H-Max Distance Measure of Bipolar Neutrosophic Sets and an Application to Medical Diagnosis

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Abstract: A single-valued neutrosophic set is one of the advanced fuzzy sets that is capable of handling complex real-world information satisfactorily. A development of single-valued neutrosophic set and fuzzy bipolar set, called a bipolar neutrosophic set, was introduced. Distance measures between fuzzy sets and advanced fuzzy sets are important tools in diagnostics and prediction problems. Sometimes they are defined without considering the condition of the inclusion relation on sets. In decision-making applications, this condition should be required (here it is called the inference of the measure). Moreover, in many cases, a distance measure capable of discriminating between two nearly identical objects is considered an effective measure. Motivated by these observations, in this paper, a new distance measure is proposed in a bipolar neutrosophic environment. Furthermore, an entropy measure is also developed by the similarity between two sets of mutual negation. Finally, an application to medical diagnosis is presented to illustrate the effective applicability of the proposed distance measure, where entropy values are used to characterize noises of different attributes.

Keywords: neutrosophic distance; similarity measure; bipolar neutrosophic sets; entropy measure; medical diagnosis

1. Introduction

In 1965, the concept of a fuzzy set (FS) was introduced by Zadeh [1] to handle uncertainty of information in real-world inference systems. According to him, the degree of membership (positivity) of an element u to a FS on a universe U is one value $\mu(u)$, where $\mu(u) \in \left[0,1\right]$. The theory of FSs has reached a huge amount of achievements in a variety of application areas. However, in many real-life problems, the presence of negativity cannot be ignored. In 1983, Atanassov [2] proposed the concept of an intuitionistic fuzzy set (IFS) by considering the membership degree $\mu(u)$ as well as the non-membership degree $\nu(u)$ with the condition on their sum which is $\mu(u)+\nu(u) \le 1$. The theory and applications of IFSs have been strongly developed such as studies on logical operators [3-5] and applications in decision making [6-10].

From a philosophical perspective on the existence of the field of neutrosophy, Smarandache considers that using IFSs to treat indeterminate and inconsistent is not satisfactory enough. In 1999, Smarandache [11] introduced the concept of neutrosophic set (NS). He named its three characteristic functions the truth membership function, the indeterminacy-membership function, and falsity-membership function, denoted by $T(u)$, $I(u)$, and $F(u)$, respectively. Their outputs are real standard or nonstandard subsets of $]^{-}0,1^{+}[$. From the requirement of practical applications about representing the featured degrees by real values, Wang et al [12] provided the definition of single-valued neutrosophic sets (SVNSs). Cuong [13] also proposed the concept of picture fuzzy set (PFS) as a particular case of NSs. Some results on PFSs can be found in [14-19]. Because of the independent existence between the considered property and its corresponding implicit antagonist, Deli et al. [20] introduced the concept of bipolar neutrosophic sets (BNSs). This is a generalization of SVNSs and bipolar fuzzy sets [21]. In a BNS X, $T^{\square}(u), I^{\square}(u), F^{\square}(u)$ represent the characteristic degrees of an element $u \in U$ corresponding to X and $T^{\approx}(u), I^{\approx}(u), F^{\approx}(u)$ represent characteristic degrees of u to some implicit counter-property corresponding to X. Some research on NSs and BNSs and their applications can be found in [22-36].

The advanced fuzzy distance measures are known as effective tools for solving decision-making problems [6-10, 13, 37]. Some of distance measures of SVNSs were proposed such as Hausdorff distance [38], Cosine similarity measures [39], and the distance measures of Ye [40], Aydoğdu [41], Huang [26], and Ngan et al. [42]. In 2018, Vakkas [43] et al. introduced similarity measures of BNSs and their application to decision-making problems. Vakkas's measure was defined without considering the condition of the inclusion relation on sets. In decision-making applications, this condition (in this paper, it is called the inference of the measure) should be required. Moreover, Vakkas's proposal does not imply cross-evaluation, which is necessary to distinguish the differences and was discussed in intuitionistic fuzzy and single-value neutrosophic environments [7,10,42]. Motivated by these observations, in this paper, a new distance measure set that includes cross-evaluation and the inference of the measure is first proposed in a bipolar neutrosophic environment. Furthermore, an entropy measure is also developed by the similarity between two sets of mutual negation. Finally, an application to medical diagnosis on the UCI dataset is presented to illustrate the effective applicability of the proposed distance measure, where entropy values of different attribute sets are used to characterize their noises.

The next sections of the paper are distributed content as follows. Some basic concepts and the related measure formulas are presented in Section 2. In Section 3, the proposals on the distance measure, the similarity measure, and the entropy measure on BNSs are introduced. In Section 4, an application to medical diagnosis given to show the effectiveness of the proposed distance measure. Finally, Section 5 shows the conclusions of the study.

2. Preliminaries

Definition 1. [25] A NS X on a universe set U is characterized by three feature functions including a truth-membership function, $T_X : U \rightarrow]^{-}0,1^{+}[$, an indeterminacy-membership function, $I_X : U \rightarrow]^{-}0,1^{+}[$, and a falsity-membership function, $F_X : U \rightarrow]^{-}0,1^{+}[$, where

$$^{-}0 \le \sup_{U} T_X(z) + \sup_{U} I_X(z) + \sup_{U} F_X(z) \le 3^{+},\ z \in U. \tag{1}$$

Definition 2. [20] A BNS X on U is defined by the form as follows:

$$X = \left\{ < z, T_X^{\square}(z), I_X^{\square}(z), F_X^{\square}(z), T_X^{\approx}(z), I_X^{\approx}(z), F_X^{\approx}(z) > \mid z \in U \right\} \text{ or}$$

$$X = < T_X^{\square}, I_X^{\square}, F_X^{\square}, T_X^{\approx}, I_X^{\approx}, F_X^{\approx} >, \tag{2}$$

where $T_X^{\square}, I_X^{\square}, F_X^{\square} : U \rightarrow [0,1]$, and $T_X^{\approx}, I_X^{\approx}, F_X^{\approx} : U \rightarrow [-1,0]$.

Denoted by $BNS(U)$ the set of all BNSs on U.

Definition 3. [20] Let X_1 and X_2 be two BNSs on U, then

- $X_1 \subseteq X_2$ if and only if $T_1^{\square}(z) \le T_2^{\square}(z), I_1^{\square}(z) \ge I_2^{\square}(z), F_1^{\square}(z) \ge F_2^{\square}(z),\ T_1^{\approx}(z) \ge T_2^{\approx}(z),\ I_1^{\approx}(z) \le I_2^{\approx}(z),$ and $F_1^{\approx}(z) \le F_2^{\approx}(z)$.
- $X_1 = X_2$ if and only if $T_1^{\square}(z) = T_2^{\square}(z), I_1^{\square}(z) = I_2^{\square}(z), F_1^{\square}(z) = F_2^{\square}(z),\ T_1^{\approx}(z) = T_2^{\approx}(z), I_1^{\approx}(z) = I_2^{\approx}(z),$ and $F_1^{\approx}(z) = F_2^{\approx}(z)$.
- $X^c = \left\{ < z, F^{\square}(z), 1 - I^{\square}(z), T^{\square}(z), F^{\approx}(z), -1 - I^{\approx}(z), T^{\approx}(z) > \mid z \in U \right\}$.

Definition 4. [43] A similarity measure of BNSs is a $S : \left(BNS\left(U\right)\right)^2 \to \left[0,1\right]$ mapping satisfying

1. $0 \le S\left(X_1, X_2\right) \le 1$,
2. $S\left(X_1, X_2\right) = S\left(X_2, X_1\right)$,
3. $S\left(X_1, X_2\right) = 1$ for $X_1 = X_2$, where $X_1, X_2 \in BNS\left(U\right)$.

In 2018, Vakkas et al. [43] proposed a similarity measure of BNSs as follows:

$$S_V\left(X_1, X_2\right) = \alpha S_{V1}\left(X_1, X_2\right) + \left(1 - \alpha\right) S_{V2}\left(X_1, X_2\right), \tag{3}$$

where $\alpha \in \left[0,1\right]$,

$$S_{V1}\left(X_1, X_2\right) = \sum_{i=1}^{n} \omega_i \left(\frac{\begin{bmatrix} \left(T_{X_1}^{\square}(z_i) T_{X_2}^{\square}(z_i) + I_{X_1}^{\square}(z_i) I_{X_2}^{\square}(z_i) + F_{X_1}^{\square}(z_i) F_{X_2}^{\square}(z_i)\right) \\ -\left(T_{X_1}^{\approx}(z_i) T_{X_2}^{\approx}(z_i) + I_{X_1}^{\approx}(z_i) I_{X_2}^{\approx}(z_i) + F_{X_1}^{\approx}(z_i) F_{X_2}^{\approx}(z_i)\right) \end{bmatrix}}{\begin{matrix} 2\Big[\left(T_{X_1}^{\square\,2}(z_i) + I_{X_1}^{\square\,2}(z_i) + F_{X_1}^{\square\,2}(z_i)\right) + \left(T_{X_2}^{\square\,2}(z_i) + I_{X_2}^{\square\,2}(z_i) + F_{X_2}^{\square\,2}(z_i)\right) \\ -\left(T_{X_1}^{\approx\,2}(z_i) + I_{X_1}^{\approx\,2}(z_i) + F_{X_1}^{\approx\,2}(z_i)\right) - \left(T_{X_2}^{\approx\,2}(z_i) + I_{X_2}^{\approx\,2}(z_i) + F_{X_2}^{\approx\,2}(z_i)\right)\Big] \end{matrix}} \right),$$

and

$$S_{V2}\left(X_1, X_2\right) = \sum_{i=1}^{n} \omega_i \left(\frac{\begin{bmatrix} \left(T_{X_1}^{\square}(x_i) T_{X_2}^{\square}(x_i) + I_{X_1}^{\square}(x_i) I_{X_2}^{\square}(x_i) + F_{X_1}^{\square}(x_i) F_{X_2}^{\square}(x_i)\right) \\ -\left(T_{X_1}^{\approx}(x_i) T_{X_2}^{\approx}(x_i) + I_{X_1}^{\approx}(x_i) I_{X_2}^{\approx}(x_i) + F_{X_1}^{\approx}(x_i) F_{X_2}^{\approx}(x_i)\right) \end{bmatrix}}{\begin{matrix} 2\Big[\sqrt{T_{X_1}^{\square\,2}(x_i) + I_{X_1}^{\square\,2}(x_i) + F_{X_1}^{\square\,2}(x_i)} \times \sqrt{T_{X_2}^{\square\,2}(x_i) + I_{X_2}^{\square\,2}(x_i) + F_{X_2}^{\square\,2}(x_i)} \\ -\sqrt{T_{X_1}^{\approx\,2}(x_i) + I_{X_1}^{\approx\,2}(x_i) + F_{X_1}^{\approx\,2}(x_i)} \times \sqrt{T_{X_2}^{\approx\,2}(x_i) + I_{X_2}^{\approx\,2}(x_i) + F_{X_2}^{\approx\,2}(x_i)}\Big] \end{matrix}} \right).$$

Note that: Vakkas's proposal is without considering the condition related to the inclusion relation on sets. Some other measures are built based on the triangle inequality condition instead of the condition related to the inclusion relation on sets, such as the Hamming distance and the Euclidean distance [44, 45].

In 2021, by reasoning about the need for the cross-evaluation, Ngan et al. [42] defined the H-max distance measure on SVNSs by

$$\begin{aligned} d_{HN}\left(X_1, X_2\right) = \sum_{i=1}^{n} \chi_i \Big(& \alpha_1 \left|T_{X_1}(z_i) - T_{X_2}(z_i)\right| + \alpha_2 \left|I_{X_1}(z_i) - I_{X_2}(z_i)\right| + \alpha_3 \left|F_{X_1}(z_i) - F_{X_2}(z_i)\right| \\ & + \alpha_4 \left|\max\left\{T_{X_1}(z_i), I_{X_2}(z_i)\right\} - \max\left\{I_{X_1}(z_i), T_{X_2}(z_i)\right\}\right| \\ & + \alpha_5 \left|\max\left\{T_{X_1}(z_i), F_{X_2}(z_i)\right\} - \max\left\{F_{X_1}(z_i), T_{X_2}(z_i)\right\}\right| \Big) \end{aligned} \tag{4}$$

where $\alpha_k \in \left(0,1\right)$, $\sum_{k=1}^{5} \alpha_k = 1$, $\chi_i \in \left[0,1\right]$, $\sum_{i=1}^{n} \chi_i = 1$.

3. H-max bipolar neutrosophic weighted measure

Now, the provided definition of distance measures of BNSs includes the inference condition. Furthermore, a specific distance measure, called H-max bipolar neutrosophic weighted measure, is introduced based on the formula of d_{HN} proposed by Ngan et al. [42].

Definition 5. For all $X_1, X_2, X_3 \in BNS(U)$ where $U = \{z_1, ..., z_n\}$, then a distance measure of BNSs is $d : (BNS(U))^2 \rightarrow [0,1]$ mapping satisfying

1. $d(X_1, X_2) = d(X_2, X_1)$,
2. $d(X_1, X_2) = 0$ if and only if $X_1 = X_2$,
3. If $X_1 \subseteq X_2 \subseteq X_3$, then $d(X_1, X_2) \le d(X_1, X_3)$ and $d(X_2, X_3) \le d(X_1, X_3)$.

Definition 6. Let $X_1, X_2 \in BNS(U)$ where $U = \{z_1, ..., z_n\}$ and

$$X_1 = \left\{ < z, T^{\square}_{X_1}(z), I^{\square}_{X_1}(z), F^{\square}_{X_1}(z), T^{\approx}_{X_1}(z), I^{\approx}_{X_1}(z), F^{\approx}_{X_1}(z) > \mid z \in U \right\},$$
$$X_2 = \left\{ < z, T^{\square}_{X_2}(z), I^{\square}_{X_2}(z), F^{\square}_{X_2}(z), T^{\approx}_{X_2}(z), I^{\approx}_{X_2}(z), F^{\approx}_{X_2}(z) > \mid z \in U \right\}.$$

Then, the formula of H-max bipolar neutrosophic weighted distance measure between X_1 and X_2 is as follows

$$d_{H-BN}(X_1, X_2) = \lambda d_{H-BN1}(X_1, X_2) + (1-\lambda) d_{H-BN2}(X_1, X_2), \tag{5}$$

where

$$\begin{aligned} d_{H-BN1}(X_1, X_2) = \sum_{i=1}^{n} \chi_i^{\square} \Big(& \alpha_1^{\square} \left| T^{\square}_{X_1}(z_i) - T^{\square}_{X_2}(z_i) \right| + \alpha_2^{\square} \left| I^{\square}_{X_1}(z_i) - I^{\square}_{X_2}(z_i) \right| + \alpha_3^{\square} \left| F^{\square}_{X_1}(z_i) - F^{\square}_{X_2}(z_i) \right| \\ & + \alpha_4^{\square} \left| \max\left\{ T^{\square}_{X_1}(z_i), I^{\square}_{X_2}(z_i) \right\} - \max\left\{ I^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i) \right\} \right| \\ & + \alpha_5^{\square} \left| \max\left\{ T^{\square}_{X_1}(z_i), F^{\square}_{X_2}(z_i) \right\} - \max\left\{ F^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i) \right\} \right| \Big), \end{aligned}$$

$$\begin{aligned} d_{H-BN2}(X_1, X_2) = \sum_{i=1}^{n} \chi_i^{\approx} \Big(& \alpha_1^{\approx} \left| T^{\approx}_{X_1}(z_i) - T^{\approx}_{X_2}(z_i) \right| + \alpha_2^{\approx} \left| I^{\approx}_{X_1}(z_i) - I^{\approx}_{X_2}(z_i) \right| + \alpha_3^{\approx} \left| F^{\approx}_{X_1}(z_i) - F^{\approx}_{X_2}(z_i) \right| \\ & + \alpha_4^{\approx} \left| \min\left\{ T^{\approx}_{X_1}(z_i), I^{\approx}_{X_2}(z_i) \right\} - \min\left\{ I^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i) \right\} \right| \\ & + \alpha_5^{\approx} \left| \min\left\{ T^{\approx}_{X_1}(z_i), F^{\approx}_{X_2}(z_i) \right\} - \min\left\{ F^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i) \right\} \right| \Big), \end{aligned}$$

where $\alpha_k^{\square}, \alpha_k^{\approx} \in (0,1)$, $\sum_{k=1}^{5} \alpha_k^{\square} = 1$, $\sum_{k=1}^{5} \alpha_k^{\square} = 1$, $\chi_i^{\square}, \chi_i^{\approx} \in [0,1]$, $\sum_{i=1}^{n} \chi_i^{\approx} = 1$, and $\lambda \in (0,1)$.

Proposition 1. d_{H-BN} satisfies the following properties for all $X_1, X_2, X_3 \in BNS(U)$.

1. $0 \le d_{H-BN}(X_1, X_2) \le 1$,
2. $d_{H-BN}(X_1, X_2) = 0$ if and only if $X_1 = X_2$,
3. $d_{H-BN}(X_1, X_2) = d_{H-BN}(X_2, X_1)$,
4. $d_{H-BN}(X_1, X_2) \le d_{H-BN}(X_1, X_3)$ and $d_{H-BN}(X_2, X_3) \le d_{H-BN}(X_1, X_3)$ if $X_1 \subseteq X_2 \subseteq X_3$.

Proof

1. Apparently, for all $i = 1, ..., n$,

$$\left| T^{\square}_{X_1}(z_i) - T^{\square}_{X_2}(z_i) \right|, \quad \left| I^{\square}_{X_1}(z_i) - I^{\square}_{X_2}(z_i) \right|, \quad \left| F^{\square}_{X_1}(z_i) - F^{\square}_{X_2}(z_i) \right| \in [0,1],$$

$$\left|\max\left\{T^{\square}_{X_1}(z_i), I^{\square}_{X_2}(z_i)\right\} - \max\left\{I^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i)\right\}\right| \in [0,1],$$
$$\left|\max\left\{T^{\square}_{X_1}(z_i), F^{\square}_{X_2}(z_i)\right\} - \max\left\{F^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i)\right\}\right| \in [0,1],$$

and

$$\left|T^{\approx}_{X_1}(z_i) - T^{\approx}_{X_2}(z_i)\right|, \left|I^{\approx}_{X_1}(z_i) - I^{\approx}_{X_2}(z_i)\right|, \left|F^{\approx}_{X_1}(z_i) - F^{\approx}_{X_2}(z_i)\right| \in [0,1],$$
$$\left|\min\left\{T^{\approx}_{X_1}(z_i), I^{\approx}_{X_2}(z_i)\right\} - \min\left\{I^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i)\right\}\right| \in [0,1],$$
$$\left|\min\left\{T^{\approx}_{X_1}(z_i), F^{\approx}_{X_2}(z_i)\right\} - \min\left\{F^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i)\right\}\right| \in [0,1].$$

Hence, $0 \le d_{H-BN}\left(X_1, X_2\right) \le 1$.

2. Clearly, $d_{H-BN}\left(X_1, X_2\right) = 0 \Leftrightarrow \begin{cases} T^{\square}_{X_1} = T^{\square}_{X_2}, T^{\approx}_{X_1} = T^{\approx}_{X_2} \\ I^{\square}_{X_1} = I^{\square}_{X_2}, I^{\approx}_{X_1} = I^{\approx}_{X_2} \\ F^{\square}_{X_1} = F^{\square}_{X_2}, F^{\approx}_{X_1} = F^{\approx}_{X_2} \end{cases} \Leftrightarrow X_1 = X_2$.

3. It can be seen that d_{H-BN} has the symmetry property.

4. Let $X_1 \subseteq X_2 \subseteq X_3$ then for all $i = 1,...,n$,

$$T^{\square}_{X_1}(z_i) \le T^{\square}_{X_2}(z_i) \le T^{\square}_{X_3}(z_i), I^{\square}_{X_1}(z_i) \ge I^{\square}_{X_2}(z_i) \ge I^{\square}_{X_3}(z_i),$$
$$F^{\square}_{X_1}(z_i) \ge F^{\square}_{X_2}(z_i) \ge F^{\square}_{X_3}(z_i), T^{\approx}_{X_1}(z_i) \ge T^{\approx}_{X_2}(z_i) \ge T^{\approx}_{X_3}(z_i),$$
$$I^{\approx}_{X_1}(z_i) \le I^{\approx}_{X_2}(z_i) \le I^{\approx}_{X_3}(z_i), \text{ and } F^{\approx}_{X_1}(z_i) \le F^{\approx}_{X_2}(z_i) \le F^{\approx}_{X_3}(z_i).$$

These lead to

$$\left|T^{\square}_{X_1} - T^{\square}_{X_2}\right| \le \left|T^{\square}_{X_1} - T^{\square}_{X_3}\right|, \left|I^{\square}_{X_1} - I^{\square}_{X_2}\right| \le \left|I^{\square}_{X_1} - I^{\square}_{X_3}\right|, \left|F^{\square}_{X_1} - F^{\square}_{X_2}\right| \le \left|F^{\square}_{X_1} - F^{\square}_{X_3}\right|,$$
$$\left|T^{\approx}_{X_1} - T^{\approx}_{X_2}\right| \le \left|T^{\approx}_{X_1} - T^{\approx}_{X_3}\right|, \left|I^{\approx}_{X_1} - I^{\approx}_{X_2}\right| \le \left|I^{\approx}_{X_1} - I^{\approx}_{X_3}\right|, \left|F^{\approx}_{X_1} - F^{\approx}_{X_2}\right| \le \left|F^{\approx}_{X_1} - F^{\approx}_{X_3}\right|.$$

Moreover,

$$\max\left\{T^{\square}_{X_3}, I^{\square}_{X_1}\right\} \ge \max\left\{T^{\square}_{X_2}, I^{\square}_{X_1}\right\} \ge \max\left\{T^{\square}_{X_1}, I^{\square}_{X_2}\right\} \ge \max\left\{T^{\square}_{X_1}, I^{\square}_{X_3}\right\},$$
$$\min\left\{T^{\approx}_{X_3}, I^{\approx}_{X_1}\right\} \le \min\left\{T^{\approx}_{X_2}, I^{\approx}_{X_1}\right\} \le \min\left\{T^{\approx}_{X_1}, I^{\approx}_{X_2}\right\} \le \min\left\{T^{\approx}_{X_1}, I^{\approx}_{X_3}\right\},$$

$$\max\left\{T^{\square}_{X_3}, F^{\square}_{X_1}\right\} \ge \max\left\{T^{\square}_{X_2}, F^{\square}_{X_1}\right\} \ge \max\left\{T^{\square}_{X_1}, F^{\square}_{X_2}\right\} \ge \max\left\{T^{\square}_{X_1}, F^{\square}_{X_3}\right\},$$
$$\min\left\{T^{\approx}_{X_3}, F^{\approx}_{X_1}\right\} \le \min\left\{T^{\approx}_{X_2}, F^{\approx}_{X_1}\right\} \le \min\left\{T^{\approx}_{X_1}, F^{\approx}_{X_2}\right\} \le \min\left\{T^{\approx}_{X_1}, F^{\approx}_{X_3}\right\}.$$

Hence,

$$\left|\max\left\{T^{\square}_{X_2}, I^{\square}_{X_1}\right\} - \max\left\{T^{\square}_{X_1}, I^{\square}_{X_2}\right\}\right| \le \left|\max\left\{T^{\square}_{X_3}, I^{\square}_{X_1}\right\} - \max\left\{T^{\square}_{X_1}, I^{\square}_{X_3}\right\}\right|,$$
$$\left|\min\left\{T^{\approx}_{X_2}, I^{\approx}_{X_1}\right\} - \min\left\{T^{\approx}_{X_1}, I^{\approx}_{X_2}\right\}\right| \le \left|\min\left\{T^{\approx}_{X_3}, I^{\approx}_{X_1}\right\} - \min\left\{T^{\approx}_{X_1}, I^{\approx}_{X_3}\right\}\right|,$$

$$\left|\max\left\{T^{\square}_{X_2}, F^{\square}_{X_1}\right\} - \max\left\{T^{\square}_{X_1}, F^{\square}_{X_2}\right\}\right| \le \left|\max\left\{T^{\square}_{X_3}, F^{\square}_{X_1}\right\} - \max\left\{T^{\square}_{X_1}, F^{\square}_{X_3}\right\}\right|,$$
$$\left|\min\left\{T^{\approx}_{X_2}, F^{\approx}_{X_1}\right\} - \min\left\{T^{\approx}_{X_1}, F^{\approx}_{X_2}\right\}\right| \le \left|\min\left\{T^{\approx}_{X_3}, F^{\approx}_{X_1}\right\} - \min\left\{T^{\approx}_{X_1}, F^{\approx}_{X_3}\right\}\right|.$$

Thus, $d_{H-BN}\left(X_1, X_2\right) \le d_{H-BN}\left(X_1, X_3\right)$. Similarly, $d_{H-BN}\left(X_2, X_3\right) \le d_{H-BN}\left(X_1, X_3\right)$ is proven. □

Definition 7. Let $X_1, X_2 \in BNS(U)$ where $U = \{z_1, ..., z_n\}$. Then, the formula of H-max bipolar neutrosophic weighted similarity measure between X_1 and X_2 is as follows

$$s_{H-BN}(X_1, X_2) = 1 - d_{H-BN}(X_1, X_2). \tag{6}$$

Proposition 2. s_{H-BN} satisfies the following properties, for all $X_1, X_2, X_3 \in BNS(U)$:

1. $0 \le s_{H-BN}(X_1, X_2) \le 1$,
2. $s_{H-BN}(X_1, X_2) = 1$ if and only if $X_1 = X_2$,
3. $s_{H-BN}(X_1, X_2) = s_{H-BN}(X_2, X_1)$,
4. $s_{H-BN}(X_1, X_2) \ge s_{H-BN}(X_1, X_3)$ and $s_{H-BN}(X_2, X_3) \ge s_{H-BN}(X_1, X_3)$ if $X_1 \subseteq X_2 \subseteq X_3$.

Remark 1. The proposed distance measure overcomes the limitations of the Hamming distance, the Euclidean distance [44, 45], and Vakkas's proposal [43]. Specifically,

- The proposed measure d_{H-BN} includes cross-evaluations:

$$\left|\max\left\{T^{\square}_{X_1}(z_i), I^{\square}_{X_2}(z_i)\right\} - \max\left\{I^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i)\right\}\right|,$$
$$\left|\max\left\{T^{\square}_{X_1}(z_i), F^{\square}_{X_2}(z_i)\right\} - \max\left\{F^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i)\right\}\right|,$$
$$\left|\min\left\{T^{\approx}_{X_1}(z_i), I^{\approx}_{X_2}(z_i)\right\} - \min\left\{I^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i)\right\}\right|,$$
$$\left|\min\left\{T^{\approx}_{X_1}(z_i), F^{\approx}_{X_2}(z_i)\right\} - \min\left\{F^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i)\right\}\right|.$$

- The proposed measure satisfies the property related to the inclusion relation, i.e., the property 4 in Proposition 1.

Example 1. Let $U = \{z_1, ..., z_n\}$. Put

$$X_1 = <0_U, 0.01_U, 1_U, -0.15_U, 0_U, -0.8_U>,$$
$$X_2 = <0.79_U, 0.01_U, 0.61_U, -0.79_U, 0_U, -0.61_U>,$$
$$X_3 = <0.8_U, 0_U, 0.6_U, -0.8_U, 0_U, -0.6_U>.$$

Then, $X_1, X_2, X_3 \in BNS(U)$ and $X_1 \subset X_2 \subset X_3$. By the similarity measure of Vakkas et al. [43] and choosing specific values for the parameters, we have

$$S_V(X_1, X_3) = \frac{1}{2} S_{V1}(X_1, X_3) + \frac{1}{2} S_{V2}(X_1, X_3),$$
$$S_V(X_2, X_3) = \frac{1}{2} S_{V1}(X_2, X_3) + \frac{1}{2} S_{V2}(X_2, X_3),$$

where,

$$S_{V1}(X_1, X_3) = \frac{(0\times 0.8 + 0.01\times 0 + 1\times 0.6) - ((-0.15)(-0.8) + 0 + (-0.8)\times(-0.6))}{2\Big[(0^2 + 0.01^2 + 1^2) + (0.8^2 + 0^2 + 0.6^2) - ((-0.15)^2 + 0^2 + (-0.8)^2) - ((-0.8)^2 + 0^2 + (-0.6)^2)\Big]} = 0,$$

$$S_{V2}(X_1, X_3) = \frac{(0\times 0.8 + 0.01\times 0 + 1\times 0.6) - ((-0.15)(-0.8) + 0 + (-0.8)\times(-0.6))}{2\Big[\sqrt{0^2 + 0.01^2 + 1^2} \times \sqrt{0.8^2 + 0^2 + 0.6^2} - \sqrt{(-0.15)^2 + 0^2 + (-0.8)^2} \times \sqrt{(-0.8)^2 + 0^2 + (-0.6)^2}\Big]} = 0,$$

$$S_{V1}\left(X_2,X_3\right)=\frac{\left(0.79\times0.8+0.01\times0+0.61\times0.6\right)-\left(\left(-0.79\right)\times\left(-0.8\right)+0+\left(-0.61\right)\left(-0.6\right)\right)}{2\left[\left(0.79^2+0.01^2+0.61^2\right)+\left(0.8^2+0^2+0.6^2\right)-\left(\left(-0.79\right)^2+0^2+\left(-0.61\right)^2\right)-\left(\left(-0.8\right)^2+0^2+\left(-0.6\right)^2\right)\right]}=0,$$

$$S_{V2}\left(X_2,X_3\right)=\frac{\left(0.79\times0.8+0.01\times0+0.61\times0.6\right)-\left(\left(-0.79\right)\times\left(-0.8\right)+0+\left(-0.61\right)\left(-0.6\right)\right)}{2\left[\sqrt{0.79^2+0.01^2+0.61^2}\times\sqrt{0.8^2+0^2+0.6^2}-\sqrt{\left(-0.79\right)^2+0^2+\left(-0.61\right)^2}\times\sqrt{\left(-0.8\right)^2+0^2+\left(-0.6\right)^2}\right]}=0.$$

The obtained calculation results are $S_V\left(X_1,X_3\right)=0$ and $S_V\left(X_2,X_3\right)=0$.

Now, from Definition 6 and choosing specific values for the parameters, we have

$$d_{H-BN}\left(X_1,X_3\right)=\frac{1}{2}d_{H-BN1}\left(X_1,X_3\right)+\frac{1}{2}d_{H-BN2}\left(X_1,X_3\right),$$

$$d_{H-BN}\left(X_2,X_3\right)=\frac{1}{2}d_{H-BN1}\left(X_2,X_3\right)+\frac{1}{2}d_{H-BN2}\left(X_2,X_3\right),$$

where

$$d_{H-BN1}\left(X_1,X_3\right)=\frac{1}{5}\Big(\left|0-0.8\right|+\left|0.01-0\right|+\left|1-0.6\right| +\left|\max\left\{0,0\right\}-\max\left\{0.01,0.8\right\}\right|+\left|\max\left\{0,0.6\right\}-\max\left\{1,0.8\right\}\right|\Big)=0.482,$$

$$d_{H-BN2}\left(X_1,X_3\right)=\frac{1}{5}\Big(\left|0.15-0.8\right|+\left|0-0\right|+\left|0.8-0.6\right| +\left|\min\left\{-0.15,0\right\}-\min\left\{0,-0.8\right\}\right|+\left|\min\left\{-0.15,-0.6\right\}-\min\left\{-0.8,-0.8\right\}\right|\Big)=0.34,$$

$$d_{H-BN1}\left(X_2,X_3\right)=\frac{1}{5}\Big(\left|0.79-0.8\right|+\left|0.01-0\right|+\left|0.61-0.6\right| +\left|\max\left\{0.79,0\right\}-\max\left\{0.01,0.8\right\}\right|+\left|\max\left\{0.79,0.6\right\}-\max\left\{0.61,0.8\right\}\right|\Big)=0.01,$$

$$d_{H-BN2}\left(X_2,X_3\right)=\frac{1}{5}\Big(\left|0.79-0.8\right|+\left|0-0\right|+\left|0.61-0.6\right| +\left|\min\left\{-0.79,0\right\}-\min\left\{0,-0.8\right\}\right|+\left|\min\left\{-0.79,-0.6\right\}-\min\left\{-0.61,-0.8\right\}\right|\Big)=0.008.$$

Hence,

$$d_{H-BN}\left(X_1,X_3\right)=0.411>d_{H-BN}\left(X_2,X_3\right)=0.009$$
$$\left(s_{H-BN}\left(X_1,X_3\right)=0.589<s_{H-BN}\left(X_2,X_3\right)=0.991\right).$$

In this case, by observation we can also see that X_2 and X_3 are almost the same. In addition, since $X_1\subset X_2\subset X_3$, it can be deduced that the difference between X_1 and X_3 is greater than the that between X_2 and X_3. The proposed distance measure is likely to properly represent this assessment and inference and overcomes the limitation of the proposal of Vakkas et al. [43].

Example 2. Let $U=\left\{z_1,...,z_n\right\}$. Put

$$X_1 = <0.4_U,0_U,0.4_U,-0.8_U,0_U,-0.8_U>,$$
$$X_2 = <0.5_U,0_U,0.5_U,-0.7_U,0_U,-0.7_U>,$$
$$X_3 = <0.4_U,0_U,0.6_U,-0.6_U,0_U,-0.8_U>.$$

Then, $X_1,X_2,X_3\in BNS\left(U\right)$, $X_1\not\subseteq X_2,X_2\not\subseteq X_1$, and $X_3\subset X_2$.

The Hamming distance [44, 45] on $BNS\left(U\right)$ can be defined as follows:

$$d_{Ham}\left(X_1,X_2\right)=\frac{1}{6}\sum_{i=1}^{n}\Big(\left|T^{\square}_{X_1}(z_i)-T^{\square}_{X_2}(z_i)\right|+\left|I^{\square}_{X_1}(z_i)-I^{\square}_{X_2}(z_i)\right|+\left|F^{\square}_{X_1}(z_i)-F^{\square}_{X_2}(z_i)\right|$$
$$+\left|T^{\approx}_{X_1}(z_i)-T^{\approx}_{X_2}(z_i)\right|+\left|I^{\approx}_{X_1}(z_i)-I^{\approx}_{X_2}(z_i)\right|+\left|F^{\approx}_{X_1}(z_i)-F^{\approx}_{X_2}(z_i)\right|\Big).$$

The Euclidean distance [44, 45] on $BNS(U)$ can be defined as follows:

$$d_{Eucl}\left(X_1,X_2\right)=\sum_{i=1}^{n}\left(\frac{1}{6}\Big(\left|T^{\square}_{X_1}(z_i)-T^{\square}_{X_2}(z_i)\right|^2+\left|I^{\square}_{X_1}(z_i)-I^{\square}_{X_2}(z_i)\right|^2+\left|F^{\square}_{X_1}(z_i)-F^{\square}_{X_2}(z_i)\right|^2\right.$$
$$\left.+\left|T^{\approx}_{X_1}(z_i)-T^{\approx}_{X_2}(z_i)\right|^2+\left|I^{\approx}_{X_1}(z_i)-I^{\approx}_{X_2}(z_i)\right|^2+\left|F^{\approx}_{X_1}(z_i)-F^{\approx}_{X_2}(z_i)\right|^2\Big)\right)^{\frac{1}{2}}.$$

Some of the calculation results obtained are as follows:

$$d_{Ham}\left(X_1,X_2\right)=d_{Ham}\left(X_3,X_2\right)=\frac{4}{6}\ ,$$

$$d_{Eucl}\left(X_1,X_2\right)=d_{Eucl}\left(X_3,X_2\right)=\frac{\sqrt{6}}{30},$$

$$d_{H-BN}\left(X_1,X_2\right)=0.06<d_{H-BN}\left(X_3,X_2\right)=0.08$$

Clearly, in this case, because of cross-evaluations, the proposed measure can distinguish the difference better than two related measures.

Definition 8. For $E:BNS(U)\to[0,1]$ mapping, if the following conditions are satisfied then E is an entropy measure of BNSs.

1. $E(X)=0$ if and only if X or X^c is a crisp set,
2. $E(X)=E(X^c)$; $E(X)=1$ if and only if $X=X^c$,
3. $E(X_1)\le E(X_2)$ if $X_1 \text{ Ð } X_2$, i.e., if $T^{\square}_{X_1}\le T^{\square}_{X_2}$, $F^{\square}_{X_1}\ge F^{\square}_{X_2}$, $T^{\approx}_{X_1}\ge T^{\approx}_{X_2}$, $F^{\approx}_{X_1}\le F^{\approx}_{X_2}$ for $T^{\square}_{X_2}\le F^{\square}_{X_2}$, $T^{\approx}_{X_2}\ge F^{\approx}_{X_2}$, $I^{\square}_{X_1}=I^{\square}_{X_2}=0.5_U$, $I^{\approx}_{X_1}=I^{\approx}_{X_2}=-0.5_U$; and $T^{\square}_{X_1}\ge T^{\square}_{X_2}$, $F^{\square}_{X_1}\le F^{\square}_{X_2}$, $T^{\approx}_{X_1}\le T^{\approx}_{X_2}$, $F^{\approx}_{X_1}\ge F^{\approx}_{X_2}$ for $T^{\square}_{X_2}\ge F^{\square}_{X_2}$, $T^{\approx}_{X_2}\le F^{\approx}_{X_2}$, $I^{\square}_{X_1}=I^{\square}_{X_2}=0.5_U$, $I^{\approx}_{X_1}=I^{\approx}_{X_2}=-0.5_U$.

Proposition 3. Let $X\in BNS(U)$, where $U=\{z_1,...,z_n\}$, then $s_{H-BN}(X,X^c)$ is an entropy measure of X.

Proof.

1. If X be a crisp set, i.e., $T^{\square}_X=1_U, I^{\square}_X=F^{\square}_X=0_U, T^{\approx}_X=I^{\approx}_X=0_U, F^{\approx}_X=-1_U$, or $T^{\square}_X=I^{\square}_X=0_U, F^{\square}_X=1_U, T^{\approx}_X=-1_U, I^{\approx}_X=F^{\approx}_X=0_U$, then, $s_{H-BN}(X,X^c)=0.$ Similarly, if X^c is a crisp set, then $s_{H-BN}(X,X^c)=0.$ If $s_{H-BN}(X,X^c)=0,$ then it's not hard to show that X or X^c is a crisp set.
2. From Proposition 2, we obtain that $E(X)=E(X^c)$; $s_{H-BN}(X,X^c)=1$ if and only if $X=X^c$.
3. Let $X_1 \text{ Ð } X_2$, assume that $T^{\square}_{X_1}\le T^{\square}_{X_2}$, $F^{\square}_{X_1}\ge F^{\square}_{X_2}$, $T^{\approx}_{X_1}\ge T^{\approx}_{X_2}$, $F^{\approx}_{X_1}\le F^{\approx}_{X_2}$ for $T^{\square}_{X_2}\le F^{\square}_{X_2}$, $T^{\approx}_{X_2}\ge F^{\approx}_{X_2}$, $I^{\square}_{X_1}=I^{\square}_{X_2}=0.5_U$, $I^{\approx}_{X_1}=I^{\approx}_{X_2}=-0.5_U$, then

$$T^{\square}_{X_1}\le T^{\square}_{X_2}\le F^{\square}_{X_2}\le F^{\square}_{X_1},$$
$$T^{\approx}_{X_1}\ge T^{\approx}_{X_2}\ge F^{\approx}_{X_2}\ge F^{\approx}_{X_1},$$

$$\max\left\{T^{\square}_{X_1},0.5_U\right\}\le\max\left\{T^{\square}_{X_2},0.5_U\right\}\le\max\left\{F^{\square}_{X_2},0.5_U\right\}\le\max\left\{F^{\square}_{X_1},0.5_U\right\},$$

$$\min\left\{T^{\approx}_{X_1},-0.5_U\right\}\ge\min\left\{T^{\approx}_{X_2},-0.5_U\right\}\ge\min\left\{F^{\approx}_{X_2},-0.5_U\right\}\ge\min\left\{F^{\approx}_{X_1},-0.5_U\right\},$$

$$d_{H-BN}\left(X_t, X_t^{c}\right) = \lambda \sum_{i=1}^{n} \chi_i^{\square} \left(\left(\omega_1^{\square} + \omega_3^{\square} + \omega_5^{\square}\right) \left| T_{X_t}^{\square}(z_i) - F_{X_t}^{\square}(z_i) \right| + \omega_4^{\square} \left| \begin{matrix} \max\left\{T_{X_t}^{\square}(z_i), 0.5\right\} \\ -\max\left\{0.5, F_{X_t}^{\square}(z_i)\right\} \end{matrix} \right| \right)$$

$$+\left(1-\lambda\right) \sum_{i=1}^{n} \chi_i^{\approx} \left(\left(\omega_1^{\approx} + \omega_3^{\approx} + \omega_5^{\approx}\right) \left| T_{X_t}^{\approx}(z_i) - F_{X_t}^{\approx}(z_i) \right| + \omega_4^{\approx} \left| \begin{matrix} \min\left\{T_{X_t}^{\approx}(z_i), -0.5\right\} \\ -\min\left\{-0.5, F_{X_t}^{\approx}(z_i)\right\} \end{matrix} \right| \right), \ t = 1,2.$$

Therefore, $d_{H-BN}\left(X_1, X_1^{c}\right) \geq d_{H-BN}\left(X_2, X_2^{c}\right)$ and then $s_{H-BN}\left(X_1, X_1^{c}\right) \leq s_{H-BN}\left(X_2, X_2^{c}\right)$.

Similarly, the remaining case is proved. $\square$

4. An application of the H-Max Bipolar Neutrosophic Distance Measure to medical diagnosis

4.1. The H-BN method

A diagnostic problem is stated as follows:

- A medical dataset includes
  - m records of m corresponding patients P_i, $i = 1,2,...,m$,
  - n attributes (symptoms) A_j, $j = 1,2...,n$, of a disease D,
  - k disease classes labeled C_t, $t = 1,2,...,k$, of D.
- The problem is to build a diagnostic system with
  - the inputs are the symptoms of a patient,
  - the output is a disease label.

The proposed method:

Inspired by the diagnostic method introduced in [42] by Ngan et al, the proposed method (H-BN) includes four steps as follows.

- **Step 1.** Built two relation matrices in the bipolar neutrosophic environment:
  - Matrix 1 (M1) presents the relations between the symptoms and patients (P_i and A_j are the i^th row and the j^th column of M1, respectively, $i = 1,...,m; j = 1,...,n$),
  - Matrix 2 (M2) shows the relations between the symptoms and the disease or the classification results. Specifically, M2 is a $k \times n$ matrix (C_t is the t^th row of M2, $t = 1,...,k$).
- **Step 2.** Find the entropies $E\left(A_j\right)$ of the symptoms A_j.
- **Step 3.** Calculate the similarity $s_{H-BN}\left(P_i, C_t\right)$ between the symptoms of P_i and the disease classes C_t, where $E\left(A_j\right)$ is put in the weight of A_j.
- **Step 4.** Diagnose the i^th patient by finding the highest similarity value $\hat{s}_{H-BN}\left(P_i, C_t\right) = s_{H-BN}\left(P_i, C_{t_0}\right)$, $t_0 \in \left[1, k\right]$. The output is t_0.

4.2. Numeric example

In this section, we use the data in the numerical example in [42] on 5 male patients (aged about 30) of Indian Liver Patient Dataset (ILPD) taken from UCI. In the dataset described in Table 1, there are 2 diagnosis labels which are La-I (liver patient) and La-II (non-liver patient). In Table 1, the considered attributes (A_1 - A_7) are Alkaline Phosphotase, Alamine Aminotransferase, Aspartate Aminotransferase, Total Bilirubin, Direct Bilirubin, Albumin, and Albumin and Globulin Ratio.

Table 1. Data of 5 male patients of the ILPD dataset.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	**Class**
P_1	1.3	0.4	482	102	80	3.3	0.9	La-I
P_2	0.8	0.2	198	26	23	4	1	La-II
P_3	0.9	0.2	518	189	17	2.3	0.7	La-I
P_4	3.8	1.5	298	102	630	3.3	0.8	La-I
P_5	0.8	0.2	156	12	15	3.7	1.1	La-II

The steps of the proposed algorithm are implemented as follows:

- **Step 1:** Input data is fuzzified by the following fuzzification functions selected by experts.

$$R^{\prec}_{\alpha,\beta}(z)=\begin{cases}0 & z\le\alpha\\ \dfrac{z-\alpha}{\beta-\alpha} & \alpha<z\le\beta\\ 1 & \beta<z\end{cases},\qquad L^{\prec}_{\alpha',\beta'}(z)=\begin{cases}1 & z\le\alpha'\\ \dfrac{z-\beta'}{\alpha'-\beta'} & \alpha'<z\le\beta'\\ 0 & \beta'<z\end{cases}$$

$$R^{\succ}_{\rho,\ell}(z)=\begin{cases}-1 & z\le\rho\\ \dfrac{z-\ell}{\ell-\rho} & \rho<z\le\ell\\ 0 & \ell<z\end{cases},\qquad L^{\succ}_{\rho',\ell'}(z)=\begin{cases}0 & z\le\rho'\\ \dfrac{z-\rho'}{\rho'-\ell'} & \rho'<z\le\ell'\\ -1 & \ell'<z\end{cases}$$

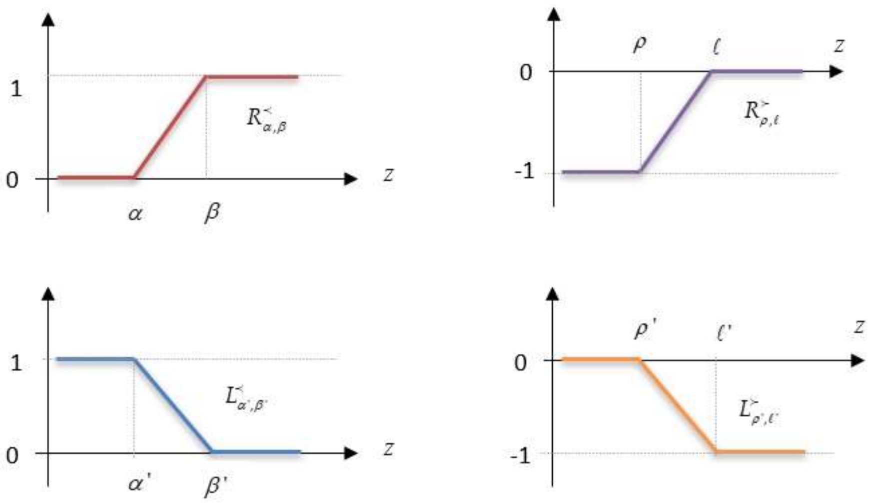

Figure 1. The fuzzification functions are illustrated by graphs.

Specifically, the symptoms on patients are represented as the following BNSs.

$$A_1=<T_1^{\square}(z),I_1^{\square}(z),F_1^{\square}(z),T_1^{\approx}(z),I_1^{\approx}(z),F_1^{\approx}(z)>=$$
$$=<R^{\prec}_{1.2,5.3}(z),L^{\prec}_{0.2,3}(z),L^{\prec}_{0.6,4}(z),R^{\succ}_{0.9,5}(z),L^{\succ}_{0.5,3.5}(z),L^{\succ}_{0.3,4.5}(z)>$$

$$A_2=<T_2^{\square}(z),I_2^{\square}(z),F_2^{\square}(z),T_2^{\approx}(z),I_2^{\approx}(z),F_2^{\approx}(z)>=$$
$$=<R^{\prec}_{0.4,2.3}(z),L^{\prec}_{0.1,1}(z),L^{\prec}_{0.15,1.5}(z),R^{\succ}_{0.2,2}(z),L^{\succ}_{0.2,1.2}(z),L^{\succ}_{0.3,2}(z)>$$

$$A_3 = < T_3^{\square}(z), I_3^{\square}(z), F_3^{\square}(z), T_3^{\approx}(z), I_3^{\approx}(z), F_3^{\approx}(z) >=$$
$$= < R^{\prec}_{140,486}(z), L^{\prec}_{80,250}(z), L^{\prec}_{100,400}(z), R^{\succ}_{110,450}(z), L^{\succ}_{90,300}(z), L^{\succ}_{110,420}(z) >$$

$$A_4 = < T_4^{\square}(z), I_4^{\square}(z), F_4^{\square}(z), T_4^{\approx}(z), I_4^{\approx}(z), F_4^{\approx}(z) > =$$
$$= < R^{\prec}_{33,119}(z), L^{\prec}_{5,60}(z), L^{\prec}_{30,100}(z), R^{\succ}_{25,90}(z), L^{\succ}_{10,70}(z), L^{\succ}_{40,95}(z) >$$

$$A_5 = < T_5^{\square}(z), I_5^{\square}(z), F_5^{\square}(z), T_5^{\approx}(z), I_5^{\approx}(z), F_5^{\approx}(z) > =$$
$$= < R^{\prec}_{33,100}(z), L^{\prec}_{10,90}(z), L^{\prec}_{23,95}(z), R^{\succ}_{33,100}(z), L^{\succ}_{10,90}(z), L^{\succ}_{23,95}(z) >$$

$$A_6 = < T_6^{\square}(z), I_6^{\square}(z), F_6^{\square}(z), T_6^{\approx}(z), I_6^{\approx}(z), F_6^{\approx}(z) > =$$
$$= < L^{\prec}_{2.2,3.5}(z), R^{\prec}_{2,4}(z), R^{\prec}_{3,5}(z), L^{\succ}_{2.3,3.3}(z), R^{\succ}_{2.2,4.1}(z), R^{\succ}_{2.8,5.2}(z) >$$

$$A_7 = < T_7^{\square}(z), I_7^{\square}(z), F_7^{\square}(z), T_7^{\approx}(z), I_7^{\approx}(z), F_7^{\approx}(z) > =$$
$$= < L^{\prec}_{0.5,1}(z), R^{\prec}_{0.3,1.5}(z), R^{\prec}_{0.8,2.5}(z), L^{\succ}_{0.6,1.1}(z), R^{\succ}_{0.2,1}(z), R^{\succ}_{0.7,2.8}(z) >$$

Two bipolar neutrosophic relation matrices M1 and M2 are placed in Tables 2 and 3.

Table 2. The relations between the symptoms and patients are presented.

(M1)	A_1	A_2	A_3	A_4	A_5	A_6	A_7
P_1	<0.02,0.6, 0.8,-0.9, -0.3,-0.2>	<0,0.6, 0.8,-0.9, -0.2,-0.06>	<0.9,0, 0,0, -1,-1>	<0.8,0, 0,0, -1,-1>	<0.7,0.1, 0.2,-0.3, -0.9,-0.8>	<0.1,0.6, 0.1,-1, -0.4,-0.8>	<0.2,0.5, 0.08,-0.6, -0.1,-0.9>
P_2	<0,0.7, 0.9,-1, -0.1,-0.1>	<0,0.8, 0.9,-1, 0,0>	<0.1,0.3, 0.6,-0.7, -0.5,-0.3>	<0,0.6, 1,-1, -0.3,0>	<0,0.8, 1,-1, -0.2,0>	<0,1, 0.5,-1, -0.05,-0.5>	<0,0.5, 0.1,-0.8, 0,-0.85>
P_3	<0,0.7, 0.9,-1, -0.1,-0.1>	<0,0.8, 0.9,-1, 0,0>	<1,0, 0,0, -1,-1>	<1,0, 0,0, -1,-1>	<0,0.9, 1,-1, -0.09,0>	<0.9,0.1, 0,0, -0.9,-1>	<0.6,0.3, 0,-0.2, -0.4,-1>
P_4	<0.6,0, 0.05,-0.3, -1,-0.8>	<0.5,0, 0,-0.3, -1,-0.7>	<0.4,0, 0.3,-0.4, -1,-0.6>	<0.8,0, 0,0, -1,-1>	<1,0, 0,0, -1,-1>	<0.1,0.6, 0.1,-1, -0.4,-0.8>	<0.4,0.4, 0,-0.4, -0.25,-0.95>
P_5	<0,0.7, 0.9,-1, -0.1,-0.1>	<0,0.8, 0.9,-1, 0,0>	<0.04,0.5, 0.8,-0.9, -0.3,-0.1>	<0,0.8, 1,-1, -0.03,0>	<0,0.9, 1,-1, -0.06,0>	<0,0.8, 0.3,-1, -0.2,-0.6>	<0,0.6, 0.2,-1, 0,-0.8>

Table 3. The relations between the symptoms and the classification results are shown.

(M2)	A_1	A_2	A_3	A_4	A_5	A_6	A_7
La-I	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>
La-II	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>

- **Step 2:** Finding the entropies $E\left(A_j\right)=s_{H-BN}(A_j,A_j^c)=1-d_{H-BN}(A_j,A_j^c)$ with $\chi_i^{\square}=\chi_i^{\approx}=\omega_j^{\square}=\omega_j^{\approx}=\frac{1}{5}\ (i,j=1,\ldots,5)$ and $\lambda=\frac{1}{2}$:

$$E\left(A_1\right)=0.27,\quad E\left(A_2\right)=0.2,\ E\left(A_3\right)=0.33,\quad E\left(A_4\right)=0.08,$$

$$E\left(A_5\right)=0.13,\quad E\left(A_6\right)=0.55,\ E\left(A_7\right)=0.68.$$

- **Step 3:** Calculating the similarities $S(i-I)=s_{H-BN}\left(P_i,(\text{La - I})\right)$ and $S\left(i-II\right)=s_{H-BN}\left(P_i,(\text{La - II})\right)$ with $\omega_j^{\square}=\omega_j^{\approx}=\frac{1}{5}\ (i,j=1,\ldots,5)$, $\lambda=\frac{1}{2}$, and $\chi_j^{\square}=\chi_j^{\approx}=\frac{E\left(A_j\right)}{\sum_{j=1}^{5}E\left(A_j\right)}$. The obtained results include: $\chi_1^{\square}=0.12,\chi_2^{\square}=0.09,\chi_3^{\square}=0.15,\chi_4^{\square}=0.035,\chi_5^{\square}=0.06,\chi_6^{\square}=0.245,\chi_7^{\square}=0.3,$

$$S(1-I)=0.49>S(1-II)=0.475,\quad S(2-I)=0.2<S(2-II)=0.75,$$
$$S(3-I)=0.642>S(3-II)=0.327,\quad S(4-I)=0.63>S(4-II)=0.33,$$
$$S(5-I)=0.186<S(5-II)=0.788.$$

- **Step 4.** The outputs are decided as follows: The outputs of P_1,P_2,P_3,P_4, and P_5 are La-I, La-II, La-I, La-I, and La-II, respectively. These decisions and the last column of Table 1 are the same.

4.3. Experiment

In this part, we test the proposed method on the ILPD dataset on Matlab programming with the evaluation criteria on accuracy is Mean Absolute Error (MAE) and the speed of the algorithms is measured in seconds (sec). Also on this data, Ngan et al. [8] tested 14 other diagnostic methods, denoted by M_{SK1-1}, M_{SK1-2}, M_{SK1-3}, M_{SK1-4}, M_{SK2}, M_{WX}, M_{VS}, M_{ZJ}, M_{W}, M_{J}, M_{SA}, $M_{H-\max}$, M_{C-QDM}, and M_{P-QDM}, based on the considered intuitionistic fuzzy distance measures (see Table 4).

Table 4. MAEs and Sec of the considered methods on the ILPD dataset.

Methods	MAE	Sec
M_{SK1-1}	0.3195	0.6177
M_{SK1-2}	0.3158	0.4427
M_{SK1-3}	0.3316	0.4827
M_{SK1-4}	0.2918	0.4602
M_{SK2}	0.2902	0.6527
M_{WX}	0.3227	0.4427
M_{VS}	0.2893	0.5552
M_{ZJ}	0.3096	0.5602
M_{W}	0.2915	0.8452
M_{J}	0.289	1.2077
M_{SA},	0.3031	0.8102
$M_{H-\max}$	0.2848	0.51
M_{C-QDM}	0.2836	0.155
M_{P-QDM}	0.2831	0.469

H-BN	0.2729	0.559770

In Table 4, it can be observed that the MAE value of the proposed method (H-BN), which is 0.2729, is less (better) than those of the other algorithms on the ILPD datasets. Figure 2 shows the MAE values of the considered methods on the ILPD dataset, where the heights of the vertical bars present the MAE values of the corresponding algorithms. The heights of the H-BN method (green bars) are lower than those of the remaining bars, that means, it is the best algorithm in terms of accuracy of the considered algorithms on the ILPD dataset. We note that the computation time of our algorithms is very close to the computation time of the other methods.

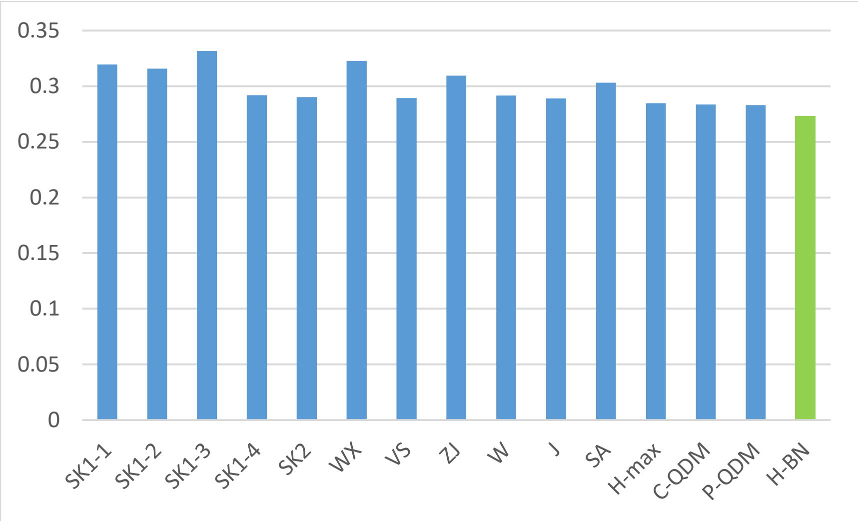

Figure 2. MAEs of the considered methods on the ILPD dataset.

5. Conclusions

In this paper, based on the H-max distance measure on IFSs and SVNSs, a new distance measure on BNSs is introduced to overcome the limitations of the related measures by including cross-evaluations and satisfying the condition of inference of a distance measure. Furthermore, a bipolar neutrosophic entropy measure and its basic properties are presented and proven. In addition, an application to medical diagnosis is shown to illustrate the effective applicability of the measures. There, the proposed diagnostic method called H-BN, a numerical example and real experiment are clarified in detail. In the future, we will test the proposed diagnostic method on other real datasets taken from UCI. Furthermore, we will develop the distance measure for interval-valued bipolar neutrosophic sets.

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Appendix

Source code and dataset of this paper can be found at this link:
https://sourceforge.net/projects/hbn-datasets-code/.

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