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Some New Type of Lacunary Statistically Convergent Sequences In Neutrosophic Normed Space

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Abstract. The idea of statistical convergence was introduced by Fast [H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244] afterwards studied by many authors. In [J.A. Fridy, C. Orhan, Lacunary statistical convergence, Pacific J. Math. 160 (1993) 43–51],Fridy and Orhan proposed the concept of lacunary statistical convergence. In present paper, we introduce lacunary statistically convergent in neutrosophic normed space (brifly, NNS). We define the concept of lacunary statistical Cauchy sequence in NNS and derive the relation between statistical completness and completeness in NNS . We give some basic properties of these concepts.
Keywords: NNS, t-norm,t-conorm, Statistical convergence,Lacunary statistical convergence,Lacunary statistical Cauchy and Lacunary statistical completeness.

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1. Introduction

Zadeh [6] was the first who introduced the theory of fuzzy sets. It has a very influential impact on all the scientific fields and is quite necessary for the real- life situations. Atanassov [9] generalized the concepts of fuzzy sets, known as intuitionistic fuzzy sets. One of the dominant problems in fuzzy topology is to obtain an appropriate hypothesis of the fuzzy metric space. Moreover, Park [2] used George and Veeramani's [1] thought of applying t-norm and t-conorm to fuzzy metric spaces for defining intuitionistic fuzzy metric spaces and studying its fundamental features. After a while, Smarandache [10] introduced the notion of neutrosophic sets (NS), which is a different kind of the notion of the classical set theory.by adding an intermediate membership function. This set is a formal setting trying to measure the truth, indeterminacy, and falsehood. Smarandache [16] further studied the differences between intuitionistic fuzzy set and neutrosophic set and the corresponding relations between these two sets. The basic differences are as follows:

(i) Neutrosophic set can distinguish between relative truth = 1 and absolute truth = 1^+. This has application in philosophy. For this reason, the unitary standard interval $[0, 1]$ used in intuitionistic fuzzy set has been extended to the unitary non-standard interval $]^-0, 1^+[$ in neutrosophic set.

(ii) In neutrosophic set, there is no condition on $\mathcal{T}$(truth), $\mathcal{H}$(indeterminacy) and $\mathcal{F}$(falsehood) other than they are subsets of $]^-0, 1^+[$, therefore:

$$^-0 \leq \inf \mathcal{T} + \inf \mathcal{H} + \inf \mathcal{F} \leq \sup \mathcal{T} + \sup \mathcal{H} + \sup \mathcal{F} \leq 3^+.$$

(iii) In neutrosophic set, the components T(truth), H(indeterminacy) and F(falsehood) can also be non-standard subsets included in the unitary non-standard interval $]^-0, 1^+[$, not only standard subsets, included in the unitary standard interval $[0, 1]$ as in intuitionistic fuzzy logic.

Neutrosophic sets are more effective and flexible because it handles, besides independent components, also partially dependent and partially independent components, while intuitionistic fuzzy sets cannot deal with these. Further, Smarandache [17–19] investigated neutroalgebra which is generalization of partial algebra , neutroalgebraic structures and antialgebraic structures. Moreover, Bera and Mahapatra [11] defined neutrosophic soft linear spaces (NSLSs). In [11] neutrosophic norm, Cauchy sequence in NSNLS, the convexity of NSNLS, metric in NSNLS were studied. There has been much progress in the study of neutrosophic theory in different fields by various authors.

Fast [13] proposed the concept of statistical convergence and later on studied by many researchers. Friday and Orhan [15] have investigated the theory of lacunary statistical convergence. Later on, the concepts of statistical convergence of double sequences have been analyzed in IFNS by Mursaleen and Mohiuddin [12]. Quite recently, Kirisci and Simsek [7] introduced the notion of neutrosophic normed space and statistical convergence. Since neutrosophic normed space is a natural generalization of IFNS and statistical convergence.

In the present paper we will study lacunary statistical convergence and lacunary statistical Cauchy in neutrosophic normed space. we will the study the concept of statistical completeness which would provide a ordinary framework to study the completeness of neutrosophic normed space. We outline the present work as follows. In Section 2, we recall some basic definitions related to the neutrosophic normed space.In Section 3, in this paper we porposed lacunary statistical convergence in NNS and prove our main results. Finally, Section 4 is devoted to introduce a recent concept, i.e. (lacunary) statistical completeness and find its relation with completeness of NNS.

2. Preliminaries

Throughtout this article, $\mathbb{N}$ will denote the set of natural numbers. Using definitions of continuous t-norm and continuous t-conorm (see [14]), Kirisci and Simsek [7] proposed the notion of NNS which is defined as follows:

Definition 2.1. [14] Given an operation $\star : [0,1] \times [0,1] \longrightarrow [0,1]$ then it is called continuous t-norm if it satisfies the following conditions:
(a) $\star$ is associative and commutative,
(b) $\star$ is continuous,
(c) $c \star 1 = c$ for all $c \in [0,1]$,
(d) $c \star d \leq f \star g$ whenever $c \leq f$ and $d \leq g$ for each $c, d, f, g \in [0,1]$.

Definition 2.2. [14] Given an operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ then it is called continuous t-conorm if it satisfies the following conditions:
(a) $\diamond$ is associative and commutative,
(b) $\diamond$ is continuous,
(c) $c \diamond 0 = c$ for all $c \in [0,1]$,
(d) $c \diamond d \leq f \diamond g$ whenever $c \leq f$ and $d \leq g$ for each $c, d, f, g \in [0,1]$.

From above definitions, we note that if we choose $0 < e_1, e_2 < 1$ with $e_1 > e_2$, then there exist $0 < e_3, e_4 < 1$ such that $e_1 * e_3 \geq e_2, e_1 \geq e_4 \diamond e_2$. Further, if we choose $e_5 \in (0,1)$, then there exist $e_6, e_7 \in (0,1)$ such that $e_6 * e_6 \geq e_5$ and $e_7 \diamond e_7 \leq e_5$.

Definition 2.3. [5] The intuitionistic fuzzy set A which is a subset of non-empty set X is an ordered triplet defined by

$$A = \{< x, \mathcal{T}(x),\ \mathcal{F}(x) >: x \in X\},$$

where $\mathcal{T}(x), \mathcal{F}(x) : X \to [0,1]$ represent the degree of membership and degree of nonmembership respectively in such a way that

$$0 \leq \mathcal{T}(x) + \mathcal{F}(x) \leq 1$$

Also, $1 - \mathcal{T}(x) - \mathcal{F}(x)$ is called degree of hesitancy. The intuitionistic fuzzy components $\mathcal{T}(x), \mathcal{F}(x)$ and degree of hesitancy are dependent on each other.

Definition 2.4. [10] Let A be a subset of non-empty set X. Then,

$$A_{NS} = \{< x, \mathcal{T}(x), \mathcal{I}(x), \mathcal{F}(x) >: x \in X\},$$

where $\mathcal{T}(x), \mathcal{I}(x), \mathcal{F}(x) : X \to [0,1]$ represent the degree of truth-membership, degree of indeterminacy-membership, and degree of false-nonmembership respectively in such a way that

$$0 \leq \mathcal{T}(x) + \mathcal{I}(x) + \mathcal{F}(x) \leq 3.$$

The neutrosophic components $\mathcal{T}(x), \mathcal{I}(x)$ and $\mathcal{F}(x)$ are independent of each other.

Definition 2.5. [20, 21] The complement of an interval neutrosophic set P is denoted by P_{-} and is defined by
$\mathcal{T}_{P_{-}}(x) = \mathcal{F}_P(x);$
$\inf \mathcal{H}_{P_{-}}(x) = 1 - \sup \mathcal{H}_P(x);\ \sup \mathcal{H}_{P_{-}}(x) = 1 - \inf \mathcal{H}_P(x);$
$\mathcal{F}_{P_{-}}(x) = \mathcal{T}_P(x);\ \forall\ x \in X.$

Definition 2.6. [8] Let P and R be two neutrosophic sets in a non-empty set X.Then,

(a) $P \subset R \iff \mathcal{T}_P(x) \leq \mathcal{T}_R(x), \mathcal{H}_P(x) \leq \mathcal{H}_R(x), \mathcal{F}_P(x) \geq \mathcal{F}_R(x)\ \forall\ x \in X$
(b) $P = R \iff \mathcal{T}_P(x) = \mathcal{T}_R(x), \mathcal{H}_P(x) = \mathcal{H}_R(x), \mathcal{F}_P(x) = \mathcal{F}_R(x)\ \forall\ x \in X$
(c) $P \cap R = \{\langle x, \min(\mathcal{T}_P(x), \mathcal{T}_R(x)), \min(\mathcal{H}_P(x), \mathcal{H}_R(x)), \min(\mathcal{F}_P(x), \mathcal{F}_R(x))\rangle \mid x \in X\}$
(d) $P \cup R = \{\langle x, \max(\mathcal{T}_P(x), \mathcal{T}_R(x)), \max(\mathcal{H}_P(x), \mathcal{H}_R(x)), \max(\mathcal{F}_P(x), \mathcal{F}_R(x))\rangle \mid x \in X\}$
(e) $P^c = \{\langle x, \mathcal{F}_P(x), 1 - \mathcal{H}_R(x)), \mathcal{T}_P(x)\rangle \mid x \in X\}$
(f) $P \backslash R = \{\langle x, \mathcal{T}_P(x) \min \mathcal{F}_R(x), \mathcal{H}_P(x) \min 1 - \mathcal{H}_R(x), \mathcal{F}_P(x) \max \mathcal{T}_R(x)\rangle \mid x \in X\}.$

Definition 2.7. [7] Let U be a vector space and $\mathcal{N} = \{< v, \mathcal{T}(v), \mathcal{H}(v), \mathcal{F}(v) >: v \in U\}$ be a normed space(NS) such that $\mathcal{T}(v), \mathcal{H}(v), \mathcal{F}(v) : U \times \mathbb{R}^+ \to [0, 1]$. Let $*$ and $\diamond$ be continuous t-norm and t-conorm respectively. If the subsequent conditions holds, then the four-tuple $(U, \mathcal{N}, *, \diamond)$ is called neutrosophic normed space (NNS): For all $v, w \in U$ and $\eta, s > 0$ and for each $a \neq 0$,

(1) $0 \leq \mathcal{T}(u, \eta) \leq 1,\ 0 \leq \mathcal{H}(v, \eta) \leq 1,\ 0 \leq \mathcal{F}(v, \eta) \leq 1$

(2) $\mathcal{T}(v, \eta) + \mathcal{H}(v, \eta) + \mathcal{F}(v, \eta) \leq 3, (\eta \in \mathbb{R}^+)$

(3) $\mathcal{T}(v, t\eta) = 1$ (for $\eta > 0$) if and only if $v = 0$,

(4) $\mathcal{T}(cv, \eta) = \mathcal{T}(v, \frac{\eta}{|c|})$ for each $c \neq 0$,

(5) $\mathcal{T}(v, \eta) * \mathcal{T}(w, s) \leq \mathcal{T}(v + w, \eta + s)$,

(6) $\mathcal{T}(v, .)$ is continuous non-decreasing function,

(7) $\lim_{\eta\to\infty} \mathcal{T}(v,\eta) = 1$

(8) $\mathcal{H}(v,\eta) = 0$ (for $\eta > 0$) if and only if $v = 0$,

(9) $\mathcal{H}(cv,\eta) = \mathcal{H}(v, \frac{\eta}{|c|})$ for each $c \neq 0$,

(10) $\mathcal{H}(v,\eta) * \mathcal{H}(v,s) \geq \mathcal{H}(v+w,\eta+s)$,

(11) $\mathcal{H}(v,.)$ is continuous non-increasing function,

(12) $\lim_{\eta\to\infty} \mathcal{H}(v,\eta) = 1$

(13) $\mathcal{F}(v,\eta) = 0$ (for $\eta > 0$) if and only if $v = 0$,

(14) $\mathcal{F}(cv,\eta) = \mathcal{F}(v, \frac{\eta}{|c|})$ for each $c \neq 0$,

(15) $\mathcal{F}(v,\eta) * \mathcal{F}(v,\mathcal{T}) \geq \mathcal{F}(v+w,\eta+s)$,

(16) $\mathcal{F}(v,.)$ is continuous non-increasing function,

(17) $\lim_{\eta\to\infty} \mathcal{F}(v,\eta) = 1$

(18) If $\eta \leq 0$, then $\mathcal{T}(v,\eta) = 0, \mathcal{H}(v,\eta) = 1$ and $\mathcal{F}(v,\eta) = 1$.

In this case, $\mathcal{N} = (\mathcal{T},\mathcal{H},\mathcal{F})$ is said to be neutrosophic norm (NN).

Example 2.8. [7] Let $(U, \|.\|)$ be a normed space. Given the operations $*$ and $\diamond$ in such a way that: $v * w = vw$, $v \diamond w = v + w - vw$. For $\eta > \|v\|$ and $\eta > 0$

$$\mathcal{T}(v,\eta) = \frac{\eta}{\eta + \|v\|}, \mathcal{H}(v,\eta) = \frac{\|v\|}{\eta + \|v\|} \text{ and } \mathcal{F}(v,\eta) = \frac{\|v\|}{\eta} \tag{1}$$

for all $v, w \in U$. If we take $\eta \leq \|v\|$, then $\mathcal{T}(v,\eta) = 0, \mathcal{H}(v,\eta) = 1$ and $\mathcal{F}(v,\eta) = 1$. Then $(U, \mathcal{N}, *, \diamond)$ is NNS in such a way that $\mathcal{N} : U \times \mathbb{R}^+ \to [0,1]$.

Example 2.9. Let $(U = \mathbb{R}, \|.\|)$ be a normed space where $\|x\| = |x|\ \forall\ x \in \mathbb{R}$. Give the operations $*$ and $\diamond$ in such a way that: $v*w = \min\{v,w\}$ and $v \diamond w = \max\{v,w\}.\forall\ v,w\ \in [0,1]$ and Define,

$$\mathcal{T}(v,\eta) = \frac{\eta}{\eta + k\|v\|}, \mathcal{H}(v,t) = \frac{k\|v\|}{\eta + \|v\|} \text{ and } \mathcal{F}(v,\eta) = \frac{k\|v\|}{\eta} \tag{2}$$

where $k > 0$ Then $A = \{(v,\eta), \mathcal{T}(v,\eta), \mathcal{H}(v,\eta), \mathcal{F}(v,\eta) : (v,\eta) \in U \times \mathbb{R}^+\}$ is a NNS on U

Definition 2.10. [7] Let $(U, \mathcal{N}, *, \diamond)$ be a NNS. Then, the sequence (a_n) is said to be convergent to $\xi \in X$ with respect to the NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$ if for each $\epsilon, \ \eta > 0$, there exists $N \in \mathbb{N}$, in such a manner that

$$\mathcal{T}(a_n - \xi, \eta) > 1 - \epsilon, \mathcal{H}(a_n - \xi, \eta) < \epsilon \text{ and } \mathcal{F}(a_n - \xi, \eta) < \epsilon \tag{3}$$

for all $n \geq N$, i.e.,

$$\lim_{n\to\infty} \mathcal{T}(a_n - \xi, \eta) = 1, \lim_{n\to\infty} \mathcal{H}(a_n - \xi, \eta) = 0 \text{ and } \lim_{n\to\infty} \mathcal{F}(a_n - \xi, \eta) = 0.$$

In such case, we denote $\mathcal{N} - \lim a_n = \xi$.

Definition 2.11. [7] Let $(U, \mathcal{N}, *, \diamond)$ be a NNS. Then, the sequence (u_n) is known as Cauchy sequence with respect to the NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$ if for each $\epsilon, \ \eta > 0$, there exists $N \in \mathbb{N}$, in such a manner that

$$\mathcal{T}(a_n - a_m, \eta) > 1 - \epsilon, \mathcal{H}(a_n - a_m, \eta) < \epsilon \text{ and } \mathcal{F}(a_n - a_m, \eta) < \epsilon \tag{4}$$

for all $n, m \geq N$.

Definition 2.12. [15]

A lacunary sequence is an increasing integer sequence $\theta = \{n_r\}$ such that $n_0 = 0$ and $h_r = n_r - n_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (n_{r-1}, n_r]$ and the ratio $\frac{n_r}{n_{r-1}}$ will be abbreviated as q_r. Let $K \subseteq \mathbb{N}$. The number

$$\delta_\theta(K) = \frac{1}{h_r} \mid \{n \in I_r : n \in K\} \mid$$

is called θ-density of K, provided the limit exists.

Definition 2.13. [15] Let θ be a lacunary sequence. A sequence $a = \{a_n\}$ of numbers is said to be lacunary statistically convergent (briefly S_θ- convergent) to the number ξ if for every $\epsilon > 0$, the set $K(\epsilon)$ has θ-density zero, where

$$K(\epsilon) = \{a \in I_r : |a_n - \xi| \geq \epsilon\}.$$

In this case we write $S_\theta - \lim a = \xi$.

3. **Lacunary Statistical convergence in NNS**

In this section, we introduce lacunary statistical convergence in NNS. First, we define the subsequent definition

Definition 3.1. Let $(U, \mathcal{N}, *, \diamond)$ be NNS. A sequence (a_n) is called Lacunary statistical convergent with respect to NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$, if there exist $\xi \in U$ such that, the set

$$\Big\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n - \xi, \eta) \geq \epsilon, \mathcal{F}(a_n - \xi, \eta) \geq \epsilon\Big\}$$

has density zero, for every $\epsilon,\ \eta > 0$ or equivalently,

$$\lim_{n} \frac{1}{n} |\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n - \xi, \eta) \geq \epsilon, \mathcal{F}(a_n - \xi, \eta) \geq \epsilon\}| = 0.$$

We write $\mathcal{N}_\theta - \lim a = \xi$.

Using the above definition and properties of θ-density, we have the subsequent lemma.

Lemma 3.2. *Let $(U, \mathcal{N}, *, \diamond)$ be a NNS and θ be a lacunary sequence . Then, for each $\epsilon,\ \eta > 0$, the subsequent statements are equivalent:*

(1) $\mathcal{N}_\theta - \lim x = \xi$

(2) $\delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, t) \leq 1 - \epsilon, \mathcal{H}(a_n - \xi, \eta) \geq \epsilon, \mathcal{F}(a_n - \xi, \eta) \geq \epsilon\}\Big) = 0$

(3) $\delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) > 1 - \epsilon, \mathcal{H}(a_n - \xi, \eta) < \epsilon \textit{ and } \mathcal{F}(a_n - \xi, \eta) < \epsilon\}\Big) = 1.$

(4) $\delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) > 1 - \epsilon\}\Big) = \delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{H}(a_n - \xi, \eta) < \epsilon\}\Big) = \delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{F}(a_n - \xi, \eta) < \epsilon\}\Big) = 1$

(5) $\mathcal{N}_\theta - \lim \mathcal{T}(a_n - \xi, \eta) = 1$, $\mathcal{N}_\theta - \lim \mathcal{H}(a_n - \xi, \eta) = 0$ *and* $\mathcal{N}_\theta - \lim \mathcal{F}(a_n - \xi, \eta) = 0.$

Theorem 3.3. *Let θ be a lacunary sequence and $(U, \mathcal{N}, *, \diamond)$ be a NNS. If a sequence $a = (a_n)$ is lacunary statistically convergent with respect to NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$ then $\mathcal{N}_\theta-$ limit is unique.*

Proof. Consider, $\mathcal{N}_\theta - \lim a = \xi_1$, $\mathcal{N}_\theta - \lim a = \xi_2$ and $\xi_1 \neq \xi_2$. Given $\epsilon > 0$, $\alpha > 0$ and $(1 - \alpha) * (1 - \alpha) > 1 - \epsilon$ and $\alpha \diamond \alpha < \epsilon$ Then, for any $\eta > 0$, define the following sets as:

$$W_{\mathcal{T},1}(\alpha, \eta) = \{k \in \mathbb{N} : \mathcal{T}(a_n - \xi_1, \frac{\eta}{2}) \leq 1 - \alpha\}$$

$$W_{\mathcal{T},2}(\alpha, \eta) = \{n \in \mathbb{N} : \mathcal{T}(a_n - \xi_2, \frac{\eta}{2}) \leq 1 - \alpha\}$$

$$W_{\mathcal{H},1}(\alpha, \eta) = \{n \in \mathbb{N} : \mathcal{H}(a_n - \xi_1, \frac{\eta}{2}) \geq \alpha\}$$

$$W_{\mathcal{H},2}(\alpha, \eta) = \{n \in \mathbb{N} : \mathcal{H}(a_n - \xi_2, \frac{\eta}{2}) \geq \alpha\}$$

$$W_{\mathcal{F},1}(\alpha, \eta) = \{n \in \mathbb{N} : \mathcal{F}(a_n - \xi_1, \frac{\eta}{2}) \geq \alpha\}$$

$$W_{\mathcal{F},2}(\alpha,\eta) = \{n \in \mathbb{N} : \mathcal{F}(a_n - \xi_2, \frac{t}{2}) \geq \alpha\}$$

Since $\mathcal{N}_\theta - \lim a = \xi_1$, then using Lemma 3.2, for every $\eta > 0$, we have

$$\delta_\theta(W_{\mathcal{T},1}(\epsilon,\eta)) = \delta_\theta(W_{\mathcal{H},1}(\epsilon,\eta)) = \delta_\theta(W_{\mathcal{F},1}(\epsilon,\eta)) = 0 \tag{5}$$

Furthermore, using $\mathcal{N}_\theta - \lim a = \xi_2$, for all $\eta > 0$, we get

$$\delta_\theta(W_{\mathcal{T},2}(\epsilon,\eta)) = \delta_\theta(W_{\mathcal{H},2}(\epsilon,\eta)) = \delta_\theta(W_{\mathcal{F},2}(\epsilon,\eta)) = 0. \tag{6}$$

Now let

$$W(\epsilon,\eta) = \{(W_{\mathcal{T},1}(\epsilon,\eta) \cup W_{\mathcal{T},1}(\epsilon,\eta))\} \cap \{(W_{\mathcal{H},1}(\epsilon,\eta) \cup W_{\mathcal{H},1}(\epsilon,\eta))\} \cap \{(W_{\mathcal{F},1}(\epsilon,\eta) \cup W_{\mathcal{F},1}(\epsilon,\eta))\}$$

Then observe that $\delta_\theta(W(\epsilon,\eta)) = 0$ which implies $\delta_\theta(\mathbb{N} \backslash W(\epsilon,\eta)) = 1$ if $k \in \mathbb{N} \backslash W(\epsilon,\eta)$, then we have three possible cases.

(a) $(\{n \in \mathbb{N} \backslash W_{\mathcal{T},1}(\epsilon,\eta) \cup W_{\mathcal{T},1}(\epsilon,\eta)\})$

(b) $(\{n \in \mathbb{N} \backslash W_{\mathcal{H},1}(\epsilon,\eta) \cup W_{\mathcal{H},1}(\epsilon,\eta)\})$

(c) $(\{n \in \mathbb{N} \backslash W_{\mathcal{F},1}(\epsilon,\eta) \cup W_{\mathcal{F},1}(\epsilon,\eta)\})$.

Therefore, one obtain
$\mathcal{T}(\xi_1 - \xi_2, \eta) = \mathcal{T}(a_n - \xi_1, \frac{\eta}{2}) * \mathcal{T}(a_n - \xi_2, \frac{\eta}{2}) > (1-\alpha) * (1-\alpha)$.
Since $(1-\alpha) * (1-\alpha) > 1-\epsilon$.
It follows that $\mathcal{T}(\xi_1 - \xi_2, \eta) > 1-\epsilon$.
Since $\epsilon > 0$ was arbitrary, we get $\mathcal{T}(\xi_1 - \xi_2, \eta) = 1$ for all $\eta > 0$, which gives $\xi_1 = \xi_2$.
Contrarily (b), if $n \in \mathbb{N} \backslash W_{\mathcal{H},1}(\epsilon,\eta) \cup W_{\mathcal{H},1}(\epsilon,\eta)$. Then,
$\mathcal{H}(\xi_1 - \xi_2, \eta) \leq \mathcal{H}(a_n - \xi_1, \frac{\eta}{2}) \diamond \mathcal{H}(a_n - \xi_2, \frac{\eta}{2}) < \alpha \diamond \alpha$
Now, utilizing the fact that $\alpha \diamond \alpha < \epsilon$, it can be easily seen that
$\mathcal{H}(\xi_1 - \xi_2, \eta) < \epsilon$.
So, $\mathcal{H}(\xi_1 - \xi_2, \eta) = 0$ for all $\eta > 0$, implies $\xi_1 = \xi_2$.
and if $n \in \mathbb{N} \backslash W_{\mathcal{F},1}(\epsilon,\eta) \cup W_{\mathcal{F},1}(\epsilon,\eta)$.Then
$\mathcal{F}(\xi_1 - \xi_2, \eta) \leq \mathcal{F}(a_n - \xi_1, \frac{\eta}{2}) \diamond \mathcal{F}(a_n - \xi_2, \frac{\eta}{2}) < \alpha \diamond \alpha$
Since $\alpha \diamond \alpha < \epsilon$, it follows that.
$\mathcal{F}(\xi_1 - \xi_2, \eta) < \epsilon$. we have
$\mathcal{F}(\xi_1 - \xi_2, \eta) = 0$ for all $\eta > 0$, which implies $\xi_1 = \xi_2$.
Therefore, in all cases, we conclude that $\mathcal{N}_\theta-$ limit is unique.

□

Theorem 3.4. *Let θ be a lacunary sequence and $(U, \mathcal{N}, *, \diamond)$ be an NNS. If $\mathcal{N} - \lim a = \xi$ then $\mathcal{N}_\theta - \lim a = \xi$. but converse need not be true.*

Proof. Let $\lim a = \xi$. Then for each $\epsilon,\ \eta > 0$, there is a number $n_0 \in \mathbb{N}$ such that $\mathcal{T}(a_n - \xi, \eta) > 1 - \epsilon$, $\mathcal{H}(a_n - \xi, \eta) < \epsilon$ and $\mathcal{F}(a_n - \xi, \eta) < \epsilon$ for all $n \geq n_0$.
Hence, the set

$$\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n - \xi, \eta) \geq \epsilon, \mathcal{T}(a_n - \xi, \eta) \geq \epsilon\} \tag{7}$$

has at most finitely many terms. Since each and every finite subset of $\mathbb{N}$ has density zero and hence

$$\delta_\theta(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \epsilon \text{ or } \diamond H(a_n - \xi, \eta) \geq \epsilon, \mathcal{F}(a_n - \xi, \eta) \geq \epsilon\}) = 0. \tag{8}$$

Therefore, $\mathcal{N}_\theta - \lim a = \xi$. □

For converse, we construct the following example:

Example 3.5. Let $(U, \|.\|)$ be a NS. Consider $U = \mathbb{R}$ and for all $v, w \in [0,1]$, define $v * w = vw$ and $v \diamond w = \min\{v + w, 1\}$. Take

$$\mathcal{T}(v, \eta) = \frac{\eta}{\eta + \|v\|}, \mathcal{H}(v, \eta) = \frac{\|v\|}{\eta + \|v\|}, \mathcal{F}(v, t) = \frac{\|v\|}{\eta}$$

for all $\eta > 0$. Then $(U, \mathcal{N}, *, \diamond)$ be NNS. Define a sequence $a = (a_n)$ by,

$$a_n = \begin{cases} n, & \text{if } n_r - [\sqrt{h_r}] + 1 \leq n \leq n_r, r \in \mathbb{N} \\ 0, & otherwise. \end{cases} \tag{9}$$

Consider

$$K_r(\epsilon, \eta) = \{n \leq I_r : \mathcal{T}(a_n, \eta) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n, \eta) \geq \epsilon, \mathcal{F}(a_n, \eta) \geq \epsilon\}$$

for every $\epsilon \in (0,1)$ and for any $\eta > 0$. Then, we have

$$K_r(\epsilon, \eta) = \{n \leq I_r : \frac{\eta}{\eta + \|v\|} \leq 1 - \epsilon \text{ or } \frac{\|v\|}{\eta + \|v\|} \geq \epsilon, \frac{\|v\|}{\eta} \geq \epsilon\}$$
$$= \{n \leq I_r : \|v\| \geq \frac{\eta\epsilon}{1 - \epsilon} \text{ or } \|v\| \geq \eta\epsilon\}$$

$$\subseteq \{n \leq I_r : a_n = n\}$$

Thus,

$$\frac{1}{h_r} \mid \{n \in I_r : n \in K_r(\epsilon, \eta)\} \mid \leq \frac{\sqrt{h_r}}{h_r} \to 0 \text{ as } r \to \infty.$$

Therefore,

$$\mathcal{N}_\theta - \lim_k a_n = 0.$$

But the sequence $a = \{a_n\}$ is not convergent to 0.

Theorem 3.6. *Let $(U, \mathcal{N}, *, \diamond)$ be an NNS. Then for any lacunary sequence θ, $\mathcal{N}_\theta - \lim a_n = \xi$ iff there exists a increasing index sequence $K = \{k_1 < k_2 < \dots\} \subseteq \mathbb{N}$ while $\delta_\theta(N) = 1$, then $\mathcal{N} - \lim_j a_{n_j} = \xi$.*

Proof. Let $\mathcal{N}_\theta - \lim a_n = \xi$. For any $\eta > 0$ and $\alpha = 1, 2, 3, \dots$

$$W(\alpha, \eta) = \{n \leq k : \mathcal{T}(a_n - \xi, \eta) > 1 - \frac{1}{\alpha} \text{ and } \mathcal{H}(a_n - \xi, \eta) < \frac{1}{\alpha},\ \mathcal{F}(a_n - \xi, \eta) < \frac{1}{\alpha}\}$$

and

$$Q(\alpha, \eta) = \{n \leq k : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \frac{1}{\alpha} \text{ or } \mathcal{H}(a_n - \xi, \eta) \geq \frac{1}{\alpha},\ \mathcal{F}(a_n - \xi, \eta) \geq \frac{1}{\alpha}\}$$

Then, $\delta_\theta(Q(\alpha, \eta)) = 0$, since $\mathcal{N}_\theta - \lim a_n = \xi$. Further, for $\eta > 0$ and $\alpha = 1, 2, 3, \dots$

$$W(\alpha, \eta) \supset W(\alpha + 1, \eta)$$

and so,

$$\delta_\theta(W(\alpha, \eta)) = 1. \tag{10}$$

Now, we imply that for $n \in W(\alpha, \eta), \lim a_n = \xi$. Assume that $\lim a_n \neq \xi$ for some $n \in W(\alpha, \eta)$. Then, there is $\beta > 0$ and a +ve integer N such that $\mathcal{T}(a_n - \xi, \eta) \leq 1 - \beta$ or $\mathcal{H}(a_n - \xi, \eta) \geq \beta, \mathcal{F}(a_n - \xi, \eta) \geq \beta$ for all $n \geq N$. Let $\mathcal{T}(a_n - \xi, \eta) > 1 - \beta$ or $\mathcal{H}(a_n - \xi, \eta) < \beta, \mathcal{F}(a_n - \xi, \eta) < \beta$ for all $n > N$. Hence

$$\lim_k \frac{1}{k} \big| \{n \leq N : \mathcal{T}(a_n - \xi, \eta) > 1 - \beta \text{ and } \mathcal{H}(a_n - \xi, \eta) < \beta, \mathcal{F}(a_n - \xi, \eta) < \beta\} \big| = 0.$$

Since $\beta > \frac{1}{\alpha}$, we obtain $\delta_\theta(W(\alpha, \eta)) = 0$, which contradicts equation (10). that's why, $\mathcal{N}_\theta - \lim a_n = \xi$.

Conversely, assume that there exists an increasing index sequence $K = \{k_1 < k_2 < \dots\} \subseteq \mathbb{N}$ while $\delta_\theta(K) = 1$, then $\lim_k x_{n_k} = \xi$ i.e., there exists a $K \in \mathbb{N}$ such that $\mathcal{T}(a_n - \xi, \eta) > 1 - \alpha, \mathcal{H}(a_n - \xi, \eta) < \alpha, \mathcal{F}(a_n - \xi, \eta) < \alpha$ for every $\alpha > 0$ and $\eta > 0$. In that case

$$Q_\theta(\alpha, \eta) = \{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \alpha \text{ and } \mathcal{H}(a_n - \xi, \eta) \geq \alpha, \mathcal{F}(a_n - \xi, \eta) \geq \alpha\}$$

$$\subseteq \mathbb{N} - \{k_{N+1}, k_{N+2}, \dots\}.$$

Therefore $\delta_\theta(Q(\alpha, \eta)) \leq 1 - 1 = 0$. Hence $\mathcal{N}_\theta - \lim a_n = \xi$. □

4. Lacunary statistical Completeness in NNS

Definition 4.1. Let $(U, \mathcal{N}, *, \diamond)$ be an NNS and θ be any lacunary sequence. The sequence (a_n) is called Lacunary statistically Cauchy with respect to Neutrosophic norm(NN) in NNS U, if there exists $M = M(\epsilon)$, for every $\epsilon > 0$ and $\eta > 0$ such that, the set

$$\delta_\theta\Big(\big\{n \in \mathbb{N} : \mathcal{T}(a_n - a_M, \eta) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n - a_M, \eta) \geq \epsilon, \mathcal{F}(a_n - a_M, \eta) \geq \epsilon\big\}\Big) = 0.$$

Theorem 4.2. *Let $(U, \mathcal{N}, *, \diamond)$ be an NNS and θ be any lacunary sequence. If a sequence $\{a_n\}$ is $\mathcal{N}_\theta$-statistically convergent, then it is $\mathcal{N}_\theta$-statistically Cauchy with respect to the NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$.*

Proof. Let a sequence $a = \{a_n\}$ is a lacunary statistically convergent in NNS U. We obtained $(1-\epsilon) * (1-\epsilon) > 1 - \alpha$ and $\epsilon \diamond \epsilon < \alpha$ for a given $\epsilon > 0$ and choose $\alpha > 0$. Then, we get

$$\delta_\theta\left(W(\epsilon, \eta)\right) = \delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \frac{\eta}{2}) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n - \xi, \frac{\eta}{2}) \geq \epsilon, \mathcal{F}(a_n - \xi, \frac{\eta}{2}) \geq \epsilon\}\Big) = 0. \tag{11}$$

and so

$$\delta_\theta\Big(W^c(\epsilon, \eta)\Big) = \delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \frac{\eta}{2}) > 1 - \epsilon \text{ or } \mathcal{H}(a_n - \xi, \frac{\eta}{2}) < \epsilon, \mathcal{F}(a_n - \xi, \frac{\eta}{2}) < \epsilon\}\Big) = 1$$

for $\eta > 0$. Let $p \in W^c(\epsilon, \eta)$ then

$$\mathcal{T}(a_n - \xi, \frac{\eta}{2}) > 1 - \epsilon \text{ and } \mathcal{H}(a_n - \xi, \frac{\eta}{2}) < \epsilon, \mathcal{F}(a_n - \xi, \frac{\eta}{2}) < \epsilon.$$

Let

$$Q(\epsilon, \eta) = \{n \in \mathbb{N} : \mathcal{T}(a_n - a_m, \eta) \leq 1 - \alpha \text{ or } \mathcal{H}(a_n - a_m, \eta) \geq \alpha, \mathcal{F}(a_n - a_m, \eta) \geq \alpha\}$$

Now, we have to show that $Q(\epsilon, \eta) \subset W(\epsilon, \eta)$. Let $q \in Q(\epsilon, \eta) \setminus W(\epsilon, \eta)$. Then

$$\mathcal{T}(a_q - a_n, \eta) \leq 1 - \alpha \text{ and } \mathcal{T}(a_q - \xi, \frac{\eta}{2}) > 1 - \epsilon,$$

in particular $\mathcal{T}(a_q - \xi, \frac{\eta}{2}) > 1 - \epsilon$. At the same time,

$$1 - \alpha \geq \mathcal{T}(a_q - a_n, \eta) \geq \mathcal{T}(u_q - \xi, \frac{\eta}{2}) * \mathcal{T}(a_n - \xi, \frac{\eta}{2}) > (1 - \epsilon) * (1 - \epsilon) > 1 - \alpha$$

which is impossible. Moreover,

$$\mathcal{H}(a_q - a_n, \eta) \geq \alpha \text{ and } \mathcal{H}(a_q - \xi, \frac{\eta}{2}) < \alpha$$

in a similar way, $\mathcal{H}(a_n - \xi, \frac{\eta}{2}) < \epsilon$. Then,

$$\alpha \leq \mathcal{H}(a_q - a_n, \eta) \leq \mathcal{H}(a_q - \xi, \frac{\eta}{2}) \diamond \mathcal{H}(a_n - \xi, \frac{\eta}{2}) < \epsilon \diamond \epsilon < \alpha$$

which is impossible. Similarly,

$$\mathcal{F}(a_q - a_n, \eta) \geq \alpha \text{ and } \mathcal{F}(a_q - \xi, \frac{\eta}{2}) < \alpha$$

in particular $\mathcal{F}(a_n - \xi, \frac{t}{2}) < \epsilon$. Then,

$$\alpha \leq \mathcal{F}(a_q - a_n, \eta) \leq \mathcal{F}(a_q - \xi, \frac{\eta}{2}) \diamond \mathcal{F}(a_n - \xi, \frac{\eta}{2}) < \epsilon \diamond \epsilon < \alpha$$

which is impossible. suppose we consider, $Q(\epsilon, \eta) \subset W(\epsilon, t)$.
Then, by (11) $\delta_\theta\Big(Q(\epsilon, \eta)\Big) = 0$. Hence, sequence (a_n) is $\mathcal{N}_\theta$-Statistical Cauchy with respect to NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$. □

Definition 4.3. The NNS $(U, \mathcal{N}, \star, \diamond)$ is called statistically $(\mathcal{N}_\theta)$ complete, if every statistically $(\mathcal{N}_\theta)$,respectively) Cauchy sequence with respect to NN $(U, \mathcal{N}, \star, \diamond)$ is statistically $\mathcal{N}_\theta$, respectively) convergent with respect to NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$.

Theorem 4.4. *Let $(U, \mathcal{N}, *, \diamond)$ be a NNS and θ be any lacunary sequence. Then every sequence $a = (a_n)$ in U is $\mathcal{N}_\theta-$complete but not complete in general.*

Proof. Let (a_n) be $\mathcal{N}_\theta-$statistical Cauchy but not $\mathcal{N}_\theta-$statistical convergent in NNS. Choose $\alpha > 0$. We get $(1-\epsilon) * (1-\epsilon) > 1 - \alpha$ and $\epsilon \diamond \epsilon < \alpha$, for a given $\epsilon > 0$ and $\eta > 0$. Since (a_n) is not $\mathcal{N}_\theta$- statistical convergent in NNS.

$$\mathcal{T}(a_n - a_M, \eta) \geq \mathcal{T}(a_n - \xi, \frac{\eta}{2}) * \mathcal{T}(a_M - \xi, \frac{\eta}{2}) > (1-\epsilon) * (1-\epsilon) > 1 - \alpha$$

$$\mathcal{H}(a_n - a_M, \eta) \leq \mathcal{H}(a_n - \xi, \frac{\eta}{2}) \diamond \mathcal{H}(a_M - \xi, \frac{\eta}{2}) < \epsilon \diamond \epsilon < \alpha$$

$$\mathcal{F}(a_n - a_M, \eta) \leq \mathcal{F}(u_n - \xi, \frac{\eta}{2}) \diamond \mathcal{F}(a_n - \xi, \frac{\eta}{2}) < \epsilon \diamond \epsilon < \alpha$$

For,

$$W(\epsilon, \alpha) = \{n \in \mathbb{N}, B_{a_n - a_M}(\epsilon) \leq 1 - \alpha\}$$

Since $\delta_\theta\Big(W^C(\epsilon, \alpha)\Big) = 0$ and so $\delta_\theta\Big(W(\epsilon, \alpha)\Big) = 1$ which is in disagreement, since (a_n) was $\mathcal{N}_\theta-$ statistical Cauchy in NNS. So that (a_n) must be $\mathcal{N}_\theta-$statistical convergent in NNS. consequently, entire NNS is $\mathcal{N}_\theta-$statistically complete. □

Example 4.5. [3] Consider, $U = (0, 1]$ and

$$\mathcal{T}(u, \eta) = \frac{\eta}{\eta + |v|}, \mathcal{H}(v, \eta) = \frac{|v|}{\eta + |v|}, \mathcal{F}(u, \eta) = \frac{|v|}{\eta}$$

for all $v \in U$. Then $(U, \mathcal{N}, *, \diamond, \wedge, \vee)$ where min $= \wedge$ and max $= \vee$ is NNS but it's incomplete, since the sequence $(\frac{1}{m})$ is Cauchy with respect to $(\mathcal{T}, \mathcal{H}, \mathcal{F})$ but not convergent with regarding to the present $(\mathcal{T}, \mathcal{H}, \mathcal{F})$.

5. Conclusions

Since every standard norm defines an neutrosophic norm, our results are more general than the corresponding results in [4].The statistical convergence is a generalization of the usual convergence. Furthermore, definition provides a new techniques to investigate the completeness in the sense of statistical convergence. These are illustrated by suitable examples. Their related properties and structural characteristics have been discussed.

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