9, 0.5) $\rangle$,
$B_2 = \langle$ (0.5, 0.2, 0.9), (0.8, 0.4, 0.7) $\rangle$,
$B_1 \cup B_2 = B_3 = \langle$ (0.5, 0.5, 0.1), (0.8, 0.9, 0.5) $\rangle$ and
$B_1 \cap B_2 = B_4 = \langle$ (0.4, 0.2, 0.9), (0.2, 0.4, 0.7) $\rangle$. Then
C (B_1) = $\langle$ (0.1, 0.5, 0.4), (0.5, 0.1, 0.2) $\rangle$,
C (B_2) = $\langle$ (0.9, 0.8, 0.5), (0.7, 0.6, 0.8) $\rangle$,
C (B_3) = $\langle$ (0.1, 0.5, 0.5), (0.5, 0.1, 0.8) $\rangle$ and
C (B_4) = $\langle$ (0.9, 0.8, 0.4), (0.7, 0.6, 0.2) $\rangle$.
Therefore *NFr* (B_1) $\cap$ *NFr* (B_2) = $1_N \cap 1_N = 1_N \nsubseteq$ H = *NFr* (B_4) = *NFr* ($B_1 \cap B_2$).

Theorem 2.25 For any *NSs A* and *B* in the *NTS* X, *NFr* ($A \cap B$) $\subseteq$ (*NFr* (*A*) $\cap$ *NCl* (*B*)) $\cup$ (*NFr* (*B*) $\cap$ *NCl* (*A*)).
Proof : Let *A* and *B* be the two *NSs* in the *NTS* X. Then by Definition 2.2,
NFr ($A \cap B$) = *NCl* ($A \cap B$) $\cap$ *NCl* (C ($A \cap B$))
By Proposition 3.2 (1) [18] ,
= *NCl* ($A \cap B$) $\cap$ *NCl* (C (*A*) $\cup$ C (*B*))
By Proposition 1.18 (n) and (h) [12] ,
$\subseteq$ (*NCl* (*A*) $\cap$ *NCl* (*B*)) $\cap$ (*NCl* (C(*A*)) $\cup$ *NCl* (C (*B*)))
= [(*NCl* (*A*) $\cap$ *NCl* (*B*)) $\cap$ *NCl* (C(*A*))]
$\cup$ [(*NCl* (*A*) $\cap$ *NCl* (*B*)) $\cap$ *NCl* (C(*B*))]
Again by Definition 2.2,
= (*NFr* (*A*) $\cap$ *NCl* (*B*)) $\cup$ (*NFr* (*B*) $\cap$ *NCl* (*A*))
Hence *NFr* ($A \cap B$) $\subseteq$ (*NFr* (*A*) $\cap$ *NCl* (*B*)) $\cup$ (*NFr* (*B*) $\cap$ *NCl* (*A*)).

The converse of the above theorem needs not be true as shown by the following example.

Example 2.26 From Example 2.24, (*NFr* (B_1) $\cap$ *NCl* (B_2)) $\cup$ (*NFr* (B_2) $\cap$ *NCl* (B_1)) = ($1_N \cap 1_N$) $\cup$ ($1_N \cap 1_N$) = $1_N \cup 1_N = 1_N \nsubseteq$ H = *NFr* ($B_1 \cap B_2$).

Corollary 2.27 For any *NSs A* and *B* in the *NTS* X, *NFr* ($A \cap B$) $\subseteq$ *NFr* (*A*) $\cup$ *NFr* (*B*).
Proof : Let *A* and *B* be the two *NSs* in the *NTS* X. Then by Definition 2.2,
NFr ($A \cap B$) = *NCl* ($A \cap B$) $\cap$ *NCl* (C ($A \cap B$))
By Proposition 3.2 (1) [18] ,
= *NCl* ($A \cap B$) $\cap$ *NCl* (C (*A*) $\cup$ C (*B*))
By Proposition 1.18 (n) and (h) [12] ,
$\subseteq$ (*NCl* (*A*) $\cap$ *NCl* (*B*)) $\cap$ (*NCl* (C(*A*)) $\cup$ *NCl* (C (*B*)))
= (*NCl* (*A*) $\cap$ *NCl* (*B*) $\cap$ *NCl* (C (*A*)))
$\cup$ (*NCl* (*A*) $\cap$ *NCl* (*B*) $\cap$ *NCl* (C (*B*)))
Again by Definition 2.2,
= (*NFr* (*A*) $\cap$ *NCl* (*B*)) $\cup$ (*NCl* (*A*) $\cap$ *NFr* (*B*))
$\subseteq$ *NFr* (*A*) $\cup$ *NFr* (*B*)
Hence *NFr* ($A \cap B$) $\subseteq$ *NFr* (*A*) $\cup$ *NFr* (*B*).

The equality in the above corollary may not hold as seen in the following example.

Example 2.28 From Example 2.24, *NFr* (B_1) $\cup$ *NFr* (B_2) = $1_N \cup 1_N = 1_N \nsubseteq$ H = *NFr* (B_4) = *NFr* ($B_1 \cap B_2$).

Theorem 2.29 For any *NS A* in the *NTS* X,
(1) *NFr* (*NFr* (*A*)) $\subseteq$ *NFr* (*A*),
(2) *NFr* (*NFr* (*NFr* (*A*))) $\subseteq$ *NFr* (*NFr* (*A*)).
Proof : (1) Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 2.2,
NFr (*NFr* (*A*)) = *NCl* (*NFr* (*A*)) $\cap$ *NCl* (C (*NFr* (*A*)))
Again by Definition 2.2,
= *NCl* (*NCl* (*A*) $\cap$ *NCl* (C (*A*))) $\cap$
NCl (C (*NCl* (*A*) $\cap$ *NCl* (C (*A*))))
By Proposition 1.18 (f) [12] and by 4.2 (b) [18] ,
$\subseteq$ (*NCl* (*NCl* (*A*)) $\cap$ *NCl* (*NCl* (C (*A*))))
$\cap$ *NCl* (*NInt* (C (*A*)) $\cup$ *NInt* (*A*))
By Proposition 1.18 (f) [12] ,
= (*NCl* (*A*) $\cap$ *NCl* (C (*A*))) $\cap$ (*NCl* (*NInt* (C (*A*)))
$\cup$ *NCl* (*NInt* (A))
$\subseteq$ *NCl* (*A*) $\cap$ *NCl* (C (*A*))
By Definition 2.2,
= *NFr* (*A*)
Therefore *NFr* (*NFr* (*A*)) $\subseteq$ *NFr* (*A*).

(2) By Definition 2.2,
NFr (*NFr* (*NFr* (*A*))) = *NCl* (*NFr* (*NFr* (*A*))) $\cap$
NCl (C (*NFr* (*NFr* (*A*))))

By Proposition 1.18 (f) [12] ,
$\subseteq$ (*NFr* (*NFr* (*A*))) $\cap$ *NCl* (C (*NFr* (*NFr* (*A*))))
$\subseteq$ *NFr* (*NFr* (*A*)).
Hence *NFr* (*NFr* (*NFr* (*A*))) $\subseteq$ *NFr* (*NFr* (*A*)).

Remark 2.30 From the above theorem, the converse of (1) needs not be true as shown by the following example and no counter-example could be found to establish the irreversibility of inequality in (2).

Example 2.31 Let X = {a, b} and τ = {0_N, A, B, 1_N }. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are C (τ) = { 1_N, C, D, 0_N } where
A = ⟨ (0.8, 0.4, 0.5), (0.4, 0.6, 0.7) ⟩,
B = ⟨ (0.4, 0.2, 0.9), (0.1, 0.4, 0.9) ⟩,
C = ⟨ (0.5, 0.6, 0.8), (0.7, 0.4, 0.4) ⟩ and
D = ⟨ (0.9, 0.8, 0.4), (0.9, 0.6, 0.1) ⟩. Define
A_1 = ⟨ (0.6, 0.7, 0.8), (0.5, 0.4, 0.5) ⟩. Then
C (A_1) = ⟨ (0.8, 0.3, 0.6), (0.5, 0.6, 0.5) ⟩.
Therefore by Definition 2.2, *NFr* (A_1) = D $\nsubseteq$ C = *NFr* (*NFr* (A_1)).

Theorem 2.32 Let *A*, *B*, *C* and *D* be the *NSs* in the *NTS* X. Then ($A \cap B$) $\times$ ($C \cap D$) = ($A \times D$) $\cap$ ($B \times C$).
Proof : Let *A*, *B*, *C* and *D* be the *NSs* in the *NTS* X. Then by Definition 2.2 [12] ,
$\mu_{(A\cap B)\times(C\cap D)}(x, y)$
$= \min \{\mu_{(A\cap B)}(x), \mu_{(C\cap D)}(y)\}$
$= \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_C(y), \mu_D(y) \} \}$
$= \min \{ \min \{ \mu_A(x), \mu_D(y) \}, \min \{ \mu_B(x), \mu_C(y) \} \}$
$= \min \{ \mu_{(A\times D)}(x, y), \mu_{(B\times C)}(x, y) \}$.
Thus $\mu_{(A\cap B)\times(C\cap D)}(x, y) = \mu_{(A\times D)\cap(B\times C)}(x, y)$.
Similarly
$\sigma_{(A\cap B)\times(C\cap D)}(x, y)$
$= \min \{\sigma_{(A\cap B)}(x), \sigma_{(C\cap D)}(y)\}$
$= \min \{ \min \{ \sigma_A(x), \sigma_B(x) \}, \min \{ \sigma_C(y), \sigma_D(y) \} \}$
$= \min \{ \min \{ \sigma_A(x), \sigma_D(y) \}, \min \{ \sigma_B(x), \sigma_C(y) \} \}$
$= \min \{ \sigma_{(A\times D)}(x, y), \sigma_{(B\times C)}(x, y) \}$.
Thus $\sigma_{(A\cap B)\times(C\cap D)}(x, y) = \sigma_{(A\times D)\cap(B\times C)}(x, y)$.
And also
$\gamma_{(A\cap B)\times(C\cap D)}(x, y)$
$= \max \{ \gamma_{(A\cap B)}(x), \gamma_{(C\cap D)}(y)\}$
$= \max \{ \max \{ \gamma_A(x), \gamma_B(x) \}, \max \{ \gamma_C(y), \gamma_D(y) \} \}$
$= \max \{ \max \{ \gamma_A(x), \gamma_D(y) \}, \max \{ \gamma_B(x), \gamma_C(y) \} \}$
$= \max \{ \gamma_{(A\times D)}(x, y), \gamma_{(B\times C)}(x, y) \}$.
Thus $\gamma_{(A\cap B)\times(C\cap D)}(x, y) = \gamma_{(A\times D)\cap(B\times C)}(x, y)$.
Hence ($A \cap B$) $\times$ ($C \cap D$) = ($A \times D$) $\cap$ ($B \times C$).

Theorem 2.33 Let X_i, $i = 1, 2, \ldots, n$ be a family of neutrosophic product related *NTSs*. If each A_i is a *NS* in X_i. Then *NFr* ($\prod_{i=1}^{n} A_i$) = [*NFr* (A_1) $\times$ *NCl* (A_2) $\times \cdots \times$ *NCl* (A_n)] $\cup$ [*NCl* (A_1) $\times$ *NFr* (A_2) $\times$ *NCl* (A_3) $\times \cdots \times$ *NCl* (A_n)] $\cup \cdots \cup$ [*NCl* (A_1) $\times$ *NCl* (A_2) $\times \cdots \times$ *NFr* (A_n)].
Proof : It suffices to prove this for $n = 2$. Let A_i be the *NS* in the neutrosophic topological space X_i. Then by Definition 2.2,
NFr ($A_1 \times A_2$) = *NCl* ($A_1 \times A_2$) $\cap$ *NCl* (C ($A_1 \times A_2$))
By Proposition 4.2 (a) [18] ,
= *NCl* ($A_1 \times A_2$) $\cap$ C (*NInt* ($A_1 \times A_2$))
By Theorem 2.17 (1) and (2) [12] ,
= (*NCl* (A_1) $\times$ *NCl* (A_2)) $\cap$ C (*NInt* (A_1) $\times$ *NInt* (A_2))
= (*NCl* (A_1) $\times$ *NCl* (A_2)) $\cap$
C[(*NInt* (A_1) $\cap$ *NSCl* (A_1)) $\times$(*NInt* (A_2) $\cap$ *NCl* (A_2))]
By Lemma 2.3 (iii) [12] ,
= (*NCl* (A_1) $\times$ *NCl* (A_2)) $\cap$ [C (*NInt* (A_1) $\cap$ *NCl* (A_1)) $\times$ 1_N $\cup$ 1_N $\times$ C (*NInt* (A_2) $\cap$ *NCl* (A_2))]
= (*NCl* (A_1) $\times$ *NCl* (A_2)) $\cap$ [(*NCl* (C (A_1)) $\cup$ *NInt* (C (A_1))) $\times$ 1_N $\cup$ 1_N $\times$ (*NCl* (C (A_2)) $\cup$ *NInt* (C (A_2)))]
= (*NCl* (A_1) $\times$ *NCl* (A_2)] $\cap$ [(*NCl* (C (A_1)) $\times$ 1_N) $\cup$ (1_N $\times$ *NCl* (C (A_2)))]
= [(*NCl* (A_1) $\times$ *NCl* (A_2)) $\cap$ (*NCl* (C (A_1)) $\times$ 1_N)] $\cup$ [(*NCl* (A_1) $\times$ *NCl* (A_2)) $\cap$ (1_N $\times$ *NCl* (C (A_2)))]
By Theorem 2.32,
= [(*NCl* (A_1) $\cap$ *NCl* (C (A_1))) $\times$ (1_N $\cap$ *NCl* (A_2))] $\cup$ [(*NCl* (A_1) $\cap$ 1_N) $\times$ (*NCl* (A_2) $\cap$ *NCl* (C (A_2)))]
= (*NFr* (A_1) $\times$ *NCl* (A_2)) $\cup$ (*NCl* (A_1) $\times$ *NFr* (A_2)).
Hence *NFr* ($A_1 \times A_2$) = (*NFr* (A_1) $\times$ *NCl* (A_2)) $\cup$ (*NCl* (A_1) $\times$ *NFr* (A_2)).

III. NEUTROSOPHIC SEMI-FRONTIER

In this section, we introduce the neutrosophic semi-frontier and their properties in neutrosophic topological spaces.

Definition 3.1 Let *A* be a *NS* in the *NTS* X. Then the neutrosophic semi-frontier of *A* is defined as *NSFr* (*A*) = *NSCl* (*A*) $\cap$ *NSCl* (C (*A*)). Obviously *NSFr* (*A*) is a *NSC* set in X.

Theorem 3.2 Let *A* be a *NS* in the *NTS* X. Then the following conditions are holds :
(i) *NSCl* (*A*) = *A* $\cup$ *NInt* (*NCl* (*A*)),
(ii) *NSInt* (*A*) = *A* $\cap$ *NCl* (*NInt* (*A*)).
Proof : (i) Let *A* be a *NS* in X. Consider
NInt (*NCl* (*A* $\cup$ *NInt* (*NCl* (*A*))))
= *NInt* (*NCl* (*A*) $\cup$ *NCl* (*NInt* (*NCl* (*A*))))
= *NInt* (*NCl* (*A*))
$\subseteq$ *A* $\cup$ *NInt* (*NCl* (*A*))
It follows that *A* $\cup$ *NInt* (*NCl* (*A*)) is a *NSC* set in X.
Hence *NSCl* (*A*) $\subseteq$ *A* $\cup$ *NInt* (*NCl* (*A*)) ------------ (1)

By Proposition 6.3 (ii) [12], *NSCl* (*A*) is *NSC* set in X. We have *NInt* (*NCl* (*A*)) $\subseteq$ *NInt* (*NCl* (*NSCl* (*A*))) $\subseteq$ *NSCl* (*A*).
Thus *A* $\cup$ *NInt* (*NCl* (*A*)) $\subseteq$ *NSCl* (*A*) -------------- (2).
From (1) and (2), *NSCl* (*A*) = *A* $\cup$ *NInt* (*NCl* (*A*)).

(ii) This can be proved in a similar manner as (i).

Theorem 3.3 For a *NS A* in the *NTS* X, *NSFr* (*A*) = *NSFr* (C (*A*)).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
NSFr (*A*) = *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
= *NSCl* (C (*A*)) $\cap$ *NSCl* (*A*)
= *NSCl* (C (*A*)) $\cap$ *NSCl* (C (C (*A*)))
Again by Definition 3.1,
= *NSFr* (C (*A*))
Hence *NSFr* (*A*) = *NSFr* (C (*A*)).

Theorem 3.4 If *A* is *NSC* set in X, then *NSFr* (*A*) $\subseteq$ *A*.
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
NSFr (*A*) = *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
$\subseteq$ *NSCl* (*A*)
By Proposition 6.3 (ii) [12] ,
= *A*
Hence *NSFr* (*A*) $\subseteq$ *A*, if *A* is *NSC* in X.

The converse of the above theorem is not true as shown by the following example.

Example 3.5 Let X = { *a*, *b*, *c* } and τ = { 0_N, A, B, C, D, 1_N }. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are C (τ) = { 1_N, F, G, H, I, 0_N } where
A = ⟨ (0.5, 0.6, 0.7), (0.1, 0.8, 0.4) , (0.7, 0.2, 0.3) ⟩,
B = ⟨ (0.8, 0.8, 0.5), (0.5, 0.4, 0.2) , (0.9, 0.6, 0.7) ⟩,
C = ⟨ (0.8, 0.8, 0.5), (0.5, 0.8, 0.2) , (0.9, 0.6, 0.3) ⟩,
D = ⟨ (0.5, 0.6, 0.7), (0.1, 0.4, 0.4) , (0.7, 0.2, 0.7) ⟩,
E = ⟨ (0.8, 0.8, 0.4), (0.5, 0.8, 0.1) , (0.9, 0.7, 0.2) ⟩,
F = ⟨ (0.7, 0.4, 0.5), (0.4, 0.2, 0.1) , (0.3, 0.8, 0.7) ⟩,
G = ⟨ (0.5, 0.2, 0.8), (0.2, 0.6, 0.5) , (0.7, 0.4, 0.9) ⟩,
H = ⟨ (0.5, 0.2, 0.8), (0.2, 0.2, 0.5) , (0.3, 0.4, 0.9) ⟩,
I = ⟨ (0.7, 0.4, 0.5), (0.4, 0.6, 0.1) , (0.7, 0.8, 0.7) ⟩ and
J = ⟨ (0.4, 0.2, 0.8), (0.1, 0.2, 0.5) , (0.2, 0.3, 0.9) ⟩.
Here E and J are neutrosophic semi-open and neutrosophic semi-closed set respectively. Therefore the neutrosophic semi-open and neutrosophic semi-closed set topologies are τ_{NSO} = 0_N, A, B, C, D, E, 1_N and C $(\tau)_{NSC}$ = 1_N, F, G, H, I, J, 0_N . Therefore *NSFr* (C) = H $\subseteq$ C. But C $\notin$ C $(\tau)_{NSC}$.

Theorem 3.6 If *A* is *NSO* set in X, then *NSFr* (*A*) $\subseteq$ C (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Proposition 4.3 [12] , *A* is *NSO* set implies C (*A*) is *NSC* set in X. By Theorem 3.4, *NSFr* (C (*A*)) $\subseteq$ C (*A*) and by Theorem 3.3, we get *NSFr* (*A*) $\subseteq$ C (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.7 From Example 3.5, *NSFr* (J) = J $\subseteq$ C (J) = E. But J $\notin$ τ_{NSO}.

Theorem 3.8 Let *A* $\subseteq$ *B* and *B* $\in$ *NSC* (X) (resp., *B* $\in$ *NSO* (X)). Then *NSFr* (*A*) $\subseteq$ *B* (resp., *NSFr* (*A*) $\subseteq$ C(*B*)), where *NSC* (X) (resp., *NSO* (X)) denotes the class of neutrosophic semi-closed (resp., neutrosophic semi-open) sets in X.
Proof : By Proposition 6.3 (iv) [12] , *A* $\subseteq$ *B* , *NSCl* (*A*) $\subseteq$ *NSCl* (*B*) -------------------- (1).
By Definition 3.1,
NSFr (*A*) = *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
$\subseteq$ *NSCl* (*B*) $\cap$ *NSCl* (C (*A*)) by (1)
$\subseteq$ *NSCl* (*B*)
By Proposition 6.3 (ii) [12] ,
= *B*
Hence *NSFr* (*A*) $\subseteq$ *B*.

Theorem 3.9 Let *A* be the *NS* in the *NTS* X. Then C (*NSFr* (*A*)) = *NSInt* (*A*) $\cup$ *NSInt* (C (*A*)).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
C (*NSFr* (*A*)) = C (*NSCl* (*A*) $\cap$ *NSCl* (C (*A*)))
By Proposition 3.2 (1) [18] ,
= C (*NSCl* (*A*)) $\cup$ C (*NSCl* (C (*A*)))
By Proposition 6.2 (ii) [12] ,
= *NSInt* (C (*A*)) $\cup$ *NSInt* (*A*)
Hence C (*NSFr* (*A*)) = *NSInt* (*A*) $\cup$ *NSInt* (C (*A*)).

Theorem 3.10 For a *NS A* in the *NTS* X, then *NSFr* (*A*) $\subseteq$ *NFr* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Proposition 6.4 [12] , *NSCl* (*A*) $\subseteq$ *NCl* (*A*) and *NSCl* (C (*A*)) $\subseteq$ *NCl* (C (*A*)).
Now by Definition 3.1,
NSFr (*A*) = *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
$\subseteq$ *NCl* (*A*) $\cap$ *NCl* (C (*A*))
By Definition 2.2,
= *NFr* (*A*)
Hence *NSFr* (*A*) $\subseteq$ *NFr* (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.11 From Example 3.5, let A_1 = ⟨ (0.4, 0.1, 0.9), (0.1, 0.2, 0.6) , (0.1, 0.3, 0.9) ⟩, then
C (A_1) = ⟨ (0.9, 0.9, 0.4), (0.6, 0.8, 0.1) , (0.9, 0.7, 0.1) ⟩. Therefore *NFr* (A_1) = H ⊈ J = *NSFr* (A_1).

Theorem 3.12 For a *NS* *A* in the *NTS* X, then *NSCl* (*NSFr* (*A*)) ⊆ *NFr* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
NSCl (*NSFr* (*A*)) = *NSCl* (*NSCl* (*A*) ∩ *NSCl* (C (*A*)))
⊆ *NSCl* (*NSCl* (*A*)) ∩ *NSCl* (*NSCl* (C (*A*)))
By Proposition 6.3 (iii) [12] ,
= *NSCl* (*A*) ∩ *NSCl* (C (*A*))
By Definition 3.1,
= *NSFr* (*A*)
By Theorem 3.10,
⊆ *NFr* (*A*)
Hence *NSCl* (*NSFr* (*A*)) ⊆ *NFr* (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.13 From Example 3.5, *NFr* (A_1) = H ⊈ J = *NSCl* (*NSFr* (A_1)).

Theorem 3.14 Let *A* be a *NS* in the *NTS* X. Then *NSFr* (*A*) = *NSCl* (*A*) – *NSInt* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. By Proposition 6.2 (ii) [12] ,
C (*NSCl* (C (*A*))) = *NSInt* (*A*) and by Definition 3.1,
NSFr (*A*) = *NSCl* (*A*) ∩ *NSCl* (C (*A*))
= *NSCl* (*A*) – C (*NSCl* (C (*A*)))
by using *A* – *B* = *A* ∩ C (*B*)
By Proposition 6.2 (ii) [12] ,
= *NSCl* (*A*) – *NSInt* (*A*)
Hence *NSFr* (*A*) = *NSCl* (*A*) – *NSInt* (*A*).

Theorem 3.15 For a *NS* *A* in the *NTS* X, then *NSFr* (*NSInt* (*A*)) ⊆ *NSFr* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
NSFr (*NInt* (*A*))=*NSCl*(*NInt*(*A*))∩*NSCl*(C(*NSInt* (*A*)))
By Proposition 6.2 (i) [12] ,
=*NSCl*(*NSInt*(*A*))∩*NSCl*(*NSCl*(C(*A*)))
By Proposition 6.3 (iii) [12] ,
= *NSCl* (*NSInt* (*A*)) ∩ *NSCl* (C (*A*))
By Proposition 5.2 (ii) [12] ,
⊆ *NSCl* (*A*) ∩ *NSCl* (C (*A*))
By Definition 3.1,
= *NSFr* (*A*)
Hence *NSFr* (*NSInt* (*A*)) ⊆ *NSFr* (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.16 Let X = { *a*, *b*, *c* } and τ_{NSO} = 0_N, A, B, C, D, E, 1_N and C $(\tau)_{NSC}$ = 1_N, F, G, H, I, J, 0_N where
A = ⟨ (0.3, 0.4, 0.2), (0.5, 0.6, 0.7), (0.9, 0.5, 0.2) ⟩,
B = ⟨ (0.3, 0.5, 0.1), (0.4, 0.3, 0.2), (0.8, 0.4, 0.6) ⟩,
C = ⟨ (0.3, 0.5, 0.1), (0.5, 0.6, 0.2), (0.9, 0.5, 0.2) ⟩,
D = ⟨ (0.3, 0.4, 0.2), (0.4, 0.3, 0.7), (0.8, 0.4, 0.6) ⟩,
E = ⟨ (0.5, 0.6, 0.1), (0.6, 0.7, 0.1), (0.9, 0.5, 0.2) ⟩,
F = ⟨ (0.2, 0.6, 0.3), (0.7, 0.4, 0.5), (0.2, 0.5, 0.9) ⟩,
G = ⟨ (0.1, 0.5, 0.3), (0.2, 0.7, 0.4), (0.6, 0.6, 0.8) ⟩,
H = ⟨ (0.1, 0.5, 0.3), (0.2, 0.4, 0.5), (0.2, 0.5, 0.9) ⟩,
I = ⟨ (0.2, 0.6, 0.3), (0.7, 0.7, 0.4), (0.6, 0.6, 0.8) ⟩ and
J = ⟨ (0.1, 0.4, 0.5), (0.1, 0.3, 0.6), (0.2, 0.5, 0.9) ⟩.
Define A_1 = ⟨ (0.2, 0.3, 0.4), (0.4, 0.5, 0.6) , (0.3, 0.4, 0.8) ⟩.
Then C (A_1) = ⟨ (0.4, 0.7, 0.2), (0.6, 0.5, 0.4) , (0.8, 0.6, 0.3) ⟩. Therefore *NSFr* (A_1) = I ⊈ 0_N = *NSFr* (*NSInt* (A_1)).

Theorem 3.17 For a *NS* *A* in the *NTS* X, then *NSFr* (*NSCl* (*A*)) ⊆ *NSFr* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
NSFr(*NSCl*(*A*))=*NSCl*(*NSCl*(*A*))∩*NSCl*(C(*NSCl* (*A*)))
By Proposition 6.3 (iii) and Proposition 6.2 (ii) [12] ,
= *NSCl* (*A*) ∩ *NSCl* (*NSInt* (C (*A*)))
By Proposition 5.2 (i) [12] ,
⊆ *NSCl* (*A*) ∩ *NSCl* (C (*A*))
By Definition 3.1,
= *NSFr* (*A*)
Hence *NSFr* (*NSCl* (*A*)) ⊆ *NSFr* (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.18 From Example 3.16, let A_2 = ⟨ (0.2, 0.6, 0.2), (0.3, 0.4, 0.6) , (0.3, 0.4, 0.8) ⟩. Then C (A_2) = ⟨ (0.2, 0.4, 0.2), (0.6, 0.6, 0.3) , (0.8, 0.6, 0.3) ⟩. Therefore *NSFr* (A_2) = 1_N ⊈ 0_N = *NSFr* (*NSCl* (A_2)).

Theorem 3.19 Let *A* be the *NS* in the *NTS* X. Then *NSInt* (*A*) ⊆ *A* – *NSFr* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Now by Definition 3.1,
A – *NSFr* (*A*) = *A* − (*NSCl* (*A*) ∩ *NSCl* (C (*A*)))
= (*A* – *NSCl* (*A*)) ∪ (*A* – *NSCl* (C (*A*)))
= *A* – *NSCl* (C (*A*))
⊇ *NSInt* (*A*).
Hence *NSInt* (*A*) ⊆ *A* – *NSFr* (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.20 From Example 3.16, A_1 – *NSFr* (A_1) = ⟨ (0.2, 0.3, 0.4), (0.4, 0.3, 0.7) , (0.3, 0.4, 0.8) ⟩ $\nsubseteq 0_N$ = *NSInt* (A_1).

Remark 3.21 In general topology, the following conditions are hold :
NSFr (*A*) ∩ *NSInt* (*A*) = 0_N,
NSInt (*A*) ∪ *NSFr* (*A*) = *NSCl* (*A*),
NSInt (*A*) ∪ *NSInt* (C (*A*)) ∪ *NSFr* (*A*) = 1_N.
But the neutrosophic topology, we give counter-examples to show that the conditions of the above remark may not be hold in general.

Example 3.22 From Example 3.16, define A_1 = ⟨ (0.4, 0.6, 0.1), (0.5, 0.8, 0.3) , (0.9, 0.6, 0.2) ⟩. Then C (A_1) = ⟨ (0.1, 0.4, 0.4), (0.3, 0.2, 0.5) , (0.2, 0.4, 0.9) ⟩. Therefore *NSFr* (A_1) ∩ *NSInt* (A_1) = F ∩ D = ⟨ (0.2, 0.4, 0.3), (0.4, 0.3, 0.7), (0.2, 0.4, 0.9) ⟩ $\neq 0_N$.

NSInt (A_1) ∪ *NSFr* (A_1) = D ∪ F = ⟨ (0.3, 0.6, 0.2), (0.7, 0.4, 0.5), (0.8, 0.5, 0.6) ⟩ $\neq 1_N$ = *NSCl* (A_1).

NSInt (A_1) ∪ *NSInt* (C (A_1)) ∪ *NSFr* (A_1) = D ∪ 0_N ∪ F = ⟨ (0.3, 0.6, 0.2), (0.7, 0.4, 0.5), (0.8, 0.5, 0.6) ⟩ $\neq 1_N$.

Theorem 3.23 Let *A* and *B* be *NSs* in the *NTS* X. Then *NSFr* (*A* ∪ *B*) ⊆ *NSFr* (*A*) ∪ *NSFr* (*B*).
Proof : Let *A* and *B* be *NSs* in the *NTS* X. Then by Definition 3.1,
NSFr (*A* ∪ *B*) = *NSCl* (*A* ∪ *B*) ∩ *NSCl* (C (*A* ∪ *B*))
By Proposition 3.2 (2) [18] ,
= *NSCl* (*A* ∪ *B*) ∩ *NSCl* (C (*A*) ∩ C (*B*))
By Proposition 6.5 (i) and (ii) [12] ,
⊆ (*NSCl*(*A*) ∪ *NSCl*(*B*))∩(*NSCl*(C(*A*))∩*NSCl*(C (*B*)))
= [(*NSCl* (*A*) ∪ *NSCl* (*B*)) ∩ *NSCl* (C (*A*))] ∩ [(*NSCl* (*A*) ∪ *NSCl* (*B*)) ∩ *NSCl* (C (*B*))]
= [(*NSCl*(*A*)∩*NSCl*(C(*A*)))∪(*NSCl*(*B*)∩*NSCl*(C(*A*)))] ∩[(*NSCl*(*A*)∩*NSCl*(C(*B*)))∪(*NSCl*(*B*)∩*NSCl*(C(*B*)))]
By Definition 3.1,
= [*NSFr* (*A*) ∪ (*NSCl* (*B*) ∩ *NSCl* (C (*A*)))] ∩ [(*NSCl* (*A*) ∩ *NSCl* (C (*B*))) ∪ *NSFr*(*B*)]
= (*NSFr* (*A*) ∪ *NSFr* (*B*)) ∩ [(*NSCl* (*B*) ∩ *NSCl* (C (*A*))) ∪ (*NSCl* (*A*) ∩ *NSCl* (C (*B*)))]
⊆ *NSFr* (*A*) ∪ *NSFr* (*B*).
Hence *NSFr* (*A* ∪ *B*) ⊆ *NSFr* (*A*) ∪ *NSFr* (*B*).

The converse of the above theorem needs not be true as shown by the following example.

Example 3.24 Let X = { *a* } with τ_{NSO} = 0_N, A, B, C, D, 1_N and C $(\tau)_{NSC}$ = 1_N, E, F, G, H, 0_N where
A = ⟨ (0.6, 0.8, 0.4) ⟩,
B = ⟨ (0.4, 0.9, 0.7) ⟩,
C = ⟨ (0.6, 0.9, 0.4) ⟩,
D = ⟨ (0.4, 0.8, 0.7) ⟩,
E = ⟨ (0.4, 0.2, 0.6) ⟩,
F = ⟨ (0.7, 0.1, 0.4) ⟩,
G = ⟨ (0.4, 0.1, 0.6) ⟩ and
H = ⟨ (0.7, 0.2, 0.4) ⟩. Now we define
B_1 = ⟨ (0.7, 0.6, 0.5) ⟩,
B_2 = ⟨ (0.6, 0.8, 0.2) ⟩,
$B_1 \cup B_2 = B_3$ = ⟨ (0.7, 0.8, 0.2) ⟩ and
$B_1 \cap B_2 = B_4$ = ⟨ (0.6, 0.6, 0.5) ⟩. Then
C (B_1) = ⟨ (0.5, 0.4, 0.7) ⟩,
C (B_2) = ⟨ (0.2, 0.2, 0.6) ⟩,
C (B_3) = ⟨ (0.2, 0.2, 0.7) ⟩ and
C (B_4) = ⟨ (0.5, 0.4, 0.6) ⟩.
Therefore *NSFr* (B_1) ∪ *NSFr* (B_2) = 1_N ∪ E = $1_N \nsubseteq$ E = *NSFr* (B_3) = *NSFr* ($B_1 \cup B_2$).

Note 3.25 The following example shows that
NSFr (*A* ∩ *B*) ⊈ *NSFr* (*A*) ∩ *NSFr* (*B*) and
NSFr (*A*) ∩ *NSFr* (*B*) ⊈ *NSFr* (*A* ∩ *B*).

Example 3.26 From Example 3.24, we define
A_1 = ⟨ (0.5, 0.1, 0.9) ⟩,
A_2 = ⟨ (0.3, 0.5, 0.6) ⟩,
$A_1 \cup A_2 = A_3$ = ⟨ (0.5, 0.5, 0.6) ⟩, and
$A_1 \cap A_2 = A_4$ = ⟨ (0.3, 0.1, 0.9) ⟩. Then
C (A_1) = ⟨ (0.9, 0.9, 0.5) ⟩,
C (A_2) = ⟨ (0.6, 0.5, 0.3) ⟩,
C (A_3) = ⟨ (0.6, 0.5, 0.5) ⟩ and
C (A_4) = ⟨ (0.9, 0.9, 0.3) ⟩.
Therefore *NSFr* (A_1) ∩ *NSFr* (A_2) = F ∩ 1_N = F ⊈ G = *NSFr* (A_4) = *NSFr* ($A_1 \cap A_2$).

Also *NSFr* ($B_1 \cap B_2$) = *NSFr* (B_4) = 1_N ⊈ E = 1_N ∩ E = *NSFr* (B_1) ∩ *NSFr* (B_2).

Theorem 3.27 For any *NSs A* and *B* in the *NTS* X,
NSFr (*A* ∩ *B*) ⊆ (*NSFr* (*A*) ∩ *NSCl* (*B*)) ∪ (*NSFr* (*B*) ∩ *NSCl* (*A*)).
Proof : Let *A* and *B* be *NSs* in the *NTS* X. Then by Definition 3.1,
NSFr (*A* ∩ *B*) = *NSCl* (*A* ∩ *B*) ∩ *NSCl* (C (*A* ∩ *B*))
By Proposition 3.2 (1) [18] ,
= *NSCl* (*A* ∩ *B*) ∩ *NSCl* (C (*A*) ∪ C (*B*))
By Proposition 6.5 (ii) and (i) [12] ,
⊆ (*NSCl*(*A*)∩*NSCl* (*B*))∩(*NSCl*(C(*A*))∪*NSCl*(C(*B*)))
= [(*NSCl* (*A*) ∩ *NSCl* (*B*)) ∩ *NSCl* (C(*A*))] ∪ [(*NSCl* (*A*) ∩ *NSCl* (*B*)) ∩ *NSCl* (C(*B*))]
By Definition 3.1,
= (*NSFr* (*A*) ∩ *NSCl* (*B*)) ∪ (*NSFr* (*B*) ∩*NSCl* (*A*))
Hence *NSFr* (*A* ∩ *B*) ⊆ (*NSFr* (*A*) ∩ *NSCl* (*B*)) ∪ (*NSFr* (*B*) ∩ *NSCl* (*A*)).

The converse of the above theorem is not true as shown by the following example.

Example 3.28 From Example 3.24, ($NSFr$ (A_1) $\cap$ $NSCl$ (A_2)) $\cup$ ($NSFr$ (A_2) $\cap$ $NSCl$ (A_1)) = (F $\cap$ 1_N) $\cup$ (1_N $\cap$ F) = F $\cup$ F = F $\nsubseteq$ G = $NSFr$ ($A_1 \cap A_2$).

Corollary 3.29 For any *NSs* A and B in the *NTS* X, $NSFr\ (A \cap B) \subseteq NSFr\ (A) \cup NSFr\ (B)$.

Proof : Let A and B be *NSs* in the *NTS* X. Then by Definition 3.1,

$NSFr\ (A \cap B) = NSCl\ (A \cap B) \cap NSCl\ (\mathrm{C}\ (A \cap B))$

By Proposition 3.2 (1) [18] ,

$= NSCl\ (A \cap B) \cap NSCl\ (\ \mathrm{C}\ (A) \cup \mathrm{C}\ (B)\ )$

By Proposition 6.5 (ii) and (i) [12] ,

$\subseteq (NSCl(A) \cap NSCl(B)) \cap (NSCl(\mathrm{C}(A)) \cup NSCl(\mathrm{C}(B)))$

$= (\ NSCl\ (A) \cap NSCl\ (B) \cap NSCl\ (\mathrm{C}\ (A))\ ) \cup$
$(\ NSCl\ (A) \cap NSCl\ (B) \cap NSCl\ (\mathrm{C}\ (B))\ )$

By Definition 3.1,

$= (\ NSFr\ (A) \cap NSCl\ (B)\ ) \cup (\ NSCl\ (A) \cap NSFr\ (B))$

$\subseteq NSFr\ (A) \cup NSFr\ (B)$.

Hence $NSFr\ (A \cap B) \subseteq NSFr\ (A) \cup NSFr\ (B)$.

The equality in the above theorem may not hold as seen in the following example.

Example 3.30 From Example 3.24, $NSFr$ (A_1) $\cup$ $NSFr$ (A_2) = F $\cup$ 1_N = 1_N $\nsubseteq$ G = $NSFr$ (A_4) = $NSFr$ ($A_1 \cap A_2$).

Theorem 3.31 For any *NS* A in the *NTS* X,

(1) $NSFr\ (NSFr\ (A)) \subseteq NSFr\ (A)$,

(2) $NSFr\ (NSFr\ (NSFr\ (A))) \subseteq NSFr\ (NSFr\ (A))$.

Proof : (1) Let A be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,

$NSFr\ (NSFr\ (A))$

$= NSCl\ (NSFr\ (A)) \cap NSCl\ (\mathrm{C}\ (NSFr\ (A)))$

By Definition 3.1,

$= NSCl\ (\ NSCl\ (A) \cap NSCl\ (\mathrm{C}\ (A))\ ) \cap$
$NSCl\ (\ \mathrm{C}\ (\ NSCl\ (A) \cap NSCl\ (\mathrm{C}\ (A))\ )\ )$

By Proposition 6.3 (iii) and 6.2 (ii) [12] ,

$\subseteq (\ NSCl\ (NSCl\ (A)) \cap NSCl\ (NSCl\ (\mathrm{C}\ (A)))\ ) \cap$
$NSCl\ (\ NSInt\ (\mathrm{C}\ (A)) \cup NSInt\ (A)\ )$

By Proposition 6.3 (iii) [12] ,

$= (NSCl\ (A) \cap NSCl\ (\mathrm{C}\ (A))) \cap (NSCl\ (NSInt(\mathrm{C}\ (A)))$
$\cup NSCl\ (NSInt\ (A))$

$\subseteq NSCl\ (A) \cap NSCl\ (\mathrm{C}\ (A))$

By Definition 3.1,

$= NSFr\ (A)$

Therefore $NSFr\ (NSFr\ (A)) \subseteq NSFr\ (A)$.

(2) By Definition 3.1,

$NSFr\ (NSFr\ (NSFr\ (A))) = NSCl\ (NSFr\ (NSFr\ (A)))$
$\cap NSCl\ (\mathrm{C}\ (NSFr\ (NSFr\ (A))))$

By Proposition 6.3 (iii) [12] ,

$\subseteq (NSFr\ (NSFr\ (A))) \cap NSCl\ (\mathrm{C}\ (NSFr\ (NSFr\ (A))))$

$\subseteq NSFr\ (NSFr\ (A))$.

Hence $NSFr\ (NSFr\ (NSFr\ (A))) \subseteq NSFr\ (NSFr\ (A))$.

Remark 3.32 From the above theorem, the converse of (1) needs not be true as shown by the following example and no counter-example could be found to establish the irreversibility of inequality in (2).

Example 3.33 From Example 3.16, $NSFr$ (A_2) = 1_N $\nsubseteq$ 0_N = $NSFr$ ($NSFr$ (A_2)).

Theorem 3.34 Let X_i, $i = 1, 2, \ldots, n$ be a family of neutrosophic product related *NTSs*. If each A_i is a *NS* in X_i, then $NSFr\ (\ \prod_{i=1}^{n} A_i\ ) = [\ NSFr\ (A_1) \times NSCl\ (A_2) \times \cdots \times NSCl\ (A_n)\ ] \cup [\ NSCl\ (A_1) \times NSFr\ (A_2) \times NSCl\ (A_3) \times \cdots \times NSCl\ (A_n)\ ] \cup \cdots \cup [\ NSCl\ (A_1) \times NSCl\ (A_2) \times \cdots \times NSFr\ (A_n)\ ]$.

Proof : It suffices to prove this for $n = 2$. Let A_i be the *NS* in the neutrosophic topological space X_i.

Then by Definition 3.1,

$NSFr\ (A_1 \times A_2) = NSCl\ (A_1 \times A_2) \cap NSCl\ (\mathrm{C}\ (A_1 \times A_2))$

By Proposition 6.2 (i) [12] ,

$= NSCl\ (A_1 \times A_2) \cap \mathrm{C}\ (\ NSInt\ (A_1 \times A_2)\ )$

By Theorem 6.9 (i) and (ii) [12] ,

$= (NSCl(A_1) \times NSCl(A_2)) \cap \mathrm{C}(NSInt(A_1) \times NSInt\ (A_2))$

$= (\ NSCl\ (A_1) \times NSCl\ (A_2)\ ) \cap \mathrm{C}\ [\ (\ NSInt\ (A_1) \cap$
$NSCl\ (A_1)\ ) \times (\ NSInt\ (A_2) \cap NSCl\ (A_2)\ )\ ]$

By Lemma 2.3 (iii) [12] ,

$= (\ NSCl\ (A_1) \times NSCl\ (A_2)\ ) \cap [\ \mathrm{C}\ (\ NSInt\ (A_1) \cap NSCl\ (A_1)\ ) \times 1_N \cup 1_N \times \mathrm{C}(\ NSInt\ (A_2) \cap NSCl\ (A_2)\ )\ ]$

$= (NSCl(A_1) \times NSCl(A_2)) \cap [(NSCl(\mathrm{C}(A_1)) \cup NSInt(\mathrm{C}(A_1))) \times 1_N \cup 1_N \times (NSCl\ (\mathrm{C}\ (A_2)) \cup NSInt\ (\mathrm{C}\ (A_2))\ )\ ]$

$= (\ NSCl\ (A_1) \times NSCl\ (A_2)\ ] \cap [\ (\ NSCl\ (\mathrm{C}\ (A_1)) \times 1_N\ )$
$\cup (\ 1_N \times NSCl\ (\mathrm{C}\ (A_2))\ )\ ]$

$= [(\ NSCl\ (A_1) \times NSCl\ (A_2)\ ) \cap (NSCl\ (\mathrm{C}\ (A_1)) \times 1_N\ )]$
$\cup [(NSCl\ (A_1) \times NSCl\ (A_2)\ ) \cap (1_N \times NSCl\ (\mathrm{C}\ (A_2))\ )\ ]$

By Theorem 2.32,

$= [\ (NSCl\ (A_1) \cap NSCl\ (\mathrm{C}\ (A_1))\ ) \times (\ 1_N \cap NSCl\ (A_2))]$
$\cup [(NSCl\ (A_1) \cap 1_N\ ) \times (NSCl\ (A_2) \cap NSCl\ (\mathrm{C}\ (A_2)))]$

$= (NSFr\ (A_1) \times NSCl\ (A_2)) \cup (NSCl\ (A_1) \times NSFr\ (A_2))$

Hence $NSFr\ (A_1 \times A_2) = (\ NSFr\ (A_1) \times NSCl\ (A_2)\ ) \cup (\ NSCl\ (A_1) \times NSFr\ (A_2)\ )$.

CONCLUSION

In this paper, we studied the concepts of frontier and semi-frontier in neutrosophic topological spaces. In future, we plan to extend this neutrosophic topology concepts by neutrosophic continuous, neutrosophic semi-continuous, neutrosophic almost continuous and neutrosophic weakly continuous in neutrosophic topological spaces, and also to expand this neutrosophic concepts by nets, filters and borders.

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