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Some New Structures in Neutrosophic Metric Spaces

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Abstract: Neutrosophic sets deals with inconsistent, indeterminate and imprecise datas. The concept of Neutrosophic Metric Space (NMS) uses the idea of continuous t- norm and continuous t - conorm in intuitionistic fuzzy metric spaces. In this paper, we introduce the definition of subcompatible maps of types (J-1 and J-2). We extend the structure of weak non-Archimedian with the help of subcompatible maps of types (J-1 and J-2) in NMS. Finally, we obtain common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

Keywords: Weak non-Archimedean, NMS, Compatible map, Sub compatible, Subcompatible maps of types (J-1) and (J-2).

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1. Introduction

Fuzzy set was presented by Zadeh [22] as a class of elements with a grade of membership. Kramosil and Michalek [8] defined new notion called Fuzzy Metric Space (FMS). Later, many authors have examined the concept of fuzzy metric in various aspects. In 2013, Muthuraj and Pandiselvi [17] introduced the concept of compatible mappings of type (P-1) andtype (P-2) in generalized fuzzy metric spaces and obtains common fixed point theorems are obtained forcompatible maps of type (P-1) and type (P- 2). Since then, many authors have obtained fixed point results in fuzzy metric space using these compatible notions.

Atanassov [1] introduced and studied the notion of intuitionistic fuzzy set by generalizing the notion of fuzzy set. Park [9] defined the notion of intuitionistic fuzzy metric space as a generalization of fuzzy metric space. In 1998, Smarandache [14-16] characterized the new concept called neutrosophic logic and neutrosophic set and explored many results in it. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. Baset et al. [2] Explored the neutrosophic applications in dif and only iferent fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making.

In 2019, Kirisci et al [9] defined NMS as a generalization of IFMS and brings about fixed point theorems in complete NMS. Erduran et.al.[13] introduced the concept of weak non-Archimedean intuitionistic fuzzy metric space and proved a common fixed point theorem for a pair of generalized (φ, Ψ) – contractive mappings. Later Jeyaraman at el [19,20] proved Fixed point results in non-Archimedean generalized intuitionistic fuzzy metric spaces. In 2020, Sowndrarajan Jeyaraman and Florentin Smarandache [18] proved some fixed point results for contraction theorems in neutrosophic metric spaces.

In this paper, we introduce the definition of sub compatible maps and sub compatible maps of types (J-1) and (J-2) in weak non-Archimedean NMS and give some examples and relationship between these definitions. We extend the structure of weak non-Archimedian with the help of subcompatible maps of types (J-1 and J-2) in NMS. Thereafter, we prove common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

2. **Preliminaries**

Definition: 2.1

A binary operation * : [0, 1] x [0, 1] → [0, 1] is a continuous t-norm [CTN] if it satisfies the following conditions :

(i) * is commutative and associative,

(ii) * is continuous,

(iii) $\varepsilon_1 * 1 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,

(iv) $\varepsilon_1 * \varepsilon_2 \le \varepsilon_3 * \varepsilon_4$ whenever $\varepsilon_1 \le \varepsilon_3$ and $\varepsilon_2 \le \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, 1]$.

Definition: 2.2

A binary operation ⋄ : [0, 1] x [0, 1] → [0, 1] is a continuous t-conorm [CTC] if it satisfies the following conditions:

(i) ⋄ is commutative and associative,

(ii) ⋄ is continuous,

(iii) $\varepsilon_1 \diamond 0 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,

(iv) $\varepsilon_1 \diamond \varepsilon_2 \le \varepsilon_3 \diamond \varepsilon_4$ whenever $\varepsilon_1 \le \varepsilon_3$ and $\varepsilon_2 \le \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4 \in [0, 1]$.

Definition: 2.3

A 6-tuple $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is said to be an NMS (shortly NMS), if Σ is an arbitrary non empty set, $*$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC and Ξ, Θ *and* Υ are neutrosophic on $\Sigma^3 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \delta, \omega \in \Sigma$, $\lambda \in \mathbb{R}^+$.

1. $0 \le \Xi(\zeta, \eta, \delta, \lambda) \le 1$; $0 \le \Theta(\zeta, \eta, \delta, \lambda) \le 1$; $0 \le \Upsilon(\zeta, \eta, \delta, \lambda) \le 1$;
2. $\Xi(\zeta, \eta, \delta, \lambda) + \Theta(\zeta, \eta, \delta, \lambda) + \Upsilon(\zeta, \eta, \delta, \lambda) \le 3$;
3. $\Xi(\zeta, \eta, \delta, \lambda) = 1$ if and only if $\zeta = \eta = \delta$;
4. $\Xi(\zeta, \eta, \delta, \lambda) = \Xi(\rho(\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
5. $\Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \mu) \le \Xi(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;

6. $\Xi(\zeta, \eta, \delta, .) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous ;
7. $\lim_{\lambda\to\infty} \Xi(\zeta, \eta, \delta, \lambda) = 1$ for all $\lambda > 0$;
8. $\Theta(\zeta, \eta, \delta, \lambda) = 0$ if and only if $\zeta = \eta = \delta$;
9. $\Theta(\zeta, \eta, \delta, \lambda) = \Theta(\rho(\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
10. $\Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, \mu) \geq \Theta(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
11. $\Theta(\zeta, \eta, \delta, .) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous;
12. $\lim_{\lambda\to\infty} \Theta(\zeta, \eta, \delta, \lambda) = 0$ for all $\lambda > 0$;
13. $\Upsilon(\zeta, \eta, \delta, \lambda) = 0$ if and only if $\zeta = \eta = \delta$;
14. $\Upsilon(\zeta, \eta, \delta, \lambda) = \Upsilon(\rho(\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
15. $\Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, \mu) \geq \Upsilon(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
16. $\Upsilon(\zeta, \eta, \delta, .) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous;
17. $\lim_{\lambda\to\infty} \Upsilon(\zeta, \eta, \delta, \lambda) = 0$ for all $\lambda > 0$;
18. If $\lambda > 0$ then $\Xi(\zeta, \eta, \delta, \lambda) = 0$; $\Theta(\zeta, \eta, \delta, \lambda) = 1$; $\Upsilon(\zeta, \eta, \delta, \lambda) = 1$.

Then, (Ξ, Θ, Υ) is called an NMS on Σ. The functions Ξ, Θ *and* Υ denote degree of closedness, neturalness and non-closedness between ζ, η *and* δ with respect to λ respectively.

Example: 2.4

Let (Σ, D) be a metric space. Define $\omega * \tau = \min\{\omega, \tau\}$ and $\omega \diamond \tau = \max\{\omega, \tau\}$ and $\Xi, \Theta, \Upsilon : \Sigma^3 \times \mathbb{R}^+ \rightarrow [0, 1]$ defined by, we define $\Xi(\zeta, \eta, \delta, \lambda) = \frac{\lambda}{\lambda + D(\zeta,\eta,\delta)}$; $\Theta(\zeta, \eta, \delta, \lambda) = \frac{D(\zeta,\eta,\delta)}{\lambda + D(\zeta,\eta,\delta)}$; $\Upsilon(\zeta, \eta, \delta, \lambda) = \frac{D(\zeta,\eta,\delta)}{\lambda}$ for all $\zeta, \eta, \delta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is called NMS induced by a metric D the standard neutrosophic metric.

Remark: 2.5

In NMS $\Xi(\zeta, \eta, \delta, \lambda, .)$ is non-decreasing, $\Theta(\zeta, \eta, \delta, .)$ is non-increasing and $\Upsilon(\zeta, \eta, \delta, .)$ is decreasing for all $\zeta, \eta, \delta \in \Sigma$.

In the above definition, if the triangular inequality (v), (x) and (xv) are replaced by the following:

$$\Xi(\zeta, \eta, \delta, \max\{\lambda, \mu\}) \geq \Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \mu),$$
$$\Theta(\zeta, \eta, \delta, \min\{\lambda, \mu\}) \leq \Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, \mu),$$
$$\Upsilon(\zeta, \eta, \delta, \min\{\lambda, \mu\}) \leq \Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, \mu)$$

or equivalently

$$\Xi(\zeta, \eta, \delta, \lambda) \geq \Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \lambda),$$
$$\Theta(\zeta, \eta, \delta, \lambda) \leq \Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, \lambda),$$
$$\Upsilon(\zeta, \eta, \delta, \lambda) \leq \Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, \lambda).$$

Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is called non-Archimedean NMS. It is easy to check that the triangle inequality (NA) implies (5), (10) and (15), that is, every non-Archimedean NMS is itself an NMS.

Example:2.6

Let Σ be a non-empty set with at least two elements. Define $\Xi(\zeta, \eta, \delta, \lambda)$ by: If we define the neutrosophic set $(\Sigma, \Xi, \Theta, \Upsilon)$ by $\Xi(\zeta, \zeta, \zeta, \lambda) = 1, \Theta(\zeta, \zeta, \zeta, \lambda) = 0$ and $\Upsilon(\zeta, \zeta, \zeta, \lambda) = 0$ for all $\zeta \in \Sigma$ and $\lambda > 0$, and $\Xi(\zeta, \eta, \delta, \lambda) = 0$, $\Theta(\zeta, \eta, \delta, \lambda) = 1$ and $\Upsilon(\zeta, \eta, \delta, \lambda) = 1$, for $\zeta \neq \eta \neq \delta$ and $0 < \lambda \leq 1$, and $\Xi(\zeta, \eta, \delta, \lambda) = 1$, $\Theta(\zeta, \eta, \delta, \lambda) = 0$ and $\Upsilon(\zeta, \eta, \delta, \lambda) = 0$, for $\zeta \neq \eta \neq \delta$ and $\lambda > 1$. Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a non-Archimedean NMS with arbitrary $*$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC. Clearly $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is also an NMS.

Definition:2.7

In Definition 2.3, if the triangular inequality (v), (x) and (xv) are replaced by the following:

$\Xi(\zeta, \eta, \delta, \lambda) \geq \max\{\Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, , \lambda/2), \Xi(\zeta, \eta, \omega, \lambda/2) * \Xi(\omega, \delta, \delta, \lambda)\}$,

$\Theta(\zeta, \eta, \delta, \lambda) \leq \min\{\Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, , \lambda/2), \Theta(\zeta, \eta, \omega, \lambda/2) \diamond \Theta(\omega, \delta, \delta, \lambda)\}$,

$\Upsilon(\zeta, \eta, \delta, \lambda) \leq \min\{\Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, , \lambda/2), \Upsilon(\zeta, \eta, \omega, \lambda/2) \diamond \Upsilon(\omega, \delta, \delta, \lambda)\}$,

for all $\Xi, \Theta, \Upsilon \in \Sigma$ and $\lambda > 0$, then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$is said to be a Weak Non- Archimedean (WNA) NMS.

Obviously, every non-Archimedean NMS is itself a weak non-Archimedean NMS.

The inequality (WNA) does not imply that $\Xi(\zeta, \eta, \delta, \lambda, .)$ is non-decreasing , $\Theta(\zeta, \eta, \delta, .)$ is non-increasing and $\Upsilon(\zeta, \eta, \delta, .)$ is decreasing. Thus, a weak non-Archimedean NMS is not necessarily an NMS.

Example: 2.8

Let $\Sigma = [0, \infty)$ and define $\Xi(\zeta, \eta, \delta, \lambda)$; $\Theta(\zeta, \eta, \delta, \lambda)$ and $\Upsilon(\zeta, \eta, \delta, \lambda)$ by

$$\Xi(\zeta, \eta, \delta, \lambda) = \begin{cases} 1, & \zeta = \eta = \delta \\ \frac{\lambda}{\lambda+1}, & \zeta \neq \eta \neq \delta \end{cases},$$

$$\Theta(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta \\ \frac{1}{\lambda+1}, & \zeta \neq \eta \neq \delta \end{cases},$$

$$\Upsilon(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta \\ \lambda + 1, & \zeta \neq \eta \neq \delta \end{cases},$$

for all $\lambda > 0$. $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$.

Definition: 2.9

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$

$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0, \text{ and}$$

$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0,$$

for all $\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition: 2.10

Let Γ and Ω be self mappings of an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$. Then the mappings are said to be compatible of type (J-1), if

$$\lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$

$$\lim_{n\to\infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and}$$

$$\lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0,$$

for all $\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition: 2.11

Let Γ and Ω be self mappings of an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$. Then the mappings are said to be compatible of type (J-2), if

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$

$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and}$$

$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \lambda) = 0,$$

for all $\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition:2.12

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ into itself. The maps Γ and Ω are said to be Occasionally Weakly Compatible (OWC) if and only if there is a point $\zeta \in \Sigma$ which is a coincidence point of Γ and Ωat which Γ and Ω commute i.e., there is a point $\zeta \in \Sigma$ such that $\Gamma\zeta = \Omega\zeta$ and $\Gamma\Omega\zeta = \Omega\Gamma\zeta$.

Definition:2.13

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$. The maps Γ and Ω are said to be reciprocally continuous if $\lim_{n\to\infty} \Gamma\Omega\zeta_n = \Gamma\zeta$, $\lim_{n\to\infty} \Omega\Gamma\zeta_n, = \Omega\zeta$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

3. Types Of Subcompatible Maps In Weak Non-Archimedean NMS.

Definition:3.1

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subsequently continuous if there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfy $\lim_{n\to\infty} \Gamma\mathcal{S}\zeta_n = \Gamma\zeta$, $\lim_{n\to\infty} \Omega\Gamma\zeta_n, = \Omega\zeta$.

Clearly, if Γ and $\Omega$are continuous or reciprocally continuous, then they are subsequentially continuous, but converse is not true in general.

Example: 3.2

Let $\Sigma = [0, \infty)$ and define, for all $\lambda > 0$, $\Xi\,(\zeta, \eta, \delta, \lambda)$; $\Theta\,(\zeta, \eta, \delta, \lambda)$ and $\Upsilon\,(\zeta, \eta, \delta, \lambda)$ by

$$\Xi\,(\zeta, \eta, \delta, \lambda) = \begin{cases} 1, & \zeta = \eta = \delta, \\ \frac{\lambda}{\lambda + 1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Theta\,(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \frac{1}{\lambda + 1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Upsilon\,(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \lambda + 1, & \zeta \neq \eta \neq \delta. \end{cases}$$

Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$. Define Γ and Ωas follows:

$$\Gamma\zeta = \begin{cases} 2, & \zeta < 3 \\ \zeta, & \zeta \geq 3 \end{cases}, \quad \Omega\zeta = \begin{cases} 2\zeta - 4, & \zeta \leq 3, \\ 3, & \zeta > 3. \end{cases}$$

Clearly Γ and Ω are discontinuous at $\zeta = 3$. Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 3 - \frac{1}{n}$ for n = 1, 2 …, then $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 2$, $2 \in \Sigma$ and $\lim_{n\to\infty} \Gamma\Omega\zeta_n = 2 = \Gamma(2)$, $\lim_{n\to\infty} \Omega\Gamma\zeta_n = 0 = \Omega(2)$. Therefore, Γ and Ω are subsequentially continuous. Now, let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 3 + \frac{1}{n}$ for n = 1,2,…, then $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 3$, $3 \in \Sigma$ and $\lim_{n\to\infty} \Omega\Gamma\zeta_n = 3 \neq 2 = \Omega(3)$. Hence Γ and Ω are not reciprocally continuous.

Definition: 3.3

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subcompatible if and only if there exist a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfies

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0.$$

It is easy to see that two owc maps are subcompatible, however the converse is not true in general. It is also interesting to see the following one-way implication:

Commuting ⇒ Weakly commuting ⇒ Compatibility ⇒ Weak compatibility ⇒ OWC⇒ Sub compatibility.

Definition:3.4

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$be a weak non-Archimedean NMS. Self- maps Γ and Ωon Σ are said to be subcompatible of type (J-1) if there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfies

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda)= 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Xi\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda)= 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and,}$$
$$\lim_{n\to\infty} \Upsilon\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Clearly, if Γ and Ω are compatible of type (J-1), then they are subcompatible of type (J-1), but converse is not true in general.

Example: 3.5

Let Σ = [0, ∞). Define $\Xi\,(\zeta, \eta, \delta, \lambda)$; $\Theta\,(\zeta, \eta, \delta, \lambda)$ and $\Upsilon\,(\zeta, \eta, \delta, \lambda)$ by
$\Xi\,(\zeta, \eta, \delta, \lambda) = \frac{\lambda}{\lambda+|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}$, $\Theta\,(\zeta, \eta, \delta, \lambda) = \frac{|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}{\lambda+|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}$ and $\Upsilon\,(\zeta, \eta, \delta, \lambda) = \frac{|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}{\lambda}$
for all $\lambda > 0$. Then, $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$.

Define Γ and Ω as follows:

$$\Gamma x = \begin{cases} \zeta^2 + 1, & \zeta < 1 \\ 2\zeta - 1, & \zeta \geq 1 \end{cases}, \quad \Omega\zeta = \begin{cases} \zeta + 1, & \zeta < 1 \\ 3\zeta - 2, & \zeta \geq 1 \end{cases}.$$

Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 1 + \frac{1}{n}$, for n = 1, 2…, then $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 1$, $1 \in \Sigma$ and

$$\Gamma\Omega\zeta_n = \Gamma\left(1 + \frac{3}{n}\right) = 2\left(1 + \frac{3}{n}\right) - 1 = 1 + \left(\frac{6}{n}\right),$$
$$\Omega\Gamma\zeta_n = \Omega\left(1 + \frac{2}{n}\right) = 3\left(1 + \frac{2}{n}\right) - 2 = 1 + \left(\frac{6}{n}\right),$$

$$\Gamma\Gamma\zeta_n = \Gamma\left(1+\frac{2}{n}\right) = 2\left(1+\frac{2}{n}\right)-1 = 1+\left(\frac{4}{n}\right),$$
$$\Omega\Omega\zeta_n = \Omega\left(1+\frac{3}{n}\right) = 3\left(1+\frac{3}{n}\right)-2 = 1+\left(\frac{9}{n}\right).$$

Therefore,

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

And,

$$\lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

That is, Γ and Ω are subcompatible of type (J-1) but if we consider a sequence $\zeta_n = 1-\frac{1}{n}$ for n = 1,2,…, then $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = 2$, $2 \in \Sigma$ and

$$\Gamma\Omega\zeta_n = \Gamma\left(2-\frac{1}{n}\right) = 2\left(2-\frac{1}{n}\right)-1 = 3-\left(\frac{2}{n}\right),\ \Omega\Gamma\zeta_n = \Omega\left(\left(1-\frac{1}{n}\right)^2+1\right) = 3\left(\left(1-\frac{1}{n}\right)^2+1\right)-2,$$
$$\Gamma\Gamma\zeta_n = \Gamma\left(\left(1-\frac{1}{n}\right)^2+1\right) = \Gamma\left(1-\frac{2}{n}+\frac{1}{n^2}\right) = \left(1-\frac{2}{n}+\frac{1}{n^2}\right)^2+1,$$
$$\Omega\Omega\zeta_n = \Omega\left(2-\frac{1}{n}\right) = 3\left(2-\frac{1}{n}\right)-2 = 4-\left(\frac{3}{n}\right).$$

Therefore,

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0,$$
$$\lim_{n\to\infty} \Xi\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \neq 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \neq 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \neq 0.$$

That is, Γ and Ω are not compatible of type (J-1).

Definition: 3.6

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subcompatible of type (J-1) if and only if there exist a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfies

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

Clearly, if Γ and Ω are compatible of type (J-2), then they are subcompatible of type (J-2), but converse is not true in general.

Example: 3.7

Let Σ = [0, ∞) and define $\Xi(\zeta, \eta, \delta, \lambda)$; $\Theta(\zeta, \eta, \delta, \lambda)$ and $\Upsilon(\zeta, \eta, \delta, \lambda)$ by

$$\Xi(\zeta, \eta, \delta, \lambda) = \begin{cases} 1, & \zeta = \eta = \delta, \\ \frac{\lambda}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Theta(\zeta,\eta,\delta,\lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \frac{1}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Upsilon(\zeta,\eta,\delta,\lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \lambda+1, & \zeta \neq \eta \neq \delta. \end{cases}$$

Then, $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$. Define Γ and Ωas follows:

$$\Gamma\zeta = \zeta^2, \quad \Omega\zeta = \begin{cases} \zeta + 2, & \zeta \in [0,4] \cup (5,\infty) \\ \zeta + 12, & \zeta \in (4,5]\ . \end{cases}$$

Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 2+\frac{1}{n}$ for n = 1,2…, then $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = 4$, and $\Gamma\Gamma\zeta_n = \Gamma\left(\left(2+\frac{1}{n}\right)^2\right) = \left(2+\frac{1}{n}\right)^4$, $\Omega\Omega\zeta_n = \Omega\left(4+\frac{1}{n}\right) = 4+\frac{1}{n} + 12 = 16 + \frac{1}{n}$.

Therefore,

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

That is, Γ and Ω are subcompatible of type (J-2) but if we consider a sequence $\zeta_n = 2-\frac{1}{n}$ for n = 1, 2,…, then $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = 4$ and $\Gamma\Gamma\zeta_n = \Gamma\left(\left(2-\frac{1}{n}\right)^2\right) = \left(2-\frac{1}{n}\right)^4$, $\Omega\Omega\zeta_n = \Omega\left(4-\frac{1}{n}\right) = 4-\frac{1}{n} + 2 = 6 - \frac{1}{n}$.

Therefore,

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0.$$

That is, Γ and Ω are not compatible of type (J-2).

Preposition: 3.8

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \to \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps if and only if they are not subcompatible of type (J-1).

Proof:

Suppose Γ and Ω are subcompatible, then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 0.$$

Since Γ and Ωare subsequentially continuous, we have

$\lim_{n\to\infty}\Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty}\Gamma\Gamma\zeta_n$, $\lim_{n\to\infty}\Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty}\Omega\Omega\zeta_n$.

Thus, from the inequality (WNA), for all $\lambda > 0$,

$$\Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) * \Xi\,(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Theta(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Upsilon\,(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$

and it follows that

$$\Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,$$

$$\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0.$$

That is, for all $\lambda > 0$,

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

By the same way,

$$\lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Consequently, Γ and Ω are subcompatible of type (J-1).

Conversely, suppose that Γ and Ω are subcompatible of type (J-1), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1, \ \lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0 \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \ \lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0 \text{ and } \lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n\to\infty} \Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty} \Gamma\Gamma\zeta_n, \ \lim_{n\to\infty} \Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty} \Omega\Omega\zeta_n.$$

Now, from the inequality (WNA), for all $\lambda > 0$,

$$\Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \geq \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) * \Xi(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \leq \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \diamond \Theta(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \leq \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \diamond \Upsilon(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$

and, it follows that,

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \geq 1 * 1 = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$

which implies that

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ω are subcompatible.

Preposition: 3.9

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \to \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps if and only if they are not subcompatible of type (J-2).

Proof:

Suppose Γ and Ω are subcompatible, then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \delta$, $\delta \in \Sigma$ and satisfy

$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1$, $\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$, and $\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$.

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n\to\infty} \Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty} \Gamma\Gamma\zeta_n, \quad \lim_{n\to\infty} \Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty} \Omega\Omega\zeta_n.$$

Thus, from the inequality (WNA),

$$\begin{aligned}
\Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\geq \Xi\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) * \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2)\\
&\geq \Xi\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) * \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2) *\\
&\qquad \Xi(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/4),\\
\Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq \Theta\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2)\\
&\leq \Theta\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2) \diamond\\
&\qquad \Theta\,(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/4) \text{ and}\\
\Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq \Upsilon\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2)\\
&\leq \Upsilon\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2) \diamond\\
&\qquad \Upsilon\,(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/4),
\end{aligned}$$

for all $\lambda > 0$, and, it follows that, for all $\lambda > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,
\end{aligned}$$

which implies that,

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.
\end{aligned}$$

Consequently, Γ and Ω are subcompatible of type (J-2). Conversely, suppose that Γ and Ω are subcompatible of type (J-2), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.
\end{aligned}$$

Now, from the inequality (WNA), we have

$$\begin{aligned}
\Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\geq \Xi\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) * \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\geq \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) * \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\qquad * \Xi\,(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/4),\\
\Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\leq \Theta\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Theta(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\leq \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\qquad \diamond \Theta\,(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/4) \text{ and}\\
\Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\leq \Upsilon\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\leq \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\qquad \diamond \Upsilon\,(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/4),
\end{aligned}$$

and, it follows that, for all $\lambda > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \geq 1 * 1 * 1 = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \leq 0 \diamond 0 \diamond 0 = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \leq 0 \diamond 0 \diamond 0 = 0,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ω are subcompatible.

Preposition: 3.10

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \to \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps of type (J-1) if and only if they are subcompatible of type (J-2).

Proof:

Suppose Γ and Ωare subcompatible of type (J-1), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfy

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and,}$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Xi\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and,}$$
$$\lim_{n\to\infty} \Upsilon\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n\to\infty}\Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty}\Gamma\Gamma\zeta_n,\ \lim_{n\to\infty}\Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty}\Omega\Omega\zeta_n.$$

Thus, from the inequality (WNA),

$$\Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq \Xi\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) * \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Theta\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Upsilon\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$

and, it follows that

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$

which implies that

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ωare subcompatible of type (J-2).

Conversely, suppose that Γ and Ω are subcompatible of type (J-2), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

Now, from the inequality (WNA), we have

$$\Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq \Xi\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) * \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Theta\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Upsilon\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$

and, it follows that

$$\lim_{n\to\infty}(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,$$
$$\lim_{n\to\infty}\Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$
$$\lim_{n\to\infty}\Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$

which implies that, for all $\lambda > 0$,

$$\lim_{n\to\infty}(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

By the same way, we obtain that

$$\lim_{n\to\infty}(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ω are subcompatible of type (J-1).

4. Main Theorems

Theorem: 4.1

Let Γ, Λ, Ω and H be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ and let the pairs (Γ, Ω) and (Λ, H) are subcompatible maps of type(J-1) and subsequentially continuous.

$$\Xi\,(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \geq \psi\,(\min\{\Xi\,(\Omega\zeta, H\eta, H\eta, \lambda), \Xi\,(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi\,(\Lambda\eta, H\eta, H\eta, \lambda), \tfrac{1}{2}[\Xi\,(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi\,(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \quad (4.1.1)$$

$$\Theta(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \phi(\max\{\Theta(\Omega\zeta, H\eta, H\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta\,(\Lambda\eta, H\eta, H\eta, \lambda), \tfrac{1}{2}[\Theta\,(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta\,(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \quad (4.1.2)$$

$$\Upsilon(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \varphi\,(\max\{\Upsilon\,(\Omega\zeta, H\eta, H\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon\,(\Lambda\eta, H\eta, H\eta, \lambda), \tfrac{1}{2}[\Upsilon\,(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon\,(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \quad (4.1.3)$$

for all $\zeta, \eta \in \Sigma$, $\lambda > 0$, where $\psi, \phi, \varphi : [0,1] \to [0,1]$ are continuous functions such that $\psi(s) > s$, $\phi(s) < s$ and $\varphi(s) < s$ for each $s \in (0,1)$. Then Γ, Λ, Ω and H have a unique common fixed point in Σ.

Proof

Since the pairs (Γ, Ω) and (Λ, H) are subcompatible maps of type (J-1) and subsequentially continuous, then there exist two sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \delta$, $\delta \in \Sigma$ and satisfy

$$\lim_{n\to\infty}\Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = \Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = \Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = \Upsilon(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 0,$$
$$\lim_{n\to\infty}\Xi\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = \Xi(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = \Theta\,(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = \Upsilon(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 0.$$

$\lim_{n\to\infty}\Lambda\zeta_n = \lim_{n\to\infty}H\zeta_n = \omega$, $\omega \in \Sigma$, and

$$\lim_{n\to\infty}\Xi\,(\Lambda H\eta_n, HH\eta_n, HH\eta_n, \lambda) = \Xi(\Lambda\omega, H\omega, H\omega, \lambda) = 1,$$

$\lim_{n\to\infty} \Theta$ (ΛΗη_n, ΗΗη_n, ΗΗη_n, λ) = Θ (Λω, Ηω, Ηω, λ) = 0,
$\lim_{n\to\infty} \Upsilon$ (ΛΗη_n, ΗΗη_n, ΗΗη_n, λ) = ϒ(Λω, Ηω, Ηω, λ) = 0,
$\lim_{n\to\infty} \Xi$ (ΗΛη_n, ΛΛη_n, ΛΛη_n, λ) = Ξ(Ηω, Λω, Λω, λ) = 1,
$\lim_{n\to\infty} \Theta$ (ΗΛη_n, ΛΛη_n, ΛΛη_n, λ) = Θ (Ηω, Λω, Λω, λ) = 0,
$\lim_{n\to\infty} \Upsilon$ (ΗΛη_n, ΛΛη_n, ΛΛη_n, λ) = ϒ (Ηω, Λω, Λω, λ) = 0.

Therefore, Γδ = Ωδ and Λω = Ηω, that is δ is a coincidence point of Γ and Ω, ω is a coincidence point of Λ and H.Now, we prove that δ = ω. By using (3.1) for ζ = ζ_n and η = η_n, we get
Ξ(Γζ_n, Λη_n, Λη_n, λ) ≥ ψ (min {Ξ(Ωζ_n, Ηη_n, Ηη_n, λ), Ξ(Γζ_n, Ωζ_n, Ωζ_n, λ), Ξ (Λη_n, Ηη_n, Ηη_n, λ), $\frac{1}{2}$[Ξ (Λη_n, Ωζ_n, Ωζ_n, λ) + Ξ (Γζ_n, Ηη_n, Ηη_n, λ)]}),
Θ (Γζ_n, Λη_n, Λη_n, λ) ≤ ϕ (max {Θ (Ωζ_n, Ηη_n, Ηη_n, λ) , Θ(Γζ_n, Ωζ_n, Ωζ_n, λ), Θ (Λη_n, Ηη_n, Ηη_n, λ), $\frac{1}{2}$[Θ (Λη_n, Ωζ_n, Ωζ_n, λ) + Θ (Γζ_n, Ηη_n, Ηη_n, λ)]}).
ϒ(Γζ_n, Λη_n, Λη_n, λ) ≤ φ(max {ϒ(Ωζ_n, Ηη_n, Ηη_n, λ) , ϒ(Γζ_n, Ωζ_n, Ωζ_n, λ), ϒ (Λη_n, Ηη_n, Ηη_n, λ), $\frac{1}{2}$[ϒ (Λη_n, Ωζ_n, Ωζ_n, λ) + ϒ (Γζ_n, Ηη_n, Ηη_n, λ)]}).

Taking the limit n → ∞, we have
Ξ (δ, ω, ω, λ) ≥ ψ (min { Ξ (δ, ω, ω, λ), Ξ (δ, δ, δ, λ) , Ξ (ω, ω, ω, λ), $\frac{1}{2}$ [Ξ (ω, δ, δ, λ) + Ξ (δ, ω, ω, λ)]}),
Θ (δ, ω, ω, λ) ≤ ϕ (max { Θ(δ, ω, ω, λ), Θ (δ, δ, δ, λ) , Θ (ω, ω, ω, λ), $\frac{1}{2}$ [Θ (ω, δ, δ, λ) + Θ (δ, ω, ω, λ)]}),
ϒ(δ, ω, ω, λ) ≤ φ (max { ϒ(δ, ω, ω, λ), ϒ(δ, δ, δ, λ), ϒ(ω, ω, ω, λ), $\frac{1}{2}$ [ϒ(ω, δ, δ, λ) + ϒ(δ, ω, ω, λ)]}),
that is,

Ξ (δ., ω, ω, λ) ≥ ψ (Ξ (δ, ω, ω, λ)) > Ξ (δ, ω, ω, λ),
Θ (δ., ω, ω, λ) ≤ ϕ (Θ (δ, ω, ω, λ)) < Θ (δ, ω, ω, λ),
ϒ (δ., ω, ω, λ) ≤ φ (ϒ (δ, ω, ω, λ)) < ϒ (δ, ω, ω, λ),

which yield δ = ω.

Again using (3.1) for ζ = δ and η = η_n, we obtain
Ξ (Γδ, Λη_n, Λη_n, λ) ≥ ψ (min {Ξ (Ωδ, Ηη_n, Ηη_n, λ), Ξ (Γδ, Ωδ, Ωδ, λ), Ξ (Λη_n, Ηη_n, Ηη_n, λ), $\frac{1}{2}$[Ξ (Λη_n, Ωδ, Ωδ, λ) + Ξ (Γδ, Ηη_n, Ηη_n, λ)]}),
Θ {Γδ, Λη_n, Λη_n, λ) ≤ ϕ (max {Θ (Ωδ, Ηη_n, Ηη_n, λ), Θ (Γδ, Ωδ, Ωδ, λ), Θ (Λη_n, Ηη_n, Ηη_n, λ), $\frac{1}{2}$[Θ (Λη_n, Ωδ, Ωδ, λ) + Θ (Γδ, Ηη_n, Ηη_n, λ)]}).
ϒ{Γδ, Λη_n, Λη_n, λ) ≤ φ(max {ϒ(Ωδ, Ηη_n, Ηη_n, λ), ϒ(Γδ, Ωδ, Ωδ, λ), ϒ(Λη_n, Ηη_n, Ηη_n, λ), $\frac{1}{2}$[ϒ(Λη_n, Ωδ, Ωδ, λ) + ϒ (Γδ, Ηη_n, Ηη_n, λ)]}).

Taking the limit as n → ∞, we have,
Ξ(Γδ, ω, ω, λ) ≥ ψ (min {Ξ(Ωδ, ω, ω, λ), Ξ(Γδ, Ωδ, Ωδ, λ), Ξ (ω, ω, ω, λ), $\frac{1}{2}$[Ξ (ω, Ωδ, Ωδ, λ) + Ξ (Γδ, ω, ω, λ)]}),
Θ(Γδ, ω, ω, λ) ≤ ϕ (max {Θ(Ωδ, ω, ω, λ), Θ(Γδ, Ωδ, Ωδ, λ), Θ (ω, ω, ω, λ), $\frac{1}{2}$[Θ (ω, Ωδ, Ωδ, λ) + Θ (Γδ, ω, ω, λ)]}),
ϒ(Γδ, ω, ω, λ) ≤ φ (max {ϒ(Ωδ, ω, ω, λ), ϒ(Γδ, Ωδ, Ωδ, λ), ϒ (ω, ω, ω, λ), $\frac{1}{2}$[ϒ (ω, Ωδ, Ωδ, λ) + ϒ (Γδ, ω, ω, λ)]}).

That is,

Ξ(Γδ, ω, ω, λ) ≥ ψ (Ξ(Γδ, ω, ω, λ)) > Ξ(Γδ, ω, ω, λ),
Θ(Γδ, ω, ω, λ) ≤ ϕ (Θ(Γδ, ω, ω, λ)) < Θ(Γδ, ω, ω, λ),
ϒ(Γδ, ω, ω, λ) ≤ φ (ϒ(Γδ, ω, ω, λ)) < ϒ(Γδ, ω, ω, λ).

which yield Γδ = ω = δ.

Therefore δ = ω is a common fixed point of Γ, Λ, Ω and H.

For uniqueness, suppose that there exist another fixed point u of Γ, Λ, Ω and H.

Then from (3.1), we have

$$\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda) \geq \psi(\min\{\Xi(\Omega\delta, Hu, Hu, \lambda), \Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Xi(\Lambda u, Hu, Hu, \lambda), \tfrac{1}{2}[\Xi(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Xi(\Gamma\delta, Hu, Hu, \lambda)]\})$$
$$= \psi(\min\{\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 1, \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \tfrac{1}{2}[\Xi(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\})$$
$$= \psi(\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda)$$
$$> \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda),$$

$$\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda) \leq \phi(\max\{\Theta(\Omega\delta, Hu, Hu, \lambda), \Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Theta(\Lambda u, Hu, Hu, \lambda), \tfrac{1}{2}[\Theta(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Theta(\Gamma\delta, Hu, Hu, \lambda)]\})$$
$$= \phi(\max\{\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 0, \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \tfrac{1}{2}[\Theta(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\})$$
$$= \phi(\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda)$$
$$< \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda),$$

$$\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda) \leq \varphi(\max\{\Upsilon(\Omega\delta, Hu, Hu, \lambda), \Upsilon(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Upsilon(\Lambda u, Hu, Hu, \lambda), \tfrac{1}{2}[\Upsilon(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Upsilon(\Gamma\delta, Hu, Hu, \lambda)]\})$$
$$= \varphi(\max\{\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 0, \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \tfrac{1}{2}[\Upsilon(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\})$$
$$= \varphi(\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda)$$
$$< \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda),$$

which yield δ = u. Therefore, uniqueness follows.

If we put Ω = H in Theorem 3.1, we get the following result.

Corollary: 4.2

Let Γ, Λ, and Ω be self-maps of a weak non-Archimedean NMS (Σ, Ξ, Θ, ϒ, ∗, ⋄) and let the pairs (Γ, Ω) and (Λ, Ω) are subcompatible maps of type (J-1) and subsequentially continuous. If

$$\Xi(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \geq \psi(\min\{\Xi(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \tfrac{1}{2}[\Xi(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \quad (4.2.1)$$
$$\Theta(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \phi(\max\{\Theta(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \tfrac{1}{2}[\Theta(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \quad (4.2.2)$$
$$\Upsilon(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \varphi(\max\{\Upsilon(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \tfrac{1}{2}[\Upsilon(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \quad (4.2.3)$$

for all ζ, η ∈ Σ, $\lambda > 0$, where $\psi, \phi, \varphi : [0,1] \to [0,1]$ are continuous functions such that $\psi(s) > s$, φ(s) < s and $\varphi(s) < s$ for each s ∈ (0,1). Then Γ, Λ and Ω have a unique common fixed point in Σ.

If we put Γ = Λ and Ω = H in Theorem 4.1, we get the following result.

Corollary: 4.3

Let Γ and Ω be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ and let the pairs (Γ, Ω) is subcompatible maps of type (J-1) and subsequentially continuous. If

$$\Xi(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \geq \psi\,(\min\{\Xi(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Xi(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.1)$$

$$\Theta(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \leq \phi\,(\max\{\Theta(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Theta(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.2)$$

$$\Upsilon(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \leq \varphi\,(\max\{\Upsilon(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Upsilon(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.3)$$

for all $\zeta, \eta \in \Sigma$, $\lambda > 0$, where $\psi, \phi, \varphi : [0,1] \to [0,1]$ are continuous functions such that $\psi(s) > s$, $\varphi(s) < s$ and $\varphi(s) < s$ for each $s \in (0,1)$. Then Γ and Ω have a unique common fixed point in Σ.

5. Conclusion

In this work, we obtained new structure of weak non-Archimedian with the help of subcompatible maps of types (J-1) and (J-2) in NMS. Also, we proved common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

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