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Generalized b Closed Sets and Generalized b Open Sets in Fuzzy Neutrosophic bi-Topological Spaces

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Abstract: In this paper, the authors study and introduce a new class of sets called generalized b-closed sets and its complement generalized b-open sets via fuzzy neutrosophic bi-topological spaces. Also, we prove some theorem related to this definitions. Then, we investigate the relations between the new defined sets by hand and some other fuzzy neutrosophic sets on the other hand. Some applications and many examples are presented and discussed.in fuzzy neutrosophic bi-topological spaces**.**

Keywords: fuzzy neutrosophic set; fuzzy neutrosophic bi-topology; fuzzy neutrosophic b-open set; fuzzy neutrosophic b-closed set; fuzzy neutrosophic generalized b-closed sets.

1. Introduction

At the beginning use of the concept of fuzzy sets "FS" was submitted by L. Zadeh's conference paper in 1965 [1] where each element had a degree of membership. Then many extension done by several studies. Intuitionistic fuzzy set "IFS" was one of the extension proved and known by K .Atanassov in 1983 [2- 4], when he has proved the degree of membership of an item of any set in"FS" and added a degree of non-membership in "IFS". Then many studies are being on the generalizations of the notion of "IFS", one of them proved was by F. Smarandache in 2005 [5,6], when he developed something else in membership and added indeterminacy membership between the last two membership and non-membership which were known in "IFS" and called it neutrosophic sets "NSs". After that, A Salama et.al. in 2014 [7,8] introduced neutrosophic topological spaces "NTSs".

The term of neutrosophic sets "NSs" was defined with membership, non-membership not specified degree. In the last three year ago, Veereswari [9] submitted his paper in fuzzy neutrosophic topological spaces "FNTSs" to be the solution and representation of the problems different fields where he takes all values between the closed interval 0 and 1 instead of the unitary non-standard interval]-0,1+[in NSs.

In this work, as generalized of the work of R.K. Al-Hamido [10] and the last papers which studied by F. Mohammed [11-13], we have identified a new category of fuzzy neutrosophic sets

"FNSs" called fuzzy neutrosophic generalized b-closed sets in fuzzy neutrosophic bi-topological spaces. Finally, on the basis of our manster's we will discuss some new characteristics and apply it. Finally, there are many application of NSs in many fields see [14-19], so before we ended our work we added some applications based in our new sets via fuzzy neutrosophic bi-topological spaces.

2. Preliminaries:

In this part of our study, we will refer to some basic definitions and operations which are useful in our work.

Definition 2.1 [9]: Let U be a non-empty fixed set. The fuzzy neutrosophic set "FNS" μ_N is an object having the form $\mu_N=\{< u, \lambda\mu_N(u), \gamma\mu_N(u), V\mu_N(u) > : u\in U\}$ where the functions $\lambda\mu_N(u), \gamma\mu_N(u), V\mu_N(u): U\rightarrow[0,1]$ denote the degree of membership function (namely $\lambda\mu_N(u)$), the degree of indeterminacy function (namely $\gamma\mu_N(u)$) and the degree of non-membership function (namely $V\mu_N(u)$) respectively of each element u$\in$ U to the set μ_N and $0 \leq \lambda_{\mu N}(u)+ \gamma_{\mu N}(u) +V_{\mu N}(u) \leq 3$, for each u$\in$ U.

Remark 2.2: FNS $\mu_N = \{< u, \lambda_{\mu N}(u), \sigma_{\mu N}(u), V_{\mu N}(u) >: u \in U\}$ can be identified to an ordered triple $< u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ *in [0,1] on U.*

Lemma 2.3 [9]: Let U be a non-empty set and the "FNS" μ_N and γ_N be in the form $\mu_N = \{< u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} > \}$ and $\gamma_N=\{< u, \lambda_{\gamma N}, \sigma_{\gamma N}, V_{\gamma N} >\}$ on U. Then,

i. $\mu_N \subseteq \gamma_N$ iff $\lambda_{\mu N} \leq \lambda_{\gamma N}, \sigma_{\mu N} \leq \sigma_{\gamma N}$ and $V_{\mu N} \geq V_{\gamma N}$,
ii. $\mu_N = \gamma_N$ iff $\mu_N \subseteq \gamma_N$ and $\gamma_N \subseteq \mu_N$,
iii. $(\mu_N)^c =\{< u, V_{\mu N}, 1-\sigma_{\mu N}, \lambda_{\mu N} >\}$,
iv. $\mu_N \cup \gamma_N =\{< u, Mx(\lambda_{\mu N}, \lambda_{\gamma N}), Mx(\sigma_{\mu N}, \sigma_{\gamma N}), Mn(V_{\mu N}, V_{\gamma N}) >\}$,
v. $\mu_N \cap \gamma_N =\{< u, Mn(\lambda_{\mu N}, \lambda_{\gamma N}), Mn(\sigma_{\mu N}, \sigma_{\gamma N}), Mx(V_{\mu N}, V_{\gamma N}) >\}$,
vi. $0_N = \{< u, 0, 0, 1>\}$ and $1_N = \{ < u, 1, 1, 0 >\}$.

Definition 2.4 [9]: Fuzzy neutrosophic topology (for short, FNT) on a non-empty set U is a family T_N of fuzzy neutrosophic subset in U satisfying the following axioms:

i. $0_N, 1_N \in T_N$,
ii. $\mu_{N1} \cap \mu_{N2} \in T_N \ \forall \mu_{N1}, \mu_{N2} \in T_N$,
iii. $\cup \mu_{Nj} \in T_N, \ \forall \{ \mu_{Nj} : j \in J\} \subseteq T_N$.

In this case the pair (U, T_N) is called fuzzy neutrosophic topological space (for short, FNTS). The elements of T_N are called fuzzy neutrosophic-open sets (for short, FN-OS). The complement of FN-OS in the FNTS (U, T_N) is called fuzzy neutrosophic- closed set (for short, FN-CS).

Definition 2.5 [9]: Let (U, T_N) is FNTS and $\mu_N = < u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ is FNS in U. Then the fuzzy neutrosophic-closure (for short, FN-Cl) and the fuzzy neutrosophic-interior (for short, FN-In) of μ_N are defined by:

$$FN\text{-}Cl\,(\mu_N) = \cap \{ \gamma_N : \gamma_N \text{ is FN-CS in U and } \mu_N \subseteq \gamma_N \},$$

$$FN\text{-}In(\mu_N) = \cup\{\gamma_N : \gamma_N \text{ is FN-OS in } U \text{ and } \gamma_N \subseteq \mu_N\}.$$

Now, the FN-Cl (μ_N) is FN-CS and FN-In(μ_N) is FN-OS in U.

Further,

i. μ_N is FN-CS in U iff $FN\text{-}Cl(\mu_N) = \mu_N$,

ii. μ_N is FN-OS in U iff $FN\text{-}In(\mu_N) = \mu_N$.

Definition 2.6: Let (U_N, T_{N1}, T_{N2}) is FNTS and $\mu_N = < u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ is FNS in U_N. Then the fuzzy neutrosophic semi-closure (resp. fuzzy neutrosophic Pre-closure and fuzzy neutrosophic α-closure) of μ_N and denoted by FN-SCl (μ_N) (resp. $FN\text{-}PCl(\mu_N)$ and $FN\text{-}\alpha Cl(\mu_N)$ are defined by:

$$FN\text{-}SCl(\mu_N) = \cap\{\gamma_N : \gamma_N \text{ is FN-SCS set in } U \text{ and } \mu_N \subseteq \gamma_N\} = \mu_N \cup FN\text{-}In(FN\text{-}Cl(\mu_N)),$$

$$FN\text{-}PCl(\mu_N) = \cap\{\gamma_N : \gamma_N \text{ is FN-PCS set in } U \text{ and } \mu_N \subseteq \gamma_N\} = \mu_N \cup FN\text{-}Cl(FN\text{-}In(\mu_N)),$$

$$FN\text{-}\alpha Cl(\mu_N) = \cap\{\gamma_N : \gamma_N \text{ is FN-}\alpha\text{CS set in } U \text{ and } \mu_N \subseteq \gamma_N\} = \mu_N \cup FN\text{-}Cl(FN\text{-}In(FN\text{-}Cl(\mu_N))),$$

Definition 2.7 [11, 12]: FNS λ_N in FNTS (U, T_N) is called**:**

i. Fuzzy neutrosophic-regular open set (FN-ROS) if $\mu_N = FN\text{-}In(FN\text{-}Cl(\mu_N))$,

ii. Fuzzy neutrosophic-regular closed set (FN-RCS) if $\mu_N = FN\text{-}Cl(FN\text{-}In(\mu_N)$,

iii. Fuzzy neutrosophic-semi open set (FN-SOS) if $\mu_N \subseteq FN\text{-}Cl(FN\text{-}In(\mu_N))$,

iv. Fuzzy neutrosophic-semi closed set(FN-SCS) if $FN\text{-}In(FN\text{-}Cl(\mu_N)) \subseteq \mu_N$,

v. Fuzzy neutrosophic pre-open set(FN-POS) if $\mu_N \subseteq FN\text{-}In(FN\text{-}Cl(\mu_N))$,

vi. Fuzzy neutrosophic pre-closed set(FN-PCS) if $FN\text{-}Cl(FN\text{-}In(\mu_N)) \subseteq \mu_N$,

vii. Fuzzy neutrosophic-α-open set(FN-αOS) if $\mu_N \subseteq FN\text{-}In(FN\text{-}Cl(FN\text{-}In(\mu_N)))$,

viii. Fuzzy neutrosophic-α-closed set(FN-αCS) if FN-Cl(FN-In(FN-Cl(μ_N))) ⊆ μ_N,

ix. Fuzzy neutrosophic generalized closed set (FN-GCS) if $FN\text{-}Cl(K \subseteq N)$ whenever K ⊆ N and N is a FN-OS,

x. Fuzzy neutrosophic generalized pre closed set (FN-GPCS) if $FN\text{-}PCl(K) \subseteq N$, whenever K⊆ N and N is a FN-OS,

xi. Fuzzy neutrosophic α generalized closed set (FN-αGCS) *if* $FN\alpha\text{-}Cl(K) \subseteq N$ whenever K⊆ N and N is a FN-OS,

xii. Fuzzy neutrosophic generalized semi closed set (FN-GSCS) if $FN\text{-}SCl(K) \subseteq N$, whenever K⊆ N and N is a FN-OS.

Definition 2.8 [13]: A fuzzy neutrosophic set K in FNTs (U, T_N) is called fuzzy neutrosophic b-closed set (FN-b-CS) set if and only if $FN\text{-}In(FN\text{-}Cl(K)) \vee FN\text{-}Cl(FN\text{-}In(K)) \leq K$.

Definition 2.9 [13]: Let U_N be a non-empty set and (U, T_{N1}), (U, T_{N2}) be two topological spaces then, the triple (U_N, T_{N1}, T_{N2}) is a fuzzy neutrosophic bi-topological space (for short, FN-bi-TS).

3. Generalized b-Open Sets and Generalized b-Closed Sets in Fuzzy Neutrosophic bi-Topological Spaces

In this section, we generalized our work [13] and study the concept of generalized b-closed sets and generalized b-open sets based of fuzzy neutrosophic bi- topological spaces and introduced it after giving the definition of fuzzy neutrosophic bi- topological spaces as follows:

Definition 3. 1: Let U be a non-empty set and T_{N1}, T_{N2} be two topologies on FNTS (U, T_N), then the triple (U, T_{N1}, T_{N2}) is a fuzzy neutrosophic bi- topological space (for short, FN-bi-TS).

Definition 3.2: Let U be a non-empty set and T_{N1}, T_{N2} be two topologies on FNTS (U, T_N). A subset A of U is called fuzzy neutrosophic open set (for short, FN-OS) set if $A \in T_{N1} \cup T_{N2}$. A is called fuzzy neutrosophic closed set (for short, FN-CS) if 1_N-A is FN-OS.
Note: In this work we refer to $T_{N1} \cup T_{N2}$ by T_N.

Example 3.3: Let $U = \{k_1, k_2\}$, $T_{N1} = \{0_N, 1_N\}$, $T_{N2} = \{0_N, 1_N, E_1\}$ and, $T_N = \{0_N, E_1, 1_N\}$ be a FN-bi-TS on U,
Where, $E_1 = < u, (k_{1(0.2)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.3)}, k_{2(0.5)}, k_{2(0.7)}) >$.
Then the neutrosophic set $Z = < u, (k_{1(0.7)}, k_{1(0.5)}, k_{1(0.3)}), (k_{2(0.6)}, k_{2(0.5)}, k_{2(0.4)}) >$ is a FN-b-CS in U.

Definition 3.4: Let (U, T_N) be any FN-bi-TS and $\mu_N = < u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ be FNS in U. Then the fuzzy neutrosophic-b-closure (for short, FN-bCl) and the fuzzy neutrosophic-b-interior (for short, FN-bIn) of μ_N are defined by:

$$FN\text{-}bCl\ (\mu_N) = \cap \{ \gamma_N : \gamma_N \text{ is FN-bCS in U and } \mu_N \subseteq \gamma_N \},$$
$$FN\text{-}bIn\ (\mu_N) = \cup \{ \gamma_N : \gamma_N \text{ is FN-bOS in U and } \gamma_N \subseteq \mu_N \}.$$

Definition 3.5: Let (U,T_N) be a FN-bi-TS, then, for each $\mu_1, \lambda_1 \in I^U$ the fuzzy set μ_1 is called fuzzy neutrosophic- generalized b-open set (for short, FN-gb-OS) set *if* $\mu_1 \leq FN\text{-}bIn\ (\lambda_1)$ such that $\mu_1 \leq \lambda_1$ and μ_1 is FN-CS.

Theorem 3.6: A fuzzy neutrosophic set Z of FN-bi-TS (U, T_N) is a FN-gb-OS *iff* $N \subseteq FN\text{-}bIn(Z)$ whenever N is a FN-CS and $N \subseteq Z$.
Proof: Necessity : Suppose Z is a FN-gb-OS in FN-bi-TS (U, T_N) and let E be a FN-CS and $N \subseteq Z$.
Then $H^c = 1_N$-H is a FN-OS in U such that $Z^c = 1_N\text{-}Z \subseteq N^c = 1_N$-N
$\Rightarrow$ 1_N-Z is a FN-gb-CS and FN-bCl(1_N-Z) $\subseteq$ 1_N-N ,
Hence, (1_N-FN-bIn(Z)) $\subseteq$ 1_N-N $\Rightarrow$ N $\subseteq$ FN-bIn(Z).
Sufficiency: Let Z be any FNS of U and let N $\subseteq$ FN-bIn(Z) whenever, N is a FN-CS and N $\subseteq$ Z.

Theorem 3.7: Let (U, T_N) be FN-bi-TS, then:

(1) Every FN-CS is a FN-gb-CS,

(2) Every FN-αCS is a FN-gb-CS,

(3) Every FN-PCS is a FN-gb-CS,

(4) Every FN-b-CS is a FN-gb-CS,

(5) Every FN-RCS is a FN-gb-CS,

(6) Every FN-GCS is a FN-gb-CS,

(7) Every FN-αGCS is a FN-gb-CS,

(8) Every FN-GPCS is a FN-gb-CS

(9) Every FN-SCS is a FN-gb-CS.

(10) Every FN-GSCS is FN-gb-CS.

Proof : (1): Let Z $\subseteq$ N and N be a FN-CS in FN-bi-TS (U, T_N) with FN-bCl(Z) $\subseteq$ FN-Cl(Z).
But, FN-bCl(Z) = Z $\subseteq$ N. Therefore, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(2): Let Z $\subseteq$ N and N $\in$ T_N,$\Rightarrow$ Z is a FN-αCl(Z) = Z. Therefore, FN-bCl(Z) $\subseteq$ FN-αCl(Z) = Z $\subseteq$ N.
Hence, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(3): Let Z $\subseteq$ N and N $\in$ T_N .
Since Z is a FN-PCS, and FN-Cl(FN-In(Z)) $\subseteq$ Z.
Therefore, FNCl(FN-In(Z)) ∩ FN- In(FN-Cl(Z)) $\subseteq$ FN-Cl(Z) ∩ FN-Cl(FN-In(Z)) $\subseteq$ Z.
$\Rightarrow$ FN-bCl(Z) $\subseteq$ N. Hence, Z is a FN-gb-CS in U.

(4): Let Z $\subseteq$ N and N be a FN-OS in FN-bi-TS (U, T_N)
$\Rightarrow$ Z is a FN-b-CS and FN-bCl(Z) = Z.
Therefore, FN-bCl(Z) = Z $\subseteq$ N. Hence, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(5): Let Z $\subseteq$ N and N $\in$ T_N and let Z be a FN-RCS.
But, FN-Cl(FN-In(Z)) = Z $\Rightarrow$ FN-Cl(Z) = FN-Cl(FN-In(Z)). Therefore, FN-Cl(Z) = Z.
Hence, Z is a FN-CS in U. By (1), we get Z is a FN- gb-CS in FN-bi-TS (U, T_N).

(6): Let Z $\subseteq$ N and N $\in$ T_N $\Rightarrow$ Z is a FN-GCS, FN-Cl(Z) $\subseteq$ N.
Therefore, FN-bCl(Z) $\subseteq$ FN-Cl(Z).
But, FN-bCl(Z) $\subseteq$ N. Hence, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(7): Let Z $\subseteq$ N and N $\in$ T_N $\Rightarrow$ Z is a FN-αGCS.
But, FN-αCl(Z) $\subseteq$ N. Therefore, FNbCl(Z) $\subseteq$ FN-αCl(Z),
So, FN-bCl(Z) $\subseteq$ N. Hence, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(8): Let Z $\subseteq$ N and N $\in$ T_N $\Rightarrow$ Z is a FN-gp-CS and FN-PCl(Z) $\subseteq$ N.
Therefore, FNbCl(Z) $\subseteq$ FN-pCl(Z), so FN-bCl(Z) $\subseteq$ N.

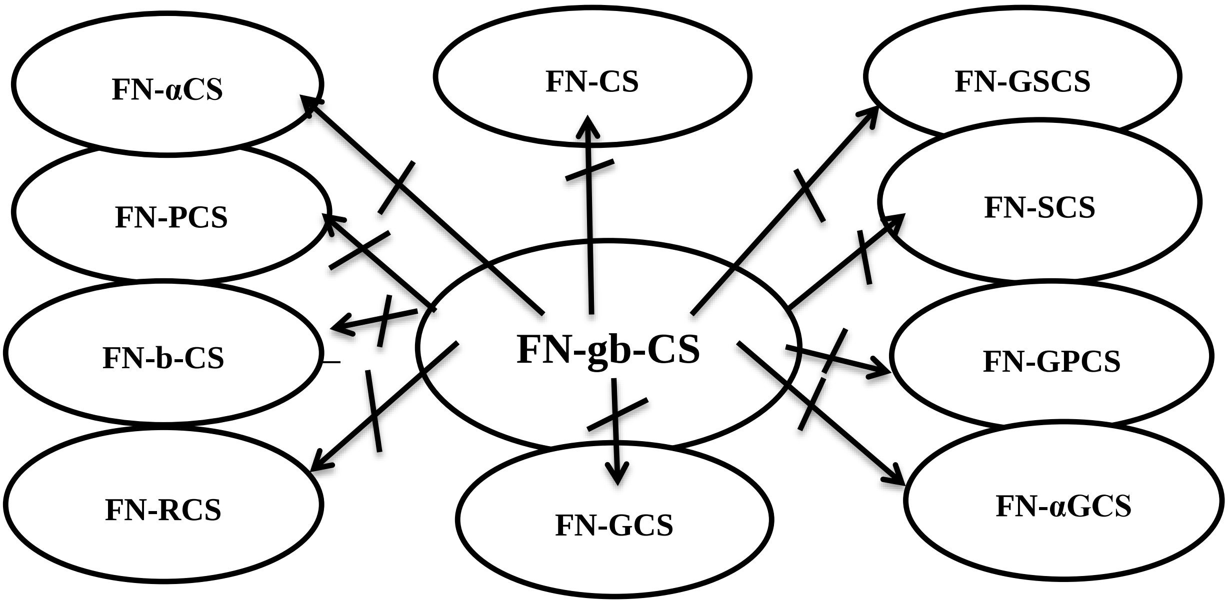
Hence, Z is a FN-gb-clos. set in FN-bi-TS (U, T_N).

(9): Let Z ⊆ N and N ∈ T_N ⟹ Z is a FN-SCS.

But, FN-bCl(Z) ⊆ FN-SCl(Z) ⊆ N. Therefore, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(10): Obivious

Proposition 3.8: The converse of theorem 3.7 is not true in general for all cases and we can see it in (Diagram 1.)



(Diagram 1)

Example 3.9: (i): Let $U = \{ k_1, k_2 \}$, $T_{N1} = \{ 0_N, 1_N\}$, $T_{N2} = \{0_N, 1_N, E_1\}$.

Then, $T_N = \{ 0_N, E_1, 1_N \}$ is a FN-bi-TS on U ,

1- Take $E_1 = < u, (k_{1(0.3)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.2)}, k_{2(0.5)}, k_{2(0.7)}) >$.

Then, the FNS $"Z" = < u, (k_{1(0.5)}, k_{1(0.5)}, k_{1(0.4)}), (k_{2(0.6)}, k_{2(0.5)}, k_{2(0.3)})$ is a FN-gb-CS but, not a FN-CS in U ⟹ FN-Cl("Z") = E_1 ≠ "Z".

2- Let $E_1 = < u, (k_{1(0.3)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.2)}, k_{2(0.5)}, k_{2(0.8)}) >$ **.** Then, the FNS "Z" = $< u, (k_{1(0.5)}, k_{1(0.5)}, k_{1(0.3)}), (k_{2(0.6)}, k_{2(0.5)}, k_{2(0.3)}) >$ is a FN-αCS in U⟹ $FN\text{-}Cl(FN\text{-}Cl(Z)) = 1_N\text{-} E_1 \nsubseteq$ "Z".

3- Let $E_1 = < u, (k_{1(0.9)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.3)}, k_{2(0.5)}, k_{2(0.7)}) >$.

Then, the FNS $"Z" = < u, (k_{1(0.4)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$ is a FN-gb-CS but, not a FN-PCS in U ⟹ FN-Cl(FN-In("Z")) = $E_1 \nsubseteq$ "Z".

4- Let $E_1 = < u, (k_{1(0.6)}, k_{1(0.5)}, k_{1(0.4)}), (k_{2(0.8)},k_{2(0.5)},k_{2(0.2)}) >$.

Then, the FNS "Z" = $< u, (k_{1(0.8)}, k_{1(0.5)}, k_{1(0.2)}), (k_{2(0.9)}, k_{2(0.5)}, k_{2(0.1)}) >$ is a FN-gb-CS but, not a FN-b-CS in FN-bi-TS (U, T_N).$\Longrightarrow$ FN-RCS is a FN-gb-CS but, not a FN-b-CS in FN-bi-TS (U, T_N),

$\Longrightarrow$ FN-Cl(FN-In("Z")) ∩ FN-In(FN-Cl("Z")) = $1_N \nsubseteq$ "Z".

5- Let $E_1 = < u, (k_{1(0.2)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.4)}, k_{2(0.5)}, k_{2(0.6)}) >$.

Then, the *FNS* "Z" = $< u, (k_{1(0.7)}, k_{1(0.5)}, k_{1(0.3)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$ is a FN-gb-CS but, not a FN-RCS in *FN-bi-TS (U, T_N)* $\Longrightarrow$ *FN-Cl(FN-In("Z")) = 1_N-$E_1 \neq$ "Z".*

6- Let $E_1 = < u, (k_{1(0.2)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.4)}, k_{2(0.5)}, k_{2(0.6)}) >$.

Then, the FNS "Z" = $< u, (k_{1(0.1)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.3)}, k_{2(0.5)}, k_{2(0.7)}) >$ is a FN-gb-CS but, not a FN-GCS in *FN-bi-TS (U, T_N).* $\Longrightarrow$ *FN-Cl("Z") = $E_1^c \nsubseteq E_1$.*

7- Let $E_1 = < u, (k_{1(0.5)}, k_{1(0.5)}, k_{1(0.4)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$.

Then, the FNS "Z" = $< u, (k_{1(0.5)}, k_{1(0.5)}, k_{1(0.5)}), (k_{2(0.3)}, k_{2(0.3)}, k_{2(0.7)}) >$ is a FN-gb-CS but, not a FN-αGCS in FN-bi-TS (U, T_N). $\Longrightarrow$ FN-Cl(FN-In(FN-Cl("Z"))) = $1_N \nsubseteq E_1$.

8- Let $E_1 = < u, (k_{1(0.9)}, k_{1(0.5)}, k_{1(0.1)}), (k_{2(0.7)}, k_{2(0.5)}, k_{2(0.2)}) >$.

Then, the FNS "Z" = $< u, (k_{1(0.7)}, k_{1(0.5)}, k_{1(0.3)}), (k_{2(0.6)}, k_{2(0.5)}, k_{2(0.4)}) >$ is a FN-gb-CS but, not a FN-SCS in FN-bi-TS (U, T_N), $\Longrightarrow$ *FN-In(FN-Cl("Z")) = $1_N \nsubseteq$ "Z".*

9- Let $E_1 = < u, (k_{1(0.8)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.0)}, k_{2(0.5)}, k_{2(0.1)}) >$.

Then, the *FNS* "Z" = $< u, (k_{1(0.6)}, k_{1(0.5)}, k_{1(0.5)}), (k_{2(0.2)}, k_{2(0.5)}, k_{2(0.3)}) >$ is a FN-gb-CS but, not a FN-GSCS in FN-bi-TS (U, T_N), $\Longrightarrow$ *FN-In(FN-Cl("Z")) = $1_N \nsubseteq$ "Z".*

10- Let $U = \{ k1, k2 \}, T_{N1} = \{0_N, E_1\}, T_{N2} = \{0_N, 1_N, E_1, E_2\} = T_N$ be a FN-bi-TS on U.

Where, $E_1 = < u, (k_{1(0.2)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.3)}, k_{2(0.5)}, k_{2(0.7)}) >$,

$E_2 = < u, (k_{1(0.4)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$.

Then, the *FNS* "Z" = $< u, (k_{1(0.4)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$ *is a FN-gb-CS but, not a FN-GPCS in U* $\Longrightarrow$ *FN-PCl("Z) = 1_N-$E_2 \nsubseteq E_2$.*

Theorem 3.10: The union of any two FN-gb-CS need not be a FN-gb-CS in general as seen from the following example:

Example 3.11: Let $U = \{ k_1, k_2 \}, T_{N1} = \{ 0_N, E_1\}$ and $T_{N2} = \{ 0_N, 1_N, E_1\} = T_N$ be a FNT on U, where

$E_1 = < u, (k_{1(0.6)}, k_{1(0.5)}, k_{1(0.4)}), (k_{2(0.8)}, k_{2(0.5)}, k_{2(0.2)}) >$.

Then, the FNS "Z" = $< u, (k_{1(0.1)}, k_{1(0.5)}, k_{1(0.9)}), (k_{2(0.8)}, k_{2(0.5)}, k_{2(0.2)}) >$,

$M = < u, (k_{1(0.6)}, k_{1(0.5)}, k_{1(0.4)}), (k_{2(0.7)}, k_{2(0.5)}, k_{2(0.3)}) >$ is a FN-gb-CS but, Z ∩ M is not a FN-gb-CS in U

$\Longrightarrow$ *FN-bCl("Z"∩M) = $1_N \nsubseteq E_1$.*

Theorem 3.12: If Z is a FN-gb-CS in FN-bi-TS (U, T_N) , such that Z ⊆ M ⊆ FN-bCl(Z) then, M is a FN-gb-CS in (U, T_N)

Proof : Let M be any FNS in a FN-bi-TS (U, T_N), such that M ⊆ N and N ∈ T_N ⟹ Z ⊆ N , since Z is a FN-gb-CS and FN-bCl(Z) ⊆ N.

By hypothesis, we have FN-bCl(M) ⊆ FNbCl(FN-bCl(Z)) = FN-bCl(Z) ⊆ N.

Hence, M is FN-gb-CS in U.

Theorem 3.13: If Z is a FN-b-OS and FN-gb-CS in FN-bi-TS (U, T_N), then Z is a FN-b-CS.

Proof : Since Z is a FN-b-OS and FN-gb-CS in FN-bi-TS (U, T_N) such that FN-bCl(Z) ⊆ Z.

But, Z ⊆ FN-bCl(Z) .

Thus, FN-bCl(Z) = Z and hence, Z is FN-b-CS in FN-bi-TS (U, T_N).

Definition 3.14: A fuzzy neutrosophic set Z is said to be a fuzzy neutrosophic generalized b open set (FN-gb-OS) in FN-bi-TS (U, T_N). If the complement 1_N-Z is a FN-gb-CS in U. The family of all FN-gb-OS of FN-bi-TS (U, T_N) is denoted by FN-gb-O (U).

Example 3.15: Let $U = \{ k_1, k_2 \}, T_{N1} = \{0_N, E_1\}, T_{N2} = \{0_N, 1_N, E_1\} = T_N$ be FN-bi-TS on U, where

$E_1 = < u, (k_{1(0.3)}, k_{1(0.5)}, k_{1(0.7)}), (k_{2(0.4)}, k_{2(0.5)}, k_{2(0.6)}) >$.

Then, the FNS $Z = < u, (k_{1(0.4)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$ is a FN-gb-OS in U.

4. Some Applications of Generalized b-Closed Sets in Fuzzy Neutrosophic bi-Topological Spaces

In [14] they propose two models for solving Neutrosophic Goal Programming Problem (NGPP), and in [15-19], we can see many applications of neutrosophic so, we will try in our study to give some application of our new studies concepts.

Definition 4.1: A FN-bi-TS (U, T_N) is called:

i. a fuzzy neutrosophic b$\frac{1}{2}$ space (for short, FN-b$\frac{1}{2}$S) if every FN-bCS is a FN-CS.

ii. a fuzzy neutrosophic gb$\frac{1}{2}$ space (for short, FN-gb$\frac{1}{2}$S) if every FN-gb-CS is a FN-CS.

iii. a fuzzy neutrosophic gbU_b space (FN-gb$_b$S) if every FN-gb-CS is a FN-b-CS.

Theorem 4.2: Every FN-gb$\frac{1}{2}$S is a FN-gbU_b S in any FN-bi-TS (U, T_N),.

Proof : Let (U, T_N) be a FN-gb$\frac{1}{2}$S and let Z be any FN-gb-CS in FN-bi-TS (U, T_N), By hypothesis, Z is a FN-CS in U.

Since every FN-CS is a FN-b-CS in U. Hence, (U, T_N) , is a FN-gbU_b S.

The converse of above theorem need not be true in general as seen from the following example:

Example 4.3: Let $U = \{ k_1, k_2 \}$, $T_{N1} = T_N = \{0_N, 1_N, E_1\}$ *and* $T_{N2} = \{ 0_N, 1_N\}$ be a FNT on U, where,

$E1 = < u, (k_{1(0.9)}, k_{1(0.5)}, k_{1(0.9)}), (k_{2(0.1)}, k_{2(0.5)}, k_{2(0.1)}) >$.

Then, the FNS "Z" $= < u, (k_{1(0.2)}, k_{1(0.5)}, k_{1(0.3)}), (k_{2(0.8)}, k_{2(0.5)}, k_{2(0.7)}) >$ is a FN-gbU_b S but, not a FN-gb$\frac{1}{2}$ S.

Theorem 4.4: Let (U, T_N) be a FN-bi-TS and (U, T_N). A FN-gb$\frac{1}{2}$S. Then we have the following statement:

i- Any union of FN-gb-CS is a FN-gb-CS.
ii- Any intersection of any FN-gb-OS is a FN-gb-OS.

Proof : (i) Let $\{N_i\}_{i \in J}$ be a collection of FN-gb-CS in a FN-gb$\frac{1}{2}$S, (U, T_N).

Therefore, every FN-gb-CS is a FN-CS.
But, the union of FN-CS is a FN-CS. Hence, the union of FN-gb-CS is a FN-gb-CS in U.

(ii) It can be proved by taking complement in **(i)**.

Theorem 4.5: A FN-bi-TS (U, T_N) is a FN-gbU_b S if and only if FN-gb(U) = FNb-O (U)

Proof : **Necessity** : Let "Z" be a FN-gb-OS in a FN-bi-TS (U, T_N). Then, 1_N-Z is a FN-gb-CS.
By hypothesis , 1_N-Z is a FN-b-CS in U. Therefore, Z is a FN-b-OS
Hence, FN-gb-O(U) = FNb-O (U).

Sufficiency : Let Z be a FN-gb-CS in any FN-bi-TS (U, T_N). Then, 1_N-Z is a FN-gb-OS in U.
By hypothesis , 1_N-Z is a FN-b-OS in U.
Therefore, Z is a FN-b-CS in U. Hence, (U, T_N) is a FN-gbU_bS.

Theorem 4.8: A FN-bi-TS (U, T_N) is a FN-gb$\frac{1}{2}$ if and only if FN-gb-O(U) = FN-O(U).

Proof : **Necessity** : Let Z be a FN-gb-OS in a FN-bi-TS (U, T_N). Then 1_N-Z is a FN-gb-CS in U.
By hypothesis, 1_N-Z is a FN-CS in U. Therefore, Z is a FN-OS in U.
Hence, FN-gb-O(U) = FN-O(U)

Sufficiency : Let Z be a FN-gb-CS. Then, 1_N-Z is a FN-gb-OS in U. By hypothesis, 1_N-Z is a FN-OS in U. Therefore, Z is a FN-CS in U. Hence, (U, T_N) is a FN-gb$\frac{1}{2}$.

5. Conclusions

In this paper, the new concept of a new class of sets was studied and called fuzzy neutrosophic generalized b-closed sets and its complement fuzzy neutrosophic generalized b-open sets. We investigated the relations between fuzzy neutrosophic generalized b closed sets and other fuzzy neutrosophic sets such as α closed sets, regular closed sets, semi closed sets pre closed sets, generalized closed sets, b closed sets, α generalized closed sets and semi generalized closed sets based of fuzzy neutrosophic bi-topological spaces and applied some new spaces to be applications of the new defined sets.

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