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An Introduction To Refined Neutrosophic Number Theory

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Abstract: Number theory is concerned with properties of integers and Diophantine equations. The objective of this paper is dedicated to introduce the basic concepts in refined neutrosophic number theory such as division, divisors, congruencies, and Pell's equation in the refined neutrosophic ring of integers $Z(I_1, I_2)$. Also, algorithms to solve refined neutrosophic linear congruencies and refined neutrosophic Pell's equation will be presented and discussed.

Keywords: refined neutrosophic integer, refined Pell's equation, neutrosophic congruence , neutrosophic Diophantine equation.

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1. Introduction

Neutrosophy is a new kind of generalized logic proposed by F.Smarandache [12,36]. It becomes a useful tool in many areas of science such as number theory [16], solving equations [19], and medical studies [11,15,21]. Also, we find many applications of neutrosophic structures in statistics [14], optimization [8], and decision making [7].

On the other hand, the theory of neutrosophic rings began in [4], where Smarandache and Kandasamy defined concepts such neutrosophic ideals and homomorphisms. These notions were handled widely by Agboola, et.al in [5,6,10]. Where homomorphisms and AH-substructures were studied [3,13,17]. More and more application of neutrosophic sets and their generalizations can be found in [25-35].

Recently, there is an arising interesting by the number theoretical concepts in neutrosophic ring of integers, where Ceven et.al defined and studied division and primes in Z(I) [2], Sankari et.al solved the linear Diophantine equations in Z(I) and $Z(I_1, I_2)$ [16]. Also, in [1], we find algorithms to solve neutrosophic Pell's equation and neutrosophic linear congruencies. In addition, Euler's famous theorem was proved in Z(I).

In this work, we extend the study to the case of refined neutrosophic ring of integers, where we determine algorithms and conditions for division, congruencies, and Pell's equation. In addition, we prove that there are no primes in $Z(I_1, I_2)$.

2. Preliminaries

Definition 2.1: [4]

Let R be a ring, I be the indeterminacy with property $I^2 = I$, then the neutrosophic ring is defined as follows:

$R(I) = \{a + bI; a, b \in R\}$.

Definition 2.2: [4]

Let R(I) be a neutrosophic ring, it is called commutative if and only if $xy = yx \forall\ x, y \in R(I)$..

Definition 2.3: [5]

The element I can be split into two indeterminacies I_1 , I_2 with conditions:

$I_1{}^2 = I_1 , I_2{}^2 = I_2 , I_1 I_2 = I_2 I_1 = I_1$.

Definition 2.4: [5]

If X is a set then $X(I_1, I_2) = \{(a, bI_1, cI_2): a , b , c \in X \}$ is called the refined neutrosophic set generated by X , I_1, I_2.

Definition 2.5: [5]

Let (R,+,×) be a ring, (R(I_1, I_2) , + ,×) is called a refined neutrosophic ring generated by R ,I_1, I_2.

Example 2.6: [6]

The refined neutrosophic ring of integers is $Z(I_1, I_2)$=$\{(a, bI_1, cI_2); a, b, c \in Z\}$.

Definition 2.7: [20]

Pell's equation is the Diophantine equation with form $X^2 - DY^2 = N; D, N \in Z$.

Theorem 2.8: [20]

If the equation $X^2 - DY^2 = 1$ has a solution, then $D > 0$ *and* D is square free.

Theorem 2.9: [20]

$Z[\sqrt{d_1}]$ is an integral domain.

Theorem 2.10: [2]

Let Z(I)= $\{a + bI; a, b \in Z\}$ the neutrosophic ring of integers. Then primes in Z(I) have one of the following forms:

$x = \pm p + (\pm 1 \pm p)I$ *or* $x = \pm 1 + (\pm p \pm 1)I$; *p is any prime in Z.*

Definition 2.11: [16]

Let $Z(I) = \{a + bI;\ a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation with two variables is defined as follows:

$AX + BY = C; A, B, C \in Z(I)$.

Theorem 2.12: [16]

Let $Z(I) = \{a + bI;\ a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation $AX + BY = C$ with two variables $X = x_1 + x_2I, Y = y_1 + y_2I$, where $A = a_1 + a_2I, B = b_1 + b_2I$ is equivalent to the following two classical Diophantine equations:

(1) $a_1x_1 + b_1y_1 = c_1$.

$$(2)(a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2.$$

Definition 2.13: [16]

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in Z\}$ be the refined neutrosophic ring of integers. The refined neutrosophic linear Diophantine equation with two variables is defined as follows:

$AX + BY = C; A, B, C \in Z(I_1, I_2)$.

Theorem 2.14: [16]

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in Z\}$ be the refined neutrosophic ring of integers,

$AX + BY = C; A, B, C \in Z(I_1, I_2)$ be a refined neutrosophic linear Diophantine equation, where

$X = (x_0, x_1I_1, x_2I_2), Y = (y_0, y_1I_1, y_2I_2), A = (a_0, a_1I_1, a_2I_2)$,

$B = (b_0, b_1I_1, b_2I_2), C = (c_0, c_1I_1, c_2I_2)$. Then $AX + BY = C$ is equivalent to the following three Diophantine equations:

(1) $a_0x_0 + b_0y_0 = c_0$.

$$(2)(a_0 + a_2)(x_0 + x_2) + (b_0 + b_2)(y_0 + y_2) = c_0 + c_2.$$

$(3)(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2.$

3. Refined neutrosophic number theory

Definition 3.1: (Division)

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in Z\}$ the refined neutrosophic ring of integers. For any $x, y \in Z(I_1, I_2)$, we say that $x|y$ if there is $r \in Z(I_1, I_2);\ r.x = y$.

Theorem 3.2: (Form of division in $Z(I_1, I_2)$)

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in Z\}$ the refined neutrosophic ring of integers, $x = (x_0, x_1I_1, x_2I_2), y = (y_0, y_1I_1, y_2I_2)$ be two arbitrary elements in $Z(I_1, I_2)$. Then $x|y$ if and only if

$$x_0|y_0, x_0 + x_2|y_0 + y_2, x_0 + x_1 + x_2|y_0 + y_1 + y_2.$$

Proof:

Suppose that $x|y$ in $Z(I_1, I_2)$, then there is $r = (r_0, r_1I_1, r_2I_2) \in Z(I_1, I_2)$ such that $r.x = y$(*).

By easy computing to equation (*) we get the following equivalent equations:

(a) $r_0x_0 = y_0$, i.e.$x_0|y_0$.

(b) $r_0x_2 + r_2x_2 + r_2x_0 = y_2$.

(c) $r_0x_1 + r_2x_1 + r_1x_0 + r_1x_1 + r_1x_2 = y_1$.

By adding equation (a) to (b) we get (**) $(r_0 + r_2)(x_0 + x_2) = y_0 + y_2, i.e. x_0 + x_2|y_0 + y_2$.

Now, we add equation (**) to (c) to get $(r_0 + r_1 + r_2)(x_0 + x_1 + x_2) = y_0 + y_1 + y_2, i.e.$

$x_0 + x_1 + x_2|y_0 + y_1 + y_2$.

For the converse, we assume that $x_0|y_0, x_0 + x_2|y_0 + y_2, x_0 + x_1 + x_2|y_0 + y_1 + y_2$.

There are

$a, b, c \in Z;\ ax_0 = y_0, b(x_0 + x_2) = y_0 + y_2, c(x_0 + x_1 + x_2) = y_0 + y_1 + y_2$.

We put

$r_0 = a, r_2 = b - a, r_1 = c - b$.

Now, we get $r = (r_0, r_1I_1, r_2I_2) \in Z(I_1, I_2), and r.x = y,$

hence $x|y$.

Definition 3.3: (Congruence)

Let $x=(x_0,x_1I_1,x_2I_2), y=(y_0,y_1I_1,y_2I_2), z=(z_0,z_1I_1,z_2I_2)$ be three elements in $Z(I_1,I_2)$. We say that $x\equiv y(modz)$ if and only if $z|x-y$.

We say that $z=\gcd(x,y)$ if and only if $z|x\ and\ z|y, and\ for\ every\ c|x\ and\ c|y, we\ have\ c|z.$

x,y are called relatively prime in Z(I) if and only if $\gcd(x,y)=(1,0,0)$.

Theorem 3.4: (Form of congruencies in $Z(I_1,I_2)$)

Let $x=(x_0,x_1I_1,x_2I_2), y=(y_0,y_1I_1,y_2I_2), z=(z_0,z_1I_1,z_2I_2)$ be three elements in $Z(I_1,I_2)$. Then $x\equiv y(modz)$ if and only if

$x_0\equiv y_0(modz_0), x_0+x_2\equiv y_0+y_2(modz_0+z_2), x_0+x_1+x_2\equiv y_0+y_1+y_2(modz_0+z_1+z_2).$

Proof:

We assume that $x\equiv y(modz)$, then $z|x-y$. By Theorem 3.2, we find that $z_0|x_0-y_0, (z_0+z_2)|(x_0+x_2)-(y_0+y_2), (z_0+z_1+z_2)|(x_0+x_1+x_2)-(y_0+y_1+y_2)$, thus

$x_0\equiv y_0(modz_0), x_0+x_2\equiv y_0+y_2(modz_0+z_2), x_0+x_1+x_2\equiv y_0+y_1+y_2(modz_0+z_1+z_2).$

The converse is trivial.

Example 3.5:

$(1,I_1,2I_2)\equiv(3,-I_1,0)(mod(2,-I_1,I_2))$, that is because

$1\equiv 3(mod2), 1+2=3\equiv(3+0)(mod\ 3), 1+1+2=4\equiv(3-1+0)(mod2\ ).$

Theorem 3.6: (Form of GCD)

Let $x=(x_0,x_1I_1,x_2I_2), y=(y_0,y_1I_1,y_2I_2)$be two elements in $Z(I_1,I_2)$. Then

$\text{r}=\gcd(x,y)=(m,nI_1,tI_2); m=\gcd(x_0,y_0), m+n+t=\gcd(x_0+x_1+x_2,y_0+y_1+y_2), m+t=\gcd(x_0+x_2,y_0+y_2).$

Proof:

It is clear that $r|x\ and\ r|y$. Let $z=(z_0,z_1I_1,z_2I_2)$ be a common divisor of $x\ and\ y$, then

(a) $z_0|x_0, z_0|y_0,$ hence $z_0|m$.

(b) $z_0+z_2|x_0+x_2 and z_0+z_2|y_0+y_2,$ hence $z_0+z_2|m+t$.

(c) $z_0+z_1+z_2|x_0+x_1+x_2 and z_0+z_1+z_2|y_0+y_1+y_2,$ hence $z_0+z_1+z_2|m+n+t$.

According to the previous discussion, we get $z|r$. Thus $\mathrm{r} = \gcd(x,y) = (m, nI_1, tI_2)$.

Example 3.7:

Let $x = (2, -I_1, 3I_2), y = (1,3I_1, I_2)$, then $\gcd(x,y) = (1,0,0)$.

Theorem 3.8: (Euclidian division theorem in $Z(I_1, I_2)$)

Let $x = (x_0, x_1I_1, x_2I_2), y = (y_0, y_1I_1, y_2I_2)$be two elements in $Z(I_1, I_2)$.

There are two corresponding elements $q = (q_0, q_1I_1, q_2I_2), r = (r_0, r_1I_1, r_2I_2) \in Z(I_1, I_2); x = qy + r$.

Proof:

By classical division in Z, we can find $s_0, p_0, s_1, p_1, s_2, p_2$such that

$x_0 = y_0s_0 + p_0, (x_0 + x_2) = s_2(y_0 + y_2) + p_2, (x_0 + x_1 + x_2) = s_1(y_0 + y_1 + y_2) + p_1$.

By putting $q_0 = s_0, q_1 = s_1 - s_2, q_2 = s_2 - s_0, r_0 = p_0, r_1 = p_1 - p_2, r_2 = p_2 - p_0$, we get

$x = qy + r$.

Example 3.9:

Consider $x = (2, I_1, -I_2), y = (1,2I_1, 2I_2)$, then we have $q = (2,0, -2I_2), r = (0, I_1, I_2)$, where

$x = qy + r$.

Remark 3.10: (Solvability of a linear congruence in $Z(I_1, I_2)$)

To solve a linear congruence $x \equiv y(modz)$. We should take its corresponding equivalent linear congruencies according to Theorem 3.4. Then we can find its solution easily.

Example 3.11:

Consider the following refined neutrosophic linear congruence

$x \equiv (2,3I_1, I_2)(mod(1, I_1, 4I_2))$.

The equivalent system of congruencies is

$x_0 \equiv 2(mod\ 1)(I), x_0 + x_2 \equiv 3(mod\ 5)(II), x_0 + x_1 + x_2 \equiv 6(mod\ 6)(III)$.

The congruence (I) has a solution $x_0 = 1$. (II) has a solution $x_0 + x_2 = 3, hence x_2 = 2$.

(III) has a solution $x_0 + x_1 + x_2 = 6, hence x_1 = 3$. Thus the solution of the refined neutrosophic linear congruence is $x = (1,3I_1, 2I_2)$. It is easy to check that $(1, I_1, 4I_2)|[(1,3I_1, 2I_2) - (2,3I_1, I_2)]$.

Definition 3.12:

We define $p = (a, bI_1, cI_2)$ to be a refined neutrosophic prime integer if and only if p is not divided by any other neutrosophic integer different from (1,0,0) and p.

Remark 3.13:

Definition 3.12 is different from the definition of prime elements in a ring, where p is called prime element if it has the following property:

If $p = rq$, then r or $q$must be a unit.

Theorem 3.14:

$Z(I_1, I_2)$ has no refined neutrosophic primes.

Proof:

Let $p = (a, bI_1, cI_2)$ be any refined neutrosophic integer different from $(1,2I_1, -2I_2)$, we have:

$r = (1,2I_1, -2I_2)$is a divisor of p, that is because $1|a, 1-2|a+c, 1+2-2|a+b+c$, which is different from (1,0,0) and p. Hence p can not be a refined neutrosophic prime.

If p=$(1,2I_1, -2I_2)$, we have $(1, -2I_1, 0)$ as a divisor different from p and (1,0,0), thus there are no refined neutrosophic primes.

The question about the structure of prime elements in the refined neutrosophic ring of integers is still open. It depends on the structure of the group of units in the refined neutrosophic ring of integers.

Definition 3.16. (Linear Combination in $Z(I\ 1\ , I\ 2\ ))$

Let u,v be non-zero refined neutrosophic integers. Then any refined neutrosophic integer that can be written in the form $ux + vy$ where $x, y \in Z(I\ 1\ , I\ 2\ )$ is called a linear combination of u and v.

Example 3.17:

Let (2,2I_1 ,8I_2),(8,3I_1 ,7I_2) ∈ Z(I 1 ,I 2), we can find refined neutrosophic integers in $Z(I\ 1\ , I\ 2\ )$ that can be written as a linear combination of (2,2I_1 ,8I_2), and (8,3I_1 ,7I_2).

To see this , Let $A(I\ 1\ , I\ 2\ )$ be the set of all linear combinations of (2,2I_1 ,8I_2), and (8,3I_1 ,7I_2).

Then

A(I 1 ,I 2) = (2,2I_1 ,8I_2)(x_0 ,x_1I_1 ,x_2I_2) + (8,3I_1 ,7I_2)(y_0 ,y_1I_1 ,y_2I_2)}

where (x_0 ,x_1I_1 ,x_2I_2),(y_0 ,y_1I_1 ,y_2I_2) ∈ $Z(I\ 1\ , I\ 2\ )$.

Now, let (m_0 ,m_1I_1 ,m_2I_2) = (2,2I_1 ,8I_2)(x_0 ,x_1I_1 ,x_2I_2) + (8,3I_1 ,7I_2)(y_0 ,y_1I_1 ,y_2I_2) for some (x_0 ,x_1I_1 ,x_2I_2) and (y_0 ,y_1I_1 ,y_2I_2).

Since

gcd((2,2I_1 ,8I_2),(8,3I_1 ,7I_2)) = (2, I_1,3 I_2).

Then

(m_0 ,m_1I_1 ,m_2I_2) = (2,2I 1 ,8I 2)(x_0 ,x_1I_1 ,x_2I_2) + (8,3I 1 ,7I 2)(y_0 ,y_1I_1 ,y_2I_2)

= (2, I_1,3 I_2)[(1,0I_1 ,I_2)(x_0 ,x_1I_1 ,x_2I_2) + (4,0I_1 ,−I_2)(y_0 ,y_1I_1 ,y_2I_2)].

We see that (2, I_1,3 I_2)|(m_0 ,m_1I_1 ,m_2I_2), whatever the values of (x_0 ,x_1I_1 ,x_2I_2) and (y_0 ,y_1I_1 ,y_2I_2).

Hence, (2, I_1,3 I_2)|(m_0 ,m_1I_1 ,m_2I_2) for all (m_0 ,m_1I_1 ,m_2I_2) ∈ $A(I\ 1\ , I\ 2\ )$. Thus, every member of $A(I\ 1\ , I\ 2\ )$ is a multiple of (2, I_1,3 I_2).

This observation is recorded in the following theorem.

Theorem 3.15:

Let u = (a_0 ,a_1I_1 ,a_2I_2),v = (b_0 ,b_1I_1 ,b_2I_2) and w = (g_0 ,g_1I_1 ,g_2I_2) be non-zero refined neutrosophic integers and let $w = gcd(u, v)$. Then every linear combination of u and v is a multiple of w. That is,

$w|up + vq$,

for all p = (p_0 ,p_1I_1 ,p_2I_2),q = (q_0 ,q_1I_1 ,q_2I_2) ∈ $Z(I\ 1\ , I\ 2\ )$.

Proof:

The proof is similar to the classical case.

4. Refined neutrosophic Pell's equation

Definition 4.1:

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in Z\}$ be the refined neutrosophic ring of integers. Refined Neutrosophic Pell's equation is defined as follows:

$X^2 - DY^2 = C; X = (x_0, x_1I_1, x_2I_2), Y = (y_0, y_1I_1, y_2I_2), D = (d_0, d_1I_1, d_2I_2), C = (c_0, c_1I_1, c_2I_2)$.

Where $c_i, d_i, x_i, y_i \in Z$.

Theorem 4.2:

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in Z\}$ be the refined neutrosophic ring of integers, $X^2 - DY^2 = C$ be a refined neutrosophic Pell's equation. Then it is equivalent to the following three classical Pell's equations:

(a) $x_0{}^2 - d_0 y_0{}^2 = c_0$.

(b) $(x_0 + x_2)^2 - (d_0 + d_2)(y_0 + y_2)^2 = c_0 + c_2$.

(c) $(x_0 + x_1 + x_2)^2 - (d_0 + d_1 + d_2)(y_0 + y_1 + y_2)^2 = c_0 + c_1 + c_2$.

Proof:

We compute:

$X^2 = (x_0{}^2, [x_0x_1 + x_1x_0 + x_1x_1 + x_1x_2 + x_1x_2]I_1, [x_0x_2 + x_2x_2 + x_2x_0]I_2)$=

$(x_0{}^2, [x_1{}^2 + 2x_0x_1 + 2x_1x_2]I_1, [x_2{}^2 + 2x_0x_2]I_2)$,

$DY^2 = (d_0, d_1I_1, d_2I_2).(y_0{}^2, [y_1{}^2 + 2y_0y_1 + 2y_1y_2]I_1, [y_2{}^2 + 2y_0y_2]I_2)$=

$(d_0y_0{}^2, [d_0y_1{}^2 + 2d_0y_0y_1 + 2d_0y_1y_2 + d_1y_0{}^2 + d_1y_1{}^2 + 2d_1y_0y_1 + 2d_1y_1y_2 + d_1y_2{}^2 + 2d_1y_0y_2 + d_2y_1{}^2 + 2d_2y_0y_1 + 2d_2y_1y_2]I_1, [d_0y_2{}^2 + 2d_0y_0y_2 + d_2y_0{}^2 + d_2y_2{}^2 + 2d_2y_0y_2]I_2)$.

Now we have:

$x_0{}^2 - d_0y_0{}^2 = c_0$. (Equation (a)).

(*)$x_2{}^2 + 2x_0x_2 - (d_0y_2{}^2 + 2d_0y_0y_2 + d_2y_0{}^2 + d_2y_2{}^2 + 2d_2y_0y_2) = c_2$.

(**)$x_1{}^2 + 2x_0x_1 + 2x_1x_2 - (d_0y_1{}^2 + 2d_0y_0y_1 + 2d_0y_1y_2 + d_1y_0{}^2 + d_1y_1{}^2 + 2d_1y_0y_1 + 2d_1y_1y_2 + d_1y_2{}^2 + 2d_1y_0y_2 + d_2y_1{}^2 + 2d_2y_0y_1 + 2d_2y_1y_2) = c_1$.

By adding (a) to (*) we get:

$(x_0 + x_2)^2 - (d_0 + d_2)(y_0 + y_2)^2 = c_0 + c_2$. (Equation (b)).

By adding (b) to (**) we get:

$(x_0 + x_1 + x_2)^2 - (d_0 + d_1 + d_2)(y_0 + y_1 + y_2)^2 = c_0 + c_1 + c_2$.

The converse is clear.

Remark 4.3:

To solve a refined neutrosophic Pell's equation, follow these steps:

(1) Write the equivalent system of classical Pell's equations.

(2) Solve equation (a).

(3) Solve (b).

(4) Solve (c).

(5) Compute x_2, y_2, and then x_1, y_1.

Example 4.5:

Consider the following refined neutrosophic Pell's equation $X^2 - (2,0,I_2)Y^2 = (1,-6I_1,3I_2)$.

The equivalent system is:

(a) ${x_0}^2 - 2{y_0}^2 = 1$.

(b)$(x_0 + x_2)^2 - 3(y_0 + y_2)^2 = 4$.

(c) $(x_0 + x_1 + x_2)^2 - 3(y_0 + y_1 + y_2)^2 = -2$.

Equation (a) has a solution $x_0 = 3, y_0 = 2$. Equation (b) has a solution $y_0 + y_2 = 2, x_0 + x_2 = 4$. Equation (c) has a solution $x_0 + x_1 + x_2 = 5, y_0 + y_1 + y_2 = 3$. Thus $y_2 = 0, x_2 = 1, y_1 = 1, x_1 = 1$, so $X = (3, I_1, I_2), Y = (2, I_1, 0)$.

5. Open questions

There are many open problems come to light according to this research. This section is devoted to present some important questions in the refined neutrosophic number theory.

Problem 1: Determine the form of prime elements in $Z(I_1, I_2)$.

Problem 2: Define Euler's function in $Z(I_1, I_2)$. Is Euler's Theorem still true in the case of refined neutrosophic integers.

Problem 3: Find an easy algorithm to solve a refined neutrosophic non linear congruence in a similar way to refined neutrosophic Pell's equation.

Problem 4: Find the form of the fundamental theorem in arithmetic in $Z(I_1, I_2)$.

4. Conclusions

In this article, we have established the basic theory of refined neutrosophic integers. Many important concepts and conditions about division, gcd, and congruencies in $Z(I_1, I_2)$. Also, refined neutrosophic Pell's equation was studied and we gave an algorithm to solve this kind of non linear Diophantine equations.

We have listed four open new problems concerning the refined neutrosophic number theory, their solution may lead to a big progression in neutrosophic number theory.

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