Neutrosophic Sets and Systems, Vol. 35, 2020

University of New Mexico

HESITANT Triangular Neutrosophic Numbers and Their Applications to MADM

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ABSTRACT: Hesitant neutrosophic sets can accomodate more uncertainty compare to hesitant fuzzy sets and hesitant intuitionistic sets. On the other hand, triangular neutrosophic numbers are often used by the decision makers to evaluate their opinion in multi-attribute group decision making problems. Based on the combination of triangular neutrosophic numbers and hesitant neutrosophic sets, in this paper, we propose hesitant triangular neutrosophic numbers. Also, we discuss various types of operations between them including some properties. Then, we propose various types of hesitant triangular neutrosophic weighted aggregation operators to aggregate the hesitant triangular neutrosophic information. Furthermore, we introduce score of hesitant triangular neutrosophic numbers to ranking the hesitant triangular neutrosophic numbers. Based on the hesitant triangular neutrosophic weighted aggregation operators and score of hesitant triangular neutrosophic numbers, we develop a multi attribute decision making (MADM) approach, in which the evaluation values of alternatives on the attribute are represented in terms of hesitant triangular neutrosophic numbers and the alternatives are ranked according to the values of the score of hesitant triangular neutrosophic numbers to select the most desirable one. Finally, we give a practical example, including a comparision study with the other existing method, for enterprise resource planning system selection to verify the application and effectiveness of the proposed method.

Keywords: Neutrosophic sets**,** hesitant triangular neutrosophic numbers, aggregation operators, score value, decision making.

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1. INTRODUCTION

In our real life, most of the mathematical problems do not contain exact or complete information about the given mathematicalmodeling. Therefore, fuzzy set theory by introduced Zadeh **[01]** is a proper tool to process inexact information because it allows the partial belongings of an element in a set with a membership function. Atanassov **[02]** generalized fuzzy sets to intuitionistic fuzzy sets by adding a non-membership function to overcome problems that contain incomplete information. In case of fuzzy sets and intuitionistic fuzzy sets, the membership (or non-membership) value of an element in a set is a unique value in the closed interval [0, 1]. But since 2009, researchers begin to investigate, what if the membership (non-membership) value of an element in a set is a discrete finite subset of [0, 1]. In order to tackle this situation, Torra **[03]** proposed the concept of a hesitant fuzzy set, which as an extension of a fuzzy set arises from our hesitation among a few different values lying between the number 0 and 1. Thus the hesitant fuzzy set can more accurately reflect the people's hesitancy in stating their preferences over objectives compared to the fuzzy set and its classical extensions. Beg and Rashid **[04]** introduced the concept of intuitionistic hesitant fuzzy sets by merging the concept of intuitionistic fuzzy sets and hesitant fuzzy sets.Various researchers have analyzed the decision making problems under fuzzy, hesitant fuzzy, intuitionistic fuzzy and intuitionistic hesitant fuzzy environment in Li **[05]**, Ye **[06]**, Xia and Xu **[07]**, Xu and Xia **[08]**, Wei et al. **[09]**, Xu and Xia **[10]**, Xu and Xia **[11]**, Xu and Zhang **[12]**, Chen et al. **[13]**, Qian et al. **[14]**, Yu **[15]**, Yu **[16]**, Ye **[17]**, Shi et al. **[18]**, Pathinathan and Johnson **[19]**, Joshi and Kumar **[20]**, Liu **[21]**, Nehi **[22]**, Zhang **[23]**, Chen and Huang **[24]**, Yang et al. **[25]**, Lan et al. **[26]** and Zhang et al. **[27]**.

Although intuitionistic fuzzy sets naturally include hesitancy degree to handle uncertain information, it cannot manage indeterminate information properly because it is dependent on memebership and non-membership degrees. To handle this situation, Smarandache **[28]** introduced the neutrosophic set which is basically a powerful general formal framework that generalizes the concept of the classical set, fuzzy set, intuitionistic fuzzy set. A neutrosophic set is characterized explicitly by truth-membership function, indeterminacy-membership function and falsity membership function and it has applications on image segmentation in Gou and Cheng **[29],** Gou and Sensur **[30]**, on clustering analysis in Karaaslan **[31]**, on medical diagnosis problem in Ansari et al. **[32]** etc.The neutrosophic set theory have also studied in Wang et al. **[33]**, Wang et al. **[34]**, Gou et al. **[35]**, Ye **[36]**, Sun et al. **[37]**, Ye **[38]** and Abdel Basset et al. **[39]**. The neutrosophic set cannot represent uncertain, imprecise, incomplete and inconsistent information with a few different values assigned by truth-membership degree, indeterminacy-membership degree and falsity-membership degree due to doubts of decision maker. In such a situation, all the decision making algorithms based on neutrosophic sets are difficult to use for such a decision making problem with three kinds of hesitancy information that exists in the real world. To overcome this situation, Ye **[40]** introduced the concept of hesitant neutrosophic sets which is characterized by three membership degrees, namely-truth membership degrees, indeterminacy membership degrees and falsity membership degrees which is a few different values lying between the number 0 and 1.

Aggregation operators play a vital role in many fields such as decision making, supply chain, personnel evaluation and financial investment to solve multi-criteria group decision making problems. A series of aggregation operatorsin Xia et al. **[41]**, Wang et al. **[42]**, Zhao et al. **[43]**, and Peng **[44]** were developed based on fuzzy and hesitant fuzzy information and those were applied in solving decision-making problems. Xu **[45]**, Wan and Dong **[46]**, Wan et al. **[47]** and Xu and Yager **[48]** presented an averaging and geometric aggregation operators for aggregating the different intuitionistic fuzzy sets based information. Wang and Liu **[49]** proposed some Einstein weighted geometric operators for intuitionistic fuzzy sets. Liu et al. **[50]** proposed some generalized neutrosophic number Hamacher aggregation operators. Liu and Wang **[51]** defined few neutrosophic normalized, weighted Bonferroni mean operators.Chen and Ye **[52]** used single-valued neutrosophic dombi weighted aggregation operators for solving a multiple attribute decision-making problem. Some more aggregation operators on neutrosophic environment can be found in Zhao et al. **[53]**, Liu and Shi **[54]** and Liu and Tang **[55]**.

Since Smarandache put forward the concept of neutrosophic sets, the neutrosophic number is given by Şubaş **[56]** subsequently, and it has been made much deeper by many authors in Abdel-Basset **[57]**. As a special neutrosophic number,Şubaş gave two special forms of single valued neutrosophic numbers such as single valued trapezoidal neutrosophic numbers and single valued triangular neutrosophic numbers on the real number set R. Now the theory of neutrosphic number has become the fundamental of neutrosophic decision making. For example; Deli and Şubaş **[58]** introduced the concepts of cut sets of neutrosophic numbers and also they applied to single valued trapezoidal neutrosophic numbers and triangular neutrosophic numbers. Finally they presented a ranking method by defining the values and ambiguities ofneutrosophic numbers. Also, by using the value and ambiguity index, Biswas et al. **[59]** presented a multi-attribute decision making method. Broumi et al. **[60]** gave an application shortest path problem under triangular fuzzy neutrosophic numbers. Deli and Şubaş **[61]** developed an approach to handle multicriteriadecision making problems under the single valued triangular neutrosophic numbers. Also, they presented some new geometric operators including weighted geometric operator, ordered weighted geometric operator and ordered hybrid weighted geometric operator. Ye **[62]**, Biswas et al. **[63]** and Deli **[64]** proposed some weighted arithmetic operators and weighted geometric operators to present some multi attribute decision making methods. Karaaslan **[65]** introduced Gaussiansingle valued neutrosophic numbers and applied to a multi attribute decision making. Öztürk **[66]** and Deli and Öztürk **[67, 68]** initiated concept of distance measure based on cut sets, magnitude function, 1. and 2. centroid point and 1. and 2. score function. Deli **[69]** defined concept of centroid point based on single valued trapezoidal neutrosophic numbers and examine several useful properties. Also, he developed hamming ranking value and Euclidean ranking value of single valued trapezoidal neutrosophic numbers. Chakraborty et al. **[70]** presented a decision making method by introducing different forms of triangular neutrosophic numbers including de-neutrosophication techniques. Fan et al. **[71]** defined linguistic neutrosophic number Einstein sum, linguistic neutrosophic number Einstein product, and linguistic neutrosophic number Einstein exponentiation operations based on the Einstein operation and used them to develop some MADM problems. Garg and Nancy **[72]** introduced some linguistic single valued neutrosophic power aggregation operators and presented their applications to group decision making process. Zhao et al. **[73]** developed induced choquet integral aggregation operators with single valued neutrosophic uncertain linguistic numbers. Recently, Deli and Karaaslan **[74]** defined generalized trapezoidal hesitant fuzzy numbers and Deli **[75]** presented a TOPSIS method formulti-criteria decision making problems by using the numbers. Some more trapezoidal/triangular hesitant fuzzy numbers can be found in Zhang et al. **[76]** and Ye **[77]**.

Motivated by the idea of triangular neutrosophic number, hesitant neutrosophic set and aggregation operators, the aim of this present article is:

(1) To present the idea of hesitant triangular neutrosophic numbers.

(2) To define few operations between hesitant triangular neutrosophic numbers and study their basic properties.

(3) To develop a few weighted aggregation operators such as hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-1, hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-2, hesitant triangular neutrosophic weighted geometric aggregation operator of type-1 and hesitant triangular neutrosophic weighted geometric aggregation operator of type-2.

(4) To propose a decision making method based on the hesitant triangular neutrosophic weighted aggregation operators to handle multicriteria decision making problems with hesitant triangular neutrosophic information.

To do so, the rest of the article is arranged as follows:

In section 2, we review some basic concepts. In Section 3, we propose hesitant triangular neutrosophic number and illustrate it with an example. Also, we discuss various types of operations between them including some properties. In section 4, we propose various types of hesitant triangular neutrosophic weighted aggregation operators to aggregate the hesitant triangular neutrosophic information. Furthermore, we introduce the score of a hesitant triangular neutrosophic number to ranking the hesitant triangular neutrosophic numbers. In section 5, based on the hesitant triangular neutrosophic weighted aggregation operators and score of hesitant triangular neutrosophic numbers, we develop a multi attribute decision making approach, in which the evaluation values of alternatives on the attribute are represented in terms of hesitant triangular neutrosophic numbers and the alternatives are ranked according to the values of the score of hesitant triangular neutrosophic numbers to select the best (most desirable) one. Also, we present a practical example for enterprise resource planning system selection to demonstrate the application and effectiveness of the proposed method. Section 6 is devoted for comparative study. In final section, we present the conclusion of the study.

2. PRELIMINARIES:

A neutrosophic set is a part of neutrosophy which studies the origin, nature and scope of neutralities as well as their interactions with different ideational spectra and is a powerful general formal framework that generalizes the traditional mathematical tools such as fuzzy sets and intuitionistic fuzzy sets.

Definition 1: [34] A single-valued neutrosophic set A on universe set E is given by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\}$$

where $T_A: E \to [0,1]$, $I_A: E \to [0,1]$, and $F_A: E \to [0,1]$ satisfy the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$, for every $x \in E$. The functions T_A, I_A, and F_A define the degree of truth-membership function, indeterminacy-membership function and falsity-membership function, respectively.

Definition 2: [52] $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\}$ and $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle : x \in E\}$ be two single-valued neutrosophic sets and $\lambda \neq 0$. Then,

1. $$A+B=\{< x, 1-\frac{1}{1+\left\{\left(\frac{T_A(x)}{1-T_A(x)}\right)^p+\left(\frac{T_B(x)}{1-T_B(x)}\right)^p\right\}^{\frac{1}{p}}}, \frac{1}{1+\left\{\left(\frac{1-I_A(x)}{I_A(x)}\right)^p+\left(\frac{1-I_B(x)}{I_B(x)}\right)^p\right\}^{\frac{1}{p}}}, \frac{1}{1+\left\{\left(\frac{1-F_A(x)}{F_A(x)}\right)^p+\left(\frac{1-F_B(x)}{F_B(x)}\right)^p\right\}^{\frac{1}{p}}} >: x \in E\}$$

2. $$A\times B=\{< x, \frac{1}{1+\left\{\left(\frac{1-T_A(x)}{T_A(x)}\right)^p+\left(\frac{1-T_B(x)}{T_B(x)}\right)^p\right\}^{\frac{1}{p}}}, 1-\frac{1}{1+\left\{\left(\frac{I_A(x)}{1-I_A(x)}\right)^p+\left(\frac{I_B(x)}{1-I_B(x)}\right)^p\right\}^{\frac{1}{p}}}, 1-\frac{1}{1+\left\{\left(\frac{F_A(x)}{1-F_A(x)}\right)^p+\left(\frac{F_B(x)}{1-F_B(x)}\right)^p\right\}^{\frac{1}{p}}} >: x \in E\}$$

3. $$\lambda.A=\{< x, 1-\frac{1}{1+\left\{\lambda\left(\frac{T_A(x)}{1-T_A(x)}\right)^p\right\}^{\frac{1}{p}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-I_A(x)}{I_A(x)}\right)^p\right\}^{\frac{1}{p}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-F_A(x)}{F_A(x)}\right)^p\right\}^{\frac{1}{p}}} >: x \in E\}$$

4. $A^{\lambda}=\{<x,\dfrac{1}{1+\left\{\lambda\left(\dfrac{1-T_A(x)}{T_A(x)}\right)^p\right\}^{\frac{1}{p}}},1-\dfrac{1}{1+\left\{\lambda\left(\dfrac{I_A(x)}{1-I_A(x)}\right)^p\right\}^{\frac{1}{p}}},\ 1-\dfrac{1}{1+\left\{\lambda\left(\dfrac{F_A(x)}{1-F_A(x)}\right)^p\right\}^{\frac{1}{p}}}>:x\in E\}$

By combining single-valued neutrosophic sets and hesitant fuzzy sets, Ye (2015a) introduced the single-valued neutrosophic hesitant fuzzy set as a further generalization of the concepts of fuzzy set, intuitionistic fuzzy set, single-valued neutrosophic set. He also developed single-valued neutrosophic hesitant fuzzy weighted averaging operator and single-valued neutrosophic hesitant fuzzy weighted geometric operator and applied them to solve a multiple-attribute decision- making problem.

Definition 3: **[40]** A hesitant neutrosophicset on universe set E is given by

$$\mathrm{N}=\left\{\langle \mathrm{x},\tilde{\mathrm{T}}_{\mathrm{N}}(\mathrm{x}),\tilde{\mathrm{I}}_{\mathrm{N}}(\mathrm{x}),\tilde{\mathrm{F}}_{\mathrm{N}}(\mathrm{x})\rangle:\mathrm{x}\in\mathrm{E}\right\}$$

in which $\tilde{\mathrm{T}}_{\mathrm{N}}(\mathrm{x}),\tilde{\mathrm{I}}_{\mathrm{N}}(\mathrm{x})$ and $\tilde{\mathrm{F}}_{\mathrm{N}}(\mathrm{x})$are three sets of some values in [0,1], denoting the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees of the element $\mathrm{x}\in\mathrm{E}$ to the set N, respectively, with the conditions $0\le\delta\ ,\gamma\ ,\eta\ \le 1$and $0\le\delta^{+}+\gamma^{+}+\eta^{+}\le 3$, where

$\delta\in\tilde{\mathrm{T}}_{\mathrm{N}}(\mathrm{x}),\ \ \gamma\in\tilde{\mathrm{I}}_{\mathrm{N}}(\mathrm{x}),\eta\ \in\tilde{\mathrm{F}}_{\mathrm{N}}(\mathrm{x}),\ \delta^{+}\in\tilde{\mathrm{T}}_{\mathrm{N}}^{+}(\mathrm{x})=\cup_{\ \delta\in\tilde{\mathrm{T}}_{\mathrm{N}}(\mathrm{x})}\max\{\delta\},\gamma^{+}\in$
$\tilde{\mathrm{I}}_{\mathrm{N}}^{+}(\mathrm{x})=\cup_{\gamma\in\tilde{\mathrm{I}}_{\mathrm{N}}(\mathrm{x})}\max\{\gamma\}$, and$\eta^{+}\ \in\tilde{\mathrm{F}}_{\mathrm{N}}^{+}(\mathrm{x})=\cup_{\eta\ \in\tilde{\mathrm{F}}_{\mathrm{N}}(\mathrm{x})}\max\{\eta\ \}$, for $\mathrm{x}\in\mathrm{E}$.

For $\mathrm{N}_1=\left\{\langle \mathrm{x},\tilde{\mathrm{T}}_{\mathrm{N}_1}(\mathrm{x}),\tilde{\mathrm{I}}_{\mathrm{N}_1}(\mathrm{x}),\tilde{\mathrm{F}}_{\mathrm{N}_1}(\mathrm{x})\rangle:\mathrm{x}\in\mathrm{E}\right\}$ and $\mathrm{N}_2=\left\{\langle \mathrm{x},\tilde{\mathrm{T}}_{\mathrm{N}_2}(\mathrm{x}),\tilde{\mathrm{I}}_{\mathrm{N}_2}(\mathrm{x}),\tilde{\mathrm{F}}_{\mathrm{N}_2}(\mathrm{x})\rangle:\mathrm{x}\in\mathrm{E}\right\}$ be two hesitant neutrosophicsets and $\lambda\neq 0$. Then,

1. $N_1\oplus N_1=\{<x,\tilde{T}_{N_1}(x)\oplus\tilde{T}_{N_2}(x),\tilde{I}_{N_1}(x)\oplus\tilde{I}_{N_2}(x),\tilde{F}_{N_1}(x)\oplus\tilde{F}_{N_2}(x)>:x\in X\}$
$$=\bigcup_{\delta_1\in\tilde{T}_{N_1}(x),\delta_2\in\tilde{T}_{N_2}(x),\gamma_1\in\tilde{I}_{N_1}(x),\gamma_2\in\tilde{I}_{N_2}(x),\eta_1\in\tilde{F}_{N_1}(x),\eta_2\in\tilde{F}_{N_2}(x)}\{<x,\{\delta_1+\delta_2-\delta_1.\delta_2\},\{\gamma_1.\gamma_2\},\{\eta_1\eta_2\}>:x\in X\}$$

2. $N_1\otimes N_1=\{<x,\tilde{T}_{N_1}(x)\otimes\tilde{T}_{N_2}(x),\tilde{I}_{N_1}(x)\otimes\tilde{I}_{N_2}(x),\tilde{F}_{N_1}(x)\otimes\tilde{F}_{N_2}(x)>:x\in X\}$
$$=\bigcup_{\delta_1\in\tilde{T}_{N_1}(x),\delta_2\in\tilde{T}_{N_2}(x),\gamma_1\in\tilde{I}_{N_1}(x),\gamma_2\in\tilde{I}_{N_2}(x),\eta_1\in\tilde{F}_{N_1}(x),\eta_2\in\tilde{F}_{N_2}(x)}\{<x,\{\delta_1.\delta_2\},\{\gamma_1+\gamma_2-\gamma_1.\gamma_2\},\{\eta_1+\eta_2-\eta_1\eta_2\}>:x\in X\}$$

3. $\lambda.N_1=\bigcup_{\delta_1\in\tilde{T}_{N_1}(x),\gamma_1\in\tilde{I}_{N_1}(x),\eta_1\in\tilde{F}_{N_1}(x)}\{<x,\{1-(1-\delta_1)^{\lambda}\},\{\gamma_1^{\lambda}\},\{\eta_1^{\lambda}\}>:x\in X\}(\lambda>0)$

4. $N_1^{\lambda}=\bigcup_{\delta_1\in\tilde{T}_{N_1}(x),\gamma_1\in\tilde{I}_{N_1}(x),\eta_1\in\tilde{F}_{N_1}(x)}\{<x,\{\delta_1^{\lambda}\},\{1-(1-\gamma_1)^{\lambda}\},\{1-(1-\eta_1)^{\lambda}\}>:x\in X\}(\lambda>0)$

Definition 4: [56] Let $a_1\le b_1\le c_1$ such that $a_1,b_1,c_1\in R.$ A triangular neutrosophic number $\tilde{A}=\langle(a_1,b_1,c_1);w_{\tilde{A}},u_{\tilde{A}},y_{\tilde{A}}\rangle$ is a special neutrosophic set on the real number set R, whose truth-membership function $\mu_{\tilde{A}}:R\to\lfloor 0,w_{\tilde{A}}\rfloor$, indeterminacy-membership function $\nu_{\tilde{A}}:R\to\lfloor u_{\tilde{A}},1\rfloor$ and falsity-membership function $\lambda_{\tilde{A}}:R\to\lfloor y_{\tilde{A}},1\rfloor$ are given as follows;

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{(x-a_1)w_{\tilde{A}}}{b_1-a_1}, & a_1\le x\le b_1\\ \dfrac{(c_1-x)w_{\tilde{A}}}{c_1-b_1}, & b_1\le x\le c_1\\ 0, & otherwise\end{cases}\quad,\quad \nu_{\tilde{A}}(x)=\begin{cases}\dfrac{b_1-x+u_{\tilde{A}}\left(x-a_1\right)}{b_1-a_1}, & a_1\le x\le b_1\\ \dfrac{x-b_1+u_{\tilde{A}}\left(c_1-x\right)}{c_1-b_1}, & b_1\le x\le c_1\\ 1, & otherwise\end{cases}$$

$$\lambda_{\tilde{A}}(x)=\begin{cases}\dfrac{b_1-x+y_{\tilde{A}}(x-a_1)}{b_1-a_1}, & a_1\le x\le b_1\\ \dfrac{x-b_1+y_{\tilde{A}}(c_1-x)}{c_1-b_1}, & b_1\le x\le c_1\\ 1, & otherwise\end{cases}$$

Since triangular neutrosophic numbers (**[56]**, **[58]**) is a special case of trapozidial neutrosophic numbers (Ye 2017), operations of trapozidial neutrosophic numbers (Ye 2015b, 2017) based on algebraic sum and algebraic product for triangular neutrosophic numbers can be given as;

If $\tilde{A}=\langle(a_1,b_1,c_1);w_{\tilde{A}},u_{\tilde{A}},y_{\tilde{A}}\rangle$ and $\tilde{B}=\langle(a_2,b_2,c_2);w_{\tilde{B}},u_{\tilde{B}},y_{\tilde{B}}\rangle$ be two triangular neutrosophic numbers and $\gamma\neq 0$, then we have

1. $\tilde{A}+\tilde{B}=\langle(a_1+a_1,b_1+b_2,c_1+c_2);w_{\tilde{A}}+w_{\tilde{B}}-w_{\tilde{A}}.w_{\tilde{B}},u_{\tilde{A}}.u_{\tilde{B}},y_{\tilde{A}}.y_{\tilde{B}}\rangle$
2. $\tilde{A}.\tilde{B}=\langle(a_1a_2,b_1b_2,c_1c_2);w_{\tilde{A}}.w_{\tilde{B}},u_{\tilde{A}}+u_{\tilde{B}}-u_{\tilde{A}}.u_{\tilde{B}},y_{\tilde{A}}+y_{\tilde{B}}-y_{\tilde{A}}.y_{\tilde{B}}\rangle$
3. $\lambda\tilde{A}=\langle(\gamma a_1,\gamma b_1,\gamma c_1);1-(1-w_{\tilde{A}})^{\lambda},u_{\tilde{A}}{}^{\lambda},y_{\tilde{A}}{}^{\lambda}\rangle$
4. $\tilde{A}^{\lambda}=\langle(a_1{}^{\lambda},b_1{}^{\lambda},c_1{}^{\lambda});w_{\tilde{A}}{}^{\lambda},1-(1-u_{\tilde{A}})^{\lambda},1-(1-y_{\tilde{A}})^{\lambda}\rangle$

Definition 5: [56] Let $\tilde{A}=\langle(a,b,c);w_{\tilde{A}},u_{\tilde{A}},y_{\tilde{A}}\rangle$ be atriangular neutrosophic number. Then,score function of $\tilde{A}$, is denoted by $S_Y(\tilde{A})$, is defined as:

$$S_Y(\tilde{A})=\frac{1}{8}[a+b+c]\times(2+\mu_{\tilde{A}}-\nu_{\tilde{A}}-\gamma_{\tilde{A}})$$

Definition 6: [61] Let $\tilde{A}_j=\langle(a_j,b_j,c_j,);w_{\tilde{A}_j},u_{\tilde{A}_j},y_{\tilde{A}_j}\rangle$ $(j=1,2,\dots,n)$ be a collection of triangular neutrosophic numbers. Then,

1. Triangular neutrosophic weighted arithmetic operatoris defined as;

$$N_{ao}(\tilde{A}_1,\tilde{A}_2,\dots,\tilde{A}_n)=\sum_{j=1}^{n}w_j\tilde{A}_j$$

2. Triangular neutrosophic weighted geometric operatoris defined as;

$$N_{go}(\tilde{A}_1,\tilde{A}_2,\dots,\tilde{A}_n)=\prod_{j=1}^{n}\tilde{A}_j{}^{w_j}$$

where, $w=(w_1,w_2,\dots,w_n)^T$ is a weight vector associated with the N_{ao} or N_{go} operator, for every j ($j=1,2,\dots,n$) and $w_j\in[0,1]$with$\sum_{j=1}^{n}w_j=1$.

3. HESITANT TRIANGULAR NEUTROSOPHIC NUMBERS:

In this section, the concept of a hesitant triangular neutrosophic number is presented on the basis of the combination of triangular neutrosophic numbers and hesitant fuzzy sets as a further generalization of the concep ttriangular neutrosophic numbers. A hesitant triangular neutrosophic number is a special hesitant neutrosophic set on the real number R, whose truth-membership function, indeterminacy-membership function and falsity-membership function are expressed by several possible functions.

Definition 7. Let $a_1\le b_1\le c_1$ such that $a_1,b_1,c_1\in R$, $w_{\tilde{a}}^i\in[0,1](i\in I_m=\{1,2,\dots,m\})$, $u_{\tilde{a}}^i\in[0,1](i\in I_n=\{1,2,\dots,n\})$ and $y_{\tilde{a}}^i\in[0,1](i\in I_k=\{1,2,\dots,k\})$. A hesitant triangular neutrosophic number $\tilde{a}=<(a_1,b_1,c_1);\{w_{\tilde{a}}^i:i\in I_m\},\{u_{\tilde{a}}^j:j\in I_n\},\{y_{\tilde{a}}^l:l\in I_k\}>$ is a special hesitant neutrosophic set on the real number R, whose truth-membership functions $\mu_{\tilde{a}}^{HTRI}:R\to\left[0,w_{\tilde{a}}^i\right](i\in I_m)$, indeterminacy-membership function $\gamma_{\tilde{a}}^{HTRI}:R\to\left[0,u_{\tilde{a}}^j\right](j\in I_n)$ and falsity-membership function $\delta_{\tilde{a}}^{HTRI}:R\to\left[0,y_{\tilde{a}}^l\right](l\in I_k)$ are given as follows;

$$\mu_{\tilde{a}}^{HTRI}=\begin{cases}\bigcup_{\{\alpha:\alpha\in\{w_{\tilde{a}}^{i}:i\in I_m\}\}}\{\frac{(x-a_1)\alpha}{b_1-a_1}\}, & a_1\le x<b_1\\ \{\alpha:\alpha\in\{w_{\tilde{a}}^{i}:i\in I_m\}\}, & x=b_1\\ \bigcup_{\{\alpha:\alpha\in\{w_{\tilde{a}}^{i}:i\in I_m\}\}}\{\frac{(c_1-x)\alpha}{c_1-b_1}\}, & b_1<x\le c_1\\ \{0\}, & \text{otherwise}\end{cases},$$

$$\gamma_{\tilde{a}}^{HTRI}=\begin{cases}\bigcup_{\{\beta:\beta\in\{u_{\tilde{a}}^{j}:j\in I_n\}\}}\{\frac{b_1-x+\beta(x-a_1)}{b_1-a_1}\}, & a_1\le x<b_1\\ \{\beta:\beta\in\{u_{\tilde{a}}^{j}:j\in I_n\}\}, & x=b_1\\ \bigcup_{\{\beta:\beta\in\{u_{\tilde{a}}^{j}:j\in I_n\}\}}\{\frac{x-b_1+\beta(c_1-x)}{c_1-b_1}\}, & b_1<x\le c_1\\ \{1\}, & \text{otherwise}\end{cases}$$

$$\delta_{\tilde{a}}^{HTRI}=\begin{cases}\bigcup_{\{\lambda:\lambda\in\{y_{\tilde{a}}^{l}:l\in I_k\}\}}\{\frac{b_1-x+\lambda(x-a_1)}{b_1-a_1}\}, & a_1\le x<b_1\\ \{\lambda:\lambda\in\{y_{\tilde{a}}^{l}:l\in I_k\}\}, & x=b_1\\ \bigcup_{\{\lambda:\lambda\in\{y_{\tilde{a}}^{l}:l\in I_k\}\}}\{\frac{x-b_1+\lambda(c_1-x)}{c_1-b_1}\}, & b_1<x\le c_1\\ \{1\}, & \text{otherwise}\end{cases}$$

Example 8. $\tilde{a}=<(1,2,5);\{0.8,0.9\},\{0.4,0.5,0.6\},\{0.4\}>$ is a hesitant triangular neutrosophic number whose truth membership function, indeterminacy membership function and falsity membership functionare given respectively by:

$$\mu_{\tilde{a}}^{HTRI}(x)=\begin{cases}\{0.8(x-1),0.9(x-1)\}, & 1\le x<2\\ \{0.8,0.9\}, & x=2\\ \{0.8\frac{(5-x)}{3},0.9\frac{(5-x)}{3}\}, & 2<x\le 5\\ \{0\}, & \text{otherwise}\end{cases},\qquad \delta_{\tilde{a}}^{HTRI}(x)=\begin{cases}\{1.6-0.6x\}, & 1\le x<2\\ \{0.4\}, & x=2\\ \{0.2x\}, & 2<x\le 3\\ \{1\}, & \text{otherwise}\end{cases}$$

$$\gamma_{\tilde{a}}^{HTRI}(x)=\begin{cases}\{1.6-0.6x,1.5-0.5x,1.4-0.4x\}, & 1\le x<2\\ \{0.4,0.5,0.6\}, & x=2\\ \{\frac{0.6x}{3},0.\frac{5x+0.5}{3},\frac{0.4x+1}{3}\}, & 2<x\le 5\\ \{1\}, & \text{otherwise}\end{cases}$$

4. OPERATIONS ON HESITANT TRIANGULAR NEUTROSOPHIC NUMBERS:

In this section, we introduce various operations between hesitant triangular neutrosophic numbers and demonstrate their basic properties.

Definition 9. Let $\tilde{a}=<(a_1,b_1,c_1);\{w_{\tilde{a}}^{i}:i\in I_{m_1}\},\{u_{\tilde{a}}^{j}:j\in I_{n_1}\},\{y_{\tilde{a}}^{l}:l\in I_{k_1}\}>$ and $\tilde{b}=<(a_2,b_2,c_2);\{w_{\tilde{b}}^{i}:i\in I_{m_2}\},\{u_{\tilde{b}}^{j}:j\in I_{n_2}\},\{y_{\tilde{b}}^{l}:l\in I_{k_2}\}>$ be two hesitant triangular neutrosophic numbers and $\sigma>0$, then

1. $\tilde{a}\oplus'\tilde{b}=<(a_1+a_2,b_1+b_2,c_1+c_2);\{\alpha_1+\alpha_2-\alpha_1\alpha_2:\alpha_1\in\{w_{\tilde{a}}^{i}:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^{i}:i\in I_{m_2}\}\},$
$\{\beta_1\beta_2:\beta_1\in\{u_{\tilde{a}}^{j}:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^{j}:j\in I_{n_2}\}\},\ \{\lambda_1\lambda_2:\lambda_1\in\{y_{\tilde{a}}^{l}:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^{l}:l\in I_{k_2}\}\}>$

$$2.\ a\otimes'\tilde{b}=<(a_1a_2,b_1b_2,c_1c_2);\{\alpha_1\alpha_2:\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\}\},\{\beta_1+\beta_2-\beta_1\beta_2:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\}\},\{\lambda_1+\lambda_2-\lambda_1\lambda_2:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\}\}>$$

$$3.\ \sigma\odot'\tilde a=<(\sigma a,\sigma b,\sigma c);\{1-(1-\alpha)^\sigma:\alpha\in\{w_{\tilde a}^i:i\in I_{m_1}\}\},\{\beta^\sigma:\beta\in\{u_{\tilde a}^j:j\in I_{n_1}\}\},\{\lambda^\sigma:\lambda\in\{y_{\tilde a}^l:l\in I_{k_1}\}\}>$$

$$4.\ \sigma*'\tilde a=<(a^\sigma,b^\sigma,c^\sigma);\{\alpha^\sigma:\alpha\in\{w_{\tilde a}^i:i\in I_{m_1}\}\},\{1-(1-\beta)^\sigma:\beta\in\{u_{\tilde a}^j:j\in I_{n_1}\}\},\{1-(1-\lambda)^\sigma:\lambda\in\{y_{\tilde a}^l:l\in I_{k_1}\}\}>$$

Theorem 10. Let $\tilde a=<(a_1,b_1,c_1);\{w_{\tilde a}^i:i\in I_{m_1}\},\{u_{\tilde a}^j:j\in I_{n_1}\},\{y_{\tilde a}^l:l\in I_{k_1}\}>$, $\tilde b=<(a_2,b_2,c_2);\{w_{\tilde b}^i:i\in I_{m_2}\},\{u_{\tilde b}^j:j\in I_{n_2}\},\{y_{\tilde b}^l:l\in I_{k_2}\}>$ and $\tilde c=<(a_3,b_3,c_3);\{w_{\tilde c}^i:i\in I_{m_3}\},\{u_{\tilde c}^j:j\in I_{n_3}\},\{y_{\tilde c}^l:l\in I_{k_3}\}>$ be three hesitant triangular neutrosophic numbers and $\sigma,\sigma_1,\sigma_2>0$, then

1. $\tilde a\oplus'\tilde b=\tilde b\oplus'\tilde a$
2. $\tilde a\otimes'\tilde b=\tilde b\otimes'\tilde a$
3. $.\tilde a\oplus'(\tilde b\oplus'\tilde c)=(\tilde a\oplus'\tilde b)\oplus'\tilde c$
4. $\tilde a\otimes'(\tilde b\otimes'\tilde c)=(\tilde a\otimes'\tilde b)\otimes'\tilde c$
5. $\sigma\odot'(\tilde a\oplus'\tilde b)=(\sigma\odot'\tilde a)\oplus'(\sigma\odot'\tilde b)$
6. $\sigma*'(\tilde a\otimes'\tilde b)=(\sigma*'\tilde a)\otimes'(\sigma*'\tilde b)$
7. $\sigma*'(\tilde a\otimes'\tilde b)=(\sigma*'\tilde a)\otimes'(\sigma*'\tilde b)$
8. $(\sigma_1+\sigma_2)*'\tilde a=(\sigma_1*'\tilde a)\otimes'(\sigma_2*'\tilde a)$

Proof:1-2 straight forward.

$$\begin{aligned}3.\ \tilde a\oplus'(\tilde b\oplus'\tilde c)=&<(a_1,b_1,c_1);w_{\tilde a},u_{\tilde a},y_{\tilde a}>\oplus'<a_2+a_3,b_2+b_3,c_2+c_3\ ;\{\alpha_2+\alpha_3-\alpha_2\alpha_3:\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\},\\&\alpha_3\in\{w_{\tilde c}^i:i\in I_{m_3}\}\},\{\beta_2\beta_3:\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\},\beta_3\in\{u_{\tilde c}^j:j\in I_{n_3}\}\},\{\lambda_2\lambda_3:\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\},\\&\lambda_3\in\{y_{\tilde c}^l:l\in I_{k_3}\}\}>\\=&<(a_1+(a_2+a_3),b_1+(b_2+b_3),c_1+(c_2+c_3));\{\alpha_1+(\alpha_2+\alpha_3-\alpha_2\alpha_3)-\alpha_1(\alpha_2+\alpha_3-\alpha_2\alpha_{31}):\\&\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\},\alpha_3\in\{w_{\tilde c}^i:i\in I_{m_3}\}\},\{\beta_1(\beta_2\beta_3):\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:\\&j\in I_{n_2}\},\beta_3\in\{u_{\tilde c}^j:j\in I_{n_3}\}\},\{\lambda_1(\lambda_2\lambda_3):\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\},\lambda_3\in\{y_{\tilde c}^l:l\in I_{k_3}\}\}>\\=&<(a_1+a_2+a_3,b_1+b_2+b_3,c_1+c_2+c_3);\{\alpha_1+\alpha_2+\alpha_3-\alpha_2\alpha_3-\alpha_1\alpha_2+\alpha_1\alpha_3-\alpha_1\alpha_2\alpha_3):\\&\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\},\alpha_3\in\{w_{\tilde c}^i:i\in I_{m_3}\}\},\{\beta_1\beta_2\beta_3:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\\&\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\},\beta_3\in\{u_{\tilde c}^j:j\in I_{n_3}\}\},\{\lambda_1\lambda_2\lambda_3:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\},\\&\lambda_3\in\{y_{\tilde c}^l:l\in I_{k_3}\}\}>\end{aligned}\tag{1}$$

and

$$\begin{aligned}(\tilde a\oplus'\tilde b)\oplus'\tilde c=&<(a_1+a_2,b_1+b_2,c_1+c_2);\{\alpha_1+\alpha_2-\alpha_1\alpha_2:\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\}\},\\&\{\beta_1\beta_2:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\}\},\{\lambda_1\lambda_2:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\}\}>\\&\oplus'<(a_3,b_3,c_3);\{w_{\tilde c}^i:i\in I_{m_3}\},\{u_{\tilde c}^j:j\in I_{n_3}\},\{y_{\tilde c}^l:l\in I_{k_3}\}>\\=&<((a_1+a_2)+a_3,(b_1+b_2)+b_3,(c_1+c_2)+c_3);\{(\alpha_1+\alpha_2-\alpha_1\alpha_2)+\alpha_3-(\alpha_1+\beta_1-\alpha_1\beta_1)\alpha_3:\\&\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\},\alpha_3\in\{w_{\tilde c}^i:i\in I_{m_3}\}\},\{(\beta_1\beta_2)\beta_3:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:\\&j\in I_{n_2}\}\beta_3\in\{u_{\tilde c}^j:j\in I_{n_3}\}\},\{(\lambda_1\lambda_2)\lambda_3:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\},\lambda_3\in\{y_{\tilde c}^l:l\in I_{k_3}\}\}>\\=&<(a_1+a_2+a_3,b_1+b_2+b_3,c_1+c_2+c_3);\{\alpha_1+\alpha_2+\alpha_3-\alpha_2\alpha_3-\alpha_1\alpha_2+\alpha_1\alpha_3-\alpha_1\alpha_2\alpha_3):\\&\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\},\alpha_3\in\{w_{\tilde c}^i:i\in I_{m_3}\}\},\{\beta_1\beta_2\beta_3:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\\&\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\},\beta_3\in\{u_{\tilde c}^j:j\in I_{n_3}\}\},\{\lambda_1\lambda_2\lambda_3:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\},\\&\lambda_3\in\{y_{\tilde c}^l:l\in I_{k_3}\}\}>\end{aligned}\tag{2}$$

Hence from eq. 1-2, we have, $\tilde a\oplus'(\tilde b\oplus'\tilde c)=(\tilde a\oplus'\tilde b)\oplus'\tilde c$.

4. Proof is similar to 3

5. $\sigma\odot'(\tilde{a}\oplus'\tilde{b})=\sigma\odot'<(a_1+a_2,b_1+b_2,c_1+c_2);\{\alpha_1+\alpha_2-\alpha_1\alpha_2:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},$

$\{\beta_1\beta_2:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},\{\lambda_1\lambda_2:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}>$

$=<(\sigma(a_1+a_2),\sigma(b_1+b_2),\sigma(c_1+c_2));\{1-(1-(\alpha_1+\alpha_2-\alpha_1\alpha_2))^\sigma:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},$

$\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},\{(\beta_1\beta_2)^\sigma:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},\{(\lambda_1\lambda_2)^\sigma:$

$\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}>$

$=<(\sigma a_1+\sigma a_2,\sigma b_1+\sigma b_2,\sigma c_1+\sigma c_2);\{1-(1-\alpha_1)^\sigma(1-\alpha_2)^\sigma:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},$

$\{\beta_1{}^\sigma\beta_2{}^\sigma:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},\{\lambda_1{}^\sigma\lambda_2{}^\sigma:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}> \quad (3)$

and

$(\sigma\odot'\tilde{a})\oplus'(\sigma\odot'\tilde{b})=<(\sigma a_1,\sigma b_1,\sigma c_1);\{1-(1-\alpha_1)^\sigma:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},\{\beta_1{}^\sigma:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},$

$\{\lambda_1{}^\sigma:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>\oplus'<(\sigma a_2,\sigma b_2,\sigma c_2);\{1-(1-\alpha_2)^\sigma:\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},$

$\{\beta_2{}^\sigma:\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},\{\lambda_2{}^\sigma:\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}>$

$=<(\sigma a_1+\sigma a_2,\sigma b_1+\sigma b_2,\sigma c_1+\sigma c_2);\{(1-(1-\alpha_1)^\sigma)+(1-(1-\alpha_2)^\sigma)-$

$(1-(1-\alpha_1)^\sigma)(1-(1-\alpha_2)^\sigma):\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},\{\beta_1{}^\sigma\beta_2{}^\sigma:\beta_1\in$

$\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},\{\lambda_1{}^\sigma\lambda_2{}^\sigma:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}>$

$=<(\sigma a_1+\sigma a_2,\sigma b_1+\sigma b_2,\sigma c_1+\sigma c_2);\{1-(1-\alpha_1)^\sigma(1-\alpha_2)^\sigma:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},$

$\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},\{\beta_1{}^\sigma\beta_2{}^\sigma:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},\{\lambda_1{}^\sigma\lambda_2{}^\sigma:$

$\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}> \quad (4)$

Hence from eq. 3-4, we have $\sigma\odot'(\tilde{a}\oplus'\tilde{b})=(\sigma\odot'\tilde{a})\oplus'(\sigma\odot'\tilde{b})$.

6. Proof is similar to 5

7. $(\sigma_1+\sigma_2)\odot'\tilde{a}=<((\sigma_1+\sigma_2)a_1,(\sigma_1+\sigma_2)b_1,(\sigma_1+\sigma_2)c_1);\{1-(1-\alpha_1)^{\sigma_1+\sigma_2}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},\{\beta_1{}^{\sigma_1+\sigma_2}:$

$\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},\{\lambda_1{}^{\sigma_1+\sigma_2}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>$

$=<(\sigma_1a_1+\sigma_2a_1,\sigma_1b_1+\sigma_2b_1,\sigma_1c_1+\sigma_2c_1);\{1-(1-\alpha_1)^{\sigma_1}(1-\alpha_1)^{\sigma_2}:\alpha_1\in w_{\tilde{a}}\},$

$\{\beta_1{}^{\sigma_1}\beta_1{}^{\sigma_2}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},\{\lambda_1{}^{\sigma_1}\lambda_1{}^{\sigma_2}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}> \quad (5)$

And

$(\sigma_1\odot'\tilde{a})\oplus'(\sigma_2\odot'\tilde{a})=<(\sigma_1a_1,\sigma_1b_1,\sigma_1c_1);\{1-(1-\alpha_1)^{\sigma_1}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},\{\beta_1{}^{\sigma_1}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},$

$\{\lambda_1{}^{\sigma_1}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>\oplus'<(\sigma_2a_1,\sigma_2b_1,\sigma_2c_1);\{1-(1-\alpha_1)^{\sigma_2}:\alpha_1\in\{w_{\tilde{a}}^i:$

$i\in I_{m_1}\}\},\{\beta_1{}^{\sigma_2}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},\{\lambda_1{}^{\sigma_2}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>$

$=<(\sigma_1a_1+\sigma_2a_1,\sigma_1b_1+\sigma_2b_1,\sigma_1c_1+\sigma_2c_1);\{(1-(1-\alpha_1)^{\sigma_1})+(1-(1-\alpha_1)^{\sigma_2})-$

$(1-(1-\alpha_1)^{\sigma_1})(1-(1-\alpha_1)^{\sigma_2}):\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},\{\beta_1{}^{\sigma_1}\beta_1{}^{\sigma_2}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},$

$\{\lambda_1{}^{\sigma_1}\lambda_1{}^{\sigma_2}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>$

$=<(\sigma_1a_1+\sigma_2a_1,\sigma_1b_1+\sigma_2b_1,\sigma_1c_1+\sigma_2c_1);\{1-(1-\alpha_1)^{\sigma_1}(1-\alpha_1)^{\sigma_2}:\alpha_1\in w_{\tilde{a}}\},$

$\{\beta_1{}^{\sigma_1}\beta_1{}^{\sigma_2}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},\{\lambda_1{}^{\sigma_1}\lambda_1{}^{\sigma_2}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}\} \quad (6)$

Hencefrom eq. 5-6, we have $(\sigma_1+\sigma_2)\odot'\tilde{a}=(\sigma_1\odot'\tilde{a})\oplus'(\sigma_2\odot'\tilde{a})$.

8. Proof is similar to 7

Definition 11. Let $\tilde{a}=<(a_1,b_1,c_1);\{w_{\tilde{a}}^i:i\in I_{m_1}\},\{u_{\tilde{a}}^j:j\in I_{n_1}\},\{y_{\tilde{a}}^l:l\in I_{k_1}\}>$ and $\tilde{b}=<(a_2,b_2,c_2);\{w_{\tilde{b}}^i:i\in I_{m_2}\},\{u_{\tilde{b}}^j:j\in I_{n_2}\},\{y_{\tilde{b}}^l:l\in I_{k_2}\}>$ be two hesitant triangular neutrosophic numbers and $\sigma>0$, then

1. $\tilde{a}\oplus''\tilde{b}=<(a_1+a_2,b_1+b_2,c_1+c_2);\{1-\frac{1}{1+\left\{\left(\frac{{\alpha_1}^2}{1-{\alpha_1}^2}\right)^2+\left(\frac{{\alpha_2}^2}{1-{\alpha_2}^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},$

$$\{\frac{1}{1+\left\{\left(\frac{1-{\beta_1}^2}{{\beta_1}^2}\right)^2+\left(\frac{1-{\beta_2}^2}{{\beta_2}^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-{\lambda_1}^2}{{\lambda_1}^2}\right)^2+\left(\frac{1-{\lambda_2}^2}{{\lambda_2}^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}>$$

2. $\tilde{a}\otimes''\tilde{b}=<(a_1a_2,b_1b_2,c_1c_2);\{\frac{1}{1+\left\{\left(\frac{1-{\alpha_1}^2}{{\alpha_1}^2}\right)^2+\left(\frac{1-{\alpha_2}^2}{{\alpha_2}^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},$

$$\{1-\frac{1}{1+\left\{\left(\frac{{\beta_1}^2}{1-{\beta_1}^2}\right)^2+\left(\frac{{\beta_2}^2}{1-{\beta_2}^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},$$

$$\{1-\frac{1}{1+\left\{\left(\frac{{\lambda_1}^2}{1-{\lambda_1}^2}\right)^2+\left(\frac{{\lambda_2}^2}{1-{\lambda_2}^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}\}>$$

3. $\sigma\odot''\tilde{a}=<(\sigma a_1,\sigma b_1,\sigma c_1);\{1-\frac{1}{1+\left\{\sigma\left(\frac{\alpha^2}{1-\alpha^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},\{\frac{1}{1+\left\{\sigma\left(\frac{1-\beta^2}{\beta^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},$

$$\{\frac{1}{1+\left\{\sigma\left(\frac{1-\lambda^2}{\lambda^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>$$

4. $\sigma *''\tilde{a}=<({a_1}^{\sigma},{b_1}^{\sigma},{c_1}^{\sigma});\{\frac{1}{1+\left\{\sigma\left(\frac{1-\alpha^2}{\alpha^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},\{1-\frac{1}{1+\left\{\sigma\left(\frac{\beta^2}{1-\beta^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},$

$$\{1-\frac{1}{1+\left\{\sigma\left(\frac{\lambda^2}{1-\lambda^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>$$

Theorem 12. Let $\tilde{a}=<(a_1,b_1,c_1);\{w_{\tilde{a}}^i:i\in I_{m_1}\},\{u_{\tilde{a}}^j:j\in I_{n_1}\},\{y_{\tilde{a}}^l:l\in I_{k_1}\}>$, $\tilde{b}=<(a_2,b_2,c_2);\{w_{\tilde{b}}^i:i\in I_{m_2}\},\{u_{\tilde{b}}^j:j\in I_{n_2}\},\{y_{\tilde{b}}^l:l\in I_{k_2}\}>$ and $\tilde{c}=<(a_3,b_3,c_3);\{w_{\tilde{c}}^i:i\in I_{m_3}\},\{u_{\tilde{c}}^j:j\in I_{n_3}\},\{y_{\tilde{c}}^l:l\in I_{k_3}\}>$ be three hesitant triangular neutrosophic numbers and $\sigma,\sigma_1,\sigma_2>0$, then

1. $\tilde{a} \oplus'' \tilde{b} = \tilde{b} \oplus'' \tilde{a}$
2. $\tilde{a} \otimes'' \tilde{b} = \tilde{b} \otimes'' \tilde{a}$
3. $\tilde{a} \oplus'' (\tilde{b} \oplus'' \tilde{c}) = (\tilde{a} \oplus'' \tilde{b}) \oplus'' \tilde{c}$
4. $\tilde{a} \otimes'' (\tilde{b} \otimes'' \tilde{c}) = (\tilde{a} \otimes'' \tilde{b}) \otimes'' \tilde{c}$
5. $\sigma \odot'' (\tilde{a} \oplus'' \tilde{b}) = (\sigma \odot'' \tilde{a}) \oplus'' (\sigma \odot'' \tilde{b})$
6. $\sigma *'' (\tilde{a} \otimes'' \tilde{b}) = (\sigma *'' \tilde{a}) \otimes'' (\sigma *'' \tilde{b})$
7. $(\sigma_1 + \sigma_2) \odot'' \tilde{a} = (\sigma_1 \odot'' \tilde{a}) \oplus'' (\sigma_2 \odot'' \tilde{a})$
8. $(\sigma_1 + \sigma_2) *'' \tilde{a} = (\sigma_1 *'' \tilde{a}) \otimes'' (\sigma_2 *'' \tilde{a})$

Proof:1.-2. Straight forward.

3. $\tilde{a} \oplus'' (\tilde{b} \oplus'' \tilde{c})$

$$= < (a_1, b_1, c_1); \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \{y_{\tilde{a}}^l : l \in I_{k_1}\} > \oplus'' < (a_2 + a_3, b_2 + b_3, c_2 + c_3);$$

$$\{1 - \frac{1}{1 + \left\{\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2 + \left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \alpha_3 \in \{w_{\tilde{c}}^i : i \in I_{m_3}\}\},$$

$$\{\frac{1}{1 + \left\{\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2 + \left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2\right\}^{\frac{1}{2}}} : \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}, \beta_3 \in \{u_{\tilde{c}}^j : j \in I_{n_3}\}\},$$

$$\{\frac{1}{1 + \left\{\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2 + \left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}, \lambda_3 \in \{y_{\tilde{c}}^l : l \in I_{k_3}\}\} >$$

$$= < (a_1 + (a_2 + a_3), b_1 + (b_2 + b_3), c_1 + (c_2 + c_3));$$

$$\{1 - \frac{1}{1 + \left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2 + \left(\frac{1 - \frac{1}{1 + \left\{\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2 + \left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2\right\}^{\frac{1}{2}}}}{1 - \left(1 - \frac{1}{1 + \left\{\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2 + \left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2\right\}^{\frac{1}{2}}}\right)}\right)^2\right\}^{\frac{1}{2}}} : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \alpha_3 \in \{w_{\tilde{c}}^i : i \in I_{m_3}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2+\left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2+\left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\},\beta_3\in\{u_{\tilde{c}}^j:j\in I_{n_3}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2+\left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2+\left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\},\lambda_3\in\{y_{\tilde{c}}^l:l\in I_{k_3}\}\}>$$

$$=<(a_1+a_2+a_3,b_1+b_2+b_3,c_1+c_2+c_3);\{1-\frac{1}{1+\left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2+\left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\},\alpha_3\in\{w_{\tilde{c}}^i:i\in I_{m_3}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2+\left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\},\ \beta_3\in\{u_{\tilde{c}}^j:j\in I_{n_3}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2+\left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\},\lambda_3\in\{y_{\tilde{c}}^l:l\in I_{k_3}\}\}> \quad (7)$$

$$(\tilde{a}\oplus''\tilde{b})\oplus''\tilde{c}$$

$$=<(a_1+a_2,b_1+b_2,c_1+c_2);\{1-\frac{1}{1+\left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}>\oplus''<(a_3,b_3,c_3);w_{\tilde{c}},u_{\tilde{c}},y_{\tilde{c}}>$$

$$
\begin{aligned}
&=<((a_1+a_2)+a_3,(b_1+b_2)+b_3,(c_1+c_2)+c_3);\\
&\{1-\frac{1}{\left[1+\left\{\left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2+\left(\frac{1-\frac{1}{\left[1+\left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}\right]}}{1-\left(1-\frac{1}{\left[1+\left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}\right]}\right)}\right)^2\right\}\right]^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\},\alpha_3\in\{w_{\tilde{c}}^i:i\in I_{m_3}\}\},\\
&\{\frac{1}{\left[1+\left\{\left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2+\left(\frac{1-\frac{1}{\left[1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}\right]}}{\frac{1}{\left[1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}\right]}}\right)^2\right\}\right]^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\},\beta_3\in\{u_{\tilde{c}}^j:j\in I_{n_3}\}\},\\
&\{\frac{1}{\left[1+\left\{\left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2+\left(\frac{1-\frac{1}{\left[1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}\right]}}{\frac{1}{\left[1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}\right]}}\right)^2\right\}\right]^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\},\lambda_3\in\{y_{\tilde{c}}^l:l\in I_{k_3}\}\}>
\end{aligned}
$$

$$
\begin{aligned}
&=<(a_1+a_2+a_3,b_1+b_2+b_3,c_1+c_2+c_3);\{1-\frac{1}{\left[1+\left\{\left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2+\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}\right]^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},\\
&\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\},\alpha_3\in\{w_{\tilde{c}}^i:i\in I_{m_3}\}\},\{\frac{1}{\left[1+\left\{\left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2+\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}\right]^{\frac{1}{2}}}:\\
&\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\},\beta_3\in\{u_{\tilde{c}}^j:j\in I_{n_3}\},\{\frac{1}{\left[1+\left\{\left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2+\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}\right]^{\frac{1}{2}}}:\\
&\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\},\lambda_3\in\{y_{\tilde{c}}^l:l\in I_{k_3}\}\}> \qquad (8)
\end{aligned}
$$

Hence from eq. 7-8, we have, $\tilde{a}\oplus''(\tilde{b}\oplus''\tilde{c})=(\tilde{a}\oplus''\tilde{b})\oplus''\tilde{c}$.

4. Proof is similar to 3.

5. $\sigma \odot'' (\tilde{a} \oplus'' \tilde{b})$

$$= \sigma \odot'' < (a_1 + a_2, b_1 + b_2, c_1 + c_2); \{1 - \frac{1}{1 + \left\{ \left(\frac{\alpha_1^2}{1-\alpha_1^2} \right)^2 + \left(\frac{\alpha_2^2}{1-\alpha_2^2} \right)^2 \right\}^{\frac{1}{2}}} : \alpha_1 \in \{ w_{\tilde{a}}^i : i \in I_{m_1} \}, \alpha_2 \in \{ w_{\tilde{b}}^i : i \in I_{m_2} \}\},$$

$$\{\frac{1}{1 + \left\{ \left(\frac{1-\beta_1^2}{\beta_1^2} \right)^2 + \left(\frac{1-\beta_2^2}{\beta_2^2} \right)^2 \right\}^{\frac{1}{2}}} : \beta_1 \in \{ u_{\tilde{a}}^j : j \in I_{n_1} \}, \beta_2 \in \{ u_{\tilde{b}}^j : j \in I_{n_2} \}\},$$

$$\{\frac{1}{1 + \left\{ \left(\frac{1-\lambda_1^2}{\lambda_1^2} \right)^2 + \left(\frac{1-\lambda_2^2}{\lambda_2^2} \right)^2 \right\}^{\frac{1}{2}}} : \lambda_1 \in \{ y_{\tilde{a}}^l : l \in I_{k_1} \}, \lambda_2 \in \{ y_{\tilde{b}}^l : l \in I_{k_2} \}\} >$$

$$= < (\sigma a_1 + \sigma a_2, \sigma b_1 + \sigma b_2, \sigma c_1 + \sigma c_2); \{1 - \frac{1}{1 + \left\{ \sigma \left(\frac{1 - \frac{1}{1 + \left\{ \left(\frac{\alpha_1^2}{1-\alpha_1^2} \right)^2 + \left(\frac{\alpha_2^2}{1-\alpha_2^2} \right)^2 \right\}^{\frac{1}{2}}}}{\frac{1}{1 + \left\{ \left(\frac{\alpha_1^2}{1-\alpha_1^2} \right)^2 + \left(\frac{\alpha_2^2}{1-\alpha_2^2} \right)^2 \right\}^{\frac{1}{2}}}} \right)^2 \right\}^{\frac{1}{2}}} : \alpha_1 \in \{ w_{\tilde{a}}^i : i \in I_{m_1} \},$$

$$\alpha_2 \in \{ w_{\tilde{b}}^i : i \in I_{m_2} \}\},$$

$$\{\frac{1}{1 + \left\{ \sigma \left(\frac{1 - \frac{1}{1 + \left\{ \left(\frac{1-\beta_1^2}{\beta_1^2} \right)^2 + \left(\frac{1-\beta_2^2}{\beta_2^2} \right)^2 \right\}^{\frac{1}{2}}}}{\frac{1}{1 + \left\{ \left(\frac{1-\beta_1^2}{\beta_1^2} \right)^2 + \left(\frac{1-\beta_2^2}{\beta_2^2} \right)^2 \right\}^{\frac{1}{2}}}} \right)^2 \right\}^{\frac{1}{2}}} : \beta_1 \in \{ u_{\tilde{a}}^j : j \in I_{n_1} \}, \beta_2 \in \{ u_{\tilde{b}}^j : j \in I_{n_2} \}\},$$

$$\{\frac{1}{1 + \left\{ \sigma \left(\frac{1 - \frac{1}{1 + \left\{ \left(\frac{1-\lambda_1^2}{\lambda_1^2} \right)^2 + \left(\frac{1-\lambda_2^2}{\lambda_2^2} \right)^2 \right\}^{\frac{1}{2}}}}{\frac{1}{1 + \left\{ \left(\frac{1-\lambda_1^2}{\lambda_1^2} \right)^2 + \left(\frac{1-\lambda_2^2}{\lambda_2^2} \right)^2 \right\}^{\frac{1}{2}}}} \right)^2 \right\}^{\frac{1}{2}}} : \lambda_1 \in \{ y_{\tilde{a}}^l : l \in I_{k_1} \}, \lambda_2 \in \{ y_{\tilde{b}}^l : l \in I_{k_2} \}\} >$$

$$=<(\sigma a_1+\sigma a_2,\sigma b_1+\sigma b_2,\sigma c_1+\sigma c_2);\{1-\frac{1}{1+\left\{\sigma\left(\left(\frac{{\alpha_1}^2}{1-{\alpha_1}^2}\right)^2+\left(\frac{{\alpha_2}^2}{1-{\alpha_2}^2}\right)^2\right)\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},$$

$$\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},\{\frac{1}{1+\left\{\sigma\left(\left(\frac{1-{\beta_1}^2}{{\beta_1}^2}\right)^2+\left(\frac{1-{\beta_2}^2}{{\beta_2}^2}\right)^2\right)\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},$$

$$\{\frac{1}{1+\left\{\sigma\left(\left(\frac{1-{\lambda_1}^2}{{\lambda_1}^2}\right)^2+\left(\frac{1-{\lambda_2}^2}{{\lambda_2}^2}\right)^2\right)\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}>$$

$$=<(\sigma a_1+\sigma a_2,\sigma b_1+\sigma b_2,\sigma c_1+\sigma c_2);\{1-\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{{\alpha_1}^2}{1-{\alpha_1}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{{\alpha_1}^2}{1-{\alpha_1}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{{\alpha_2}^2}{1-{\alpha_2}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{{\alpha_2}^2}{1-{\alpha_2}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{1-{\beta_1}^2}{{\beta_1}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{1-{\beta_1}^2}{{\beta_1}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{1-{\beta_2}^2}{{\beta_2}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{1-{\beta_2}^2}{{\beta_2}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{1-{\lambda_1}^2}{{\lambda_1}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{1-{\lambda_1}^2}{{\lambda_1}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{1-{\lambda_2}^2}{{\lambda_2}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{1-{\lambda_2}^2}{{\lambda_2}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}>$$

$$=<(\sigma a_1,\sigma b_1,\sigma c_1);\{1-\frac{1}{1+\left\{\sigma\left(\frac{{\alpha_1}^2}{1-{\alpha_1}^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},\{\frac{1}{1+\left\{\sigma\left(\frac{1-{\beta_1}^2}{{\beta_1}^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},$$

$$\{\frac{1}{1+\left\{\sigma\left(\frac{1-\lambda_1{}^2}{\lambda_1{}^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>\oplus''<\ \sigma a_2,\sigma b_2,\sigma c_2\ ;\{1-\frac{1}{1+\left\{\sigma\left(\frac{\alpha_2{}^2}{1-\alpha_2{}^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},$$

$$\{\frac{1}{1+\left\{\sigma\left(\frac{1-\beta_2{}^2}{\beta_2{}^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},\{\frac{1}{1+\left\{\sigma\left(\frac{1-\lambda_2{}^2}{\lambda_2{}^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}>$$

$$=(\sigma\odot''\tilde{a})\oplus''(\sigma\odot''\tilde{b})$$

6. Proof is similar to 5

7. $(\sigma_1+\sigma_2)\odot''\tilde{a}$

$$=<((\sigma_1+\sigma_2)a_1,(\sigma_1+\sigma_2)b_1,(\sigma_1+\sigma_2)c_1);\{1-\frac{1}{1+\left\{(\sigma_1+\sigma_2)\left(\frac{\alpha_1{}^2}{1-\alpha_1{}^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},$$

$$\{\frac{1}{1+\left\{(\sigma_1+\sigma_2)\left(\frac{1-\beta_1{}^2}{\beta_1{}^2}\right)^2\right\}^{\frac{1}{2}}}\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},\{\frac{1}{1+\left\{(\sigma_1+\sigma_2)\left(\frac{1-\lambda_1{}^2}{\lambda_1{}^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}$$

$$=<((\sigma_1+\sigma_2)a_1,(\sigma_1+\sigma_2)b_1,(\sigma_1+\sigma_2)c_1);$$

$$\{1-\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma_1\left(\frac{\alpha_1{}^2}{1-\alpha_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_1\left(\frac{\alpha_1{}^2}{1-\alpha_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma_2\left(\frac{\alpha_1{}^2}{1-\alpha_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_2\left(\frac{\alpha_1{}^2}{1-\alpha_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma_1\left(\frac{1-\beta_1{}^2}{\beta_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_1\left(\frac{1-\beta_1{}^2}{\beta_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma_2\left(\frac{1-\beta_1{}^2}{\beta_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_2\left(\frac{1-\beta_1{}^2}{\beta_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma_2\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_2\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>$$

$$=<(\sigma_1a_1,\sigma_1b_1,\sigma_1c_1);\{1-\frac{1}{1+\left\{\sigma_1\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},\{\frac{1}{1+\left\{\sigma_1\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},$$

$$\{\frac{1}{1+\left\{\sigma_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>\oplus''<(\sigma_2a_1,\sigma_2b_1,\sigma_2c_1);\{1-\frac{1}{1+\left\{\sigma_2\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},\{\frac{1}{1+\left\{\sigma_2\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},\{\frac{1}{1+\left\{\sigma_2\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>$$

$$=(\sigma_1\odot''\tilde{a})\oplus''(\sigma_2\odot''\tilde{a})$$

8. Proof is similar to 7.

5. HESITANT TRIANGULAR NEUTROSOPHIC WEIGHTED AGGREGATION OPERATORS:

This section deals with various types of hesitant triangular neutrosophic weighted aggregation operators along with their basic properties.

Definition 13: Let $\tilde{a}_j=<(a_j,b_j,c_j);\{w_{\tilde{a}_j}^i:i\in I_{m_j}\},\{u_{\tilde{a}_j}^r:r\in I_{n_j}\},\{y_{\tilde{a}_j}^l:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-1 ($HTNWAAO_{T_1}$ for short) is defined as:

$$HTNWAAO_{T_1}\ \tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n\ =(w_1\odot'\tilde{a}_1)\oplus'(w_2\odot'\tilde{a}_2)\oplus'(w_3\odot'\tilde{a}_3)\oplus'........\oplus'(w_n\odot'\tilde{a}_n)$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……, *n*) such that $w_j\geq 0$ and $\sum_{j=1}^{n}w_j=1$.

Theorem 14: Let $\tilde{a}_j=<(a_j,b_j,c_j);\{w_{\tilde{a}_j}^i:i\in I_{m_j}\},\{u_{\tilde{a}_j}^r:r\in I_{n_j}\},\{y_{\tilde{a}_j}^l:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangularneutrosophic numbers. Then $HTNWAAO_{T_1}\ \tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n$ is a hesitant triangular neutrosophic number and

$$HTNWAAO_{T_1}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)=<(\sum_{j=1}^{n}w_ja_j,\sum_{j=1}^{n}w_jb_j,\sum_{j=1}^{n}w_jc_j),\{1-\prod_{j=1}^{n}(1-\alpha_{jk})^{w_j}:$$

$$\alpha_j\in\{w_{\tilde{a}_j}^i:i\in I_{m_j}\}\},\{\prod_{j=1}^{n}\beta_{jm}{}^{w_j}:\beta_j\in\{u_{\tilde{a}_j}^r:r\in I_{n_j}\}\},\{\prod_{j=1}^{n}\lambda_{jr}{}^{w_j}:\lambda_j\in\{y_{\tilde{a}_j}^l:l\in I_{k_j}\}\}>$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……,n) such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.

Proof: Let us prove the result using the method of mathematical induction. For n=2, $HTNWAO_{T_1}(\tilde{a}_1, \tilde{a}_2)$

$$= (w_1 \odot' \tilde{a}_1) \oplus' (w_2 \odot' \tilde{a}_2)$$

$$=<(w_1a_1, w_1b_1, w_1c_1); \{1-(1-\alpha_1)^{w_1} : \alpha_1 \in \{w^i_{\tilde{a}_1} : i \in I_{m_1}\}\}, \{\beta_1^{w_1} : \beta_1 \in \{u^r_{\tilde{a}_1} : r \in I_{n_1}\}\},$$
$$\{\lambda_1^{w_1} : \lambda_1 \in \{y^l_{\tilde{a}_1} : l \in I_{k_1}\}\} > \oplus' < (w_2a_2, w_2b_2, w_2c_2); \{1-(1-\alpha_2)^{w_2} : \alpha_2 \in \{w^i_{\tilde{a}_2} : i \in I_{m_2}\}\},$$
$$\{\beta_2^{w_2} : \beta_2 \in \{u^r_{\tilde{a}_2} : j \in I_{n_2}\}\}, \{\lambda_2^{w_2} : \lambda_2 \in \{y^l_{\tilde{a}_2} : l \in I_{k_2}\}\}>$$

$$=<(w_1a_1 + w_2a_2, w_1b_1 + w_2b_2, w_1c_1 + w_2c_2); (1-(1-\alpha_1)^{w_1}) + (1-(1-\alpha_2)^{w_2}) -$$
$$\{(1-(1-\alpha_1)^{w_1}) \times (1-(1-\alpha_2)^{w_2}) : \alpha_1 \in \{w^i_{\tilde{a}_1} : i \in I_{m_1}\}, \alpha_2 \in \{w^i_{\tilde{a}_2} : i \in I_{m_2}\}\}, \{\beta_1^{w_1}\beta_2^{w_2} :$$
$$\beta_1 \in \{u^r_{\tilde{a}_1} : r \in I_{n_1}\}, \beta_2 \in \{u^r_{\tilde{a}_2} : j \in I_{n_2}\}\}, \{\lambda_1^{w_1}\lambda_2^{w_2} : \lambda_1 \in \{y^l_{\tilde{a}_1} : l \in I_{k_1}\}, \lambda_2 \in \{y^l_{\tilde{a}_2} : l \in I_{k_2}\}\}>$$

$$=<(w_1a_1 + w_2a_2, w_1b_1 + w_2b_2, w_1c_1 + w_2c_2); \{1-(1-\alpha_1)^{w_1}(1-\alpha_2)^{w_2} : \alpha_1 \in \{w^i_{\tilde{a}_1} : i \in I_{m_1}\},$$
$$\alpha_2 \in \{w^i_{\tilde{a}_2} : i \in I_{m_2}\}\}, \{\beta_1^{w_1}\beta_2^{w_2} : \beta_1 \in \{u^r_{\tilde{a}_1} : r \in I_{n_1}\}, \beta_2 \in \{u^r_{\tilde{a}_2} : j \in I_{n_2}\}\}, \{\lambda_1^{w_1}\lambda_2^{w_2} :$$
$$\lambda_1 \in \{y^l_{\tilde{a}_1} : l \in I_{k_1}\}, \lambda_2 \in \{y^l_{\tilde{a}_2} : l \in I_{k_2}\}\}>$$

$$=<(\sum_{j=1}^{2} w_j a_j, \sum_{j=1}^{2} w_j b_j, \sum_{j=1}^{2} w_j c_j); \{1-\prod_{j=1}^{2}(1-\alpha_j)^{w_j} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \{\prod_{j=1}^{2}\beta_j^{w_j} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},$$
$$\{\prod_{j=1}^{2}\lambda_j^{w_j} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\}>$$

Thus the result is true for n=2. Let us assume that the result is true for n=s. Then $HTNWAO_{T_1}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_s)$

$$=<(\sum_{j=1}^{s} w_j a_j, \sum_{j=1}^{s} w_j b_j, \sum_{j=1}^{s} w_j c_j); \{1-\prod_{j=1}^{s}(1-\alpha_j)^{w_j} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \{\prod_{j=1}^{s}\beta_j^{w_j} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},$$
$$\{\prod_{j=1}^{s}\lambda_j^{w_j} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\}>$$

Now for n=s+1, we have, $HTNWAO_{T_1}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_{s+1})$

$$=<(\sum_{j=1}^{s} w_j a_j, \sum_{j=1}^{s} w_j b_j, \sum_{j=1}^{s} w_j c_j); \{1-\prod_{j=1}^{s}(1-\alpha_j)^{w_j} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \prod_{j=1}^{s}\beta_j^{w_j} : \{\beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},$$
$$\{\prod_{j=1}^{s}\lambda_j^{w_j} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\}> \oplus' (w_{s+1} \odot \tilde{a}_{s+1})$$

$$=<(\sum_{j=1}^{s} w_j a_j, \sum_{j=1}^{s} w_j b_j, \sum_{j=1}^{s} w_j c_j), \{1-\prod_{j=1}^{s}(1-\alpha_j)^{w_j} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \prod_{j=1}^{s}\beta_j^{w_j} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},$$
$$\{\{\prod_{j=1}^{s}\lambda_j^{w_j} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\}> \oplus' < (w_{s+1}a_{s+1}, w_{s+1}b_{s+1}, w_{s+1}c_{s+1}); \{1-(1-\alpha_{(s+1)})^{w_{s+1}} :$$
$$\alpha_{s+1} \in \{w^i_{\tilde{a}_{s+1}} : i \in I_{m_{s+1}}\}\} \{\beta_{s+1}^{w_{s+1}} : \beta_{s+1} \in \{u^r_{\tilde{a}_{s+1}} : r \in I_{n_{s+1}}\}\}, \{\lambda_{s+1}^{w_{s+1}} : \lambda_{s+1} \in \{y^l_{\tilde{a}_{s+1}} : l \in I_{k_{s+1}}\}\}>$$

$$=<(\sum_{j=1}^{s} w_j a_j + w_{s+1}a_{s+1}, \sum_{j=1}^{s} w_j b_j + w_{s+1}b_{s+1}, \sum_{j=1}^{s} w_j c_j + w_{s+1}c_{s+1});$$
$$\{(1-\prod_{j=1}^{s}(1-\alpha_j)^{w_j}) + (1-(1-\alpha_{(s+1)})^{w_{s+1}}) - (1-\prod_{j=1}^{s}(1-\alpha_j)^{w_j}) \times (1-(1-\alpha_{(s+1)})^{w_{s+1}}) : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\},$$

$$\alpha_{s+1}\in\{w^{i}_{\tilde a_{s+1}}:i\in I_{m_{s+1}}\}\},\{(\prod_{j=1}^{s}\beta_j^{w_j})\beta_{s+1}^{w_{s+1}}:\beta_j\in\{u^{r}_{\tilde a_j}:r\in I_{n_j}\},\beta_{s+1}\in\{u^{r}_{\tilde a_{s+1}}:r\in I_{n_{s+1}}\}\},$$

$$\{(\prod_{j=1}^{s}\lambda_j^{w_j})\lambda_{s+1}^{w_{s+1}}:\lambda_j\in\{y^{l}_{\tilde a_j}:l\in I_{k_j}\},\lambda_{s+1}\in\{y^{l}_{\tilde a_{s+1}}:l\in I_{k_{s+1}}\}\}>$$

$$=<(\sum_{j=1}^{s}w_ja_j+w_{s+1}a_{s+1},\sum_{j=1}^{s}w_jb_j+w_{s+1}b_{s+1},\sum_{j=1}^{s}w_jc_j+w_{s+1}c_{s+1});\{1-((\prod_{j=1}^{s}(1-\alpha_j)^{w_j})\alpha_{s+1}^{w_{s+1}}):\alpha_j\in\{w^{i}_{\tilde a_j}:i\in I_{m_j}\},$$

$$\alpha_{s+1}\in\{w^{i}_{\tilde a_{s+1}}:i\in I_{m_{s+1}}\}\},\{(\prod_{j=1}^{s}\beta_j^{w_j})\beta_{s+1}^{w_{s+1}}:\beta_j\in\{u^{r}_{\tilde a_j}:r\in I_{n_j}\},\beta_{s+1}\in\{u^{r}_{\tilde a_{s+1}}:r\in I_{n_{s+1}}\}\},$$

$$\{(\prod_{j=1}^{s}\lambda_j^{w_j})\lambda_{s+1}^{w_{s+1}}:\lambda_j\in\{y^{l}_{\tilde a_j}:l\in I_{k_j}\},\lambda_{s+1}\in\{y^{l}_{\tilde a_{s+1}}:l\in I_{k_{s+1}}\}\}>$$

$$=<(\sum_{j=1}^{s+1}w_ja_j,\sum_{j=1}^{s+1}w_jb_j,\sum_{j=1}^{s+1}w_jc_j),\{1-\prod_{j=1}^{s+1}(1-\alpha_j)^{w_j}:\alpha_j\in\{w^{i}_{\tilde a_j}:i\in I_{m_j}\}\},\{\prod_{j=1}^{s+1}\beta_j^{w_j}:\beta_j\in\{u^{r}_{\tilde a_j}:r\in I_{n_j}\}\},$$

$$\{\prod_{j=1}^{s+1}\lambda_j^{w_j}:\lambda_j\in\{y^{l}_{\tilde a_j}:l\in I_{k_j}\}\}>$$

Thus the result is true for $n=s+1$ also. Hence by the principle of mathematical induction, the result is true for any natural number n.

Theorem 15: Let $\tilde a_j=<(a_j,b_j,c_j);\{w^{i}_{\tilde a_j}:i\in I_{m_j}\},\{u^{r}_{\tilde a_j}:r\in I_{n_j}\},\{y^{l}_{\tilde a_j}:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number $\tilde\theta$, we have,

(i) $HTNWAAO_{T_1}\ \tilde\theta\oplus'\tilde a_1,\tilde\theta\oplus'\tilde a_2,\tilde\theta\oplus'\tilde a_3,......,\tilde\theta\oplus'\tilde a_n\ =\tilde\theta\oplus' HTNWAAO_{T_1}\ \tilde a_1,\tilde a_2,\tilde a_3,......,\tilde a_n$

(ii) $HTNWAAO_{T_1}\ \tilde a_1,\tilde a_2,\tilde a_3,......,\tilde a_n\ =\tilde\theta$ if $\tilde a_j=\tilde\theta$ for each j

Proof: Straight Forward.

Definition 16: Let $\tilde a_j=<(a_j,b_j,c_j);\{w^{i}_{\tilde a_j}:i\in I_{m_j}\},\{u^{r}_{\tilde a_j}:r\in I_{n_j}\},\{y^{l}_{\tilde a_j}:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophic weighted geometric aggregation operator of type-1 ($HTNWGAO_{T_1}$ for short) is defined as:

$$HTNWGAO_{T_1}(\tilde a_1,\tilde a_2,\tilde a_3,......,\tilde a_n)=(w_1*'\tilde a_1)\otimes'(w_2*'\tilde a_2)\otimes'(w_3*'\tilde a_3)\otimes'........\otimes'(w_n*'\tilde a_n)$$

where w_j is the weight of $\tilde a_j$ (j=1,2,3,……,n) such that $w_j\ge 0$ and $\sum_{j=1}^{n}w_j=1$.

Theorem 17: Let $\tilde a_j=<(a_j,b_j,c_j);\{w^{i}_{\tilde a_j}:i\in I_{m_j}\},\{u^{r}_{\tilde a_j}:r\in I_{n_j}\},\{y^{l}_{\tilde a_j}:l\in I_{k_j}\}>$ (j=1,2,3,…..,n) be a collection of hesitant triangularneutrosophic numbers. Then $HTNWGAO_{T_1}(\tilde a_1,\tilde a_2,\tilde a_3,......,\tilde a_n)$ is a hesitant triangular neutrosophic number and $HTNWGAO_{T_1}(\tilde a_1,\tilde a_2,\tilde a_3,......,\tilde a_n)$

$$=<(\prod_{j=1}^{n}a_j^{w_j},\prod_{j=1}^{n}b_j^{w_j},\prod_{j=1}^{n}c_j^{w_j});\prod_{j=1}^{n}\alpha_j^{w_j}:\alpha_j\in\{w^{i}_{\tilde a_j}:i\in I_{m_j}\}\},\{1-\prod_{j=1}^{n}(1-\beta_j)^{w_j}:\beta_j\in\{u^{r}_{\tilde a_j}:r\in I_{n_j}\}\},$$

$$\{1-\prod_{j=1}^{n}(1-\lambda_j)^{w_j}:\lambda_j\in\{y^{l}_{\tilde a_j}:l\in I_{k_j}\}\}>$$

where w_j is the weight of $\tilde a_j$ (j=1,2,3,……,n) such that $w_j\ge 0$ and $\sum_{j=1}^{n}w_j=1$.

Proof: Similar to the proof of Theorem **14.**

Theorem 18: Let $\tilde a_j=<(a_j,b_j,c_j);\{w^{i}_{\tilde a_j}:i\in I_{m_j}\},\{u^{r}_{\tilde a_j}:r\in I_{n_j}\},\{y^{l}_{\tilde a_j}:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number $\tilde\theta$, we have,

(i) $HTNWGAO_{T_1}\ \tilde{\theta}\oplus'\tilde{a}_1,\tilde{\theta}\oplus'\tilde{a}_2,\tilde{\theta}\oplus'\tilde{a}_3,......,\tilde{\theta}\oplus'\tilde{a}_n\ =\tilde{\theta}\oplus' HTNWGAO_{T_1}\ \tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n$

(ii) $HTNWGAO_{T_1}\ \tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n\ =\tilde{\theta}$ if $\tilde{a}_j=\tilde{\theta}$ for each j

Proof: Straight forward .

Definition 19: Let $\tilde{a}_j=<(a_j,b_j,c_j);\{w^i_{\tilde{a}_j}:i\in I_{m_j}\},\{u^r_{\tilde{a}_j}:r\in I_{n_j}\},\{y^l_{\tilde{a}_j}:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-2 is denoted by $HTNWAAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)$ and is defined by:

$$HTNWAAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)=(w_1\odot''\tilde{a}_1)\oplus''(w_2\odot''\tilde{a}_2)\oplus''(w_3\odot''\tilde{a}_3)\oplus''........\oplus''(w_n\odot''\tilde{a}_n)$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……,n) such that $w_j\geq 0$ and $\sum_{j=1}^{n}w_j=1$.

Theorem 20: Let $\tilde{a}_j=<(a_j,b_j,c_j);\{w^i_{\tilde{a}_j}:i\in I_{m_j}\},\{u^r_{\tilde{a}_j}:r\in I_{n_j}\},\{y^l_{\tilde{a}_j}:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then $HTNWAAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)$ is a hesitant triangular neutrosophic number and $HTNWAAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)$

$$=<(\sum_{j=1}^{n}w_ja_j,\sum_{j=1}^{n}w_jb_j,\sum_{j=1}^{n}w_jc_j),\{1-\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_j\in\{w^i_{\tilde{a}_j}:i\in I_{m_j}\}\},\{\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}}:$$
$$\beta_j\in\{u^r_{\tilde{a}_j}:r\in I_{n_j}\}\},\{\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_j\in\{y^l_{\tilde{a}_j}:l\in I_{k_j}\}\}>$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……,n) such that $w_j\geq 0$ and $\sum_{j=1}^{n}w_j=1$.

Proof: For n=2, we have,

$$HTNWAAO_{T_2}(\tilde{a}_1,\tilde{a}_2)$$
$$=(w_1\odot''\tilde{a}_1)\oplus''(w_2\odot''\tilde{a}_2)$$
$$=<(w_1a_1,w_1b_1,w_1c_1);\{1-\frac{1}{1+\left\{w_1\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w^i_{\tilde{a}_1}:i\in I_{m_1}\}\},\frac{1}{1+\left\{w_1\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}:\{\beta_1\in\{u^r_{\tilde{a}_1}:r\in I_{n_1}\}\},$$
$$\{\frac{1}{1+\left\{w_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y^l_{\tilde{a}_1}:l\in I_{k_1}\}\}>\oplus''<(w_2a_2,w_2b_2,w_2c_2);\{1-\frac{1}{1+\left\{w_2\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}:$$
$$\{\alpha_2\in\{w^i_{\tilde{a}_2}:i\in I_{m_2}\}\},\{\frac{1}{1+\left\{w_2\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_2\in u_{\tilde{a}_2}\},\{\frac{1}{1+\left\{w_2\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_2\in\{y^l_{\tilde{a}_2}:l\in I_{k_2}\}\}>$$

$$=<(w_1a_1+w_2a_2, w_1b_1+w_2b_2, w_1c_1+w_2c_2);$$

$$\{1-\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}}{1-1+\frac{1}{1+\left\{w_1\left(\frac{\alpha_{1k}^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}}{1-1+\frac{1}{1+\left\{w_2\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}_1}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{a}_2}^i:i\in I_{m_2}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_1\left(\frac{1-\beta_{1m}^2}{\beta_{1m}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_2\left(\frac{1-\beta_{2m}^2}{\beta_{2m}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}_1}^r:r\in I_{n_1}\},\beta_2\in\{u_{\tilde{a}_2}^r:j\in I_{n_2}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_2\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}_1}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{a}_2}^l:l\in I_{k_2}\}\}>$$

$$=<(w_1a_1+w_2a_2, w_1b_1+w_2b_2, w_1c_1+w_2c_2);\{1-\frac{1}{1+\left\{w_1\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+w_2\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}_1}^i:i\in I_{m_1}\},$$

$$\alpha_2\in\{w_{\tilde{a}_2}^i:i\in I_{m_2}\}\},\{\frac{1}{1+\left\{w_1\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+w_2\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}_1}^r:r\in I_{n_1}\},\beta_2\in\{u_{\tilde{a}_2}^r:j\in I_{n_2}\}\},$$

$$\{\frac{1}{1+\left\{w_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+w_2\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}_1}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{a}_2}^l:l\in I_{k_2}\}\}>$$

$$=<(\sum_{j=1}^{2}w_ja_j,\sum_{j=1}^{2}w_jb_j,\sum_{j=1}^{2}w_jc_j);\{1-\frac{1}{1+\left\{\sum_{j=1}^{2}w_j\left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_j\in\{w_{\tilde{a}_j}^i:i\in I_{m_j}\}\},\{\frac{1}{1+\left\{\sum_{j=1}^{2}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\beta_j \in \{u^r_{\tilde a_j} : r \in I_{n_j}\}\}, \{\frac{1}{1+\left\{\sum_{j=1}^{2} w_j \left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_j \in \{y^l_{\tilde a_j} : l \in I_{k_j}\}\} >$$

Thus the result is true for n=2. Let us assume that the result is true for n=s.

Then we have, $HTNWAAO_{T_2}(\tilde a_1, \tilde a_2, \tilde a_3, ..., \tilde a_s)$

$$= < (\sum_{j=1}^{s} w_j a_j, \sum_{j=1}^{s} w_j b_j, \sum_{j=1}^{s} w_j c_j), \{1-\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_j \in \{w^i_{\tilde a_j} : i \in I_{m_j}\}\}, \{\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}} :$$

$$\beta_j \in \{u^r_{\tilde a_j} : r \in I_{n_j}\}\}, \{\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_j \in \{y^l_{\tilde a_j} : l \in I_{k_j}\}\} >$$

Now for n=s+1, we have, $HTNWAO_{T_2}(\tilde a_1, \tilde a_2, \tilde a_3, ..., \tilde a_{s+1})$

$$= < (\sum_{j=1}^{s} w_j a_j, \sum_{j=1}^{s} w_j b_j, \sum_{j=1}^{s} w_j c_j); \{1-\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_j \in \{w^i_{\tilde a_j} : i \in I_{m_j}\}\},$$

$$\{\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}} : \beta_j \in \{u^r_{\tilde a_j} : r \in I_{n_j}\}\}, \{\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_j \in \{y^l_{\tilde a_j} : l \in I_{k_j}\}\} >$$

$$\oplus'' < (w_{s+1} a_{s+1}, w_{s+1} b_{s+1}, w_{s+1} c_{s+1}); \{1-\frac{1}{1+\left\{w_{s+1} \left(\frac{\alpha_{(s+1)}^2}{1-\alpha_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_{s+1} \in \{w^i_{\tilde a_{s+1}} : i \in I_{m_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{w_{s+1} \left(\frac{1-\beta_{(s+1)}^2}{\beta_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}} : \beta_{s+1} \in \{u^r_{\tilde a_{s+1}} : r \in I_{n_{s+1}}\}\}, \{\frac{1}{1+\left\{w_{s+1} \left(\frac{1-\lambda_{(s+1)}^2}{\lambda_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_{s+1} \in \{y^l_{\tilde a_{s+1}} : l \in I_{k_{s+1}}\}\} >$$

$$= < (w_1 a_1 + w_2 a_2, w_1 b_1 + w_2 b_2, w_1 c_1 + w_2 c_2),$$

$$\{1-\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{\alpha_{jk}^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}}}{1-1+\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2 + \left(\frac{1-\frac{1}{1+\left\{w_{s+1} \left(\frac{\alpha_{(s+1)}^2}{1-\alpha_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}{1-1+\frac{1}{1+\left\{w_{s+1} \left(\frac{\alpha_{(s+1)}^2}{1-\alpha_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}} : \alpha_j \in \{w^i_{\tilde a_j} : i \in I_{m_j}\}, \alpha_{s+1} \in \{w^i_{\tilde a_{s+1}} : i \in I_{m_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{\left(1-\frac{\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(1-\frac{\frac{1}{1+\left\{w_{s+1}\left(\frac{1-\beta_{(s+1)}^2}{\beta_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_{s+1}\left(\frac{1-\beta_{(s+1)}^2}{\beta_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\beta_j\in\{u_{\tilde{a}_j}^r:r\in I_{n_j}\},\beta_{s+1}\in\{u_{\tilde{a}_{s+1}}^r:r\in I_{n_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{\left(1-\frac{\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(1-\frac{\frac{1}{1+\left\{w_{s+1}\left(\frac{1-\lambda_{(s+1)}^2}{\lambda_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_{s+1}\left(\frac{1-\lambda_{(s+1)}^2}{\lambda_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\lambda_j\in\{y_{\tilde{a}_j}^l:l\in I_{k_j}\},\lambda_{s+1}\in\{y_{\tilde{a}_{s+1}}^l:$$

$$l\in I_{k_{s+1}}\}\}>$$

$$=<(\sum_{j=1}^{s+1}w_ja_j,\sum_{j=1}^{s+1}w_jb_j,\sum_{j=1}^{s+1}w_jc_j),\{1-\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2+w_{s+1}\left(\frac{\alpha_{(s+1)}^2}{1-\alpha_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\alpha_j\in\{w_{\tilde{a}_j}^i:i\in I_{m_j}\},\alpha_{s+1}\in\{w_{\tilde{a}_{s+1}}^i:i\in I_{m_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2+w_{s+1}\left(\frac{1-\beta_{(s+1)}^2}{\beta_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_j\in\{u_{\tilde{a}_j}^r:r\in I_{n_j}\},\beta_{s+1}\in\{u_{\tilde{a}_{s+1}}^r:r\in I_{n_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2+w_{s+1}\left(\frac{1-\lambda_{(s+1)}^2}{\lambda_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_j\in\{y_{\tilde{a}_j}^l:l\in I_{k_j}\},\lambda_{s+1}\in\{y_{\tilde{a}_{s+1}}^l:l\in I_{k_{s+1}}\}\}>$$

$$=<(\sum_{j=1}^{s+1}w_ja_j,\sum_{j=1}^{s+1}w_jb_j,\sum_{j=1}^{s+1}w_jc_j);\{1-\frac{1}{1+\left\{\sum_{j=1}^{s+1}w_j\left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_j\in\{w_{\tilde{a}_j}^i:i\in I_{m_j}\},\alpha_{s+1}\in\{w_{\tilde{a}_{s+1}}^i:i\in I_{m_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{\sum_{j=1}^{s+1}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_j\in\{u_{\tilde{a}_j}^r:r\in I_{n_j}\},\beta_{s+1}\in\{u_{\tilde{a}_{s+1}}^r:r\in I_{n_{s+1}}\}\},\{\frac{1}{1+\left\{\sum_{j=1}^{s+1}w_j\left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\lambda_j\in\{y_{\tilde{a}_j}^l:l\in I_{k_j}\},\lambda_{s+1}\in\{y_{\tilde{a}_{s+1}}^l:l\in I_{k_{s+1}}\}\}>$$

Thus the result is true for n=s+1 also. Hence by the principle of mathematical induction, the result is true for any natural number n.

Theorem 21: Let $\tilde{a}_j =<(a_j,b_j,c_j);\{w^i_{\tilde{a}_j} : i\in I_{m_j}\},\{u^r_{\tilde{a}_j} : r\in I_{n_j}\},\{y^l_{\tilde{a}_j} : l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number $\tilde{\theta}$, we have,

(i) $HTNWAAO_{T_2}\ \tilde{\theta}\oplus''\tilde{a}_1,\tilde{\theta}\oplus''\tilde{a}_2,\tilde{\theta}\oplus''\tilde{a}_3,......,\tilde{\theta}\oplus''\tilde{a}_n\ =\tilde{\theta}\oplus'' HTNWAAO_{T_2}\ \tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n$

(ii) $HTNWAAO_{T_2}\ \tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n\ =\tilde{\theta}$ if $\tilde{a}_j=\tilde{\theta}$ for each j

Proof: Straight forward .

Definition 22: Let $\tilde{a}_j =<(a_j,b_j,c_j);\{w^i_{\tilde{a}_j} : i\in I_{m_j}\},\{u^r_{\tilde{a}_j} : r\in I_{n_j}\},\{y^l_{\tilde{a}_j} : l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophicweighted geometric aggregation operator of type-2 is denoted by $HTNWGAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)$ and is defined by:

$$HTNWGAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)=(w_1 *''\tilde{a}_1)\otimes''(w_2 *''\tilde{a}_2)\otimes''(w_3 *''\tilde{a}_3)\otimes''........\otimes''(w_n *''\tilde{a}_n)$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……, n) such that $w_j\geq 0$ and $\sum_{j=1}^{n} w_j=1$.

Theorem 23: Let $\tilde{a}_j =<(a_j,b_j,c_j);\{w^i_{\tilde{a}_j} : i\in I_{m_j}\},\{u^r_{\tilde{a}_j} : r\in I_{n_j}\},\{y^l_{\tilde{a}_j} : l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then $HTNWGAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)$ is a hesitant triangular neutrosophic number and

$$HTNWGAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)$$
$$=<(\prod_{j=1}^{n} a_j^{\,w_j},\prod_{j=1}^{n} b_j^{\,w_j},\prod_{j=1}^{n} c_j^{\,w_j});\{\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{1-\alpha_j^2}{\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_j\in\{w^i_{\tilde{a}_j} : i\in I_{m_j}\}\},\{1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{\beta_j^2}{1-\beta_j^2}\right)^2\right\}^{\frac{1}{2}}} :$$
$$\beta_j\in\{u^r_{\tilde{a}_j} : r\in I_{n_j}\}\},\ \{1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{\lambda_j^2}{1-\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_j\in\{y^l_{\tilde{a}_j} : l\in I_{k_j}\}>$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……,n) such that $w_j\geq 0$ and $\sum_{j=1}^{n} w_j=1$.

Proof: Similar to the proof of Theorem 20.

Theorem 24: Let $\tilde{a}_j =<(a_j,b_j,c_j);\{w^i_{\tilde{a}_j} : i\in I_{m_j}\},\{u^r_{\tilde{a}_j} : r\in I_{n_j}\},\{y^l_{\tilde{a}_j} : l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number $\tilde{\theta}$, we have,

(i) $HTNWGAO_{T_2}\ \tilde{\theta}\otimes''\tilde{a}_1,\tilde{\theta}\otimes''\tilde{a}_2,\tilde{\theta}\otimes''\tilde{a}_3,......,\tilde{\theta}\otimes''\tilde{a}_n\ =\tilde{\theta}\otimes'' HTNWGAO_{T_2}\ \tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n$

(ii) $HTNWGAO_{T_2}\ \tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n\ =\tilde{\theta}$ if $\tilde{a}_j=\tilde{\theta}$ for each j

Proof: Straight forward .

Definition 25: Let $\tilde{a}=<(a_1,b_1,c_1);\{w^i_{\tilde{a}} : i\in I_m\},\{u^j_{\tilde{a}} : j\in I_n\},\{y^l_{\tilde{a}} : l\in I_k\}>$ be a hesitant triangular neutrosophic number. Then the score of $\tilde{a}$ is defined by:

$$S(\tilde{a})=\frac{1}{3\times\max\{a_1,b_1,c_1\}}(a_1+b_1+c_1)\times\left[2+\frac{1}{m}\sum_{j=1}^{m}\alpha_j-\frac{1}{n}\sum_{j=1}^{n}\beta_j-\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right]$$

Where $\alpha_j\in\{w^i_{\tilde{a}_j}:i\in I_{m_j}\},\beta_j\in\{u^r_{\tilde{a}_j}:r\in I_{n_j}\},\lambda_j\in\{y^l_{\tilde{a}_j}:l\in I_{k_j}\}$

If $\tilde{a}_j=<(a_j,b_j,c_j);\{w^i_{\tilde{a}_j}:i\in I_{m_j}\},\{u^r_{\tilde{a}_j}:r\in I_{n_j}\},\{y^l_{\tilde{a}_j}:l\in I_{k_j}\}>(j=1,2)$ be two hesitant triangular neutrosophic numbers, then, the comparison method is given as;

I. If $S(\tilde{a}_1)>S(\tilde{a}_2)$ then $\tilde{a}_1\succ\tilde{a}_2$

II. If $S(\tilde{a}_1)=S(\tilde{a}_2)$ then $\tilde{a}_1=\tilde{a}_2$

5. APPLICATION OF HESITANT TRAPEZOIDAL NEUTROSOPHIC NUMBERS:

In this section, we apply the weighted aggregation operators and the score function of hesitant triangular neutrosophic numbers to the multi-attribute decision-making problem with hesitant triangular neutrosophic information.

Let $X=\{A_1,A_2,A_3,.....,A_m\}$ be a set of alternatives, $A=\{c_1,c_2,c_3,.....,c_n\}$ be a set of attributes and $w=\{w_1,w_2,w_3,.....,w_n\}$ be a set of weights (w_j is the weight of attribute c_j (j=1,2,3,……,n) such that $w_j\geq 0$ and $\sum_{j=1}^{n}w_j=1$.) In this case, the characteristic of the alternative $A_i(i=1,2,...,m)$ on attribute c_j $(j=1,2,...,n)$ is represented by the following form of a hesitant triangular neutrosophic number:

$A_{ij}=<(a_{ij},b_{ij},c_{ij});\{w^p_{\tilde{a}_{ij}}:p\in I_{m_{ij}}\},\{u^r_{\tilde{a}_{ij}}:r\in I_{n_{ij}}\},\{y^l_{\tilde{a}_{ij}}:l\in I_{k_{ij}}\}>$.

Now, we construct a multi-attribute decision making method by the following algorithm:

- **ALGORITHM:**

Step-1: Express the evaluation results of the expert based on the alternative $A_i(i=1,2,...,m)$ on attribute $c_j\,(1,2,\dots,n)$ in terms of hesitant triangular neutrosophic numbers x_{ij} as a $m\times n$ Table.

Step-2: Compute the aggregation values $g_i^{T_k^A}$ $(i=1,2,...,m)$ $(k=1,2)$ *or* $g_i^{T_k^G}$ $(i=1,2,...,m)$ $(k=1,2)$ of $A_i(i=1,2,...,m)$ as;

$$g_i^{T_k^A}=HTNWAAO_{T_k}(A_{i1},A_{i2},...,A_{in})\ (i=1,2,...,m)\ (k=1,2)$$

or

$$g_i^{T_k^G}=HTNWGAO_{T_k}(A_{i1},A_{i2},...,A_{in})\ (i=1,2,...,m)\ (k=1,2)$$

Step-3: Calculate the score values of $g_i^{T_k^A}$ $(i=1,2,...,m)$ $(k=1,2)$ *or* $g_i^{T_k^G}$ $(i=1,2,...,m)$ $(k=1,2)$ of $A_i(i=1,2,...,m)$ based on Definition 25.

Step-4: Rank the alternatives by using definition 25.

Example 22:

Let us consider a decision making problem adapted from Wei et al. (2017). Suppose an organisation plans to implement enterprise resource planning (ERP) system. The first step is to form a project team that consists of CIO and two senior representatives from user departments. By collecting all possible information about ERP vendors and systems, project team chooses five potential ERP systems A_i (i=1, 2, 3, 4, 5) as candidates. The company employs some external professional organizations (or experts) to aid this decision making. The project team selects four attributes to evaluate the alternatives: function and technology c_1, strategic fitness c_2, vendor's ability c_3 and vendor's reputation c_4. The five possible ERP systems A_i (i=1, 2, 3, 4, 5) are to be evaluate during the hesitant triangular neutrosophic numbers by the decision makers under the above four attributes whose weighting vector is $w=(0.3,0.3,0.2,0.2)$.

Step-1: We express the initial evaluation results of the expertforfivepossiblealternativesbased on fourattributesbythe form of hesitant triangular neutrosophic numbers, as shownin Table 1.

Table 1: The evaluation result by the expert is shown in the below table

	c_1	c_2	c_3	c_4
A_1	<(0.3,0.4,0.2); {0.5,0.6},{0.1,0.3},{0.4, 0.7}>	<(0.1,0.5,0.6);{0.3},{0.4, 0.5},{0.3,0.6}>	<(0.5,0.6,0.7);{0.2,0.4} ,{0.5},{0.3,0.4}>	<(0.4,0.6,0.7);{0.8},{0.6, 0.7},{0.1,0.2}>
A_2	<(0.2,0.3,0.4); {0.2}, {0.5,0.7},{0.4}>	<(0.4,0.6,0.6);{0.7,0.9},{ 0.2},{0.5,0.7}>	<(0.5,0.7,0.9);{0.1,0.2} , {0.6,0.8},{0.5,0.6}>	<(0.2,0.3,0.5);{0.6},{0.4, 0.5},{0.2,0.4}>
A_3	<(0.6,0.6,0.7);0.1,0.4},{0 .6,0.8},{0.4,0.5,0.7}>	<(0.3,0.4,0.4);{0.6, 0.7},{0.5},{0.5,0.8}>	<(0.1,0.2,0.3);{0.2,0.3} ,{0.4,0.6},{0.5,0.6}>	<(0.5,0.5,0.6);{0.3,0.5}, {0.3,0.5},{0.2}>
A_4	<(0.1,0.3,0.3); {0.7}, {0.6,0.9},{0.4}	<(0.5,0.5,0.5);{0.1,0.2},{ 0.4,0.7},{0.2,0.5}>	<(0.4,0.5,0.6);{0.2,0.6} ,{0.2,0.4},{0.3,0.4}>	<(0.1,0.1,0.2);{0.3,0.5}, {0.7,0.8},{0.1,0.2}>
A_5	<(0.5,0.6,0.6); {0.3,0.6}, {0.4,0.5},{0.2,0.4}>	<(0.2,0.3,0.3); {0.3,0.5},{0.4,0.5},{0.4, 0.6}>	<(0.7,0.8,0.8);{0.5,0.7} ,{0.3,0.6},{0.1,0.3}>	<(0.2,0.3,0.5); {0.4,0.5},{0.2,0.3},{0.6, 0.7,0.8}>

Step-2: We compute the aggregationvalues $g_i^{T_1^A}$ $(i=1,2,...,5)$ of $A_i\,(i=1,2,...,5)$ as;

$g_1^{T_1^A} = HTNWAAO_{T_1}(A_{11}, A_{12}, A_{13}, A_{14})$

= <(0.30,0.51,0.52); {0.494124,0.522409, 0.526880, 0.553333}, {0.299254,0.308624,0.319973,0.329992,0.416080, 0.429108, 0.444888,0.458818},{0.262529, 0.301566,0.278077,0.319426, 0.323211, 0.371272, 0.342353, 0.393260,0.310519,0.356693, 0.328909,0.377818,0.382294, 0.439141,0.412567,0.465148}>

$g_2^{T_1^A} = HTNWAAO_{T_1}(A_{21}, A_{22}, A_{23}, A_{24})$

= <(0.32,0.47,0.58); {0.468719,0.481089,0.617891,0.626787},{0.376740,0.393934, 0.399052,0.417264,0.416754, {0.389321,0.447212,0.403779, 0.435774,0.441436,0.461583},0.463821,0.430672,0.494712, 0.446666,0.513085}>

$g_3^{T_1^A} = HTNWAAO_{T_1}(A_{31}, A_{32}, A_{33}, A_{34})$

= <(0.39,0.44,0.51);{0.419636, 0.457406,0.434930,0.471704,0.467623,0.502270,0.481653, 0.515387,0.344568, 0.387223, 0.361840,0.403371,0.398762, 0.437891, 0.414606,0.452704},{0.456007,0.505058,0.494527, 0.547722,0.497111, 0.550583, 0.539102,0.597092},{0.389321, 0.448274,0.479310,0.460489, 0.416275,0.530219,0.403779, 0.464922,0.497111,0.477590, 0.431735,0.549910}>

$g_4^{T_1^A} = HTNWAAO_{T_1}(A_{41}, A_{42}, A_{43}, A_{44})$

= <(0.39,0.44,0.51);{0.398762,0.437891, 0.476592,0.510656,0.419636,0.457406, 0.494764,0.527644},{0.439833, 0.451737,0.505235,0.518910,0.520235,0.534315, 0.597593,0.613767,0.496724,0.510168, 0.570586,0.586030,0.587525, 0.603427, 0.674889,0.693156},{0.232461,0.267027,0.282842,0.246228,0.306007,0.351510, 0.372328,0.324130}>

$g_5^{T_1^A} = HTNWAAO_{T_1}(A_{51}, A_{52}, A_{53}, A_{54})$

= <(0.39,0.49,0.53);{0.365425,0.388147,0.427055, 0.447570,0.426353,0.446894, 0.482066,0.500612,0.463497,0.482708,0.515603,0.532948,0.515010, 0.532376,0.562112,0.577792},{0.328749, 0.356519,0.377634,0.409533,0.351510,0.381202,0.403779,0.437887,0.375847, 0.407595,0.431735,0.468204}, {0.267027,0.275388,0.282842,0.332644,0.343059,0.352345, 0.301566,0.311008,0.319426,0.375670, 0.387433, 0.397920,0.328749,0.339042,0.348219, 0.409533,0.422356,0.433787,0.371272,0.382897, 0.393260,0.462505, 0.476986,0.489897}>

Step-3: We calculate the score values of $g_i^{T_k^A}\ (i=1,2,...,5)$ of $A_i\ (i=1,2,...,5)$ as

$$S(A_1)=\frac{(0.30+0.51+0.52)}{3\times 0.52}\times[2+\frac{1}{4}(0.494124+0.522409+0.526880+0.553333)$$
$$-\frac{1}{8}(0.299254+0.308624+0.319973+0.329992+0.416080+0.429108+0.444888+0.458818)$$
$$-\frac{1}{16}(0.262529+0.301566+0.278077+0.319426+0.323211+0.371272+0.342353+0.393260$$
$$+0.310519+0.356693+0.328909+0.377818+0.382294+0.439141+0.412567+0.465148)$$
$$=\frac{1.33}{1.56}\times\left[2+0.524186-0.375842-0.354048\right]$$
$$=1.5297,$$

Similarly, we have; $S(A_2)=1.3244$, $S(A_3)=1.2687$, $S(A_4)=1.4110, S(A_5)=1.5235.$

Step-4: Since $S(A_1)>S(A_5)>S(A_4)>S(A_2)>S(A_3)$, So $A_1\succ A_5\succ A_4\succ A_2\succ A_3$.

Thus we conclude that A_1 is the best (most desirable) ERP system. On the other hand, if we apply the other proposed weighted aggregation operators such as $HTNWGAO_{T_1}$, $HTNWAAO_{T_2}$, $HTNWGAO_{T_2}$ for computing the best alternative(s), then step 2 of the proposed approach has been executed for each weighted aggregation operators and hence their corresponding hesitant triangular neutrosophic number has been constructed. Finally, based on these, the score values of the aggregated hesitant triangular neutrosophic numbers are computed and ranking has been done which are summarized in table-2. We can conclude from table-2 that although the ranking orders of the alternatives are slightly different; the best (most desirable) alternative is still A_1 in all cases.

Table-2: Ranking order of alternatives

Weighted aggregation operators	Ranking	Best alternative
$HTNWAAO_{T_1}$	$A_1\succ A_5\succ A_4\succ A_2\succ A_3$	A_1
$HTNWGAO_{T_1}$	$A_1\succ A_5\succ A_2\succ A_4\succ A_3$	A_1
$HTNWAAO_{T_2}$	$A_1\succ A_2\succ A_5\succ A_3\succ A_4$	A_1
$HTNWGAO_{T_2}$	$A_1\succ A_2\succ A_3\succ A_5\succ A_4$	A_1

6. COMPARATIVE STUDY:

In order to compare the performance of the proposed method with some existing methods (Ye 2013a, Ye 2014, Ye 2015a, Ye 2015b, Liu 2016, Abdel-Basset et al. 2017, Wei et al. 2017), a comparative study is presented and their corresponding final ranking are summarized in table 3. From table-3, it is clear that although the ranking order of the alternatives are slightly different, but the best (most desirable) alternative is the same as found in the existing approaches (Ye 2013a, Ye 2014, Ye 2015a, Ye 2015b, Liu 2016, Abdel-Basset et al. 2017). Thus, our proposed method can be suitably utilized to solve the multi attribute decision making problems than the other existing methods due to the fact that more fuzziness and uncertainties are involved in our proposed approach.

Table 3: Comparative study

Existing approach	Ranking	Our proposed method	Ranking	Best alternative
Ye **[36]**	$A_2 \succ A_4 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
		$HTNWGAO_{T_1}$	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2
		$HTNWAAO_{T_2}$	$A_2 \succ A_4 \succ A_1 \succ A_3$	A_2
		$HTNWGAO_{T_2}$	$A_2 \succ A_1 \succ A_4 \succ A_3$	A_2
Ye **[17]**	$A_2 \succ A_4 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
		$HTNWGAO_{T_1}$	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2
		$HTNWAAO_{T_2}$	$A_2 \succ A_4 \succ A_1 \succ A_3$	A_2
		$HTNWGAO_{T_2}$	$A_2 \succ A_1 \succ A_4 \succ A_3$	A_2
Ye **[40]**	$A_2 \succ A_4 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
		$HTNWGAO_{T_1}$	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2
		$HTNWAAO_{T_2}$	$A_2 \succ A_4 \succ A_1 \succ A_3$	A_2
		$HTNWGAO_{T_2}$	$A_2 \succ A_1 \succ A_4 \succ A_3$	A_2
Ye **[38]**	$A_4 \succ A_2 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_4 \succ A_2 \succ A_3 \succ A_1$	A_4
		$HTNWGAO_{T_1}$	$A_4 \succ A_3 \succ A_2 \succ A_1$	A_4
		$HTNWAAO_{T_2}$	$A_4 \succ A_2 \succ A_1 \succ A_3$	A_4
		$HTNWGAO_{T_2}$	$A_4 \succ A_1 \succ A_2 \succ A_3$	A_4
Liu **[21]**	$A_4 \succ A_2 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_4 \succ A_2 \succ A_3 \succ A_1$	A_4
		$HTNWGAO_{T_1}$	$A_4 \succ A_3 \succ A_2 \succ A_1$	A_4
		$HTNWAAO_{T_2}$	$A_4 \succ A_2 \succ A_1 \succ A_3$	A_4
		$HTNWGAO_{T_2}$	$A_4 \succ A_1 \succ A_2 \succ A_3$	A_4
Abdel-Basset et al. **[57]**	$A_4 \succ A_2 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_4 \succ A_2 \succ A_3 \succ A_1$	A_4
		$HTNWGAO_{T_1}$	$A_4 \succ A_3 \succ A_2 \succ A_1$	A_4
		$HTNWAAO_{T_2}$	$A_4 \succ A_2 \succ A_1 \succ A_3$	A_4
		$HTNWGAO_{T_2}$	$A_4 \succ A_1 \succ A_2 \succ A_3$	A_4

7. CONCLUSION

In this paper, hesitant triangular neutrosophic numbers and their basic properties are presented. Also, various types of operations between the hesitant triangular neutrosophic numbers are discussed. Then, various types of hesitant triangular neutrosophic weighted aggregation operators are proposed to aggregate the hesitant triangular neutrosophic information. Furthermore, score of hesitant triangular neutrosophic numbers is proposed to ranking the hesitant triangular neutrosophic numbers.Based on the hesitant triangular neutrosophic weighted aggregation operators and score of hesitant triangular neutrosophic numbers, a multi attribute decision making method is developed, in which the evaluation values of alternatives on the attribute are represented in terms of hesitant triangular neutrosophic numbers and the alternatives are ranked according to the values of the score of hesitant triangular neutrosophic numbers to select the most desirable one. Finally, a practical example for enterprise resource planning (ERP) system selection is presented to demonstrate the application and effectiveness of the proposed method. The advantage of the proposed method is that it is more suitable for solving multi attribute decision making problems with hesitant triangular neutrosophic information because hesitant triangular neutrosophic numbers can handle indeterminate and inconsistent information and are the extensions of hesitant triangular fuzzy numbers, hesitant triangular intuitionistic fuzzy numbersas well as triangular neutrosophic numbers.

In the future, we will develop another approach called linguistic hesitant triangular neutrosophic number as a further generalization of it and this will be applied in different practical problems.

FUNDING: This research received no external funding.

ACKNOWLEDGEMENTS: Nil.

CONFLICTS OF INTEREST: The authors declare no conflict of interest.

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Received: April 11, 2020. Accepted: July 2, 2020