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Neutrosophic Vague Binary BCK/BCI-algebra

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Abstract: Ineradicable hindrances of the existing mathematical models widespread from probabilities to soft sets. These difficulties made up way for the opening of "neutrosophic set model'. Set theory of 'vague' values is an already established branch of mathematics. Complex situations which arose in problem solving, demanded more accurate models. As a result, 'neutrosophic vague' came into screen. At present, research works in this area are very few. But it is on the way of its moves. Algebra and topology are well connected, as algebra and geometry. So, anything related to geometric topology is equally important in algebraic topology too. Separate growth of algebra and topology will slow down the development of each branch. And in one sense it is imperfect! In this paper a new algebraic structure, BCK/BCI is developed for 'neutrosophic' and to 'neutrosophic vague' concept with 'single' and 'double' universe. It's sub-algebra, different kinds of ideals and cuts are developed in this paper with suitable examples where necessary. Several theorems connected to this are also got verified.

Keywords: Vague H - ideal, neutrosophic vague binary BCK/BCI - algebra, neutrosophic vague binary BCK/BCI – subalgebra, neutrosophic vague binary BCK/BCI - ideal, neutrosophic vague binary BCK/BCI p- ideal, neutrosophic vague binary BCK/BCI q - ideal, neutrosophic vague binary BCK/BCI a-ideal, neutrosophic vague binary BCK/BCI H - ideal, neutrosophic vague binary BCK/BCI - cut

Notations: NVBS : neutrosophic vague binary set, NVBSS : neutrosophic vague binary subset, NVBI : neutrosophic vague binary ideal, N BCK/BCI - algebra : neutrosophic BCK/BCI-algebra, NV BCK/BCI - algebra : neutrosophic vague BCK/BCI-algebra, NVB BCK/BCI - algebra : neutrosophic vague binary BCK/BCI - algebra, N BCK/BCI - subalgebra : neutrosophic BCK/BCI - subalgebra, NV BCK/BCI - subalgebra : neutrosophic vague BCK/BCI - subalgebra, NVB BCK/BCI – subalgebra : neutrosophic vague binary BCK/BCI - subalgebra, N BCK/BCI - ideal : neutrosophic BCK/BCI –ideal, NV BCK/BCI - ideal : neutrosophic vague BCK/BCI - ideal , NVB BCK/BCI- ideal : neutrosophic vague binary BCK/BCI - ideal, NVB BCK/BCI p-ideal : neutrosophic vague binary BCK/BCI p-ideal, NVB BCK/BCI q - ideal : neutrosophic vague binary BCK/BCI q - ideal, NVB BCK/BCI a - ideal : neutrosophic vague binary BCK/BCI a - ideal, NVB BCK/BCI H - ideal : neutrosophic vague binary BCK/BCI H - ideal

1. Introduction

Before 1990's, mathematicians and researchers made use of different mathematical models for problem solving viz. , Probability theory, Hard set theory, Fuzzy set theory, Rough set theory, Intuitionistic Fuzzy set theory etc., for problem solving. In 1993, W. L. Gau and D. J. Buehrer [16] introduced vague sets, with "truth and false" membership values as measurement tools. In 1995, Florentin Smarandache [13] introduced, "Neutrosophic set theory", in which an additional data 'uncertainty', is also got added besides 'truth and false'. In 2015, Shawkat Alkhazaleh [45] introduced 'Neutrosophic Vague' set theory, by inserting vague values, to each neutrosophic value –'truth, uncertainty & false'. With its several operations, he gave a rich explanation about the concept, in his pioneer paper itself. Neutrosophic set's main difference with Neutrosophic Vague is, with its outlook as an "interval" (imposed with certain conditions). An algebraic structure is a universal set with a set of operations applicable to that set, together with a set of axioms to be satisfied. BCK/BCI-logical algebra- is a new type of algebraic structure developed in 1966, by Yasuyuki Imai and Kiyoshi Iséki [48]. It is now found to be an active research area. MV-algebras, Boolean algebras etc. are some t-related logical algebras which extend to BCK-algebra. BCK-algebra further extends to BCI-algebra. In 2015, Samy M. Mostafa and Reham Ghanem [42] gave cubic structures of medial ideal on BCI- algebras. Paper introduced cubic medial – ideal, and it illustrates a relation between cubic medial – ideal and cubic medial BCI – ideal. Homomorphism and Cartesian product of this concept have been duly verified. In 2017, M. Kaviyarasu and K. Indira [22] gave a review on BCI/BCK-algebras and its developmental scenario. In 1999, Khalid and Ahmad [25] introduced fuzzy H- ideals in BCI-algebras. In 2007, Kordi and Moussavi [26] gave a detailed study on fuzzy ideals of BCI-algebras. In 2012, Borumand Saeid. A, Prince Williams. D. R and Kuchaki Rafsanjani [10] gave a preliminary note on anti-fuzzy BCK/BCI-subalgebra. Paper mainly contributed on generalized notion of fuzzy BCK/BCI-algebra. In 2018, based on hyper fuzzy structure, Young Bae Jun, Seok-Zun Song and Seon Jeong Kim [50] introduced length - fuzzy subalgebras (length -k-fuzzy; k = 1≤ k≤ 4) in BCK/BCI- algebras. In 2018, Anas Al-Masarwah and Abd Ghafur Ahmad [4] discussed some properties of bipolar fuzzy H-ideals in BCK/BCI--algebra. In 2019, Anas Al-Masarwah, Abd Ghafur Ahmad [5] introduced m-Polar fuzzy subalgebras, m-polar fuzzy closed ideal and m-polar fuzzy commutative ideal of m-polar fuzzy sets. They also investigated their several characterizations and theorems. In 2019, Alcheikh. M & Anas Sabouh [3] proved several theorems connected to the already existing notions of fuzzy ideal, anti-fuzzy ideal and anti-fuzzy p-ideal of BCK-algebra. In 1983, Hu. Q. P and Li. X [19] defined BCH-algebra as a generalization of BCK/BCI-algebra. In 2001, Muhammad Anwar Chaudhary and Hafiz Fakhar-ud-din [34] studied some classes of BCH- algebras. In 1998, Jun. Y. B, Roh. E. H and Kim. H. S [21] introduced BH -algebra as a generalization of BCH/BCI/BCK-algebra and they discussed it's ideals and homomorphisms. In 2001, Qun Zhang, Young Bae Jun and Eun Hwan Roh [41] studied the connection of BH- algebras with 'BCH' and 'BCK/BCI'-algebras. They defined BH_1-algebra and normal BH - algebra. In 1996, Neggers. J and Kim. H. S [38] introduce d-algebras as a generalization of BCK-algebras and proved that oriented digraphs correspond to class of edge d-algebras. They also gave several notions of d-algebra with examples and also defined direct product and direct sum of d-algebras. In 1996, Stanley Gudder [46] introduced D-algebras as a generalization of D-poset (without assuming a partial order in D poset). He explained (interval) effect algebras, based on group structure and proved several lemmas and theorems regarding to this in a deep manner. In 2012, Muhammad Anwar Chaudhry and Faisal Ali [32] introduced multipliers in d-algebras. He remarked with example that every BCK-algebra is a d-algebra but the converse does not hold, in general. He defined positive implicative d - algebra and proved related theorems. In 2005, Akram. M and Dar. K. H [2] defined Fuzzy d-algebras, Fuzzy d-ideals, Fuzzy d-subalgebras, Fuzzy d-homomorphisms. In 2014, S. R. Barbhulya. K. Dutta. Choudhury [43] defined $(\varepsilon, \varepsilon vq)$- fuzzy ideals of d - algebra, it's cartesian product, homomorphism and also investigated a few theorems. In 2002, Neggers. J and Kim. H. S [39] introduced B – algebra which is closely related to BCH/BCI/BCK - algebras. Using a digraph on algebras, they gave a connection between B - algebras and groups. They also defined commutative B - algebras. In 2006, BM - algebras are introduced by Kim. C. B and Kim. H. S [24] as a specialization of B - algebras. They proved BM - algebra as a proper subclass of B - algebras. They showed that BM - algebra is equivalent to a 0 - commutative B - algebra. In 2011, A. Borumand Saeid and A. Zarandi [11] applied Vague Set theory to BM - algebras and discussed its cuts, Artinian and Noetherian concepts. In 2017, Arsham Borumand Saeid, Hee Sik Kim and Akbar Rezaei [7] introduced BI - algebras as a generalization of (dual) implication algebra. They defined ideals and congruence relations in BI - algebras. In 2006, Hee Sik Kim and Young Hee Kim [18] studied a generalization of BCK – algebras known as BE - algebras. They discussed its filter and self - distributive property along with some theorems. Various structures formed within short periods of time, along with smaller or bigger changes are B, BE, BF, BF_1, BF_2, BG, BI , BL, BM, BN, BO, BP, BQ, BZ, CI, Coxeter - algebra, FL, FL_{ew} (bounded integral commutative residuated lattice), GK, HW, KU, PS, Q, QS, QP, RG, TM, TP, TU, BCC (or BIK), BCL, BBG, $SBL_{\neg}$, Smarandache BCH - algebra , SU, UP, Z etc. In 2006, group theory of vague sets is introduced by Hakimuddin Khan, Musheer Ahmad and Ranjit Biswas [17]. In 2008, Lee. K. J, So. K. S and Bang. K. S [27] introduced vague BCK/BCI- algebras with several theorems and propositions. It was one of the pioneer work in the area of BCK/BCI - algebraic structure with vague sets. Notion of vague ideals are introduced with properties. A condition for a vague set to become a vague ideal is also provided. Several characteristics for vague ideal are investigated and established. Arsham Borumand Saeid [6] also introduced vague BCK/BCI- algebras in 2008, but his work has been published in 2009. He discussed on cuts, subalgebras and their related theorems of vague BCK/BCI – algebra. In 2017, Jafari. A, Mariapresenti. L and Arockiarani. I [20] discussed on vague direct product in BCK- algebra. In 2002, Neggers. J and Kim Hee Sik [40] introduced, β −algebra as generalization of BCK-algebras. In 2016, B. Nageswararao, N. Ramakrishna, T. Eswarlal [37] introduced vague $\beta-$algebras, vague β −ideals, translation operators on vague β −algebras, translation operators on vague $\beta-$ ideals, vague β −ideal extension of vague β −algebra etc. In 2013, Yun Sun Hwang and Sun Shin Ahn [52] developed vague p-ideals and vague a-ideals in BCI-algebras. In 2006, neutrosophic algebraic structures are introduced by Vasantha Kandasamy. W. B and Florentin Smarandache [47]. Neutrosophic group structure, neutrosophic ring structure etc., with lots of theorems and propositions are investigated. Based on this, in 2015, A. A. A. Agboola and B. Davvaz [1] introduced neutrosophic BCI/BCK- algebras and their elementary properties. In 2018, Young Bae Jun, Seok - Zun Song, Florentin Smarandache and Hashem Bordbar [51] discussed neutrosophic quadruple BCK/BCI-algebras. Paper consists of the newly defined definition of neutrosophic quadruple BCK/BCI-number, neutrosophic quadruple BCK/BCI-ideals etc., with proper verification of inter-connected notions. In 2019, Muhiuddin. G, Smarandache. F, Young Bae Jun, [35] gave a new idea - neutrosophic Quadruple ideals in neutrosophic Quadruple BCI- algebras. In 2018, Seon Jeong Kim, Seok-Zun Song and Young Bae Jun [44] discussed generalizations of neutrosophic subalgebras in BCK/BCI--algebras based on neutrosophic points. In 2018, Muhammad Akram, Hina Gulzar, Florentin Smarandache and Saeid Broumi [34] defined single-valued neutrosophic topological K-algebras and investigated some of the properties like C_5-connected, super-connected, compact and hausdorff. They also investigated the image and pre-image of these algebras under homomorphism. In 2018, Young Bae Jun, Florentin Smarandache, Mehmat AliÖztürk [49] introduced commutative falling neutrosophic ideals in BCK-algebras. In 2017, Bijan Daavavaz, Samy M. Mostafa and Fatema F. Kareem [8] developed Neutrosophic ideals of neutrosophic KU - algebras. In 2018, Muhiuddin. G, Bordbar. H, Smarandache. F, Jun. Y. B, [36] gave certain results on $(\varepsilon, \varepsilon)$ – neutrosophic subalgebras and ideals in BCK/BCI- algebras. They defined commutative $(\varepsilon, \varepsilon)$ neutrosophic ideal and commutative falling neutrosophic ideal for a BCK-algebra. In 2019, Chul Hwan Park [12] developed neutrosophic ideals of subtraction algebras. Khadaeman. S, Zahedi. M. M, Borzooei. R. A, Jun. Y. B [23] developed neutrosophic hyper BCK-ideals in 2019. Neutrosophic sets handle uncertainty in a remarkable way. But its generalization named as plithogenic set handles uncertainty in a more powerful level than neutrosophic! Application works using plithogenic are very few as it is a recently introduced work in set theory. But, in 2019, Mohamed Abdel-Basset and Rehab Mohamed [31], used a plithogenic TOPSIS-CRISIS method to sustainability supply chain risk management in telecommunication industry. Problem is well and systematically explained with adequate assistance of diagrams like bar diagram, pie diagram etc. In 2019, Mohamed Abdel-Basset, Mai Mohamed, Mohamed Elhoseny, Le Hoang Son, Francisco Chiclana, Abd El-Nasser H Zaied [28] pointed out some draw backs of

Dice and Jaccard similarity measures in bipolar neutrosophic set with examples. They provided a cosine similarity measure and weighted cosine similarity measure methods for 'bipolar and interval-valued bipolar'- neutrosophic set. They used the above method for diagnosing bipolar disorder diseases. A computational algorithm for MADM (Multi Attribute Decision Making) has also given in the paper. In 2019, using 'neutrosophic sets', Mohamed Abdel-Basset, Mumtaz Ali, Asma Atef [30] framed a resource levelling problem to construction projects. To improve work efficiency and to minimize cost were underlying principle. For calculating activity durations, trapezoidal neutrosophic numbers were used in this model. In 2019, Mohamed Abdel-Basset, Mumtaz Ali, Asma Atef [29] designed uncertainty assessments of linear Time-Cost Tradeoffs using neutrosophic sets. In 2020, Florentin Smarandache [14] introduced neutro - algebra as a generalization of partial algebra with examples and showed their differences. Points of odds between universal algebra, neutro - algebra and anti-algebra are well explained in the paper. Neutro - functions are more useful when range or domain is not clear. Several applications to neutro functions are given with a well explanation. In 2020, Bordbar. H, Mohseni Takallo. M, Borzooei. R. A, Young Bae Jun [9], defined BMBJ - neutrosophic subalgebra in BCI/BCK – algebras. Authors introduced BMBJ neutrosophic set as a generalization of neutrosophic set. Its subalgebra, images, translations, S - extension and its application to BCI/BCK – algebra are defined and explained. Neutrosophic vague binary sets are developed by Francina Shalini. A and Remya. P. B [15] in 2019. Authors developed a neutrosophic vague set with 2 universes and discussed its properties.

In this paper, BCK/BCI-algebraic structure is introduced to neutrosophic vague binary sets and it is simply called as neutrosophic vague binary BCK/BCI - algebra. It's ideal, neutrosophic vague binary BCK/BCI – ideal is also developed. Moreover, different neutrosophic vague binary BCK/BCI-ideals like neutrosophic vague binary BCK/BCI p-ideal, neutrosophic vague binary BCK/BCI q-ideal, neutrosophic vague binary BCK/BCI a-ideal and neutrosophic vague binary BCK/BCI H-ideal are also developed and compared. Neutrosophic vague binary BCK/BCI - subalgebra, neutrosophic vague binary BCK/BCI-cut and their relationships, properties and several theorems are also investigated and illustrated with examples.

Without algebra we can't even imagine mathematics. In one sense, geometry and algebra are equally important in mathematics. Even a layman can understand geometry because it deals with lines and shapes. It's applicational use in day to day life can't neglect. But algebra is like a silent player. In geometry, for finding out the solutions to lot of situations like, to get co-ordinates of centroid or to find out solution space to equations which represents lines, ellipse, hyperbolas, etc. - common way is to adopt the method of algebra. Study of surfaces is the main concept behind topology. Topological objects can bend, twist or stretch but are not allowed to tear, since there it loses its continuity. As a result, topological objects will become non – topological! Automatically they admit lack of homeomorphism in these situations. Geometrical nature of topology needs the assistance of algebra in several circumstances. This inevitable need of a mixed strategy, produced a new branch of mathematics called 'algebraic topology'. So developmental moments in any branch connected to topology from basic sets to neutrosophic sets via "fuzzy, rough, intuitionistic fuzzy, vague, interval mathematics, soft"- will equally demand the developments of it's counterpart-algebra. Thus both of them developed equally and produced vivid outputs like fuzzy BCK/BCI algebra, intuitionistic fuzzy BCK/BCI-algebra, rough BCK/BCI-algebra, vague BCK/BCI- algebra, soft BCK/BCI algebra and so on. So to stabilize neutrosophic branch, developments in various algebraic structures like BCK/BCI, BCH, BH etc are very critical and essential. This work will be important to neutrosophic due to its 'easy way approach' than [1] to reach to the same destination.

Method given in [1] is equally good but the concept of generating element is a little bit perplexing. Since [1] is closely connected to [47], it will be helpful, to verify lot of deep ideas given in [47]. Neutrosophic 'group and loop' concepts are well defined with examples and explanations in

[47]. It is to be noted that, as per [47], neutrosophic group does not possess a direct group structure, but it always contains one! Neutrosophic vague is a mixed form of neutrosophic and vague. It draws every positives and negatives of both the aforementioned sets. Numerical calculations for 'neutrosophic' are more than 'vague' due to its additional component - uncertainty. In real life problems, complex situations demand a more clear and easily accessible method to use with - 'neutrosophic, neutrosophic vague or neutrosophic vague binary'- set values. In group theory or ring theory algebraic structure is formed in such a manner that it includes set itself as a first member of the structure, then provide various algebraic operations as example shows : $(Z, +_4)$, (mZ, +), $(R, +, \ .)$ etc. Vague BCK/BCI algebraic-structure is defined as $(\mathrm{U}, *, 0)$ by enclosing only universal set and by omitting the corresponding vague set A. But in this context, universal set and vague set are simultaneously essential and available: since problem is being to be checked for a 'vague BCK/BCI-algebra and not for mere BCK/BCI-algebra' ! Our conclusion is that, being a core object in taking a decision to the question 'vague BCK/BCI-algebra or not? ' : inclusion of vague set, 'inside the structure' is important. It will avoid more confusions while doing theoretical work! Same thing is referable to fuzzy BCK/BCI-algebra, intuitionistic BCK/BCI-algebra, neutrosophic BCK/BCI-algebra and so on. This will be useful and applicable to all other existing structures like BCH, BH, B etc., with uncertain sets.

It is hoped that, when comparing to [1], concept developed in this paper, will be more useful to common people, since it uses values directly and hence easily accessible. This method depends on vague BCK/BCI paper [6]. In this paper our primary interest is to develop BCK/BCI-algebraic concept to neutrosophic vague binary sets. For this neutrosophic BCK/BCI algebraic concept and neutrosophic vague BCK/BCI-algebraic concept are needed as a base. Since it is not developed yet, in this paper, those are also developed with neutrosophic vague binary! An alternative structure approach, to vague BCK/BCI-algebra mentioned in [6] can be given as follows: A vague BCK/BCI-algebra is a structure $\mathfrak{B}_{\mathrm{A}} = \left(\mathrm{A}, \mathrm{U}^{\mathfrak{B}_{\mathrm{A}}} = (\mathrm{U}, *, 0), *, 0\right) = \left(\mathrm{A}, \mathrm{U}^{\mathfrak{B}_{\mathrm{A}}}, *, 0\right)$, where A is the vague set under consideration and $\mathrm{U}^{\mathfrak{B}_{\mathrm{A}}} = (\mathrm{U}, *, 0)$ is the underlying BCK/BCI-algebraic structure for A with universal set U, binary operation "$*$" and with constant "0". Similarly, when A becomes fuzzy set, the structure got is fuzzy BCK/BCI-algebraic. For theoretical applications, new approach is found to be more helpful and clear. Throughout this paper, new structure is used for neutrosophic/neutrosophic vague/ neutrosophic vague binary BCK/BCI-algebra.

Primary objective of this work is to develop a BCK/BCI-algebraic structure to neutrosophic vague binary set. Along with, care is taken, to use this novel concept, in 'theoretical applications'. Secondary objective is kept as the formation of various ideals to this new concept and their verification in theory part.

Paper consists of 8 sections. 1st section, provides an introduction, in which literature review has given. 2nd paragraph gives a general format of the work. 3rd paragraph explains why this work is essential to neutrosophic branch. 4th paragraph, points out 2 limitations of existing approaches. 5th paragraph mentions the alternative approaches to the limitations. 6th paragraph gives 2 objectives for the work. 7th paragraph, clearly explains how the paper is organized. 8th paragraph, summarizes all contributions of this paper in bullets. 2nd section of the paper describes materials for the work. In 3rd, neutrosophic vague binary/ neutrosophic vague/ neutrosophic BCK/BCI -algebras are developed. In 4th, neutrosophic vague binary BCK/BCI-subalgebra and neutrosophic vague binary BCK/BCI-ideal are developed. In 5th section, various neutrosophic vague binary BCK/BCI-ideals are formed and compared using a table. In 6th section, neutrosophic vague binary/ neutrosophic vague/ neutrosophic BCK/BCI - cuts are defined. In 7th section, propositions and lemmas related to this novel concept are discussed as a theoretical application. In 8th section, a conclusion to the paper is given.

Contributions in this paper are given in bullets below:

- Vague H-ideal
- Neutrosophic Vague Binary BCK/BCI-algebra
- Neutrosophic Vague Binary BCK/BCI-subalgebra
- Neutrosophic Vague Binary BCK/BCI-ideal
- Neutrosophic Vague Binary BCK/BCI- p ideal
  Neutrosophic Vague Binary BCK/BCI- q ideal
  Neutrosophic Vague Binary BCK/BCI- a ideal
  Neutrosophic Vague Binary BCK/BCI- H ideal
- Neutrosophic vague binary BCK/BCI- cut

2. Preliminaries

Some preliminaries are given in this section

Definition 2.1 [45] (Neutrosophic Vague Set)

A neutrosophic vague set A_{NV} (NVS in short) on the universe of discourse X can be written as $A_{NV} = \{\langle x ; \hat{T}_{A_{NV}}(X) ; \hat{I}_{A_{NV}}(X), \hat{F}_{A_{NV}}(X)\rangle; x \in X\}$ whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as

$\hat{T}_{A_{NV}}(x) = [T^-, T^+]$, $\hat{I}_{A_{NV}}(x) = [I^-, I^+]$ and $\hat{F}_{A_{NV}}(x) = [F^-, F^+]$

where (1) $T^+ = (1- F^-)$; $F^+ = (1- T^-)$ and

(2) $-0 \leq T^- + I^- + F^- \leq 2^+$

$-0 \leq T^+ + I^+ + F^+ \leq 2^+$

Definition 2.2 [15] (Neutrosophic Vague Binary Set)

A neutrosophic vague binary set (NVBS in short) M_{NVB} over a common universe $\{U_1 = \{x_j / 1 \leq j \leq n\}; U_2 = \{y_k/1 \leq k \leq p\}\}$ is an object of the form

$$M_{NVB} = \left\{\langle\frac{\hat{T}_{M_{NVB}}(x_j), \ \hat{I}_{M_{NVB}}(x_j), \ \hat{F}_{M_{NVB}}(x_j)}{x_j}; \forall x_j \in U_1\rangle \langle\frac{\hat{T}_{M_{NVB}}(y_k), \ \hat{I}_{M_{NVB}}(y_k), \ \hat{F}_{M_{NVB}}(y_k)}{y_k}; \forall y_k \in U_2\rangle\right\}$$

is defined as, ($\forall x_j \in U_1$ & $\forall y_k \in U_2$)

$\hat{T}_{M_{NVB}}(x_j) = [T^-(x_j), T^+(x_j)]$, $\hat{I}_{M_{NVB}}(x_j) = [I^-(x_j), I^+(x_j)]$ and $\hat{F}_{M_{NVB}}(x_j) = [F^-(x_j), F^+(x_j)]$ and

$\hat{T}_{M_{NVB}}(y_k) = [T^-(y_k), T^+(y_k)]$, $\hat{I}_{M_{NVB}}(y_k) = [I^-(y_k), I^+(y_k)]$ and $\hat{F}_{M_{NVB}}(y_k) = [F^-(y_k), F^+(y_k)]$

where (1) $T^+(x_j) = 1- F^-(x_j)$; $F^+(x_j) = 1- T^-(x_j)$ & $T^+(y_k) = 1- F^-(y_k)$; $F^+(y_k) = 1- T^-(y_k)$

(2) $^-0 \leq T^-(x_j)+I^-(x_j)+F^-(x_j) \leq 2^+$; $^-0 \leq T^-(y_k)+I^-(y_k)+F^-(y_k) \leq 2^+$ **or** $^-0 \leq T^-(x_j)+I^-(x_j)+F^-(x_j)+T^-(y_k)+I^-(y_k)+F^-(y_k) \leq 4^+$

$^-0 \leq T^+(x_j)+I^+(x_j)+F^+(x_j) \leq 2^+$; $^-0 \leq T^+(y_k)+I^+(y_k)+F^+(y_k) \leq 2^+$ **or** $^-0 \leq T^+(x_j)+I^+(x_j)+F^+(x_j)+T^+(y_k)+I^+(y_k)+F^+(y_k) \leq 4^+$

(3) $T^-(x_j), I^-(x_j), F^-(x_j) : V(U_1) \longrightarrow [0,1]$ and $T^-(y_k), I^-(y_k), F^-(y_k) : V(U_2) \longrightarrow [0,1]$

$T^+(x_j), I^+(x_j), F^+(x_j) : V(U_1) \longrightarrow [0,1]$ and $T^+(y_k), I^+(y_k), F^+(y_k) : V(U_2) \longrightarrow [0,1]$

Here $V(U_1)$, $V(U_2)$ denotes power set of vague sets on U_1, U_2 respectively

Definition 2.3 [48] (BCI-algebra)

Let X be a non-empty set with a binary operation $*$ and a constant 0. Then (X, $*$, 0) is called a BCI-algebra if it satisfies the following conditions:

(i) $((x*y)*(x*z))*(z*y) = 0$

(ii) $((x * (x * y)) * y = 0$
(iii) $(x * x) = 0$
(iv) $(x * y) = 0$ and $(y * x) = 0$ imply $x = y$, for all $x, y, z \in X$

Remark 2.4 [48]

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $(x * y) = 0$

Remark 2.5 [48] (BCK – algebra)

If a BCI-algebra X satisfies $(0 * x) = 0$ for all $x \in X$, then we say that X is a BCK- algebra

Remark 2.6 [48]

A BCI-algebra X has the following properties:
(i) $(x * 0) = x$; $\forall x \in X$
(ii) $(x * y) * z = (x * z) * y$; $\forall x, y, z \in X$
(iii) $0 * (x * y) = (0 * x) * (0 * y)$; $\forall x, y \in X$
(iv) $x * (x * (x * y)) = (x * y)$; $\forall x, y \in X$
(v) $x \leq y \Rightarrow (x * z) \leq (y * z)$; $(z * y) \leq (z * x)$; $\forall x, y, z \in X$
(vi) $(x * z) * (y * z) \leq (x * y)$; $\forall x, y, z \in X$
(vii) $0 * (0 * ((x * z) * (y * z))) = ((0 * y) * (0 * x))$; $\forall x, y, z \in X$
(viii) $0 * (0 * (x * y)) = ((0 * y) * (0 * x))$; $\forall x, y \in X$

Definition 2.7 [2, 4, 6, 25, 52] (Sub-algebra, Ideal, p-ideal, q-ideal, a-ideal, H-ideal)

Any non-empty subset I of a BCK/BCI- algebra X is called,
- subalgebra /ideal /p-ideal /q-ideal /a-ideal /H-ideal - of X, if it satisfies the axioms given table:

	Condition 1	Condition 2
Subalgebra of X	Nil	$(x * \mathbf{y}) \in I$; $\forall x, y \in I$
BCK/BCI-subalgebra of X	Nil	$(x * \mathbf{y}) \in I$; $\forall x, y \in I$
μ be a fuzzy BCK/BCI-algebra of X	Nil	$\mu(\mathbf{x} * \mathbf{y}) \geq \mathbf{min}\{\mu(\mathbf{x}) * \mu(\mathbf{y})\}$; $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$; **μ be a fuzzy set in a BCK/BCI – algebra X**
A be a vague BCK/BCI-algebra of X	Nil	$\mathbf{V_A(x * y) \geqslant r\,min\{V_A(x), V_A(y)\}}$; $\forall \mathbf{x, y \in X}$; **i. e.,** $\mathbf{t_A(x * y) \geq min\{t_A(x), t_A(y)\}}$; $\mathbf{1 - f_A(x * y) \geq min\{1 - f_A(x), 1 - f_A(y)\}}$ **A be a vague set in a BCK/BCI – algebra X**
Ideal of X	$0 \in I$	$(x * y) \in I$ & $y \in I \Rightarrow x \in \mathbf{I}$; $\forall \mathbf{x} \in \mathbf{I}$, $\forall \mathbf{y} \in X$
μ be a fuzzy BCI- ideal of X	$\mu(0) \geq \mu(x)$	$\mu(\mathbf{x}) \geq \mathbf{min}\{\mu(\mathbf{x} * \mathbf{y}), \mu(\mathbf{y})\}$; $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$; **μ be a fuzzy set in a BCI – algebra X**
BCK/BCI - Ideal of X	$0 \in I$	$(x * y) \in I$ & $y \in I \Rightarrow x \in \mathbf{I}$; $\forall \mathbf{x} \in \mathbf{I}$, $\forall \mathbf{y} \in X$
p - ideal of X	$0 \in I$	$[(x * z) * (y * z)] \in \mathbf{I}$ & $y \in \mathbf{I} \Rightarrow x \in \mathbf{I}$; $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$
q - ideal of X	$0 \in I$	$[x * (y * z)] \in \mathbf{I}$ & $y \in \mathbf{I} \Rightarrow (x * z) \in \mathbf{I}$; $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$
a - ideal of X	$0 \in I$	$[(x * z) * (0 * y)] \in \mathbf{I}$ & $z \in \mathbf{I} \Rightarrow (y * x) \in \mathbf{I}$; $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$
H - ideal of X	$0 \in I$	$[x * (y * z)] \in \mathbf{I}$ & $y \in \mathbf{I} \Rightarrow (x * z) \in \mathbf{I}$; $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$
μ be a fuzzy BCK/BCI-ideal of X	$\mu(0) \geq \mu(x)$	$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$; $\forall x, y \in X$; μ be a fuzzy set in a BCK/BCI – algebra X
vague BCI-ideal of X	$V_A(0) \geq V_A(x)$	$V_A(x) \geq r\,min\{V_A(x * y), V_A(y)\}$; $\forall x, y \in X$; A- vague set in X

Remark 2.8 [6] (r min & r max)

Let D[0, 1] denote the family of all closed sub-intervals of [0, 1]. Now we define the refined minimum (briefly, *r min*) and an order "≤" on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of D[0, 1] as : *r min* $(D_1, D_2) = [$*min* $\{a_1, a_2\}$, *min*$\{b_1, b_2\}]$. Similarly, we can define ≥, = and *r max*. Then the concept of r min and r max could be extended to define r inf and r sup of infinite number of elements of D [0, 1]. It is a known fact that L = {D [0, 1], *r inf*, *r sup*, ≤} is a lattice with universal bounds [0, 0] and [1, 1].

Definition 2.9 [6] (Vague–cuts)

Let A be a vague set of a universe X with the true-membership function t_A and false-membership function f_A. The (α, β)-cut of the vague set A is a crisp subset $A_{(\alpha, \beta)}$ of the set X given by

$A_{(\alpha,\beta)} = \{x \in X \,/\, V_A(x) \geq [\alpha, \beta]\}$ where $\alpha \leq \beta$. Clearly, $A_{(0,0)} = X$. The (α, β)-cuts are also called vague-cuts of the vague set A.

The α-cut of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.
Note that $\mathbf{A_0} = X$ and if $\alpha \geq \beta$ then $A_\beta \subseteq A_\alpha$ and $A_{(\beta,\alpha)} = A_\alpha$.
Equivalently, we can define the α-cut as $A_\alpha = \{x \in X \,/\, \mathbf{t_A(x)} \geq \alpha\}$

Definition 2.10 [50]

Given a non-empty set X, let $B_K(X)$ and $B_I(X)$ denote the collection of all BCK-algebras and all BCI-algebras, respectively. Also, $B(X) := B_K(X) \cup B_I(X)$.

For any $(X, *, 0) \in B(X)$, a fuzzy structure (X, μ) over $(X, *, 0)$ is called a

- Fuzzy subalgebra of $(X, *, 0)$ with type 1 (briefly, 1-fuzzy subalgebra of $(X, *, 0)$ if
  $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}; \ \forall\ \mathbf{x}, \mathbf{y} \in X$

- Fuzzy subalgebra of $(X, *, 0)$ with type 2 (briefly, 2 - fuzzy subalgebra of $(X, *, 0)$ if
  $\mu(x * y) \leq \min\{\mu(x), \mu(y)\}; \ \forall\ \mathbf{x}, \mathbf{y} \in X$

- Fuzzy subalgebra of $(X, *, 0)$ with type 3 (briefly, 3-fuzzy subalgebra of $(X, *, 0)$ if
  $\mu(x * y) \geq \max\{\mu(x), \mu(y)\}; \ \forall\ \mathbf{x}, \mathbf{y} \in X$

- Fuzzy subalgebra of $(X, *, 0)$ with type 4 (briefly, 4-fuzzy subalgebra of $(X, *, 0)$
  $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}; \ \forall\ \mathbf{x}, \mathbf{y} \in X$

3. Neutrosophic vague binary BCK/BCI-algebra

In this section neutrosophic BCK/BCI-algebra is developed first, based on paper [6]. Neutrosophic BCK/BCI-algebraic structure developed in this paper is a little bit different from the definition given in paper [1]. Concept is extended to neutrosophic vague sets and to neutrosophic vague binary sets.

Definition 3.1 (Neutrosophic BCK/BCI-algebra)

A neutrosophic BCK/BCI-algebra is a structure $\mathfrak{B}_{M_N} = \left(M_N,\ U^{\mathfrak{B}_{M_N}} = (U, *, 0),\ *, 0\right) = \left(M_N, U^{\mathfrak{B}_{M_N}},\ *, 0\right)$ where,

(1) M_N is a non-empty neutrosophic set

(2) $U^{\mathfrak{B}_{M_N}} = (U,\ *,\ 0)$ is the underlying BCK/BCI- algebraic structure, to the neutrosophic set M_N with a universal set U, a binary operation "$*$" & a constant "0" . It satisfies the following axioms :

(i) $((u_x * u_y) * (u_x * u_z)) * (u_z * u_y) = 0$ (ii) $((u_x * (u_x * u_y)) * u_y = 0$ (iii) $(u_x * u_x) = 0$

(iv) $(u_x * u_y) = 0$ and $(u_y * u_x) = 0$ imply $u_x = u_y$, $\forall\ u_x, u_y, u_z \in U$ (v) $(0 * u_x) = 0\ \forall\ u_x \in U$

(3) "$*$" and "0" are taken as defined in (2)

which satisfies the following condition,

$N_{M_N}(u_x * u_y) \geqslant r \min\{N_{M_N}(u_x), N_{M_N}(u_y)\} \quad ; \quad \forall\ u_x, u_y \in U$. That is,

$T_{M_N}(u_x * u_y) \geq \min\{T_{M_N}(u_x), T_{M_N}(u_y)\}$; $I_{M_N}(u_x * u_y) \leq \max\{I_{M_N}(u_x), I_{M_N}(u_y)\}$; $F_{M_N}(u_x * u_y) \leq \max\{F_{M_N}(u_x), F_{M_N}(u_y)\}$

Definition 3.2. (Neutrosophic vague BCK/BCI-algebra)

A neutrosophic vague BCK/BCI - algebra is a structure,

$\mathfrak{B}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}_{M_{NV}}} = (U, *, 0), *, 0\right) = \left(M_{NV}, U^{\mathfrak{B}_{M_{NV}}}, *, 0\right)$, where

(1) M_{NV} is a non-empty neutrosophic vague set

(2) $U^{\mathfrak{B}_{M_{NV}}} = (U, *, 0)$ is the underlying BCK/BCI- algebraic structure to the neutrosophic vague set M_{NV} with a universal set U, a binary operation "∗" & a constant "0" satisfies the following axioms:

(i) $((u_x * u_y) * (u_x * u_z)) * (u_z * u_y) = 0$ (ii) $((u_x * (u_x * u_y)) * u_y = 0$ (iii) $(u_x * u_x) = 0$

(iv) $(u_x * u_y) = 0$ and $(u_y * u_x) = 0$ imply $u_x = u_y$, $\forall\ u_x, u_y, u_z \in U$

(v) $(0 * u_x) = 0\ \forall\ u_x \in U$

(3) "∗" and "0" are taken as defined in $U^{\mathfrak{B}_{M_{NV}}}$

which satisfies the following condition,

$$NV_{M_{NV}}(u_x * u_y) \succcurlyeq r\min\{NV_{M_{NV}}(u_x), NV_{M_{NV}}(u_y)\} \quad ; \quad \forall\ u_x, u_y \in U. \quad \text{That is,}$$

$\hat{T}_{M_{NV}}(u_x * u_y) \geq \min\{\hat{T}_{M_{NV}}(u_x), \hat{T}_{M_{NV}}(u_y)\}$; $\hat{I}_{M_{NV}}(u_x * u_y) \leq \max\{\hat{I}_{M_{NV}}(u_x), \hat{I}_{M_{NV}}(u_y)\}$; $\hat{F}_{M_{NV}}(u_x * u_y) \leq \max\{\hat{F}_{M_{NV}}(u_x), \hat{F}_{M_{NV}}(u_y)\}$

General Outline

Let $U = \{0, u_p^1, u_p^2, u_p^3, ----, ----, u_p^k, ----, ---- u_p^i\}$ be a universal set with algebraic structure $U^{\mathfrak{B}_{M_{NV}}} = (U, *, 0)$ where ∗ is the given binary operation and 0 is the constant . Let $U^{\mathfrak{B}_{M_{NV}}}$ forms a BCK/BCI-algebra. Corresponding Cayley table is given below:

∗	**0**	u_p^1	u_p^2	----	----	u_p^k	----	----	u_p^i
0	0	0	0	0	0	0	----	0	0
u_p^1	u_p^1	0	----	----	----	----	----	----	----
u_p^2	u_p^2	----	0	----	----	----	----	----	----
----	----	----	----	0	----	----	----	----	----
----	----	----	----	----	0	----	----	----	----
u_p^k	u_p^k	----	----	----	----	0	----	----	----
----	----	----	----	----	----	----	0	----	----
----	----	----	----	----	----	----	----	----	----
u_p^i	u_p^i	----	----	----	----	----	----	----	0

By taking U as underlying set, form a neutrosophic vague set $\mathbf{M_{NV}}$ with neutrosophic vague membership grades, for any $\mathbf{u_p^k} \in U$,

$$\hat{\mathbf{T}}_{\mathbf{M_{NV}}}(\mathbf{u_p^k}) = \begin{cases}[\alpha_1, \alpha_2] ; \mathbf{u_p^k} = \mathbf{0} \\ [\alpha_3, \alpha_4] ; \mathbf{u_p^k} \neq \mathbf{0}\end{cases} ; \quad \hat{\mathbf{I}}_{\mathbf{M_{NV}}}(\mathbf{u_p^k}) = \begin{cases}[\beta_1, \beta_2] ; \mathbf{u_p^k} = \mathbf{0} \\ [\beta_3, \beta_4] ; \mathbf{u_p^k} \neq \mathbf{0}\end{cases} ; \quad \hat{\mathbf{F}}_{\mathbf{M_{NV}}}(\mathbf{u_p^k}) = \begin{cases}[\gamma_1, \gamma_2] ; \mathbf{u_p^k} = \mathbf{0} \\ [\gamma_3, \gamma_4] ; \mathbf{u_p^k} \neq \mathbf{0}\end{cases}$$

∴ Corresponding neutrosophic vague set is,

$$\mathbf{M_{NV}} = \left\{\langle\frac{[\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]}{0}, \frac{[\alpha_3,\alpha_4],[\beta_3,\beta_4],[\gamma_3,\gamma_4]}{u_p^1}, \frac{[\alpha_3,\alpha_4],[\beta_3,\beta_4],[\gamma_3,\gamma_4]}{u_p^2}, --, \frac{[\alpha_3,\alpha_4],[\beta_3,\beta_4],[\gamma_3,\gamma_4]}{u_p^k}, --, \frac{[\alpha_3,\alpha_4],[\beta_3,\beta_4],[\gamma_3,\gamma_4]}{u_p^i}\rangle\right\}$$

Algebraic structure $\mathbf{\mathfrak{B}_{M_{NV}}} = (\mathbf{M_{NV}}, \mathbf{U^{\mathfrak{B}_{M_{NV}}}}, *, 0)$ is called a neutrosophic vague BCK/BCI-algebra if it satisfies, $\mathbf{NV_{M_{NV}}(u_p^k * u_q^k) \succcurlyeq r\min\{NV_{M_{NV}}(u_p^k), NV_{M_{NV}}(u_q^k)\}}$; $\forall\ \mathbf{u_p^k, u_q^k \in U}$

Remark 3.3

Different neutrosophic vague membership grades are also applicable. It is explained in the general outline of definition 3.4

Definition 3.4 (Neutrosophic vague binary BCK/BCI- algebra)

A neutrosophic vague binary BCK/BCI- algebra is a structure,
$\mathfrak{B}_{M_{NVB}} = \left(M_{NVB}, U^{\mathfrak{B}_{M_{NVB}}} = (U, *, 0), *, 0\right) = \left(M_{NVB}, U^{\mathfrak{B}_{M_{NVB}}}, *, 0\right)$, where

(1) M_{NVB} is a non-empty neutrosophic vague binary set

(2) $U^{\mathfrak{B}_{M_{NVB}}} = \left(U = \{U_1 \cup U_2\}, *, 0\right)$ is the underlying BCK/BCI - algebraic structure to the neutrosophic vague binary set M_{NVB} with a universal set $U = \{U_1 \cup U_2\}$ [where U_1 and U_2 are universes of M_{NVB} & " $\cup$ " is the usual set-theoretic union], a binary operation " $*$ " & a constant "0" satisfies the following axioms:

(i) $((u_x * u_y) * (u_x * u_z)) * (u_z * u_y) = 0$ (ii) $((u_x * (u_x * u_y)) * u_y = 0$ (iii) $(u_x * u_x) = 0$

(iv) $(u_x * u_y) = 0$ and $(u_y * u_x) = 0$ imply $u_x = u_y \ \forall \ u_x, u_y, u_z \in U$ (v) $(0 * u_x) = 0 \ \forall \ u_x \in U$

(3) "$*$" and "0" are same as defined in $U^{\mathfrak{B}_{M_{NVB}}}$

which satisfies the following condition,

$NVB_{M_{NVB}}(u_x * u_y) \succcurlyeq r \min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}, \ \forall \ u_x, u_y \in U = \{U_1 \cup U_2\}$. That is,
$\hat{T}_{M_{NVB}}(u_x * u_y) \geq \min\{\hat{T}_{M_{NVB}}(u_x), \hat{T}_{M_{NVB}}(u_y)\}$; $\hat{I}_{M_{NVB}}(u_x * u_y) \leq \max\{\hat{I}_{M_{NVB}}(u_x), \hat{I}_{M_{NVB}}(u_y)\}$; $\hat{F}_{M_{NVB}}(u_x * u_y) \leq \max\{\hat{F}_{M_{NVB}}(u_x), \hat{F}_{M_{NVB}}(u_y)\}$

Remark 3.5

(i) Every NVB BCK-algebra is NVB BCI–algebra too. Generally, converse not true! (proved: Theorem 7.3). So distinguishing between structures of these two are important! To denote NVB BCK-algebra, following structures can be used: $\mathfrak{B}^{BCK}_{M_{NVB}} = \left(M_{NVB}, U^{\mathfrak{B}^{BCK}_{M_{NVB}}}, *, 0\right)$ or simply as $\mathfrak{B}^{K}_{M_{NVB}} = \left(M_{NVB}, U^{\mathfrak{B}^{K}_{M_{NVB}}}, *, 0\right)$. Similarly, to denote NVB BCI – algebra, following structures can be used : $\mathfrak{B}^{BCI}_{M_{NVB}} = \left(M_{NVB}, U^{\mathfrak{B}^{BCI}_{M_{NVB}}}, *, 0\right)$ or simply as $\mathfrak{B}^{I}_{M_{NVB}} = \left(M_{NVB}, U^{\mathfrak{B}^{I}_{M_{NVB}}}, *, 0\right)$.

(ii) For NVB BCK algebra, notation for NVB BCK/BCI – algebra, i.e., $\mathfrak{B}_{M_{NVB}} = (M_{NVB}, U^{\mathfrak{B}_{M_{NVB}}}, *, 0)$ is used in this paper instead of using, those given in remark 3.5 (i)

(iii) Similarly structures for:

Neutrosophic :

N BCK – algebra : $\mathfrak{B}^{BCK}_{M_N} = \left(M_N, U^{\mathfrak{B}^{BCK}_{M_N}}, *, 0\right)$ or $\mathfrak{B}^{K}_{M_N} = \left(M_N, U^{\mathfrak{B}^{K}_{M_N}}, *, 0\right)$ or $\mathfrak{B}_{M_N} = (M_N, U^{\mathfrak{B}_{M_N}}, *, 0)$

N BCI – algebra : $\mathfrak{B}^{BCI}_{M_N} = \left(M_N, U^{\mathfrak{B}^{BCI}_{M_N}}, *, 0\right)$ or simply as $\mathfrak{B}^{I}_{M_N} = \left(M_N, U^{\mathfrak{B}^{I}_{M_N}}, *, 0\right)$

Neutrosophic vague:

NV BCK – algebra : $\mathfrak{B}^{BCK}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}^{BCK}_{M_{NV}}}, *, 0\right)$ or $\mathfrak{B}^{K}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}^{K}_{M_{NV}}}, *, 0\right)$ or $\mathfrak{B}_{M_{NV}} = (M_{NV}, U^{\mathfrak{B}_{M_{NV}}}, *, 0)$

NV BCI – algebra : $\mathfrak{B}^{BCI}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}^{BCI}_{M_{NV}}}, *, 0\right)$ or simply as $\mathfrak{B}^{I}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}^{I}_{M_{NV}}}, *, 0\right)$

General Outline

Let $\mathbf{U_1} = \{0, \mathbf{u_p^1}, \mathbf{u_p^2}, \mathbf{u_p^3}, ---------, \mathbf{u_p^i}\}$ and $\mathbf{U_2} = \{0, \mathbf{u_q^1}, \mathbf{u_q^2}, \mathbf{u_q^3}, ---------, \mathbf{u_q^j}\}$ be two universes under consideration. Let the combined universe $U = \{\mathbf{U_1} \cup \mathbf{U_2}\} = \{0, \mathbf{u_p^1}, \mathbf{u_p^2}, \mathbf{u_p^3}, ----, \mathbf{u_p^i}, \mathbf{u_q^1}, \mathbf{u_q^2}, \mathbf{u_q^3}, ----, \mathbf{u_q^j}\} = \{\mathbf{0}, \mathbf{u_r^1}, \mathbf{u_r^2}, \mathbf{u_r^3}, ---------, \mathbf{u_r^k}\}$ (obtained by recording once, the common elements) be a set with a binary operation $*$ and constant 0 . Let $\mathbf{U}^{\mathfrak{B}_{M_{NVB}}} = (U = \{\mathbf{U_1} \cup \mathbf{U_2}\}, *, 0)$ forms a BCK/BCI-algebra. By taking U = $\{\mathbf{U_1} \cup \mathbf{U_2}\}$ as underlying set, form a neutrosophic vague binary set $\mathbf{M_{NVB}}$. Let neutrosophic vague binary membership grades are as follows:

for any $\mathbf{u_p^k} \in \mathbf{U_2}$:

$$\begin{cases} \hat{T}_{M_{NVB}}(u_p^k) = \begin{cases} [\alpha_1^0, \alpha_2^0] \ ; \ u_p^k = 0 \\ [\alpha_1^1, \alpha_2^1] \ ; \ u_p^k = u_p^1 \\ [\alpha_1^2, \alpha_2^2] \ ; \ u_p^k = u_p^2 \\ ----- \\ ----- \\ [\alpha_1^i, \alpha_2^i] \ ; \ u_p^k = u_p^i \end{cases} \\ \hat{I}_{M_{NVB}}(u_p^k) = \begin{cases} [\beta_1^0, \beta_2^0] \ ; \ u_p^k = 0 \\ [\beta_1^1, \beta_2^1] \ ; \ u_p^k = u_p^1 \\ [\beta_1^2, \beta_2^2] \ ; \ u_p^k = u_p^2 \\ ----- \\ ----- \\ [\beta_1^i, \beta_2^i] \ ; \ u_p^k = u_p^i \end{cases} \\ \hat{F}_{M_{NVB}}(u_p^k) = \begin{cases} [\gamma_1^0, \gamma_2^0] \ ; \ u_p^k = 0 \\ [\gamma_1^1, \gamma_2^1] \ ; \ u_p^k = u_p^1 \\ [\gamma_1^2, \gamma_2^2] \ ; \ u_p^k = u_p^2 \\ ----- \\ ----- \\ [\gamma_1^i, \gamma_2^i] \ ; \ u_p^k = u_p^i \end{cases} \end{cases}$$

& for any $\mathbf{u_q^k} \in \mathbf{U_2}$:

$$\begin{cases} \hat{T}_{M_{NVB}}(u_q^k) = \begin{cases} [\delta_1^0, \delta_2^0] \ ; \ u_q^k = 0 \\ [\delta_1^1, \delta_2^1] \ ; \ u_q^k = u_q^1 \\ [\delta_1^2, \delta_2^2] \ ; \ u_q^k = u_q^2 \\ ----- \\ ----- \\ [\delta_1^j, \delta_2^j] \ ; \ u_q^k = u_q^j \end{cases} \\ \hat{I}_{M_{NVB}}(u_q^k) = \begin{cases} [\rho_1^0, \rho_2^0] \ ; \ u_q^k = 0 \\ [\rho_1^1, \rho_2^1] \ ; \ u_q^k = u_q^1 \\ [\rho_1^2, \rho_2^2] \ ; \ u_q^k = u_q^2 \\ ----- \\ ----- \\ [\rho_1^j, \rho_2^j] \ ; \ u_q^k = u_q^j \end{cases} \\ \hat{F}_{M_{NVB}}(u_q^k) = \begin{cases} [\vartheta_1^0, \vartheta_2^0] \ ; \ u_q^k = 0 \\ [\vartheta_1^1, \vartheta_2^1] \ ; \ u_q^k = u_q^1 \\ [\vartheta_1^2, \vartheta_2^2] \ ; \ u_q^k = u_q^2 \\ ----- \\ ----- \\ [\vartheta_1^j, \vartheta_2^j] \ ; \ u_q^k = u_q^j \end{cases} \end{cases}$$

From this neutrosophic vague binary set $\mathbf{M_{NVB}}$, form neutrosophic vague binary membership grade for U = $\{\mathbf{U_1} \cup \mathbf{U_2}\}$ = $\{\mathbf{0},\ \mathbf{u_r^1},\ \mathbf{u_r^2}, ---, \mathbf{u_r^k}\}$ as :

$$\text{for any } u_r^k \in U \ : \ NVB_{M_{NVB}}(u_r^k) = \begin{cases} NVB_{M_{NVB}}(u_p^k = 0) \cup NVB_{M_{NVB}}(u_q^k = 0) \ ; u_r^k \in U_1;\ u_r^k \in U_2\ u_r^k = 0 \\ NVB_{M_{NVB}}(u_p^k) \ ; \ u_r^k \in U_1; u_r^k \notin U_2;\ u_r^k \neq 0 \\ NVB_{M_{NVB}}(u_q^k) \ ; \ u_r^k \notin U_1,\ u_r^k \in U_2;\ u_r^k \neq 0 \\ NVB_{M_{NVB}}(u_p^k) \cup NVB_{M_{NVB}}(u_q^k) \ ; \ u_r^k \in U_1;\ u_r^k \in U_2 \ ; \ u_r^k \neq 0 \end{cases}$$

i.e., for any $\mathbf{u_r^k} \in \mathbf{U}$:

$$\begin{cases} \hat{T}_{M_{NVB}}(u_r^k) = \begin{cases} \max\{\hat{T}_{M_{NVB}}(u_p^k = 0), \hat{T}_{M_{NVB}}(u_q^k = 0)\} = \max\{[\alpha_1^0, \alpha_2^0], [\delta_1^0, \delta_2^0]\};\ u_r^k \in U_1;\ u_r^k \in U_2\ u_r^k = 0 \\ \hat{T}_{M_{NVB}}(u_p^k) \ ; \ u_r^k \in U_1; u_r^k \notin U_2;\ u_r^k \neq 0 \\ \hat{T}_{M_{NVB}}(u_q^k) \ ; \ u_r^k \notin U_1,\ u_r^k \in U_2;\ u_r^k \neq 0 \\ \max\{\hat{T}_{M_{NVB}}(u_p^k), \hat{T}_{M_{NVB}}(u_q^k)\};\ u_r^k \in U_1;\ u_r^k \in U_2 \ ; \ u_r^k \neq 0 \end{cases} \\ \hat{I}_{M_{NVB}}(u_r^k) = \begin{cases} \min\{\hat{I}_{M_{NVB}}(u_p^k = 0), \hat{I}_{M_{NVB}}(u_q^k = 0)\} = \max\{[\beta_1^0, \beta_2^0],\ [\rho_1^0, \rho_2^0]\}; u_r^k \in U_1;\ u_r^k \in U_2\ u_r^k = 0 \\ \hat{I}_{M_{NVB}}(u_p^k) \ ; \ u_r^k \in U_1; u_r^k \notin U_2;\ u_r^k \neq 0 \\ \hat{I}_{M_{NVB}}(u_q^k) \ ; \ u_r^k \notin U_1,\ u_r^k \in U_2;\ u_r^k \neq 0 \\ \min\{\hat{I}_{M_{NVB}}(u_p^k), \hat{I}_{M_{NVB}}(u_q^k)\};\ u_r^k \in U_1;\ u_r^k \in U_2 \ ; \ u_r^k \neq 0 \end{cases} \\ \hat{F}_{M_{NVB}}(u_r^k) = \begin{cases} \min\{\hat{F}_{M_{NVB}}(u_p^k = 0), \hat{F}_{M_{NVB}}(u_q^k = 0)\} = \min\{[\beta_1^0, \beta_2^0],\ [\rho_1^0, \rho_2^0]\}; u_r^k \in U_1; u_r^k \in U_2;\ u_r^k = 0 \\ \hat{F}_{M_{NVB}}(u_p^k) \ ; \ u_r^k \in U_1; u_r^k \notin U_2;\ u_r^k \neq 0 \\ \hat{F}_{M_{NVB}}(u_q^k) \ ; \ u_r^k \notin U_1,\ u_r^k \in U_2;\ u_r^k \neq 0 \\ \min\{\hat{F}_{M_{NVB}}(u_p^k), \hat{F}_{M_{NVB}}(u_q^k)\};\ u_r^k \in U_1;\ u_r^k \in U_2 \ ; \ u_r^k \neq 0 \end{cases} \end{cases}$$

Corresponding Cayley table is given by:

*	0	u_r^1	u_r^2	---	u_r^k
0	0	0	0	0	0
u_r^1	u_r^1	0	---	---	---
u_r^2	u_r^2	---	0	---	---
---	---	---	---	0	---
u_r^k	u_r^k	---	---	---	0

Algebraic structure $\mathfrak{B}_{M_{NVB}}$ = $\left(\mathbf{M_{NVB}},\ \mathbf{U}^{\mathfrak{B}_{M_{NVB}}},\ *,\ \mathbf{0}\right)$ is called a NVB BCK/BCI-algebra, if it satisfies:

$$NVB_{M_{NVB}}(u_r^k * u_s^k) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_r^k), NVB_{M_{NVB}}(u_s^k)\} \ ; \quad \forall \quad u_r^k,\ u_s^k \in U$$

$$\hat{T}_{M_{NVB}}(u_r^k * u_s^k) \geq \min\{\hat{T}_{M_{NVB}}(u_r^k), \hat{T}_{M_{NVB}}(u_s^k)\} \ ; \ \hat{I}_{M_{NVB}}(u_r^k * u_s^k) \leq \max\{\hat{I}_{M_{NVB}}(u_r^k), \hat{I}_{M_{NVB}}(u_s^k)\} \ ; \ \hat{F}_{M_{NVB}}(u_r^k * u_s^k) \leq \max\{\hat{F}_{M_{NVB}}(u_r^k), \hat{F}_{M_{NVB}}(u_s^k)\}$$

Remark 3.6

(i) Neutrosophic vague binary membership grade of common elements of $\mathbf{U_1}$ and $\mathbf{U_2}$ is got by taking their neutrosophic vague binary union.

For eg., let $\mathbf{U_1} = \{\mathbf{0},\mathbf{1}\}$ and $\mathbf{U_2} = \{\mathbf{0},\mathbf{1},\mathbf{2}\}$ be two universes; $\therefore$ $\{\mathbf{U_1} \cup \mathbf{U_2}\} = \{\mathbf{0},\mathbf{1},\mathbf{2}\}$; $\mathbf{U_1} \cap \mathbf{U_2} = \{\mathbf{0},\mathbf{1}\}$ $\therefore$ $\mathbf{NVB_{M_{NVB}}(0) = NVB^{U_1}_{M_{NVB}}(0) \cup NVB^{U_2}_{M_{NVB}}(0)}$; $\mathbf{NVB^{U_1}_{M_{NVB}}(0)}$ is the neutrosophic vague binary membership grade of 0 in universe 1. Similarly, to other common elements.

(ii) It is to be noted that, neutrosophic vague binary membership grade of 0 is not same in $\mathbf{U_1}$, $\mathbf{U_2}$ generally. Similarly, to other common elements!

Example 3.7

Let $\mathbf{U_1} = \{\mathbf{0},\mathbf{a}\}$ and let $\mathbf{U_2} = \{\mathbf{0},\mathbf{1},\mathbf{2}\}$ be the universes under consideration. Combined universe $\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\} = \{\mathbf{0},\mathbf{a},\mathbf{1},\mathbf{2}\}$ with $(\mathbf{U_1} \cap \mathbf{U_2}) = \{0\}$. Cayley table to the binary operation $*$ for U is given as:

$*$	0	a	1	2
0	0	0	0	0
a	a	0	0	1
1	1	a	0	1
2	2	2	2	0

Clearly, $\mathbf{U^{\mathfrak{B}_{M_{NVB}}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}, *, \mathbf{0})$ is a BCK/BCI-algebra. Let a non-empty neutrosophic vague binary set $\mathbf{M_{NVB}}$ with underlying set $\mathbf{U}$, is given as:

$$\mathbf{M_{NVB}} = \left\{\left\langle \frac{[0.3,0.8],[0.1,0.3],[0.2,0.7]}{0}, \frac{[0.2,0.3],[0.2,0.5],[0.7,0.8]}{a} \right\rangle, \left\langle \frac{[0.1,0.7],[0.7,0.8],[0.3,0.9]}{0}, \frac{[0.2,0.6],[0.5,0.7],[0.4,0.8]}{1}, \frac{[0.2,0.6],[0.5,0.7],[0.4,0.8]}{2} \right\rangle\right\}$$

$\Rightarrow \quad \forall \quad \mathbf{u^k_p} \in \mathbf{U_1}$ and $\forall \quad \mathbf{u^k_q} \in \mathbf{U_2}$,

$$\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_p}) = \begin{cases} [0.3, 0.8] & \text{if } \mathbf{u^k_p} = 0 \\ [0.2, 0.3] & \text{if } \mathbf{u^k_p} = a \text{ or } \mathbf{u^k_p} \neq 0 \end{cases} ; \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_p}) = \begin{cases} [0.1, 0.3] & \text{if } \mathbf{u^k_p} = 0 \\ [0.2, 0.5] & \text{if } \mathbf{u^k_p} = a \text{ or } \mathbf{u^k_p} \neq 0 \end{cases} ; \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_p}) = \begin{cases} [0.2, 0.7] & \text{if } \mathbf{u^k_p} = 0 \\ [0.7, 0.8] & \text{if } \mathbf{u^k_p} = a \text{ or } \mathbf{u^k_p} \neq 0 \end{cases}$$

$$\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_q}) = \begin{cases} [0.1, 0.7]; \text{ if } \mathbf{u^k_q} = 0 \\ [0.2, 0.6]; \text{ if } \mathbf{u^k_q} = \{1,2\} \text{ or } \mathbf{u^k_q} \neq 0 \end{cases} ; \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_q}) = \begin{cases} [0.7, 0.8]; \text{ if } \mathbf{u^k_q} = 0 \\ [0.5, 0.7]; \text{ if } \mathbf{u^k_q} = \{1,2\} \text{ or } \mathbf{u^k_q} \neq 0 \end{cases} ; \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_q}) = \begin{cases} [0.3, 0.9]; \text{ if } \mathbf{u^k_q} = 0 \\ [0.4, 0.8]; \text{ if } \mathbf{u^k_q} = \{1,2\} \text{ or } \mathbf{u^k_q} \neq 0 \end{cases}$$

$\therefore \mathbf{NVB_{M_{NVB}}(0)} = ([0.3, 0.8], [0.1, 0.3], [0.2, 0.7]) \cup ([0.1, 0.7], [0.7, 0.8], [0.3, 0.9]) = ([0.3, 0.8], [0.1, 0.3], [0.2, 0.7])$

$\mathbf{NVB_{M_{NVB}}(a)} = ([0.2, 0.3], [0.2, 0.5], [0.7, 0.8])$ [since a is not a common element]

$\mathbf{NVB_{M_{NVB}}(1)} = \mathbf{NVB_{M_{NVB}}(2)} = ([0.2, 0.6], [0.5, 0.7], [0.4, 0.8])$; [since 1 and 2 are not a common element]

$$\Rightarrow \mathbf{NVB_{M_{NVB}}(u^k_r)} = \begin{cases} [0.3, 0.8], [0.1, 0.3], [0.2, 0.7]; & \mathbf{u^k_r} = \mathbf{0} \\ [0.2, 0.3], [0.2, 0.5], [0.7, 0.8]; & \mathbf{u^k_r} = \{\mathbf{a}\} \text{ and } \mathbf{u^k_r} \neq \mathbf{0} \\ [0.2, 0.6], [0.5, 0.7], [0.4, 0.8]; & \mathbf{u^k_r} = \{\mathbf{1},\mathbf{2}\} \text{ and } \mathbf{u^k_r} \neq \mathbf{0} \end{cases} ; \text{(for any } \mathbf{u^k_r} \in \text{U)}$$

It is clear after verification that, $\mathfrak{B}_{\mathbf{M_{NVB}}} = \left(\mathbf{M_{NVB}}, \mathbf{U^{\mathfrak{B}_{M_{NVB}}}}, *, \mathbf{0}\right)$ is a NVB BCK/BCI- algebra.

Remark. 3.8

(1) If $\mathbf{U_1} \subseteq \mathbf{U_2}$ then $\mathbf{U} = \mathbf{U_2}$ (2) If $\mathbf{U_2} \subseteq \mathbf{U_1}$ then $\mathbf{U} = \mathbf{U_1}$

(2) The symbols $\succcurlyeq$ and $\not\succcurlyeq$ does not imply our usual $\geq$ or $\ngeq$

(3) In a Cayley table,

(i) principal diagonal elements of a BCK/BCI-algebra U is always zero, since $(\mathbf{x} * \mathbf{x}) = \mathbf{0}, \forall\, \mathbf{x} \in \mathbf{U}$

(ii) Using the property $(\mathbf{x} * \mathbf{0}) = \mathbf{x}$; $\forall\ \mathbf{x} \in \mathbf{U}$ of BCI- algebra, it is clear that $(\mathbf{0} * \mathbf{0}) = \mathbf{0}$

Every BCK-algebra is a BCI-algebra. Hence the above is true for BCK-algebra also

(iii) Body of first column of Cayley table for a BCI- algebra will be an exact copy of column of operands, by using the property $(\mathbf{x} * \mathbf{0}) = \mathbf{x}\ \forall\ \mathbf{x} \in \mathbf{U}$. But $\mathbf{1^{st}}$ row need not be!

(iv) Above is true for a BCK- algebra also, since every BCK- algebra is a BCI-algebra. In addition, for a BCK- algebra, body of first row takes only 0, using the property $(\mathbf{0} * \mathbf{x}) = \mathbf{0}$; $\forall\ \mathbf{x} \in \mathbf{U}$

Binary Operation $*$	**Row of operands** (Elements of $\mathbf{U}$)
Column of operands (Elements of $\mathbf{U}$)	Body of Cayley table (occupy with elements got after binary operation taken via column vise row operations)

4. Neutrosophic vague binary BCK/BCI-subalgebra & Neutrosophic vague binary BCK/BCI-ideal

In this section N/NV/NVB BCK/BCI- subalgebra (neutrosophic/neutrosophic vague/neutrosophic vague binary BCK/BCI – subalgebra) & N/NV/NVB BCK/BCI– ideal (neutrosophic/neutrosophic vague/neutrosophic vague binary BCK/BCI – ideal) are developed. Priority is given for developing sub-algebraic and ideal concepts to neutrosophic vague binary BCK/BCI- algebra [NVB BCK/BCI-algebra]. For neutrosophic and neutrosophic vague, things are similar.

Definition 4.1 (Neutrosophic vague binary BCK/BCI-subalgebra)

A NVBSS $\mathbf{P_{NVB}}$ of a NVB BCK/BCI-algebra $\mathfrak{B}_{\mathbf{M_{NVB}}} = \left(\mathbf{M_{NVB}},\ \mathbf{U}^{\mathfrak{B}_{\mathbf{M_{NVB}}}} = (\mathbf{U},\ *,\ \mathbf{0}),\ *,\ \mathbf{0}\right)$ is called NVB- BCK/BCI - subalgebra of $\mathfrak{B}_{\mathbf{M_{NVB}}}$ if,

$$\mathbf{NVB_{P_{NVB}}}(\mathbf{u_x} * \mathbf{u_y}) \succcurlyeq \mathbf{r\ min}\left\{\mathbf{NVB_{P_{NVB}}}(\mathbf{u_x}), \mathbf{NVB_{P_{NVB}}}(\mathbf{u_y})\right\} \quad ; \quad \forall \quad \mathbf{u_x}, \mathbf{u_y} \in \mathbf{U}$$

$\hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_x} * \mathbf{u_y}) \geq \mathbf{min}\{\hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_x}), \hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_y})\}$; $\hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_x} * \mathbf{u_y}) \leq \mathbf{max}\{\hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_x}), \hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_y})\}$; $\hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_x} * \mathbf{u_y}) \leq \mathbf{max}\{\hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_x}), \hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_y})\}$

Definition 4.2 (Neutrosophic vague binary BCK/BCI- Ideal)

A non-empty NVBSS $\mathbf{P_{NVB}}$ of a NVB BCK/BCI-algebra, $\mathfrak{B}_{\mathbf{M_{NVB}}} = \left(\mathbf{M_{NVB}},\ \mathbf{U}^{\mathfrak{B}_{\mathbf{M_{NVB}}}},\ *,\ \mathbf{0}\right)$ is called a NVB BCK/BCI- ideal of $\mathfrak{B}_{\mathbf{M_{NVB}}}$ if

(i) $\mathbf{NVB_{P_{NVB}}}(\mathbf{0}) \succcurlyeq \mathbf{NVB_{P_{NVB}}}(\mathbf{u_k})$; for any $\mathbf{u_k} \in \mathbf{U}$

i.e., $\hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{0}) \geq \hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_k})$; $\hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{0}) \leq \hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_k})$; $\hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{0}) \leq \hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_k})$

(ii) $\mathbf{NVB_{P_{NVB}}}(\mathbf{u_a}) \succcurlyeq \mathbf{r\ min}\left\{\mathbf{NVB_{P_{NVB}}}(\mathbf{u_a} * \mathbf{u_b}), \mathbf{NVB_{P_{NVB}}}(\mathbf{u_b})\right\}$; for any $\mathbf{u_a}, \mathbf{u_b} \in \mathbf{U}$

$\hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_a}) \geq \mathbf{min}\{\hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_a} * \mathbf{u_b}), \hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_b})\}$; $\hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_a}) \leq \mathbf{max}\{\hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_a} * \mathbf{u_b}), \hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_b})\}$; $\hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_a}) \leq \mathbf{max}\{\hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_a} * \mathbf{u_b}), \hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_b})\}$

Remark 4.3
For NVB BCK – ideal underlying structure will confine to BCK -algebra and for NVB BCI – ideal it will confine to BCK -algebra. For different ideals mentioned in definition 5.2, the same principle follows.
Remark 4.4
Similarly, for neutrosophic and neutrosophic vague. Only difference is with sets $\mathbf{P_N}$, $\mathbf{P_{NV}}$ instead of $\mathbf{P_{NVB}}$ in above definitions taken in order. It is trivial. Moreover, instead of $\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}$, for both of them U is applied.

5. Various neutrosophic vague binary BCK/BCI-ideals

In this section vague H-ideal is developed first. Then p-ideal, q-ideal, a-ideal and H-ideal are developed for NVB BCK/BCI-algebra $\mathfrak{B}_{\mathbf{M_{NVB}}} = \left(\mathbf{M_{NVB}}, \mathbf{U}^{\mathfrak{B}_{\mathbf{M_{NVB}}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\},\ *,\ \mathbf{0}),\ *,\ \mathbf{0}\right)$
Definition 5.1 (Vague H-ideal)

A vague set A of X is called a vague H - ideal of a BCI – algebra X if it satisfies

(i) $\mathbf{V_A}(\mathbf{0}) \succcurlyeq \mathbf{V_A}(\mathbf{x}) \quad (\forall\ \mathbf{x} \in \mathbf{X})$;

i.e., $\begin{cases} \mathbf{t_A(0) \geq t_A(x)} \\ \mathbf{1 - f_A(0) \geq 1 - f_A(x)} \end{cases}$ i.e., $\begin{cases} \mathbf{t_A(0) \geq t_A(x)} \\ \mathbf{f_A(0) \leq f_A(x)} \end{cases}$ and

(ii) $\mathbf{V_A(x*z) \succcurlyeq r\min\{V_A(x*(y*z)), V_A(y)\}}$; $(\forall\ x, y, z \in X)$; i.e., $\begin{cases} \mathbf{t_A(x*z) \geq \min\{t_A(x*(y*z)), t_A(y)\}} \\ \mathbf{1 - f_A(x*z) \geq \min\{1 - f_A(x*(y*z)), 1 - f_A(y)\}} \end{cases}$

Definition 5.2 (Comparison of different NVB BCK/BCI- ideals)

Let $\mathfrak{B}_{M_{NVB}} = (M_{NVB}, U^{\mathfrak{B}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, *, 0), *, 0)$ be a NVB BCK/BCI-algebra. Conditions for a non-empty NVBSS $\mathbf{P_{NVB}}$ of $\mathfrak{B}_{M_{NVB}}$ to become a neutrosophic vague binary BCK/BCI - p ideal, neutrosophic vague binary BCK/BCI - q ideal, neutrosophic vague binary BCK/BCI - a ideal and neutrosophic vague binary BCK/BCI - H ideal are given in the table below:

	Condition (1) ; $(\forall\ \mathbf{u_k} \in \mathbf{U})$	Condition (2) ; (for any $\mathbf{u_a}, \mathbf{u_b}, \mathbf{u_c} \in \mathbf{U}$)
NVB BCK/BCI **p**-ideal	$\mathbf{NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_k)}$	$\mathbf{NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_c) * (u_b * u_c)), NVB_{P_{NVB}}(u_b)\}}$
NVB BCK/BCI **q**-ideal	$\mathbf{NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_k)}$	$\mathbf{NVB_{P_{NVB}}(u_a * u_c) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * (u_b * u_c))), NVB_{P_{NVB}}(u_b)\}}$
NVB BCK/BCI **a**-ideal	$\mathbf{NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_k)}$	$\mathbf{NVB_{P_{NVB}}(u_b * u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(((u_a * u_c) * (0 * u_b))), NVB_{P_{NVB}}(u_c)\}}$
NVB BCK/BCI **H**-ideal	$\mathbf{NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_k)}$	$\mathbf{NVB_{P_{NVB}}(u_a * u_c) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * (u_b * u_c))), NVB_{P_{NVB}}(u_b)\}}$

6. Neutrosophic vague binary BCK/BCI - cuts

In this section N BCK/BCI-cut, NV BCK/BCI-cut and NVB BCK/BCI-cut are developed

Definition 6.1 (Neutrosophic BCK/BCI-$(\alpha, \beta, \gamma)-$ cut or Neutrosophic BCK/BCI-cut)
Let the neutrosophic set M_N is a N BCK/BCI-algebra with algebraic structure $\mathfrak{B}_{M_N} = (M_N, U^{\mathfrak{B}_{M_N}}, *, 0)$. Truth membership function, indeterminacy membership function and false membership function of M_N are T_{M_N}, I_{M_N}, F_{M_N} respectively. A neutrosophic BCK/BCI (α, β, γ) - cut of $\mathfrak{B}_{M_N}$ is a crisp subset $M_{N(\alpha,\beta,\gamma)}$ of the neutrosophic set M_N given by:

$$M_{N(\alpha,\beta,\gamma)} = \{u_k \in U \ /\ N_{M_N}(u_k) \succcurlyeq (\alpha, \beta, \gamma)\ ;\ \text{with } \alpha, \beta, \gamma \in [0,1]\}$$
$$= \{u_k \in U / T_{M_N}(u_k) \geq \alpha\ ;\ I_{M_N}(u_k) \leq \beta\ ;\ F_{M_N}(u_k) \leq \gamma\ ;\ \text{with } \alpha, \beta, \gamma \in [0,1]\}$$

Definition 6.2 (**Neutrosophic Vague BCK/BCI** $([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2])$ **- cut or Neutrosophic Vague BCK/BCI- cut)**
Let the neutrosophic vague set M_{NV} is a NV BCK/BCI-algebra with algebraic structure $\mathfrak{B}_{M_{NV}} = (M_{NV}, U^{\mathfrak{B}_{M_{NV}}} = (U, *, 0), *, 0)$. Truth membership function, indeterminacy membership function and false membership function of M_{NV} are $\hat{T}_{M_{NV}}$, $\hat{I}_{M_{NV}}$, $\hat{F}_{M_{NV}}$ respectively. A neutrosophic vague BCK/BCI $([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2])$ - cut of $\mathfrak{B}_{M_{NV}}$ is a crisp subset $M_{NV([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])}$ of the neutrosophic vague set M_{NV} given by :

$$M_{NV([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])}$$
$$= \{u_k \in U \ /\ NV_{M_{NV}}(u_k) \succcurlyeq ([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2])\ ;\ \text{where } \alpha_1 \leq \alpha_2,\ \beta_1 \leq \beta_2,\ \gamma_1 \leq \gamma_2;\ \text{with } \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in [0,1]\}$$
$$= \{u_k \in U / \ \hat{T}_{M_{NV}}(u_k) \geq [\alpha_1, \alpha_2]\ ;\hat{I}_{M_{NV}}(u_k) \leq [\beta_1, \beta_2];\ \hat{F}_{M_{NV}}(u_k) \leq [\gamma_1, \gamma_2]\}\ ;$$

i.e., $T^-(u_k) \geq \alpha_1$ and $T^+(u_k) \geq \alpha_2$; $I^-(u_k) \leq \beta_1$ and $I^+(u_k) \leq \beta_2$; $F^-(u_k) \leq \gamma_1$ and $F^+(u_k) \leq \gamma_2$

Definition 6.3 (Neutrosophic Vague Binary BCK/BCI $\langle([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]), ([\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2])\rangle$ **- cut or Neutrosophic Vague Binary BCK/BCI- cut)**
Let the NVBS M_{NVB} is a NVB BCK/BCI-algebra with algebraic structure,
$\mathfrak{B}_{M_{NVB}} = (M_{NVB}, U^{\mathfrak{B}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, *, 0), *, 0)$. Truth membership function, indeterminacy membership function and false membership function of M_{NVB} are $\hat{T}_{M_{NVB}}$, $\hat{I}_{M_{NVB}}$, $\hat{F}_{M_{NVB}}$ respectively. A neutrosophic vague binary BCK/BCI $\langle([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]), ([\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2])\rangle$ - cut of $\mathfrak{B}_{M_{NVB}}$ is a crisp subset $M_{NVB\langle([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])\rangle}$ of the NVBS M_{NVB} given by :

$$M_{NVB\langle([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])\rangle} = \{u_k \in U / \ NVB_{M_{NVB}}(u_k) \succcurlyeq \begin{cases} [\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\ ;\ \text{if } u_k \in U_1 \\ [\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2]\ ;\ \text{if } u_k \in U_2 \\ [\chi_1, \chi_2], [\phi_1, \phi_2], [\pi_1, \pi_2]\ ;\ \text{if } u_k \in U_1 \cap U_2 \end{cases}$$

$\max\{[\alpha_1, \alpha_2], [\delta_1, \delta_2]\} = [\chi_1, \chi_2]$ (say); $\min[\beta_1, \beta_2], [\rho_1, \rho_2]\} = [\phi_1, \phi_2]$ (say); $\min\{[\gamma_1, \gamma_2], [\vartheta_1, \vartheta_2] = [\pi_1, \pi_2]$ (say) ;

with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \rho_1, \rho_2, \vartheta_1, \vartheta_2, \chi_1, \chi_2, \phi_1, \phi_2, \pi_1, \pi_2 \in [0, 1]$
and $\alpha_1 \leq \alpha_2, \quad \beta_1 \leq \beta_2, \quad \gamma_1 \leq \gamma_2 ; \quad \delta_1 \leq \delta_2, \quad \rho_1 \leq \rho_2, \quad \vartheta_1 \leq \vartheta_2, \chi_1 \leq \chi_2, \phi_1 \leq \phi_2 ; \quad \pi_1 \leq \pi_2$

i.e., $\widehat{T}_{M_{NVB}}(u_k) \geq [\alpha_1, \alpha_2]$; $\hat{I}_{M_{NVB}}(u_k) \leq [\beta_1, \beta_2]$; $\widehat{F}_{M_{NVB}}(u_k) \leq [\gamma_1, \gamma_2]$
$\widehat{T}_{M_{NVB}}(u_k) \geq [\delta_1, \delta_2]$; $\hat{I}_{M_{NVB}}(u_k) \leq [\rho_1, \rho_2]$; $\widehat{F}_{M_{NVB}}(u_k) \leq [\vartheta_1, \vartheta_2]$
$\widehat{T}_{M_{NVB}}(u_k) \geq [\chi_1, \chi_2]$; $\hat{I}_{M_{NVB}}(u_k) \leq [\phi_1, \phi_2]$; $\widehat{F}_{M_{NVB}}(u_k) \leq [\pi_1, \pi_2]$

i.e., $T^-(u_k) \geq \alpha_1$ and $T^+(u_k) \geq \alpha_2$; $I^-(u_k) \leq \beta_1$ and $I^+(u_k) \leq \beta_2$; $F^-(u_k) \leq \gamma_1$ and $F^+(u_k) \leq \gamma_2$
$T^-(u_k) \geq \delta_1$ and $T^+(u_k) \geq \delta_2$; $I^-(u_k) \leq \rho_1$ and $I^+(u_k) \leq \rho_2$; $F^-(u_k) \leq \vartheta_1$ and $F^+(u_k) \leq \vartheta_2$
$T^-(u_k) \geq \chi_1$ and $T^+(u_k) \geq \chi_2$; $I^-(u_k) \leq \phi_1$ and $I^+(u_k) \leq \phi_2$; $F^-(u_k) \leq \pi_1$ and $F^+(u_k) \leq \pi_2$

Remark 6.4

(i) (a) $M_{NVB}([0,0],[1,1],[1,1]) = U$
(b) $M_{NVB}\langle([0,0],[1,1],[1,1]),([0,0],[1,1],[1,1])\rangle = U = \{U_1 \cup U_2\}$

(ii) If $[\alpha_1, \alpha_2]$ and $[\delta_1, \delta_2]$ coincides; $[\beta_1, \beta_2]$ and $[\rho_1, \rho_2]$coincides; $[\gamma_1, \gamma_2]$ and $[\vartheta_1, \vartheta_2]$ coincides, then $(\langle[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\rangle, \langle[\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2]\rangle) -$ cuts are called $(\langle[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\rangle, \langle[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\rangle)$ - cuts and is denoted by $M_{NVB([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2],[\delta_1,\delta_2])}$ instead of $M_{NVB(\langle[\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]\rangle,\langle[\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]\rangle)}$

(iii) If $(\langle[\alpha_1^*, \alpha_2^*], [\beta_1^*, \beta_2^*], [\gamma_1^*, \gamma_2^*]\rangle, \langle[\delta_1^*, \delta_2^*], [\rho_1^*, \rho_2^*], [\vartheta_1^*, \vartheta_2^*]\rangle) \geq (\langle[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\rangle, \langle[\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2]\rangle)$ then $M_{NVB(\langle[\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]\rangle,\langle[\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2]\rangle)} \subseteq M_{NVB(\langle[\alpha_1^*,\alpha_2^*],[\beta_1^*,\beta_2^*],[\gamma_1^*,\gamma_2^*]\rangle,\langle[\delta_1^*,\delta_2^*],[\rho_1^*,\rho_2^*],[\vartheta_1^*,\vartheta_2^*]\rangle)}$

7. Application

In this section theoretical application of NVB BCK/BCI algebra is developed. Various theorems and propositions are found good to this concept.

Lemma 7.1

Every NVB BCI – algebra $\mathfrak{B}_{M_{NVB}}^{BCI}$ of a BCI -algebra $U^{\mathfrak{B}_{M_{NVB}}^{BCI}}$ satisfies:
$NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall$ $u_k \in U = \{U_1 \cup U_2\}$

Proof

For a $\mathfrak{B}_{M_{NVB}}^{BCI}$, underlying BCI - algebraic structure satisfies, $(u_k * u_k) = 0, \forall u_k \in U$

[By property (iii)of definition 2.3]

$\Rightarrow \forall u_k \in U$, $NVB_{M_{NVB}}(0) = NVB_{M_{NVB}}(u_k * u_k) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_k), NVB_{M_{NVB}}(u_k)\} = NVB_{M_{NVB}}(u_k)$

[By definition 3.4]

Lemma 7.2

Every $\mathfrak{B}_{M_{NVB}}^{BCK}$ satisfies $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall$ $u_k \in U$

Proof

For a $\mathfrak{B}_{M_{NVB}}^{BCK}$, underlying BCK- algebraic structure satisfies, an additional condition, $(0 * u_k) = 0, \forall u_k \in U$ besides $(u_k * u_k) = 0, \forall u_k \in U$; [By remark 2.5]

$\Rightarrow$ Additional to, $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall$ $u_k \in U$ [by lemma 7.1], we get,
$NVB_{M_{NVB}}(0) = NVB_{M_{NVB}}(0 * u_k) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$; $\forall u_k \in U$
$\Rightarrow$ $NVB_{M_{NVB}}(0) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$; $\forall$ $u_k \in U$, for $\mathfrak{B}_{M_{NVB}}^{BCK}$
& $r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$ will depend upon the given NVBS M_{NVB}
$\Rightarrow$ $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$ and $NVB_{M_{NVB}}(0) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$
Even if $r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$ will depend upon the given NVBS, using lemma 7.1, $NVB_{M_{NVB}}(0) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$ will become $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$
$\Rightarrow$ $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$ and $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$ $\forall u_k \in U$
So, combining both, for a $\mathfrak{B}_{M_{NVB}}^{BCK}$ too, $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall u_k \in U$

Remark 7.3

Every $\mathfrak{B}_{M_{NVB}}^{BCK}$/ $\mathfrak{B}_{M_{NVB}}^{BCI}$ satisfies: $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall u_k \in U$
i.e., Every NVB BCK/BCI – algebra satisfies: $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall u_k \in U$

Theorem 7.4

Every $\mathfrak{B}^{BCK}_{M_{NVB}}$ is a $\mathfrak{B}^{BCI}_{M_{NVB}}$. But converse not true, generally. i.e., every $\mathfrak{B}^{BCI}_{M_{NVB}}$ is not a $\mathfrak{B}^{BCK}_{M_{NVB}}$ generally.

Proof

For a fixed universal set U, underlying BCK – algebraic structure of $\mathfrak{B}^{BCK}_{M_{NVB}}$ consists the underlying BCI – structure of $\mathfrak{B}^{BCI}_{M_{NVB}}$ $\Rightarrow$ Every $\mathfrak{B}^{BCK}_{M_{NVB}}$ is $\mathfrak{B}^{BCI}_{M_{NVB}}$. But converse does not hold. It is illustrated with the case (i) of remark 7.5.

Remark 7.5

Following example illustrates both the cases:

Let $U_1 = \{0\}$ and let $U_2 = \{0, 1\}$ be the universes under consideration. Combined universe $U = \{U_1 \cup U_2\} = \{0, 1\}$ with $(U_1 \cap U_2) = \{0\}$.

$\therefore$ Cayley table to the binary operation $*$ for U is given as:

$*$	0	1
0	0	1
1	1	0

BCI-algebra [fig (i)]

$*$	0	1
0	0	0
1	1	0

BCK/BCI-algebra[fig(ii)]

Clearly, $\mathbf{U}^{\mathfrak{B}^{BCI}_{M_{NVB}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}, *, \mathbf{0})$ is a BCI-algebra [fig (i)].

$\mathbf{U}^{\mathfrak{B}^{BCK}_{M_{NVB}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}, *, \mathbf{0})$ is a BCI-algebra [fig (ii)].

Case (i) : Example for a $\mathfrak{B}^{BCI}_{M_{NVB}}$ which is a $\mathfrak{B}^{BCK}_{M_{NVB}}$

Let M_{NVB} be a non-empty NVBS with $U^{\mathfrak{B}^{BCI}_{M_{NVB}}}$ as underlying algebraic structure:

$M_{NVB} = \left\{\langle\frac{[0.1,0.8],[0.1,0.5],[0.2,0.9]}{0}\rangle, \langle\frac{[0.3,0.7],[0.2,0.4],[0.3,0.7]}{0}, \frac{[0.1,0.4],[0.3,0.5],[0.6,0.9]}{1}\rangle\right\}$; Here, $(U_1 \cap U_2) = \{0\}$

$NVB_{M_{NVB}}(0) = ([0.1, 0.8], [0.1, 0.5], [0.2, 0.9]) \cup ([0.3, 0.7], [0.2, 0.4], [0.3, 0.7]) = [0.3, 0.8], [0.1, 0.4], [0.2, 0.7]$

After verification, clearly M_{NVB} is a $\mathfrak{B}^{BCI}_{M_{NVB}}$. Next question is that, - "whether $\mathfrak{B}^{BCI}_{M_{NVB}}$ is a $\mathfrak{B}^{BCK}_{M_{NVB}}$ or not " ? $\therefore$ Additional condition to be satisfied is that, for a BCK-algebra is, $(0 * 1) = 0$ from Cayley table fig (ii). Correspondingly,

$NVB_{M_{NVB}}(0-1) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(1)\} \Rightarrow NVB_{M_{NVB}}(0) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(1)\}$

$\Rightarrow [0.3, 0.8], [0.1, 0.4], [0.2, 0.7] \succcurlyeq r\min\{[0.3, 0.8], [0.1, 0.4], [0.2, 0.7], [0.1, 0.4], [0.3, 0.5], [0.6, 0.9]\}$

$\Rightarrow [0.3, 0.8], [0.1, 0.4], [0.2, 0.7] \succcurlyeq [0.1, 0.4], [0.3, 0.5], [0.6, 0.9]$

Since additional condition got satisfied, $\mathfrak{B}^{BCI}_{M_{NVB}}$ is clearly a $\mathfrak{B}^{BCK}_{M_{NVB}}$.

Case (ii) : Example for a $\mathfrak{B}^{BCI}_{P_{NVB}}$ which is not a $\mathfrak{B}^{BCK}_{P_{NVB}}$

Take binary operation and Cayley table as taken in Case (i).

Consider another NVBS $\mathbf{P}_{NVB}$ with same conditions as in case (i)

$\mathbf{P}_{NVB} = \left\{\langle\frac{[0.1,0.5],[0.2,0.5],[0.5,0.9]}{0}\rangle, \langle\frac{[0.1,0.6],[0.3,0.3],[0.4,0.9]}{0}, \frac{[0.1,0.7],[0.3,0.4],[0.3,0.9]}{1}\rangle\right\}$

$NVB_{P_{NVB}}(0) = ([0.1, 0.5], [0.2, 0.5], [0.5, 0.9]) \cup ([0.1, 0.6], [0.3, 0.3], [0.4, 0.9]) = [0.1, 0.6], [0.2, 0.3], [0.4, 0.9]$

By verification P_{NVB} is a $\mathfrak{B}^{BCI}_{P_{NVB}}$. But in this case, additional condition not got satisfied:

$NVB_{P_{NVB}}(0 * 1) \not\succcurlyeq r\min\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(1)\}$ [Since, $NVB_{P_{NVB}}(0) \not\succcurlyeq r\min\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(1)\}$.

Since, $[0.1, 0.6], [0.2, 0.3], [0.4, 0.9] \not\succcurlyeq r\min\{[0.1, 0.8], [0.2, 0.3], [0.2, 0.9], [0.1, 0.7], [0.3, 0.4], [0.3, 0.9]\}$

Since, $[0.1, 0.6], [0.2, 0.3], [0.4, 0.9] \not\succcurlyeq [0.1, 0.7], [0.3, 0.4], [0.3, 0.9]]$

In this case, clearly, $\mathfrak{B}^{BCI}_{P_{NVB}}$ is not a $\mathfrak{B}^{BCK}_{P_{NVB}}$

Theorem 7.6

Intersection of two NVB BCK/BCI -algebra remains as a NVB BCK/BCI-algebra itself.

Proof

Let M_{NVB} and P_{NVB} be two NVB BCK/BCI -algebras with structures $\mathfrak{B}_{M_{NVB}} = (M_{NVB}, U^{\mathfrak{B}_{M_{NVB}}}, *, 0)$

and $\mathfrak{B}_{P_{NVB}} = (P_{NVB}, U^{\mathfrak{B}_{P_{NVB}}}, *, 0)$ respectively, with same universal sets U_1 and U_2. So, $\forall$ $u_1, u_2 \in U$, $NVB_{(M_{NVB} \cap P_{NVB})}(u_1 * u_2) = r\min\{NVB_{M_{NVB}}(u_1 * u_2), NVB_{P_{NVB}}(u_1 * u_2)\}$
$\succcurlyeq r\min\{r\min\{NVB_{M_{NVB}}(u_1), NVB_{M_{NVB}}(u_2)\}, r\min\{NVB_{P_{NVB}}(u_1), NVB_{P_{NVB}}(u_2)\}\}$
$= r\min\{NVB_{(M_{NVB} \cap P_{NVB})}(u_1), NVB_{(M_{NVB} \cap P_{NVB})}(u_2)\}$
Therefore, $NVB_{(M_{NVB} \cap P_{NVB})}(u_1 * u_2) \succcurlyeq r\min\{NVB_{(M_{NVB} \cap P_{NVB})}(u_1), NVB_{(M_{NVB} \cap P_{NVB})}(u_2)\}$
$\Rightarrow$ $(M_{NVB} \cap P_{NVB})$ is also a NVB BCK/BCI - algebra

Proposition 7.7

Every NVB BCI - ideal P_{NVB} of a $\mathfrak{B}^{BCI}_{M_{NVB}}$ satisfies:

(i) $u_a \leq u_b \Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq NVB_{P_{NVB}}(u_b)$; $(\forall u_a, u_b \in U)$

(ii) $NVB_{P_{NVB}}(u_a * u_c) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_b) * u_c), NVB_{P_{NVB}}(u_b)\}; \forall u_a, u_b, u_c \in U$

Proof

(i) Let $u_a, u_b \in U$ be such that $u_a \leq u_b$.
Since P_{NVB} is a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$
$\Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$, [By condition (2) of definition 4.2]
$= r\min\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(u_b)\}$, take $(u_a * u_b) = 0$
$= NVB_{P_{NVB}}((u_b))$ [By lemma 7.1]
$\Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq NVB_{P_{NVB}}(u_b)$

(ii) Let P_{NVB} be a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$
$\Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$; $\forall u_a, u_b \in U$
$\Rightarrow NVB_{P_{NVB}}(u_a * u_c) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_c) * u_b), NVB_{P_{NVB}}(u_b)\}$;
[by putting $u_a = (u_a * u_c)$; $\forall u_a, u_b, u_c \in U$]
$\Rightarrow NVB_{P_{NVB}}(u_a * u_c) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_b) * u_c), NVB_{P_{NVB}}(u_b)\}$;[By property (ii)of remark 2.6]

Lemma 7.8

Let P_{NVB} be a NVB BCI-ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$. Then, $NVB_{P_{NVB}}(0 * (0 * u_k)) \succcurlyeq NVB_{P_{NVB}}(u_k)$; $\forall u_k \in U$

Proof

$NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$; for any $u_a, u_b \in U$ [By definition 4.2]
Let $u_a = (0 * (0 * u_k))$ and $u_b = u_k$.
$\therefore$ For any $u_k \in U$, $NVB_{P_{NVB}}(0 * (0 * u_k)) \succcurlyeq r\min\left\{NVB_{P_{NVB}}\left((0 * (0 * u_k)) * u_k\right), NVB_{P_{NVB}}(u_k)\right\}$
$= r\min\{NVB_{P_{NVB}}((0 * u_k) * (0 * u_k)), NVB_{P_{NVB}}(u_k)\}$; [By property (ii)of remark 2.6]
$= r\min\{NVB_{P_{NVB}}(0 * (u_k * u_k)), NVB_{P_{NVB}}(u_k)\}$; [By property (iii)of remark 2.6]
$= r\min\{NVB_{P_{NVB}}(0 * 0), NVB_{P_{NVB}}(u_k)\}$; [By condition (iii)of definition 2.3]
$= r\min\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(u_k)\}$; [By property (i)of remark 2.6]
$= NVB_{P_{NVB}}(u_k)$ [By lemma 7.1]
$\therefore$ It is concluded that, $NVB_{M_{NVB}}(0 * (0 * u_k)) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall u_k \in U$

Proposition 7.9

If the NVBS R_{NVB} of $\mathfrak{B}^{BCI}_{M_{NVB}}$ is a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$, then it satisfies: for any $u_a, u_b, u_c \in U$; $(u_a * u_b) \leq u_c \Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$

Proof

Let R_{NVB} be a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$ with $(u_a * u_b) \leq u_c$ where $u_a, u_b, u_c \in U$
$\Rightarrow NVB_{R_{NVB}}(u_a * u_b) \succcurlyeq NVB_{R_{NVB}}(u_c)$ [By proposition 7.7]
Since R_{NVB} be a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$
$\Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_a * u_b), NVB_{R_{NVB}}(u_b)\}$ for any $u_a, u_b \in U$
$\succcurlyeq r\min\{NVB_{R_{NVB}}(u_c), NVB_{R_{NVB}}(u_b)\} = r\min\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$
$\Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$ for any $u_a, u_b, u_c \in U$

Proposition 7.10

If the NVBS R_{NVB} of $\mathfrak{B}^{BCI}_{M_{NVB}}$ is a NVB BCI - algebra $\mathfrak{B}^{BCI}_{R_{NVB}}$ of then it satisfies for any $u_x, u_y, u_z \in U$ $(u_a * u_b) \leq u_c \Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$

Proof

Let R_{NVB} be a NVBS of $\mathfrak{B}^{BCI}_{M_{NVB}}$ with $(u_a * u_b) \leq u_c \Rightarrow NVB_{R_{NVB}}(u_c) \succcurlyeq NVB_{R_{NVB}}(u_a * u_b)$

R_{NVB} is a $\mathfrak{B}^{BCI}_{R_{NVB}}$ $\Rightarrow NVB_{R_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_a), NVB_{R_{NVB}}(u_b)\}$
$\Rightarrow NVB_{R_{NVB}}(u_c) \succcurlyeq NVB_{R_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_a), NVB_{R_{NVB}}(u_b)\}$
$\Rightarrow NVB_{R_{NVB}}(u_c) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\}$
$\Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_c), NVB_{R_{NVB}}(u_b)\}$; [By putting $u_c = u_a$ & $u_a = u_c$]
$\Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$;

Theorem 7.11

Let S_{NVB} be both a NVB BCI–algebra $\mathfrak{B}^{BCI}_{S_{NVB}}$ and a NVB BCI-ideal of a NVB BCI - algebra $\mathfrak{B}^{BCI}_{S_{NVB}}$. Then $NVB_{S_{NVB}}(0 * u_k) \succcurlyeq NVB_{S_{NVB}}(u_k)$ for all $u_k \in U$

Proof

Let S_{NVB} be a NVB BCI- algebra $\mathfrak{B}^{BCI}_{S_{NVB}}$
$\Rightarrow NVB_{S_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{S_{NVB}}(u_a), NVB_{S_{NVB}}(u_b)\}$; for all $u_a, u_b \in U$
$\Rightarrow NVB_{S_{NVB}}(0 * u_b) \succcurlyeq r\min\{NVB_{S_{NVB}}(0), NVB_{S_{NVB}}(u_b)\}$; [By putting $u_a = 0$]
$\Rightarrow NVB_{S_{NVB}}(0 * u_b) \succcurlyeq NVB_{S_{NVB}}(u_b)$ [By definition 4.2 (i)]
$\Rightarrow NVB_{S_{NVB}}(0 * u_k) \succcurlyeq NVB_{S_{NVB}}(u_k)$ [By putting $u_b = u_k$]
$\therefore$ For any $u_k \in U$, $NVB_{S_{NVB}}(0 * u_k) \succcurlyeq NVB_{S_{NVB}}(u_k)$

Proposition 7.12

Let T_{NVB} be a NVB BCI - ideal of a NVB BCI -algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$.
If T_{NVB} satisfies $NVB_{T_{NVB}}(u_a * u_b) \succcurlyeq NVB_{T_{NVB}}((u_a * u_c) * (u_b * u_c))$ for all $u_a, u_b, u_c \in U$,
then T_{NVB} is a NVB BCI p - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$

Proof

T_{NVB} be a NVB BCI - ideal of a NVB BCI - algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$.
$\Rightarrow NVB_{T_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{T_{NVB}}(u_a * u_b), NVB_{T_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$
$\Rightarrow NVB_{T_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{T_{NVB}}((u_a * u_c) * (u_b * u_c)), NVB_{M_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$
[From given condition]
$\Rightarrow T_{NVB}$ is a NVB BCI – p ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$ [By definition 5.2]

Proposition 7.13

Any NVB BCI - ideal D_{NVB} of a NVB BCI -algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$ is a NVB BCI -p ideal
$\Leftrightarrow NVB_{D_{NVB}}(u_a) \succcurlyeq NVB_{D_{NVB}}(0 * (0 * u_a))$; for all $u_a \in U$

Proof

Let D_{NVB} be a NVB BCI - ideal of a NVB BCI -algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$. Also let D_{NVB} is a NVB BCI -p ideal.
$\therefore NVB_{D_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{D_{NVB}}((u_a * u_c) * (u_b * u_c)), NVB_{D_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$
[By definition 5.2 of NVB BCI − p ideal]
Put $u_c = u_a$ and $u_b = 0$ in the above,
$\therefore NVB_{D_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{D_{NVB}}((u_a * u_a) * (0 * u_a)), NVB_{D_{NVB}}(0)\}$ for all $u_a, u_b, u_c \in U$
$\Rightarrow NVB_{D_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{D_{NVB}}(0 * (0 * u_a)), NVB_{D_{NVB}}(0)\}$ for all $u_a, u_b \in U$
[By condition (iii)of definition 2.3]
$= NVB_{D_{NVB}}(0 * (0 * u_a))$ for all $u_a \in U$ [By lemma 7.1]
$\Rightarrow NVB_{D_{NVB}}(u_a) \succcurlyeq NVB_{D_{NVB}}(0 * (0 * u_a))$; for all $u_a \in U$
Conversely, let a NVB BCI - ideal D_{NVB} of a NVB BCI - algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$ satisfies the given condition, $NVB_{D_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}(0 * (0 * u_a))$; for all $u_a \in U$. By lemma 7.8,
" Let P_{NVB} be a NVB BCI-ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$. Then, $NVB_{P_{NVB}}(0 * (0 * u_k)) \succcurlyeq NVB_{P_{NVB}}(u_k)$; $\forall$ $u_k \in U$"
$\Rightarrow NVB_{D_{NVB}}((u_a * u_c) * (u_b * u_c)) \preccurlyeq NVB_{M_{NVB}}\left(0 * \left(0 * ((u_a * u_c) * (u_b * u_c))\right)\right)$
[By putting $u_k = (u_a * u_c) * (u_b * u_c)$ in lemma 7.8]
$= NVB_{D_{NVB}}((0 * u_b) * (0 * u_a))$ [By property (vii)of remark 2.6]
$= NVB_{D_{NVB}}\left(0 * (0 * (u_a * u_b))\right)$; [By property (viii) of remark 2.6]
$= NVB_{D_{NVB}}(0 * (u_a * u_b))$; [By property (i) of remark 2.6]
$= NVB_{D_{NVB}}(u_a * u_b)$; [By property (i) of remark 2.6]
$\Rightarrow NVB_{D_{NVB}}(u_a * u_b) \succcurlyeq NVB_{D_{NVB}}((u_a * u_c) * (u_b * u_c))$

$\Rightarrow$ $NVB_{D_{NVB}}$ is a NVB BCI - p ideal [By proposition 7.12]

Theorem 7.14

Every NVB BCI - p ideal of a NVB BCI - algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$ is a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$.

Proof

Let M_{NVB} be a NVB BCI - p ideal of a NVB BCI - algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$. By definition,

$NVB_{M_{NVB}}(u_x) \succcurlyeq r\min\left\{NVB_{M_{NVB}}\left((u_x * u_z) * (u_y * u_z)\right), NVB_{M_{NVB}}(u_y)\right\}$ for all $u_x, u_y, u_z \in U$

Put $u_z = 0$ then the above becomes,

$NVB_{M_{NVB}}(u_x) \succcurlyeq r\min\left\{NVB_{M_{NVB}}\left((u_x * 0) * (u_y * 0)\right), NVB_{M_{NVB}}(u_y)\right\}$ for all $u_x, u_y \in U$

$= r\min\{NVB_{M_{NVB}}(u_x * u_y), NVB_{M_{NVB}}(u_y)\}$ for all $u_x, u_y \in U$

[By property (iii)of 2.3 &By property (i)of remark 2.6]

$\Rightarrow NVB_{M_{NVB}}(u_x) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x * u_y), NVB_{M_{NVB}}(u_y)\}$ for all $u_x, u_y \in U$

Obviously, M_{NVB} is a NVB BCI - ideal

Converse of this statement need not be true and it can be verified with an example and is trivial.

Theorem 7.15

Every NVB BCK/BCI H - ideal of a NVB BCK/BCI - algebra $\mathfrak{B}_{M_{NVB}}$ acts both as

(i) NVB BCK/BCI - ideal of $\mathfrak{B}_{M_{NVB}}$ (ii) NVB BCK/BCI - subalgebra $\mathfrak{B}_{M_{NVB}}$

Proof

Let I_{NVB} be a NVB BCK/BCI- H ideal of a NVB BCK/BCI – algebra $\mathfrak{B}_{M_{NVB}}$

(i) From definition of NVB BCK/BCI- H ideal,

$NVB_{I_{NVB}}(u_a * u_c) \succcurlyeq \min\{NVB_{I_{NVB}}(u_a * (u_b * u_c)), NVB_{I_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$

Put $u_c = 0$

$NVB_{I_{NVB}}(u_a * 0) \succcurlyeq \min\{NVB_{I_{NVB}}(u_a * (u_b * 0)), NVB_{M_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$

$\Rightarrow NVB_{I_{NVB}}(u_a) \succcurlyeq \min\{NVB_{I_{NVB}}(u_a * u_b), NVB_{M_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$

[Using property (i)of remark 2.6]

Since I_{NVB} is a NVB BCK/BCI- H ideal $\Rightarrow NVB_{I_{NVB}}(0) \succcurlyeq NVB_{I_{NVB}}(u_k)$; for any $u_k \in U$

$\therefore$ I_{NVB} is a NVB BCK/BCI - ideal of $\mathfrak{B}_{M_{NVB}}$ [By definition 4.2]

(ii) Let I_{NVB} be a NVB BCK/BCI- H ideal of $\mathfrak{B}_{M_{NVB}}$

$\therefore NVB_{I_{NVB}}(u_a * u_c) \succcurlyeq r\min\{NVB_{I_{NVB}}(u_a * (u_b * u_c)), NVB_{I_{NVB}}(u_b)\}$; for all $u_a, u_b, u_c \in U$

$\Rightarrow$ $NVB_{I_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{I_{NVB}}(u_a * (u_b * u_b)), NVB_{I_{NVB}}(u_b)\}$; [By putting $u_c = u_b$]

$\Rightarrow$ $NVB_{I_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{I_{NVB}}(u_a * 0), NVB_{I_{NVB}}(u_b)\}$; [By condition (iii)of definition 2.3]

$\Rightarrow$ $NVB_{I_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{I_{NVB}}(u_a), NVB_{I_{NVB}}(u_b)\}$; [By condition (i)of remark 2.6]

$\Rightarrow$ I_{NVB} be a NVB BCK/BCI - subalgebra of $\mathfrak{B}_{M_{NVB}}$

Theorem 7.16

P_{NVB} be a NVBS of a NVB BCK/BCI - algebra $\mathfrak{B}_{M_{NVB}}$. Then P_{NVB} is a NVB BCK/BCI -ideal of $\mathfrak{B}_{M_{NVB}}$ $\Leftrightarrow$ it satisfies the following conditions:

(i) $NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq NVB_{P_{NVB}}(u_b)$; $(\forall\, u_a, u_b \in U)$

(ii) $NVB_{P_{NVB}}\left(u_a * \left((u_a * u_m) * u_n\right)\right) \succcurlyeq r\min\left\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\right\}$; $(\forall\, u_a, u_m, u_n \in U)$

Proof

Let P_{NVB} be NVB BCK/BCI - ideal of $\mathfrak{B}_{M_{NVB}}$. By definition,

$NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$; $\forall$ $u_a, u_b \in U$

(i) Put $u_a = (u_a * u_b)$ and $u_b = u_a$ in the above,

$NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_b) * u_a), NVB_{P_{NVB}}(u_a)\}$

$\Rightarrow NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_a) * u_b), NVB_{P_{NVB}}(u_a)\}$; [By property (ii)of remark 2.6]

$\Rightarrow NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{P_{NVB}}(0 * u_b), NVB_{P_{NVB}}(u_a)\}$; [By condition (iii)of definition 2.3]

$\Rightarrow NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{P_{NVB}}(0 * u_b), NVB_{P_{NVB}}(u_b)\}$; [By assumption $u_a = u_b$]

$\Rightarrow$ $NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(u_b)\}$; [By remark 2.5]

$\Rightarrow$ $NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq NVB_{P_{NVB}}(u_b)$ [Using lemma 7.1]

(ii) Consider, $\left(u_a * \left((u_a * u_m) * u_n\right)\right) * u_m$

$= (u_a * u_m) * \left((u_a * u_m) * u_n\right) \leq u_n$

$$\begin{bmatrix} \text{By condition (ii)of remark 2.3, } (x * (x * y)) * y = 0 \Rightarrow x * (x * y) \le y, \text{by remark 2.4} \\ Here, \left(u_a * ((u_a * u_m) * u_n)\right) * u_m = (u_a * u_m) * ((u_a * u_m) * u_n) = ((u_a * u_m) * (u_a * u_m)) * u_n = 0 * u_n = 0. \text{ So remark 2.4 is applicable in this case} \\ \text{Since } (u_a * u_m) * ((u_a * u_m) * u_n) = 0 \text{ we have } (u_a * u_m) * ((u_a * u_m) * u_n) \le u_n \end{bmatrix}$$

Above can be written as, $\left(u_a * ((u_a * u_m) * u_n)\right) * u_m \le u_n$

$\Rightarrow NVB_{P_{NVB}}\left(\left(u_a * ((u_a * u_m) * u_n)\right) * u_m\right) \succcurlyeq NVB_{P_{NVB}}(u_n)$ [By proposition 7.7]

P_{NVB} is a NVB BCK/BCI -ideal of $\mathfrak{B}_{M_{NVB}} \Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$

Put $u_a = \left(u_a * ((u_a * u_m) * u_n)\right)$ & $u_b = u_m$ in above,

$NVB_{P_{NVB}}\left(u_a * ((u_a * u_m) * u_n)\right) \succcurlyeq r\min\left\{NVB_{P_{NVB}}\left(\left(u_a * ((u_a * u_m) * u_n)\right) * u_m\right), NVB_{P_{NVB}}(u_m)\right\}$

$= r\min\{NVB_{P_{NVB}}(u_n), NVB_{P_{NVB}}(u_m)\}$ [proved above]

$\succcurlyeq r\min\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$

$\Rightarrow NVB_{M_{NVB}}\left(u_a * ((u_a * u_m) * u_n)\right) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$; $[\forall u_a, u_m, u_n \in U]$

Conversely,

Let P_{NVB} be a NVBS of a NVB BCK/BCI – algebra $\mathfrak{B}_{M_{NVB}}$ satisfying, the given conditions,

(i) $NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq NVB_{P_{NVB}}(u_a)$; $[\forall u_a, u_b \in U]$

(ii) $NVB_{P_{NVB}}\left(u_a * ((u_a * u_m) * u_n)\right) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$; $[\forall u_a, u_m, u_n \in U]$

To prove condition (1) of a NVB BCK/BCI - ideal, take $u_b = u_a$ in (i) and (ii) respectively,

(i) $\Rightarrow NVB_{P_{NVB}}(u_a * u_a) \succcurlyeq NVB_{P_{NVB}}(u_a) \Rightarrow NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_a)$;

[By property (iii)of definition 2.3]

To prove condition (2) of a NVB BCK/BCI - ideal,

take, $NVB_{P_{NVB}}(u_a) = NVB_{P_{NVB}}(u_a * 0)$ [By property(i)of remark 2.6]

$= NVB_{P_{NVB}}\left(u_a * ((u_a * u_b) * (u_a * u_b))\right)$; [By property (iii)of definition 2.3]

$= NVB_{P_{NVB}}\left(u_a * ((u_a * (u_a * u_b)) * u_b)\right)$; [By property (ii) of remark 2.6]

$= NVB_{P_{NVB}}\left(u_a * ((u_a * u_m) * u_n)\right)$; [By putting $(u_a * u_b) = u_m$ and $u_b = u_n$]

$\succcurlyeq r\min\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$; [By condition (ii) in the assumption]

$= r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$; [By putting $(u_a * u_b) = u_m$ and $u_b = u_n$]

$\Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$

$\therefore$ P_{NVB} is a NVB BCK/BCI - ideal of $\mathfrak{B}_{M_{NVB}}$

Theorem 7.17

Let M_{NVB} be a NVB BCK/BCI - algebra $\mathfrak{B}_{M_{NVB}}$. Then any NVB BCK/BCI - cut of M_{NVB} is a crisp NVB BCK/BCI - subalgebra of $\mathfrak{B}_{M_{NVB}}$

Proof

Let for any $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \rho_1, \rho_2, \vartheta_1, \vartheta_2 \in [0, 1]$,

$M_{NVB([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}$ be a NVB BCK/BCI -cut of M_{NVB}.

Assume $u_x, u_y \in M_{NVB([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}$

$\Rightarrow NVB_{M_{NVB}}(u_x) \ge [\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]$ & $NVB_{M_{NVB}}(u_x) \ge ([\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2])$

$NVB_{M_{NVB}}(u_y) \ge [\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]$ & $NVB_{M_{NVB}}(u_y) \ge ([\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2])$

$\Rightarrow \widehat{T}_{M_{NVB}}(u_x) \ge [\alpha_1, \alpha_2]$; $\widehat{I}_{M_{NVB}}(u_x) \le [\beta_1, \beta_2]$; $\widehat{F}_{M_{NVB}}(u_x) \le [\gamma_1, \gamma_2]$ &

$\widehat{T}_{M_{NVB}}(u_x) \ge [\delta_1, \delta_2]$; $\widehat{I}_{M_{NVB}}(u_x) \le [\rho_1, \rho_2]$; $\widehat{F}_{M_{NVB}}(u_x) \le [\vartheta_1, \vartheta_2]$

$\widehat{T}_{M_{NVB}}(u_y) \ge [\alpha_1, \alpha_2]$; $\widehat{I}_{M_{NVB}}(u_y) \le [\beta_1, \beta_2]$; $\widehat{F}_{M_{NVB}}(u_y) \le [\gamma_1, \gamma_2]$

$\widehat{T}_{M_{NVB}}(u_y) \ge [\delta_1, \delta_2]$; $\widehat{I}_{M_{NVB}}(u_y) \le [\rho_1, \rho_2]$; $\widehat{F}_{M_{NVB}}(u_y) \le [\vartheta_1, \vartheta_2]$

M_{NVB} is a NVB BCK/BCI -algebra $\mathfrak{B}_{M_{NVB}} \Rightarrow NVB_{M_{NVB}}(u_x * u_y) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$

$\Rightarrow \widehat{T}_{M_{NVB}}(u_x * u_y) \ge \min\{\widehat{T}_{M_{NVB}}(u_x), \widehat{T}_{M_{NVB}}(u_y)\}$; $\widehat{I}_{M_{NVB}}(u_x * u_y) \le \max\{\widehat{I}_{M_{NVB}}(u_x), \widehat{I}_{M_{NVB}}(u_y)\}$; $\widehat{F}_{M_{NVB}}(u_x * u_y) \le \max\{\widehat{F}_{M_{NVB}}(u_x), \widehat{F}_{M_{NVB}}(u_y)\}$

$\Rightarrow (u_x * u_y) \in M_{NVB([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}$

$\Rightarrow$ NVB BCK/BCI - cut $M_{NVB([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}$ of M_{NVB} is a crisp NVB BCK/BCI - subalgebra of $\mathfrak{B}_{M_{NVB}}$

8. Conclusions

In this paper, two logical algebras viz., BCK and BCI are developed for neutrosophic vague binary sets. It's subalgebra, ideal and cuts are also got discussed. Different kinds of ideals like p ideal, q ideal, a ideal, H ideal for neutrosophic vague binary BCK/BCI -algebra have been investigated. Theorems and propositions related to this concept are verified. In this paper BCK/BCI-algebra for neutrosophic sets are firstly developed. Then it is extended to neutrosophic vague and to neutrosophic vague binary. Work can be further extended to higher concepts like its group, rings, filter, near-rings etc. Behavior differences of these two algebras in different algebraic notions have to be addressed more deeply and properly to get a correct vision. This area demands some more notice and filtering to find out its correct drawbacks. Further investigations will make it, to balance its moves to the correct direction. Medial BCI -algebra, commutative BCK-algebra, Associative BCI – algebra, BCK/BCI- homomorphisms, bounded commutative BCI-algebra are a few points have to be addressed and have to be analyzed more.

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