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Neutrosophic Beta Distribution with Properties and Applications

Rehan Ahmad Khan Sherwani[1,*], Mishal Naeem[1], Muhammad Aslam[2], Muhammad Ali Raza[3], Muhammad Abid[3], Shumaila Abbas[1]

[1] College of Statistical and Actuarial Sciences, University of the Punjab Lahore, Pakistan. rehan.stat@pu.edu.pk; mishalbutt45@gmail.com; shumaila.stat@pu.edu.pk; sana.stat@pu.edu.pk

[2] Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21551, Saudi Arabia;

magmuhammad@kau.edu.sa

[3]Govt. College University Faisalabad, Pakistan; ali.raza@gcuf.edu.pk; mabid@gcuf.edu.pk

* Correspondence: rehan.stat@pu.edu.pk

Abstract: This research is an extension of classical statistics distribution theory as the theory did not deal with the problems having ambiguity, impreciseness, or indeterminacy. An important life-time distribution called Beta distribution from classical statistics is proposed by considering the indeterminate environment and named the new proposed distribution as neutrosophic beta distribution. Various distributional properties like mean, variance, moment generating function, r-th moment order statistics that includes smallest order statistics, largest order statistics, joint order statistics, and median order statistics are derived. The parameters of the proposed distribution are estimated via maximum likelihood method. Proposed distribution is applied on two real data sets and goodness of fit is assessed through AIC and BIC criteria's. The estimates of the proposed distribution suggested a better fit than the classical form of Beta distribution and recommended to use when the data in the interval form follows a Beta distribution and have some sort of indeterminacy.

1. **Introduction**

The real world is full of ambiguity, unclear, uncertain situations, and problems and a particular value cannot be assigned to the characteristics of the statistics in such an imprecise situation [1]. In such situations, classical probability fails to provide accurate results [2]. In recent times, several developments have been made to model such imprecise situations by considering fuzzy logic and neutrosophy [3-6]. Smarandache [7] proposed neutrosophic statistics that deal with indeterminacy or some part unclear aspect present in the data. He introduced the concept of neutrosophic logic in 1995 by representing the components as T, I, F that represents a true part, undetermined part, and falsehood. Several researchers have contributed to the theory of neutrosophic statistics both methodologically and applied form e.g., Alhabib [8] design the time-series theory under indeterminacy, Aslam et al. [9-14] extended the theory of control charts and sampling plans under indeterminacy environment and presented several neutrosophic control charts and sampling plans, [1, 15, 16] applied the neutrosophic theory in engineering problems. Various researchers have been done in terms of neutrosophic probability distributions to calculate indeterminacy in real-life problems and produced better results in comparison to classical statistics. Alhasan and Smarandache [17] proposed Neutrosophic Weibull distribution in terms of neutrosophic statistics. This distribution gives more space in the applied area due to its wide applicability in classical statistics that helps in solving more problems that have been ignored in classical statistics under indeterminacy.

Neutrosophic Uniform, Neutrosophic exponential, and Neutrosophic Poisson have been developed and solved numerically [5, 18]. Normal distribution and binomial distribution in terms of neutrosophy are explored in detail through many examples by [19]. Aslam and Ahtisham [2] proposed the neutrosophic form of Raleigh distribution. In this research, we proposed an important life-time Beta distribution in the form of neutrosophy and extended the applications of the classical beta distribution when the data is in interval form and has some form of indeterminacy. Several properties are explored under the newly proposed distribution and explained the applications with the help of simulated and real-life data examples.

2. **Neutrosophic Form**

In classical data, there is the crisp value or specific values to deal with but in neutrosophic statistics, data can be in any form because indeterminacy can occur in any form and it depends upon the type of problem we are solving. The form of Neutrosophic number in terms of extension of classical statistics has a standard form and is shown as follow:

$$X = a + i$$

where, a = determined/known part of the data and, i = uncertain/ indeterminacy part of the data. a and i can be any real number. $\mu_N \in [\mu_L, \mu_U]$

3. **Some existing neutrosophic continuous probability distributions**

The followings are the extended classical distributions with a neutrosophic logic in literature:

3.1 Neutrosophic Weibull Distribution

The probability density function of neutrosophic Weibull distribution is:

$$f_N(X) = \frac{\beta_N}{\alpha_N} X^{\beta_N - 1} e^{-\left(\frac{X}{\alpha_N}\right)^{\beta_N}}, \qquad X > 0, \alpha_N > 0, \beta_N > 0 \tag{1}$$

3.2 Neutrosophic Gamma Distribution

The probability density function of the neutrosophic Gamma distribution is:

$$f(t_N) = \frac{{b_N}^{a_N}}{\Gamma a_N} {t_N}^{a_N - 1} e^{-b_N t_N}; \; t_N, b_N, a_N > 0 \tag{2}$$

3.3 Neutrosophic Exponential Distribution

The probability density function of the neutrosophic exponential distribution is:

$$f_N(x) = \lambda_N e^{-x\lambda_N} ; x>0, \lambda_N > 0 \tag{3}$$

3.4 Neutrosophic Normal Distribution

Probability density function of neutrosophic normal distribution is:

$$X_N \sim N_N(\mu_N, {\sigma^2}_N) = \frac{1}{\sigma_N\sqrt{2\pi}} \exp\left(-\frac{(x-\mu_N)^2}{2{\sigma^2}_N}\right); X, \mu_N, \sigma_N > 0 \tag{4}$$

4. **Neutrosophic Beta Distribution**

A neutrosophic Beta Distribution (N-Beta) of a continuous variable X can or cannot be a classical beta distribution of X, having or not having its mean or parameters imprecise or unclear. Consider X as the classical random variable which has a neutrosophic beta distribution having neutrosophic parameters α_N , β_N i.e. X→N-beta ($x; \alpha_N , \beta_N$), then pdf is as follow:

$$f_N(X) = \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)} \qquad where\ X > 0 \tag{5}$$

α_N , β_N are the neutrosophic shape parameters.

with cdf

$$\mathrm{F_N(X)} = \mathrm{I_x}(\alpha_N, \beta_N) = \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)} \tag{6}$$

5. **Mathematical Properties**

Various properties of neutrosophic beta distribution for the r.v X→N-beta ($x; \alpha_N , \beta_N$) have been derived and the results are shown as follow:

Mean: $\mathrm{E_N(X)} = \frac{\alpha_N}{\alpha_N + \beta_N}$ (7)

Variance: $\mathrm{V_N(X)} = \frac{\alpha_N \beta_N}{(\alpha_N + \beta_N + 1)(\alpha_N + \beta_N)^2}$ (8)

R-th Moment: $\mathrm{E_N}(X^r) = \prod_{i=0}^{r-1} \frac{(\alpha_N + i)}{(\alpha_N + \beta_N + i)}$ (9)

Moment Generating Function: $\mathrm{E_N}(e^{tx}) = \sum_{k=0}^{\infty} \frac{t^k \alpha_N{}^{(k)}}{k!\,(\alpha_N + \beta_N)^{(k)}}$ (10)

Hazard Rate Function: $h_N(X) = \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N) - \beta(X, \alpha_N, \beta_N)}$ (11)

Survival Function: $S_N(X) = 1 - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}$ (12)

R-th Order Statistics: Let $X_1, X_2, \dots, X_r$ be the random sample from N-beta and let $X_{(1)}, X_{(2)}, \dots, X_{(r)}$ be the corresponding order statistics. R^{th} order statistics of neutrosophic beta distribution can be given as:

$$f_{N_{r,n}}(x) = \frac{1}{B(r, n-r+1)} \left[\frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{r-1} \left[1 - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{n-r} * \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)} \tag{13}$$

Smallest Order Statistics: $f_{N_{1:n}}(x) = n_N \left[1 - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{n-1} * \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)}$ (14)

Largest Order Statistics: $f_{N_{n:n}}(x) = n_N \left[\frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{n-1} * \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)}$ (15)

Joint Order Statistics:

$$f_{N_{m+1:n}}(x) = \frac{n_N!}{(i-1)!\,(n_N - j)!(i-j-1)!} * \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)} * \frac{Y^{\alpha_N - 1}(1-Y)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)} * \left[\frac{\beta(Y, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)} - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{j-i-1} \left[1 - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{n_N - j} * \left[\frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{i-1} \tag{16}$$

Median Order Statistics:

$$f_{N_{m+1:n}}(x) = \frac{(2m+1)!}{m! n_N!} \left[\frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{m} \left[1 - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{m} * \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)} \tag{17}$$

6. Parameter Estimation

In this section, the parameters of neutrosophic beta distribution has been calculated through the maximum likelihood method. Estimated parameters are given below:

$$\frac{\partial lnL}{\partial \alpha_N} = \sum_{i=1}^{n} lnx_i - \frac{n.\partial \ln[\beta(\alpha_N,\beta_N)]}{\partial \alpha_N} = 0 \quad (18)$$

$$\frac{\partial lnL}{\partial \beta_N} = \sum_{i=1}^{n} \ln(1-x_i) - n.\frac{\partial \ln[\beta(\alpha_N,\beta_N)]}{\partial \beta_N} = 0 \quad (19)$$

Theoretical estimation of parameters is not possible but they can be estimated mathematical simulation.

7. APPLICATIONS

In this section parameters of the proposed distribution will be estimated with the help of real-life data examples and the goodness of fit of the proposed distribution will be assessed by using the AIC and BIC criterias.

Case Study 1: Data has been taken from [19] and related to the exceedances of "flood peaks (in m 3/s) of the Wheaton river near Carcross in Yukon Territory, Canada". Parameters of the neutrosophic beta distribution are calculated by the MLE method and to see the performance of this model goodness of fit is calculated. We assumed beta as a neutrosophic parameter and alpha as a classical parameter. For comparison purposes classical beta distribution will also be used. The results are as follow:

Table 1. Parameter estimation and goodness of fit of N-Beta (α_N, β_N).

Distribution	**MLE Estimates**		**AIC**	**BIC**
	$\hat{\alpha}$	$\hat{\beta}$		
N-beta	0.8597	[1.9450,1.276]	[278.608, 83.8750]	[362.965, 53.9857]
Beta	0.9096	1.316	398.437	446.587

It can be seen that in terms of neutrosophy, neutrosophic statistics is more accurate to give results in terms of impreciseness instead of ignoring impreciseness and uncertainty. Hence we can say that the proposed neutrosophic Beta distribution is more accurate than the classical beta distribution when the data is in interval form and contains some sort of indeterminacy.

Case Study 2: Another real-life application has been done on data of automobiles that have been taken from an automobile manufacturing company in Korea. Twenty eight uncertain data observations have been taken [20]. Parameters of the neutrosophic Beta distribution are calculated by the MLE method and to see the performance of this model goodness of fit AIC, BIC method is used. We assumed beta as a neutrosophic parameter and alpha are taken as a classical parameter. For comparison purpose, classical beta distribution is used. The results are as follow:

From the results presented in TABLE 2, it can be seen that the proposed neutrosophic Beta distribution is more accurate as compared to classical distribution.

Table 2. Parameter estimation and goodness of fit of automobile data for N-Beta (α_N, β_N).

Distribution	MLE Estimates		AIC	BIC
	$\hat{\alpha}$	$\hat{\beta}$		
N-beta	1.6386	[2.8437,0.8561]	[728.068, 96.1830]	[658.7123, 49.1298]
Beta	3.6587	9.5462	898.3621	846.8352

8. CONCLUSION

A generalization of the classical beta distribution has been proposed in the form of neutrosophic beta distribution by considering the interval form of the data occurring in many real-life situations. we derived several properties of the proposed distribution that include the measure of location, measure of spread R-th moment, survival function, hazard rate function, and common forms of order statistics. Neutrosophic beta distribution is applied to two real-life data sets to see if they behave well as compared to classical beta distribution and found better. We conclude that in real life situations having some sort of uncertainty, and indeterminacy in it and follow the beta distribution than in such situation our proposed form of the beta distribution will perform better and provide more realistic results by coping the indeterminacy of the data.

Conflicts of Interest: The authors declare no conflict of interest.

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