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Some Neutrosophic Triplet Subgroup Properties and Homomorphism Theorems in Singular Weak Commutative Neutrosophic Extended Triplet Group

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Abstract. In 2018, the study of neutrosophic triplet cosets and neutrosophic triplet quotient group of a neutrosophic extended triplet group was initiated with a follow up of the establishment of fundamental homomorphism theorems for neutrosophic extended triplet group. But some lapses in these earlier results were identified and revised through the introduction of special kind of weak commutative neutrosophic extended triplet group (WCNETG) called perfect neutrosophic extended triplet group. Furthermore, neutro-homomorphism basic theorem has been established for commutative neutrosophic extended triplet group. In this current work, the generalization and extention of the above results was done by investigating neutro-homomorphism in singular WCNETG. This was achieved with the introduction and study of some new types of NT-subgroups that are right (left) cancellative, semi-strong, and maximally normal in a singular WCNETG. For any given non-empty subset S and NT-subgroup H of a singular WCNETG X, some of these new NT-subgroups were shown to exist as non-empty neutrosophic triplet normalizer, generated subset and centralizer of S, closure of H, derived subset of X and center of X. With these, the first, second and third neutro-isomorphism and neutro-correspondence theorems were established. This finally led to the proof of the neutro-Zassenhaus Lemma (Neutro-Butterfly Theorem).

Keywords: Group; Neutrosophic Extended Triplet Group; Weakly Commutative Neutrosophic Extended Triplet Group; Isomorphism Theorems

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1. **Introduction**

. After the emergence of generalized group (completely simple semigroup), which is an algebraic structure with deep physical background in the unified guage theory and also has direct relation with isotopies (Adeniran et al. [1]), some other algebraic structures which generalize generalized groups have evolved and have been studied alongside with their applications. Among these are neutrosophic triplet group (NTG); Smarandache and Ali [7] and Jaiyéọlá and Smarandache [11], neutrosophic extended triplet group (NETG); Zhang et al. [10], neutrosophic triplet loop (NTL); Jaiyéọlá and Smarandache [3], Quasi neutrosophic triplet loops; Zhang et al. [8], Jaiyéọlá [12, 13] and generalized neutrosophic extended triplet group; Ma et al. [14]. A summary account of these past efforts was compiled and reported by Smarandache et al. [15].

Smarandache and Ali [7] introduced neutrosophic triplets in 2016 while Smarandache [16–19] introduced neutrosophic extended triplets in between 2016 and 2017. The studies of neutrosophic extended triplet group and neutrosophic extended triplet loop became more fascinating with the recent studies of Abel-Grassmann neutrosophic triplet group (loop) and Bol-Moufang types of quasi neutrosophic triplet loops (Fenyves BCI-algebras) by Zhang et al. [20], Wu and Zhang [21] and Jaiyéọlá [12,13]. The captivating discoveries in these studies are the facts that:

(1) a groupoid is a neutrosophic extended triplet group if and only if it is a completely regular semigroup;
(2) a groupoid is a weak commutative neutrosophic extended triplet group if and only if it is a Clifford semigroup (a type of completely regular semigroup);
(3) there are 540 varieties of Bol-Moufang type quasi neutrosophic triplet loops.

These discoveries established that: the theory of neutrosophic extended triplet group is associated with the theory of semigroup, the theory of weak commutative neutrosophic extended triplet group is associated with the theory of clifford semigroup and the theory of quasi neutrosophic triplet loops is expansive. Shalla and Olgun [5, 6] studied neutrosophic extended triplet group action and the Burnside's lemma, and their direct and Semi-direct products.

We now switch to the definition of a neutrosophic extended triplet group and related structures.

Definition 1.1. (Neutrosophic Extended Triplet Set-NETS)

Let X be a set together with a binary operation $*$ defined on it. Then, X is called a neutrosophic extended triplet set if for any $x \in X$, there exist a neutral of 'x' denoted by $neut(x)$ and an opposite of 'x' denoted by $anti(x)$, with $neut(x), anti(x) \in X$ such that:

$$x * neut(x) = neut(x) * x = x \quad \text{and} \quad x * anti(x) = anti(x) * x = neut(x).$$

The elements $x, neut(x)$ and $anti(x)$ are collectively referred to as neutrosophic triplet, and denote by $(x, neut(x), anti(x))$.

Remark 1.2. In a NETS X, for any $x \in X$, each of $neut(x)$ and $anti(x)$ may not be unique. This is because, in a neutrosophic triplet set $(X, *)$, an element y (resp. z) is the second (resp. third) component of a neutrosophic triplet if there exist $x, z \in X$ $(x, y \in X)$ such that $x * y = y * x = x$ and $x * z = z * x = y$. Thus, (x, y, z) is the neutrosophic triplet.

Definition 1.3. (Neutrosophic Extended Triplet Group-NETG)

Let $(X, *)$ be a neutrosophic extended triplet set. Then, $(X, *)$ is called a neutrosophic extended triplet group if $(X, *)$ is a semigroup. If in addition, $(X, *)$ obeys the commutativity law, then $(X, *)$ is called a commutative extended neutrosophic triplet group (CNETG).

Remark 1.4. In a NETG X, it was shown by Zhang et al. [9] that $neut(x)$ is unique for each $x \in X$. But, the same is not necessarily true for $anti(x)$. Thus, the set of opposites for $x \in X$ is usually denoted by $\{anti(x)\}$.

Definition 1.5. (Weak Commutative Neutrosophic Extended Triplet Group-WCNETG, Definition 4, Zhang et al. [9]; Singular NETG, Definition 6, Zhang et al. [10])

Let $(X, *)$ be a neutrosophic extended triplet group. $(X, *)$ is called a weak commutative neutrosophic extended triplet group (WCNETG) if $a * neut(b) = neut(b) * a$ for all $a, b \in X$.

A NETG is said to be singular if $|\{anti(x)\}| = 1$ for all $x \in X$.

Definition 1.6. (Neutrosophic Triplet Subgroup or NT-Subgroup)

Let $(X, *)$ be a neutrosophic extended triplet group and let $H \subseteq X$. H is called a neutrosophic triplet subgroup (NTSG) of X if $(H, *)$ is a neutrosophic extended triplet group and this is expressed as $H \leq X$. Furthermore, for any fixed $x \in X$, H is called x-normal NTSG of X, written $H \lhd_x X$ if $xy\ anti(x) \in H$ for all $y \in H$.

Lemma 1.7. *(Proposition 2, Zhang et al. [10])*

Let $(X, *)$ be a neutrosophic triplet group and let $H \subseteq X$. H is a neutrosophic triplet subgroup of X if and only if the following conditions are true.

(1) $(H, *)$ is a groupoid;

(2) $anti(x) \in H$ for all $x \in H$.

We now state some important results on singular NETG and WCNETG which are of importance to this work.

Theorem 1.8. *(Proposition 2, 3, Zhang et al. [9])*

Let $(X, *)$ be a NETG. Then $(X, *)$ is a WCNETG if and only the following conditions are true.

(1) $neut(x) * neut(y) = neut(y) * neut(x)$ *for all* $x, y \in X$.

(2) $neut(x) * neut(y) * x = x * neut(y)$ *for all* $x, y \in X$.

Hence, $neut(x) * neut(y) = neut(y * x)$ *and* $anti(x) * anti(y) \in \{anti(y * x)\}$ *for all* $x, y \in X$.

Theorem 1.9. *(Theorem 6, Zhang et al. [10])*

Let $(X, *)$ *be a singular NETG. Then*

(1) $neut(x) * anti(x) = anti(x) * neut(x) = anti(x)$ *for all* $x \in X$.

(2) $anti\big(neut(x)\big) = neut(x)$ *for all* $x \in X$.

(3) $anti\big(anti(x)\big) = x$ *for all* $x \in X$.

(4) $neut\big(anti(x)\big) = neut(x)$ *for all* $x \in X$.

Hence, $neut(x) * neut(y) = neut(y * x)$ *and* $anti(x) * anti(y) \in \{anti(y * x)\}$ *for all* $x, y \in X$.

Here are two methods of constructing a WCNETG as recently described. These new constructions will be of judicious use for illustrations and as examples in order to justify some of the results in this study.

Theorem 1.10. *(First WCNETG, Zhang et al. [20])*

Let $(G_1, *_1)$ *and* $(G_2, *_2)$ *be two groups, with identity elements* e_1 *and* e_2 *respectively, such that* $G_1 \cap G_2 = \emptyset$. *Let* $G = G_1 \cup G_2$, *and define the binary operation* $*$ *on* G *as follows:*

$$x * y = \begin{cases} x *_1 y, & \text{if } x, y \in G_1; \\ x *_2 y, & \text{if } x, y \in G_2; \\ x, & \text{if } x \in G_1,\ y \in G_2; \\ y, & \text{if } x \in G_2,\ y \in G_1 \end{cases}$$

Then, $(G, *)$ *is a WCNETG.*

Theorem 1.11. *(Second WCNETG, Zhang et al. [20])*

Let $(G_1, *_1)$ *and* $(G_2, *_2)$ *be two groups, with identity elements* e_1 *and* e_2 *respectively, such that* $G_1 \cap G_2 = \emptyset$. *Let* $G = G_1 \cup G_2$, *and define the binary operation* $*$ *on* G *as follows:*

$$x * y = \begin{cases} x *_1 y, & \text{if } x, y \in G_1; \\ x *_2 y, & \text{if } x, y \in G_2; \\ y, & \text{if } x \in G_1,\ y \in G_2; \\ x, & \text{if } x \in G_2,\ y \in G_1 \end{cases}$$

Then, $(G, *)$ *is a WCNETG.*

Remark 1.12. For easy reference, the WCNETG in Theorem 1.10 and WCNETG in Theorem 1.11 for any chosen pairs of groups will be called first WCNETG and second WCNETG respectively. It must be noted that both are singular WCNETGs.

Table 1. Group $(G_1, *_1)$

$*_1$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Table 2. Group $(G_2, *_2)$

$*_2$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	5	4	3
3	3	5	1	6	2	4
4	4	6	5	1	3	2
5	5	3	4	2	6	1
6	6	4	2	3	1	5

Table 3. First WCNETG $(G, *)$ of $(G_1, *_1)$ and $(G_2, *_2)$

$*$	e	a	b	c	1	2	3	4	5	6
e	e	a	b	c	e	e	e	e	e	e
a	a	e	c	b	a	a	a	a	a	a
b	b	c	e	a	b	b	b	b	b	b
c	c	b	a	e	c	c	c	c	c	c
1	e	a	b	c	1	2	3	4	5	6
2	e	a	b	c	2	1	6	5	4	3
3	e	a	b	c	3	5	1	6	2	4
4	e	a	b	c	4	6	5	1	3	2
5	e	a	b	c	5	3	4	2	6	1
6	e	a	b	c	6	4	2	3	1	5

Using the groups $(G_1, *_1)$ and $(G_2, *2)$ with multiplication Table 1 and Table 2, Zhang et al. [20] constructed a WCNETG $(G, *)$ with multiplication Table 3.

Bal et al. [22] initated the study of neutrosophic triplet cosets and neutrosophic triplet quotient group of a neutrosophic extended triplet group. This work was then followed up with the establishment of fundamental homomorphism theorems for neutrosophic extended triplet group by Celik et al. [23]. But, Zhang et al. [24] identified some lapses in these earlier articles and revised the results in question by introducing special kind of WCNETG called perfect NETG. On the other hand, Jaiyéọlá and Smarandache [11] also established an homomorphism for NETG which they jointly revised with some other authors in Zhang et al. [10] based on some observations in Zhang et al. [9]. By using a neutrosophic triplet subgroup of a commutative neutrosophic triplet group, Zhang et al. [25] established a new congruence relation, and then constructed the quotient structure induced by neutrosophic triplet subgroup to establish the neutro-homomorphism basic theorem.

. The aim of this current work is to generalize and extend the results in Zhang et al. [24, 25] by investigating neutro-homomorphism in singular WCNETG. This will be done with the introduction and study of some new types of NT-subgroups that are right (left) cancellative, semi-strong, and maximally normal in a singular WCNETG. For any given non-empty subset S and NT-subgroup H of a singular WCNETG X, some of these new NT-subgroups are shown to exist as non-empty neutrosophic triplet normalizer, generated subset and centralizer of S, closure of H, derived subset of X and center of X. With these, the first, second and third neutro-isomorphism and neutro-correspondence theorems are established. And finally, the neutro-Zassenhaus Lemma is established.

2. **Main Results**

2.1. *Some new results on first and second WCNETGs*

In this subsection, we shall discuss some results associated with the first and second WCNETGs, introduced in Theorem 1.10 and Theorem 1.11, which shall be found useful as examples for illustrations in latter subsections.

Lemma 2.1. *Let $(G,*)$ be the WCNETG of the groups $(G_1,*_1,e_1)$ and $(G_2,*_2,e_2)$ in Theorem 1.10 or Theorem 1.11. Let $h_i : G_i \to G_i,\ i = 1,2$ be mappings and let $h : G \to G$ be defined as*

$$h(x) = \begin{cases} h_1(x), & \text{if } x \in G_1; \\ h_2(x), & \text{if } x \in G_2 \end{cases}$$

(1) If $h_i,\ i = 1,2,$ are endomorphisms of $(G_i,*_i,e_i),\ i = 1,2,$ then h is an neutro-endomorphism of $(G,*)$.

(2) h is a neuto-monomorphism (neutro-epimorphism) of $(G,*)$ if and only if $h_i,\ i = 1,2$ are monomorphisms (epimorphisms) of $(G_i,*_i,e_i),\ i = 1,2$.

(3) h is a neuto-automorphism of $(G,*)$ if and only if $h_i,\ i = 1,2$ are automorphisms of $(G_i,*_i,e_i),\ i = 1,2$.

Proof. This is easy. □

Lemma 2.2. *Let $(G,*)$ and $(G,\circ)$ be the WCNETGs of the pair of groups $(G_1,*_1)$ and $(G_2,*_2)$, and pair of groups $(G_1,\circ_1)$ and $(G_2,\circ_2)$ respectively in Theorem 1.10 or Theorem 1.11. Let $h_i : G_i \to G_i,\ i = 1,2$ be mappings and let $h : G \to G$ be defined as*

$$h(x) = \begin{cases} h_1(x), & \text{if } x \in G_1; \\ h_2(x), & \text{if } x \in G_2 \end{cases}$$

(1) If $h_i,\ i = 1,2,$ are homomorphisms of $(G_i,*_i),\ i = 1,2$ to $(G_i,\circ_i),\ i = 1,2,$ then h is a neutro-homomorphism of $(G,*)$ to $(G,\circ)$.

(2) h is a neuto-monomorphism (neutro-epimorphism) of $(G,*)$ to $(G,\circ)$ if and only if $h_i,\ i = 1,2$ are monomorphisms (epimorphisms) of $(G_i,*_i),\ i = 1,2$ to $(G_i,\circ_i),\ i = 1,2$.

(3) h is a neuto-isomorphism of $(G,*)$ to $(G,\circ)$ if and only if $h_i,\ i = 1,2$ are isomorphisms of $(G_i,*_i),\ i = 1,2$ to $(G_i,\circ_i),\ i = 1,2$.

(4) $\ker h = \ker h_1 \cup \ker h_2$ and $\mathrm{Im}(h) = \mathrm{Im}(h_1) \cup \mathrm{Im}(h_2)$.

Proof. The proof of this is a generalization of the proof of Lemma 2.1. □

2.2. *Some new subgroupoids and NT-subgroups of a WCNETG*

We shall now introduce some new NT-subgroups of a NETG and study them in singular WCNETG.

Definition 2.3. (Neutrosophic Triplet (Lormalizer, Mormalizer, Normalizer)-NTL, NTM, NTN)

Let X be a NETG and let $\emptyset \neq S \subseteq X$.

(1) The neutrosophic triplet lormalizer (NTL) of S in X is the set defined as $L(S) = \{x \in X | xS\ anti(x) = S\}$.
(2) The neutrosophic triplet mormalizer (NTM) of S in X is the set defined as $M(S) = \{x \in X | neut(x)\ S = S\}$.
(3) The neutrosophic triplet normalizer (NTN) of S in X is the set defined as $N(S) = L(S) \cap M(S)$.

Lemma 2.4. *Let X be a singular WCNETG and $\emptyset \neq S \subseteq X$.*

(1) If $L(S) \neq \emptyset$, then $L(S)$ is a subgroupoid of X.
(2) If $L(S) \neq \emptyset$, then for any $x \in L(S)$, $neut(x) \in L(S) \Leftrightarrow anti(x) \in L(S) \Leftrightarrow neut(x)\ S = S$.
(3) If $M(S) \neq \emptyset$, then $M(S)$ is a NT-subgroup of X.

Proof.

(1) Let $x, y \in L(S)$. Then,
$$(xy)S\ anti(xy) = (xy)S\ anti(y)anti(x) = x\big(yS\ anti(y)\big)anti(x) = xS\ anti(x) = S.$$
So, $xy \in L(S)$.
(2) $neut(x)S\ anti\big(neut(x)\big) = neut(x)S\ neut(x) = neut(x)neut(x)S = neut(x)S$ while
$$anti(x)S\ anti\big(anti(x)\big) = anti(x)xS\ anti(x)anti\big(anti(x)\big) = neut(x)S\ neut\big(anti(x)\big)$$
$$= neut(x)neut\big(anti(x)\big)S = neut\big(anti(x)x\big)S = neut\big(neut(x)\big)S = neut(x)S.$$
By these two arguments, $neut(x) \in L(S) \Leftrightarrow anti(x) \in L(S) \Leftrightarrow neut(x)\ S = S$.
(3) Let $x, y \in M(S)$. Then, $neut(xy)S = neut(y)neut(x)S = neut(y)S = S \Rightarrow xy \in M(S)$. If $x \in M(S)$, then $neut\big(anti(x)\big)S = neut(x)S = S$. So, going by Lemma 1.7 , $M(S)$ is a NT-subgroup of X. □

Example 2.5. In the singular WCNETG $(G,*)$ represented by Table 3, let $S = G_0 = \{e, 1\}$. Then, $L(G_0) = \{1, 2, 3, 4, 5, 6\} = G_2$ and $(G_2,*)$ is a subgroupoid of $(G,*)$. Furthermore, $M(G_0) = \{1, 2, 3, 4, 5, 6\} = G_2$ and $(G_2,*)$ is a NT-subgroup of $(G,*)$.

Theorem 2.6. *Let H and K be NT-subgroups of a singular WCNETG X. Then, HK is a NT-subgroup of X if and only if $HK = KH$.*

Proof. Let $HK = KH$ and let $a, b \in HK$. Then, $a = h_1k_1, b = h_2k_2$ for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$. So, $ab = h_1k_1h_2k_2 = h_1h_3k_3k_2 = h_4k_4 \in HK$ where $h_3k_3 = k_1h_2$. Let $a = hk$, then, $anti(a) = anti(hk) = anti(k)anti(h) = k'h' = h''k'' \in HK$. So, HK is a NT-subgroup of X going by Lemma 1.7.

Conversely, let HK be a NT-subgroup of X and let $a \in KH$. Then, $a = kh$ for some $k \in K$ and $h \in H$. So, $anti(a) = anti(kh) = anti(h)anti(k) = h'k' \in HK \Rightarrow KH \subseteq HK$. Let $b \in HK$, then $anti(b) \in HK$. Thus, $anti(b) = hk$, $h \in H, k \in K$, and so $b = anti\big(anti(b)\big) = anti(hk) = anti(k)anti(h) = k'h' \in KH \Rightarrow HK \subseteq KH$. $\therefore$ $HK = KH$. □

Example 2.7. Consider the singular WCNETG $(G,*)$ represented by Table 3.

(1) $(G_1,*_1)$ and $(G_2,*_2)$ are groups represented by Table 1 and Table 2 respectively. Hence, they are NT-subgroups $(G_1,*)$ and $(G_2,*)$ of $(G,*)$. Now, take $H = G_1$ and $K = G_2$, then $G_1G_2 = G_1 = G_2G_1$, and hence, Theorem 2.6 is true.

(2) $G_0 = \{e, 1\}$ is a NT-subgroup but not a subgroup of G. Now, take $H = G_0$ and $K = G_1$, then $G_0G_1 = G_1 = G_1G_0$, and hence, Theorem 2.6 is true.

(3) $G_2^e = \{e, 1, 2, 3, 4, 5, 6\}$ is a NT-subgroup but not a subgroup of G. Now, take $H = G_0$ and $K = G_2$, then $G_0G_2 = G_2^e = G_2G_0$, and hence, Theorem 2.6 is true.

Theorem 2.8. *Let X be a singular WCNETG, $\emptyset \neq S \subseteq X$ and H a NT-subgroup of X.*

(1) If $N(S) \neq \emptyset$, then $N(S)$ is a NT-subgroup of X.

(2) $N(H)$ is the largest NT-subgroup of X in which H is a x-normal NT-subgroup.

(3) If K is a NT-subgroup of $N(H)$, then $H \lhd_x HK$.

Proof.

(1) $N(S) \neq \emptyset \Leftrightarrow L(S), M(S) \neq \emptyset$. Since $N(S) = L(S) \cap M(S)$, then the fact that $N(S)$ is a NT-subgroup of X follows from Lemma 2.4

(2) Let H be a NT-subgroup of X. Then, $hH\ anti(h) = H$ for all $h \in H$. Thus, $H \subseteq N(H)$ and H is a NT-subgroup of $N(H)$. By definition, $xH\ anti(x) = H$ for all $x \in N(H)$. Hence, $H \lhd_x N(H)$. Let K be an arbitrary NT-subgroup of X such that $H \lhd_x K$. Then, $kH\ anti(k) = H$ for all $k \in K$, which implies that $K \subset N(H)$. Thus, $N(H)$ is the largest NT-subgroup of X in which H is a x-normal NT-subgroup.

(3) Let K be a NT-subgroup of $N(H)$, then for all $k \in K$, $kH\ anti(k) = H$. Hence, $kH\ anti(k)k = Hk \Rightarrow kH\ neut(k) = Hk \Rightarrow k\ neut(k)H = Hk \Rightarrow kH = Hk \Rightarrow HK = KH$. Hence, by Theorem 2.6, KH is a NT-subgroup of $N(H)$ and $H \subset KH$ (since $neut(k)\ H = H,\ k \in K \subset M(H)$). Consequently, $H \lhd_x HK$. □

Example 2.9. By Example 2.5, with $S = G_0 = \{e, 1\}$, $N(G_0) = L(G_0) \cap M(G_0) = \{1, 2, 3, 4, 5, 6\} = G_2$ and $(G_2, *)$ is a NT-subgroup of $(G, *)$.

Definition 2.10. (Normal Neutrosophic Triplet Subgroup)

Let X be a NETG and let N be a NT-subgroup of X. Let $neut(x)N = N$ for all $x \in X$, then N is said to be a normal NT-subgroup of X if $xN\ anti(x) \subset N$ and this represented by $N \lhd X$.

Lemma 2.11. *Let X be a singular WCNETG, $\emptyset \neq S \subseteq X$. If $< S >$ is generated by S in X, i.e.*

$$< S >= \left\{ \prod_{i=1}^{n} x_i = x_1 x_2 \cdots x_n \mid x_i \in S \text{ or } anti(x_i) \in S,\ 1 \leq i \leq n \right\},$$

then $< S >$ is a NT-subgroup of X which contains S.

Proof. $S \subset < S >$. So, $< S > \neq \emptyset$. If $a, b \in < S >$, then $a = \prod_{i=1}^{m} x_i$ and $b = \prod_{i=1}^{n} y_i$. So,

$$ab = \prod_{i=1}^{m} x_i \prod_{i=1}^{n} y_i \in < S > \text{ and } anti(a) = anti\left(\prod_{i=1}^{m} x_i \right) = \prod_{i=1}^{m} anti\big(x_{m-i+1}\big) \in < S >$$

Let Y be any NT-subgroup of X containing S; then for all $x \in S$, $x \in Y$. So, $anti(x) \in Y$, and Y contains all finite product $\prod_{i=1}^{n} x_i$ such that $x_i \in S$ or $anti(x_i) \in S$, $1 \leq i \leq n$. Hence, $< S > \subset Y$. □

Theorem 2.12. *Let X be a singular WCNETG and N a be NT-subgroup of X. If $neut(x)N = N$ for all $x \in X$, then the following are equivalent:*

(1) $N \lhd X$.

(2) $xN\ anti(x) = N$ for all $x \in X$.

(3) $xN = Nx$ for all $x \in X$.

(4) $xNyN = (xy)N$ for all $x, y \in X$.

Proof.

1⇒2: Let $N \lhd X$ and $x \in X$. Then, $xN\ anti(x) \subset N$. Since $anti(x) \in X$, then $anti(x)N\ anti\big(anti(x)\big) \subset N \Rightarrow anti(x)N\ x \subset N$. Now, $x(anti(x)N\ x)anti(x) = (x\ anti(x))N(x\ anti(x)) = neut(x)N\ neut(x) = N\ neut(x) = neut(x)N = N$. So, $N = x(anti(x)N\ x)anti(x) \subset xN\ anti(x) \Rightarrow N \subset xN\ anti(x)$. Hence, $xN\ anti(x) = N$.

2⇒3: $xN\ anti(x) = N \Rightarrow Nx = (xN\ anti(x))x = xN\ anti(x)x = xN\ neut(x) = xN \Rightarrow Nx = xN$.

3⇒4: $xNyN = x(Ny)N = x(yN)N = (xy)NN$. Now, $NN \subset N$ since N is a groupoid. On the other hand, $N = e(n)N \subset NN$ for some $n \in N$. Hence, $NN = N$. $\therefore\ xNyN = (xy)N$.

4⇒1: $xN\ anti(x) = xN\ neut(n)anti(x) = xN\ anti(x)neut(n) \subset xN\ anti(x)N = (x\ anti(x))N = neut(x)N = N \Rightarrow xN\ anti(x) \subset N \Rightarrow N \lhd X$.□

Remark 2.13. Note that $neut(x) \in N$ for all $x \in X \Rightarrow neut(x)N \subseteq N$ but the converse is not necessarily true. For example, in the first WNCETG of Table 3, $neut(x)G_1 = neut(x)\{e, a, b, c\} = G_1$, but $neut(x) \notin G_1$ for all $x \in G_2$.

Definition 2.14. (Closure of a set)

Let X be a NETG and $\emptyset \neq S \subseteq X$ and $Y \leq X$. The closure of S in H will be defined by $Cl_H(S) = \{x \in H | xS = S\}$. If $H = X$, then this will simply be expressed as $Cl(S)$.

Lemma 2.15. *Let X be a singular WCNETG and H a NT-subgroup of X. Then*

(1) $Cl(H) \neq \emptyset$ and $Cl(H)$ is a NT-subgroup of X.

(2) $Cl(H)$ is a NT-subgroup of $N(H)$.

Proof.

(1) $Cl(H) \neq \emptyset\ \because H \subseteq Cl(H)$. Let $x, y \in Cl(H)$, then $(xy)H = x(yH) = xH = H \Rightarrow xy \in Cl(H)$.

Let $x \in Cl(H)$, then $xH = H \Rightarrow \big(neut(x)x\big)H = H \Rightarrow neut(x)(xH) = H \Rightarrow neut(x)H = H \Rightarrow neut(x) \in Cl(H)$. Furthermore, $neut(x)H = H \Rightarrow \big(anti(x)x\big)H = H \Rightarrow anti(x)(xH) = H \Rightarrow anti(x)H = H \Rightarrow anti(x) \in Cl(H)$ and so, $Cl(H)$ is a NT-subgroup of X.

(2) Let $x \in Cl(H)$, then by (1), $neut(x)H = H$. More so, $H = neut(x)H = H\ neut(x) = Hx\ anti(x) = H\ anti(x) \Rightarrow H = H\ anti(x)$. Thence, $xH\ anti(x) = H\ anti(x) = H$. $\therefore\ Cl(H)$ is a NT-subgroup of $N(H)$.□

Example 2.16. For the singular WCNETG $(G, *)$ in Table 3, $G_0 = \{e, 1\} \leq G$, even though G_0 is not a subgroup in $(G, *)$. $Cl(G_0) = \{1\} \leq G$. Furthermore, by Example 2.9, with $H = G_0 = \{e, 1\}$, $N(G_0) = L(G_0) \cap M(G_0) = \{1, 2, 3, 4, 5, 6\} = G_2$ and $(G_2, *)$ is a NT-subgroup of $(G, *)$. So, $Cl(G_0) \leq (G_2, *)$.

Definition 2.17. Let X be a NETG.

(1) If $\emptyset \neq S \subseteq X$, the set $C_X(S) = \{x \in X | xs = sx \ \forall \ s \in S\}$ will be called the centralizer of S in X.

(2) The set $Z(X) = \{x \in X | xy = yx \ \forall \ y \in X\}$ will be called the center X.

(3) Let $Y \leq X$. Then, Y is called a complete NT-subgroup of X if $neut(g)y \in Y$ for all $g \in X$ and $y \in Y$.

Lemma 2.18. *Let X be a singular WCNETG.*

(1) For any $\emptyset \neq S \subseteq X$, $C_X(S) \neq \emptyset$ and $C_X(S)$ is a complete NT-subgroup of X for which $neut(g) \in C_X(S)$ for all $g \in X$. Furthermore, $neut(g) \in Cl\big(C_X(S)\big)$ if and only if $C_X(S) \subseteq neut(g)C_X(S)$ for all $g \in X$.

(2) $C_X(X) = Z(X) \lhd X \Leftrightarrow Z(X) \subseteq neut(g)Z(X)$ for all $g \in X$.

Proof.

(1) Consider $neut(g) \in X$, for any $g \in X$. Observe that $neut(g)s = s\ neut(g)$ for all $s \in S$ implies that $neut(g) \in C_X(S)$ for any $g \in X$. So, $C_X(S) \neq \emptyset$. Furthermore, $neut(g)C_X(S) \subseteq C_X(S)$ for any $g \in X$. So, $neut(g) \in Cl\big(C_X(S)\big) \Leftrightarrow C_X(S) \subseteq neut(g)C_X(S)$ for all $g \in X$.

Let $x, y \in C_X(S)$, then $xs = sx$ and $ys = sy$ for all $s \in S$.

$$(xy)s = x(ys) = x(sy) = (xs)y = (sx)y = s(xy) \Rightarrow xy \in C_X(S).$$

$$anti(x)s = anti(x)neut\big(anti(x)\big)s = anti(x)neut(x)s = anti(x)s\ neut(x) =$$
$$anti(x)sx\ anti(x) = anti(x)xs\ anti(x) = neut(x)s\ anti(x) = s\ neut(x)anti(x) =$$
$$s\ neut(x)anti(x) \Rightarrow anti(x) \in C_X(S).$$

So, $C_X(S)$ is a complete NT-subgroup of X.

(2) $C_X(X) = \{x \in X | xg = gx \ \forall \ g \in X\} = Z(X)$. Let $x \in Z(X)$ and $g \in X$, then $gx\ anti(g) = xg\ anti(g) = x\ neut(x) \in Z(X)$. So, $C_X(X) = Z(X) \lhd X \Leftrightarrow Z(X) \subseteq neut(g)Z(X)$ based on 1.□

Example 2.19. Consider the singular WCNETG $(G, *)$ represented by Table 3.

(1) Let $S = G_0 = \{e, 1\} \leq G$. $C_G(G_0) = G \leq G$ and so, $neut(g) \in C_G(G_0)$ for all $g \in G$. $Cl(C_G(G_0)) = Cl(G) = G_2 \leq G$. Observe that $neut(g) \in Cl(C_G(G_0))$ for some $g \in G$ and so, $neut(g) \notin Cl(C_G(G_0))$ for all $g \in G$.

(2) Furthermore, $Z(G) = \{1\} \cup G_1 \leq G$, Now, $xZ(G)\ anti(x) \subset Z(G)$ for all $x \in G$. For all $x \in G_2$, note that $neut(x)Z(G) = 1 \cdot Z(G) = Z(G)$ but for all $x \in G_1$, $neut(x)Z(G) = e \cdot Z(G) \subset Z(G)$. So, $neut(x)Z(G) \neq Z(G)$ for all $x \in G$. Hence, $Z(G) \ntriangleleft G$.

(3) Given any group G with subgroup H and normal subgroup K, G is a WCNETG with complete NT-subgroup H and normal NT-subgroup K.

Definition 2.20. Let X be a NETG.

(1) If $neut(a)b = neut(a)c$ implies that $b = c$ for all $a, b \in X$, then X is said to be neutro-left cancellative.
(2) If $b\ neut(a) = c\ neut(a)$ implies that $b = c$ for all $a, b \in X$, then X is said to be neutro-right cancellative.
(3) Let H be a NT-subgroup of X. H is said to be right self cancellative in X if $xH = H$ implies $x \in H$ for all $x \in X$. This will sometimes be represented as $H \leq_{\mathrm{rsc}} X$.
(4) Let H be a NT-subgroup of X. H is said to be left self cancellative in X if $Hx = H$ implies $x \in H$ for all $x \in X$. This will sometimes be represented as $H \leq_{\mathrm{lsc}} X$.
(5) Let H be a NT-subgroup of X. H is said to be a semi-strong NT-subgroup of X if $neut(x) \in H$ for all $x \in X$. This will sometimes be represented as $H \leq_{\mathrm{ss}} X$.
(6) For $Y, Z \leq X$, Y will be said to be Z-neutro-solvable in X if for any $x \in X$ and $y \in Y$, $neut(x)y \in Z \Rightarrow y \in Z$.

Remark 2.21. In a WCNETG, neutro-left cancellation and neutro-right cancellation are equivalent. In a NETG, left self cancellation and right self cancellation are equivalent for any given normal NT-subgroup. The use of 'semi-strong' in Definition 2.20 is based on the use of 'strong' in Definition 5 of [9].

Example 2.22. Consider the singular WCNETG $(G, *)$ represented by Table 3.

(1) Based on Table 1 representing $(G_1, *_1)$, $(G_1, *)$ is a subgroup (hence, NT-subgroup)of $(G, *)$ but $G_1 \not\leq_{\mathrm{rsc}} G$ because $xG_1 = G_1 \not\Rightarrow x \in G_1$ for all $x \in G$. Similarly, $G_1 \not\leq_{\mathrm{lsc}} G$. On the hand, based on Table 2 representing $(G_2, *_2)$, $(G_2, *)$ is a subgroup (hence, NT-subgroup) of $(G, *)$. Whereas, $G_2 \leq_{\mathrm{rsc}} G$ and $G_2 \leq_{\mathrm{lsc}} G$. These difference between G_1 and G_2 shows that the notions of right self cancellation and left self cancellation NT-subgroup is peculiar in NETG and not trivial from the point of view classical group. This is because, even though, $G_0 = \{e, 1\}$ is not a subgroup of G, it is right self cancellative and left self cancellative.
(2) G_1 and G_2 are subgroups (hence, NT-subgroup) of $(G, *)$, but they are not semi-strong NT-subgroup of G because $neut(x) \notin G_1$ for all $x \in G_2$ and $neut(x) \notin G_2$ for all $x \in G_1$. Thus, the concept semi-strong NT-subgroup is peculiar in NETG and not trivial from the point of view classical group. This is because, even though $G_0 = \{e, 1\}$ is not a subgroup of G, it is a semi-strong NT-subgroup of G. In addition, despite the fact that $G_1^1 = \{1\} \cup G_1$ and $G_2^e = \{e\} \cup G_2$ are not subgroups of G, $G_1^1 \leq_{\mathrm{ss}} G$ and $G_2^e \leq_{\mathrm{ss}} G$.

(3) Since $G_2 \leq_{\rm rsc} G$, then it can be observed that $Cl(G_2) = G_2$.

(4) We shall now see that the notion of 'neutro solvability' in NETG is not subgroup biased as the case is in classical groups.

(a) Even though $G_0 = \{e, 1\}$ is a NT-subgroup of X and not a subgroup of X, it is both G_2^e-neutro solvable and G_1^1-neutro solvable in G.

(b) G_1 and G_2 are subgroups of G: G_2 is not G_1-neutro solvable in X, but G_1 is G_2-neutro solvable in G.

(c) G_1^1 and G_2^e are not subgroups of G: G_2^e is not G_1^1-neutro solvable in G, but G_1^1 is G_2^e-neutro solvable in G.

Lemma 2.23. *Let X be a NETG such that $Y, Z \leq X$.*

(1) $Y \leq_{rsc} X$ if and only if $Cl(Y) \subseteq Y$.

(2) $Cl(Y) \cap Cl(Z) \subseteq Cl(Y \cap Z) \Leftrightarrow xY \cap xZ = x(Y \cap Z)$ for all $x \in X$.

(3) Let X be a singular NETG. If any of the following is true:

(a) Y is Z-neutro-solvable in X and $Z \lhd X$ or $Z \leq_{ss} X$ or $neut(x) \in Cl(Z)$ for all $x \in X$;

(b) Z is Y-neutro-solvable in X and $Y \lhd X$ or $Y \leq_{ss} X$ or $neut(x) \in Cl(Y)$ for all $x \in X$;

then, $xY \cap xZ = x(Y \cap Z)$ for all $x \in X$ and $Cl(Y) \cap Cl(Z) \subseteq Cl(Y \cap Z)$.

Proof.

(1) Let $Y \leq_{\rm rsc} X$, then for any $x \in X$, $xY = Y \Rightarrow x \in Y$. Let $x \in Cl(Y)$, then $xY = Y \Rightarrow x \in Y$. So, $Cl(Y) \subseteq Y$.

Conversely, let $x \in Cl(Y)$, then $xY = Y$. Since $Cl(Y) \subseteq Y$, then, $x \in Y$. Thus, for any $x \in X$, $xY = Y \Rightarrow x \in Cl(Y) \Rightarrow x \in Y$. Thence, $Cl(Y) \subseteq Y$.

(2) If $Cl(Y) \cap Cl(Z) \subseteq Cl(Y \cap Z)$, then $x \in Cl(Y) \cap Cl(Z) \Rightarrow x \in Cl(Y \cap Z)$. So, $x \in Cl(Y) \Rightarrow xY = Y$ and $x \in Cl(Z) \Rightarrow xZ = Z$ for all $x \in X$ and $x(Y \cap Z) = Y \cap Z$ for all $x \in X$. Thus, $x(Y \cap Z) = Y \cap Z = xY \cap xZ = Y \cap Z$ for all $x \in X$.

Conversely, let $xY \cap xZ = x(Y \cap Z)$ for all $x \in X$, then $x \in Cl(Y) \cap Cl(Z) \Rightarrow Y \cap xZ = Y \cap Z$ for all $x \in X$ will give $x(Y \cap Z) = Y \cap Z$ for all $x \in X \Rightarrow x \in Cl(Y \cap Z)$. Therefore, $Cl(Y) \cap Cl(Z) \subseteq Cl(Y \cap Z)$.

(3) The proof of $x(Y \cap Z) \subseteq xY \cap xZ$ is routine while the proof of $xY \cap xZ \subseteq x(Y \cap Z)$ requires the conditions in (a) or (b). The last part follows from 2. □

Example 2.24.

(1) As mentioned in Example 2.22, $G_0 = \{1, e\} \leq_{\rm rsc} X$ and $Cl(G_0) = \{1\} \subset G_0$.

(2) $Cl(G_2^e) = G_2$ and $Cl(G_1^1) = \{1\}$, so $Cl(G_1^1) \cap Cl(G_2^e) = \{1\} = Cl(G_0) = Cl\big(G_1^1 \cap G_2^e\big)$.

(3) By Example 2.22(2)(4): $G_0 = \{e, 1\}$ is both G_2^e-neutro solvable and G_1^1-neutro solvable in G, and, $G_1^1 \leq_{ss} G$ and $G_2^e \leq_{ss} G$. So, $xY \cap xZ = x(Y \cap Z)$ for all $x \in X$ and $Cl(Y) \cap Cl(Z) \subseteq Cl(Y \cap Z)$ for the pairings: $Y = G_0$ and $Z = G_1^1$; $Y = G_0$ and $Z = G_2^e$.

2.3. *Neutrosophic Triplet Group Homomorphism*

Let X and Y be NETGs and let $\phi : X \to Y$. Then, ϕ is called a neutro-homomorphism if $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in X$. If a neutro-homomorphism is a mono (epi), then , it is called a neutro-monomorphism (neutro-epimorphism). If a neutro-homomorphism is a bijection, then , it is called a neutro-isomorphism. In such a case, X and Y are said to be neutro-isomorphic (or simply isomorphic) and this will be written as $X \cong Y$.

$\ker\phi = \{x \in X | \phi(x) = neut(y) \text{ for some } y \in Y\}$ and $\mathrm{Im}(\phi) = \{y \in Y | \phi(x) = y \text{ for some } x \in X\}$.

Theorem 2.25. *Let X be a singular WCNETG and $N \lhd X$. Then*

(1) $X/N = \{xN | x \in X\}$ is a group.

(2) The mapping $\phi : X \to X/N \uparrow x \mapsto xN$ is a neutro-epimorphism.

(3) Let $NT(X)$ and $NT(X/N)$ represent the set of all NTs of X and X/N respectively, i.e.

$$NT(X) = \big\{\big(x, neut(x), anti(x)\big) \mid x \in X\big\} \text{ and}$$
$$NT(X/N) = \big\{\big(xN, neut(xN), anti(xN)\big) \mid xN \in X/N\big\}.$$

Then, there exists a binary operation $\odot$ on $NT(X)$ and $NT(X/N)$, and a mapping $\alpha : NT(X) \to NT(X/N)$ such that

(a) $NT(X)$ is a singular WCNETG and $NT(X/N)$ is a group.

(b) α is a neutro-epimorphism if X/N is an abelian group.

(4) $\ker\phi = Cl(N)$ *and*

$$\ker\alpha = \Big(Cl(N), neut\big(Cl(N)\big), anti\big(Cl(N)\big)\Big) = \Big(\ker\phi, neut\big(\ker\phi\big), anti\big(\ker\phi\big)\Big).$$

Proof.

(1) **Closure:** By Theorem 2.12(4), $xNyN = (xy)N$ for all $x, y \in X$.

Associativity: By repeated use of Theorem 2.12(4), $(xNyN)zN = xN(yNzN)$ for all $x, y, z \in X$.

Identity: Let $neut(xN) = neut(x)N = N$. Then, $neut(xN)xN = neut(x)NxN = (neut(x)x)N = xN$ and $xN\ neut(xN) = xN\ neut(x)N = (x\ neut(x))N = xN$.

Inverse: Let $anti(xN) = anti(x)N$. Then, $anti(xN)xN = anti(x)NxN = (anti(x)x)N = neut(x)N = N$ and $xN\ anti(xN) = xN\ anti(x)N = (x\ anti(x))N = neut(x)N = N$.

$\therefore X/N$ is a group.

(2) By definition, ϕ is onto and for all $x, y \in X$, $\phi(xy) = (xy)N = xNyN = \phi(x)\phi(y)$. Thus, ϕ is a neutro-epimorphism.

(3) Define $\odot$ on $NT(X)$ as follows:

$$\big(x, neut(x), anti(x)\big) \odot \big(y, neut(y), anti(y)\big) = \big(xy, neut(y)neut(x), anti(y)anti(x)\big).$$

Closure: $\big(x, neut(x), anti(x)\big) \odot \big(y, neut(y), anti(y)\big) = \big(xy, neut(xy), anti(xy)\big) \in NT(X)$.

Neutral and Opposite: Define the neutral of $\big(x, neut(x), anti(x)\big)$ as follows:

$$neut\big(x, neut(x), anti(x)\big) = \big(neut(x), neut(x), neut(x)\big). \text{ Then}$$

$$neut\big(x, neut(x), anti(x)\big) = \big(neut(x), neut(neut(x)), anti(neut(x))\big) \in NT(X).$$

On the other hand, define the opposite of $\big(x, neut(x), anti(x)\big)$ as follows:

$$anti\big(x, neut(x), anti(x)\big) = \big(anti(x), neut(x), x\big). \text{ Then,}$$

$$anti\big(x, neut(x), anti(x)\big) = \big(anti(x), neut(anti(x)), anti(anti(x))\big) \in NT(X). \text{ Now}$$

$$LHS = \big(x, neut(x), anti(x)\big) \odot\ neut\big(x, neut(x), anti(x)\big) = \big(x, neut(x), anti(x)\big) \odot$$

$$\big(neut(x), neut(neut(x)), anti(neut(x))\big) = \big(x\ neut(x), neut(x\ neut(x)), anti(x\ neut(x))\big)$$

$$= \big(x, neut(x), anti(x)\big). \text{ Similarly,}$$

$$RHS = neut\big(x, neut(x), anti(x)\big) \odot \big(x, neut(x), anti(x)\big) = \big(x, neut(x), anti(x)\big).$$

$$LHS = \big(x, neut(x), anti(x)\big) \odot\ anti\big(x, neut(x), anti(x)\big) = \big(x, neut(x), anti(x)\big) \odot$$

$$\big(anti(x), neut(anti(x)), anti(anti(x))\big) = \big(x\ anti(x), neut(x\ anti(x)), anti(x\ anti(x))\big)$$

$$= \big(neut(x), neut(neut(x)), anti(neut(x))\big) = neut\big(x, neut(x), anti(x)\big). \text{ Similarly,}$$

$$RHS = anti\big(x, neut(x), anti(x)\big) \odot \big(x, neut(x), anti(x)\big) = neut\big(x, neut(x), anti(x)\big).$$

$$\therefore \Big(\big(x, neut(x), anti(x)\big), neut\big(x, neut(x), anti(x)\big), anti\big(x, neut(x), anti(x)\big)\Big)$$

forms a neutrosophic triplet for $\big(x, neut(x), anti(x)\big) \in NT(X)$ and so, $NT(X)$ is a neurotrophic triplet set.

Associativity: $LHS = \Big(\big(x, neut(x), anti(x)\big) \odot \big(y, neut(y), anti(y)\big)\Big) \odot$

$$\big(z, neut(z), anti(z)\big) = \big(xy, neut(xy), anti(xy)\big) \odot \big(z, neut(z), anti(z)\big) =$$

$$\big(xy \cdot z, neut(xy \cdot z), anti(xy \cdot z)\big). \text{ Similarly, } RHS = \big(x, neut(x), anti(x)\big) \odot$$

$$\Big(\big(y, neut(y), anti(y)\big) \odot \big(z, neut(z), anti(z)\big)\Big) = \big(x \cdot yz, neut(x \cdot yz), anti(x \cdot yz)\big)$$

So, $NT(X)$ is a NETG.

Weak Commutativity:

$$LHS = neut\big(x, neut(x), anti(x)\big) \odot \big(y, neut(y), anti(y)\big) =$$
$$\big(neut(x), neut(neut(x)), anti(neut(x))\big) \odot \big(y, neut(y), anti(y)\big) =$$
$$\big(neut(x)y, neut(neut(x)y), anti(neut(x)y)\big) =$$
$$\big(y\ neut(x), neut(y\ neut(x)), anti(y\ neut(x))\big) = \big(y, neut(y), anti(y)\big)\odot$$
$$neut\big(x, neut(x), anti(x)\big) = RHS.$$

Singularity: $anti\big(x, neut(x), anti(x)\big)$ is unique for each $\big(x, neut(x), anti(x)\big) \in NT(X)$.

$\therefore$ $NT(X)$ is a singular WCNETG.

(4) $\ker\phi = \{x \in X|\phi(x) = neut(yN),\ yN \in X/N\} = \{x \in X|\phi(x) = neut(y)N = N,\ y \in X\} = Cl(N)$.

$$\ker\alpha = \Big\{\big(x, neut(x), anti(x)\big) \in NT(X)|\big(x, neut(x), anti(x)\big) = neut\big(x, neut(x), anti(x)\big)\Big\}$$
$$= \Big\{\big(x, neut(x), anti(x)\big) \in NT(X)|\big(xN, N, anti(xN)\big) = \big(N, N, N\big)\Big\}$$
$$= \Big\{\big(x, neut(x), anti(x)\big) \in NT(X)|xN = N \text{ and } anti(xN) = N\Big\}$$
$$= \Big\{\big(x, neut(x), anti(x)\big) \in NT(X)|x \in Cl(N) \text{ or } x \in \ker\phi\Big\}$$
$$= \Big(Cl(N), neut\big(Cl(N)\big), anti\big(Cl(N)\big)\Big) = \Big(\ker\phi, neut\big(\ker\phi\big), anti\big(\ker\phi\big)\Big). \square$$

2.4. *Isomorphism Theorems for Singular WCNETG*

We are now ready to establish the first, second and third neutro-isomorphism theorems, neutro-correspondence theorem and the neutro-Zassenhaus Lemma (Neutro-Butterfly Theorem).

Theorem 2.26. *(First Neutro-Isomorphism Theorem for Singular WCNETG)*

Let X and Y be singular WCNETGs and let $\phi : X \to Y$ be a a neutro-homomorphism.

(1) (a) $\ker\phi$ is a complete NT-subgroup of X.
(b) $\ker\phi \lhd_x X$ for all $x \in X$.
(c) $\ker\phi \lhd X \Leftrightarrow \ker\phi \subset neut(x)\ker\phi$ for all $x \in X$.

(2) $\mathrm{Im}(\phi)$ is a NT-subgroup of Y and if K is a NT-subgroup of Y, then $\emptyset \neq \phi^{-1}(K)$ is a NT-subgroup of X.

(3) If Y is neutro-left (neutro-right) cancellative and $\ker\phi \subset neut(x)\ker\phi$ for all $x \in X$, then $X/\ker\phi \cong \mathrm{Im}(\phi)$. Hence, if in addition, ϕ is a neutro-epimorphism, then $X/\ker\phi \cong Y$.

Proof. Let $\phi : X \to Y$ be a neutro-homomorphism, then $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in X$.

(1) Put $y = neut(x)$ in $\phi(xy) = \phi(x)\phi(y)$ to get $\phi\big(x\ neut(x)\big) = \phi(x)\phi(neut(x)) \Rightarrow \phi(x) = \phi(x)\phi(neut(x))$. Also, put $y = neut(x)$ in $\phi(yx) = \phi(y)\phi(x)$ to get $\phi\big(neut(x)x\big) = \phi(neut(x))\phi(x) \Rightarrow \phi(x) = \phi(neut(x))\phi(x)$. Thus, $\phi(neut(x)) = neut(\phi(x))$ for all $x \in X$. So, $\ker\phi \neq \emptyset$.

Let $a, b \in \ker\phi$, then $\phi(a) = neut(g)$ and $\phi(b) = neut(h)$ for some $g, h \in Y$. Then, $\phi(ab) = \phi(a)\phi(b) = neut(g)neut(h) = neut(gh) \Rightarrow ab \in \ker\phi$.

Put $y = anti(x)$ in $\phi(xy) = \phi(x)\phi(y)$ to get $\phi\big(x\ anti(x)\big) = \phi(x)\phi(anti(x)) \Rightarrow \phi(neut(x)) = \phi(x)\phi(anti(x)) \Rightarrow neut(\phi(x)) = \phi(x)\phi(anti(x))$. Also, put $y = anti(x)$ in $\phi(yx) = \phi(y)\phi(x)$ to get $\phi\big(anti(x)x\big) = \phi(anti(x))\phi(x) \Rightarrow \phi(neut(x)) = \phi(anti(x))\phi(x) \Rightarrow neut(\phi(x)) = \phi(anti(x))\phi(x)$. Thus, $\phi(anti(x)) = anti(\phi(x))$ for all $x \in X$.

Now, let $x \in \ker\phi$, then $\phi(x) = neut(y)$ for some $y \in Y$. Using the above result, $\phi(anti(x)) = anti(\phi(x)) = anti(neut(y)) = neut(y) \Rightarrow anti(x) \in \ker\phi$ for all $x \in X$. Thus, $\ker\phi$ is a NT-subgroup of X. Furthermore, for any $g \in X$ and $x \in \ker\phi$,

$$\phi\big(gx\ anti(g)\big) = \phi(g)\phi(x)\phi(anti(g)) = \phi(g)neut(y)anti(\phi(g)) = neut(y)\phi(g)\ anti(\phi(g))$$

$$= neut(y)neut(\phi(g)) = neut\big(y\phi(g)\big) \Rightarrow gx\ anti(g) \in \ker\phi.$$

Also, for any $g \in X$, $\phi(neut(g)) = neut\big(\phi(g)\big) \Rightarrow neut(g) \in \ker\phi$. Thus, $\ker\phi$ is a complete NT-subgroup of X, $\ker\phi \lhd_x X$ for all $x \in X$ and therefore, $\ker\phi \lhd X \Leftrightarrow \ker\phi \subset neut(x)\ker\phi$ for all $g \in X$.

(2) For any $g \in X$, $\phi(neut(g)) = neut\big(\phi(g)\big) \in \mathrm{Im}(\phi)$. So, $\mathrm{Im}(\phi) \neq \emptyset$. Let $x', y' \in \mathrm{Im}(\phi)$, then $x' = \phi(x)$ and $y' = \phi(y)$. Thus, $x'\ anti(y') = \phi(x)anti(\phi(y)) = \phi(x)\phi(anti(y)) = \phi(x\ anti(y)) \in \mathrm{Im}(\phi)$. So, $\mathrm{Im}(\phi)$ is a NT-subgroup of Y.

If K is a NT-subgroup of Y, then $\emptyset \neq \phi^{-1}(K) = \{x \in X : \phi(x) \in K\}$.

Let $x, y \in \phi^{-1}(K)$, then there exist $x', y' \in K$ such that $x' = \phi(x)$ and $y' = \phi(y)$. Thus, $x'\ anti(y') = \phi(x)anti(\phi(y)) = \phi(x)\phi(anti(y)) = \phi(x\ anti(y)) \in K \Rightarrow x\ anti(y) \in \phi^{-1}(K)$. So, $\phi^{-1}(K)$ is a NT-subgroup of X.

(3) Let $\psi : X/\ker\phi \to \mathrm{Im}(\phi)\ \uparrow\ \psi\big(x\ker\phi\big) = \phi(x)$ for each $x \in X$.

Well Defined: For any $x, y \in X$,

$$x\ker\phi = y\ker\phi \Rightarrow anti\big(y\ker\phi\big)x\ker\phi = anti\big(y\ker\phi\big)y\ker\phi \Rightarrow$$

$$\big(anti(y)x\big)\ker\phi = \ker\phi \Rightarrow anti(y)xr = s,\ r, s \in \ker\phi \Rightarrow \phi\big(anti(y)xr\big) = \phi(s) \Rightarrow$$

$$\phi\big(anti(y)x\big)\phi(r) = \phi(s) \Rightarrow \phi\big(anti(y)x\big)neut(r') = neut(s'),\ r', s' \in Y \Rightarrow$$

$$\phi\big(anti(y)x\big)neut(r')anti\big(neut(r')\big) = neut(s')anti\big(neut(r')\big) \Rightarrow$$

$$\phi\big(anti(y)x\big)neut(r') = neut(s')neut(r') \Rightarrow \phi\big(anti(y)x\big)neut(r') = neut(s'r') \Rightarrow$$

$$anti\big(\phi(y)\big)\phi(x)neut(r') = neut(s'r') \Rightarrow \phi(y)anti\big(\phi(y)\big)\phi(x)neut(r') =$$

$$\phi(y)neut(s'r') \Rightarrow neut\big(\phi(y)\big)\phi(x)neut(r') = \phi(y)neut(s'r') \Rightarrow$$

$$\phi(x)neut\big(\phi(y)\big)neut(r') = \phi(y)neut(s'r') \Rightarrow \phi(x)neut\big(\phi(y)r'\big) = \phi(y)neut(s'r')$$

$$\Rightarrow \phi(x) = \phi(y) \Rightarrow \psi\big(x\ker\phi\big) = \psi\big(y\ker\phi\big).$$

without out loss of generality, we take $neut\big(\phi(y)r'\big) = neut(s'r')$ and because H is neutro-left(or neutro-right) cancellative.

One to one:

$$\psi\big(x\ker\phi\big) = \psi\big(y\ker\phi\big) \Rightarrow \phi(x) = \phi(y) \Rightarrow \phi(x)anti\big(\phi(y)\big) = \phi(y)anti\big(\phi(y)\big) \Rightarrow$$

$$\phi\big(x\ anti(y)\big) = neut\big(\phi(y)\big) \Rightarrow x\ anti(y) \in \ker\phi \Rightarrow x\ anti(y)y \in y\ker\phi \Rightarrow$$

$$x\ neut(y) \in y\ker\phi \Rightarrow x\ neut(y)\ker\phi = x\ker\phi \subseteq y\ker\phi\ker\phi = y\ker\phi \Rightarrow$$

$$x\ker\phi \subseteq y\ker\phi$$

Similarly, it can be shown that $y\ker\phi \subseteq x\ker\phi$. Thus, $x\ker\phi = y\ker\phi$.

Onto: This is obvious.

neutro-homomorphism:

$$\psi\big(x\ker\phi\cdot y\ker\phi\big) = \psi\big((xy)\ker\phi\big) = \phi(xy) = \phi(x)\phi(y) = \psi\big(x\ker\phi\big)\psi\big(y\ker\phi\big)$$

$\therefore\ X/\ker\phi \cong \mathrm{Im}(\phi)$ and if ϕ is a neutro-epimorphism, then $X/\ker\phi \cong Y$.$\square$

Example 2.27. In Lemma 2.2, consider the WCNETGs $(G,*)$ and $(G,\circ)$ of the pair of groups $(G_1,*_1,e_1)$ and $(G_2,*_2,e_2)$, and pair of groups $(G_1,\circ_1)$ and $(G_2,\circ_2)$ respectively. Let $h_i : (G_i,*_i) \to (G_i,\circ_i)$, $i = 1,2$ be homomorphisms, then $h : (G,*) \to (G,\circ)$ is a neutro-homomorphism.

(1) Recall that $\ker h = \ker h_1 \cup \ker h_2$. So, $\ker h \leq (G,*)$ since $\ker h_1$ and $\ker h_2$ are subgroups of $(G_1,*_1)$ and $(G_2,*_2)$ respectively. We need the facts that $\ker h_i = \{g \in G_i | h_i(g) = e_i\}$ for $i = 1,2$ and $\{e_1,e_2\} \leq \ker h$. Let $Y = \ker h$, then for all $g \in G$ and any $y \in Y$:

$$h\big(neut(g)y\big) = \begin{cases} e_i \in \ker h, & \text{if } g \in G_i,\ y \in \ker h_i,\ i = 1,2; \\ e_i \in \ker h \text{ or } e_j \in \ker h, & \text{if } g \in G_i,\ y \in \ker h_j,\ i,j \in \{1,2\},\ i \neq j \end{cases}$$

Then, $neut(g)y \in \ker h$ for all $g \in G$ and any $y \in Y$. Whence, $\ker h$ is a complete NT-subgroup of $(G,*)$.

(2) $\ker h \lhd G \Leftrightarrow \ker h \subset neut(g)\ker h\ \forall\ g \in G$ if and only if $\ker h \subset neut(g)\ker h\ \forall\ g \in G_1$ and $\ker h \subset neut(g)\ker h\ \forall\ g \in G_2$ if and only if $\ker h \subset e_1 * \ker h\ \forall\ g \in G_1$ and $\ker h \subset e_2 * \ker h$.

(3) Recall that $\mathrm{Im}(h) = \mathrm{Im}(h_1) \cup \mathrm{Im}(h_2)$. So, $\mathrm{Im}(h) \leq (G,\circ)$ since $\mathrm{Im}(h_1)$ and $\mathrm{Im}(h_2)$ are subgroups of $(G_1,\circ_1)$ and $(G_2,\circ_2)$ respectively.

Theorem 2.28. *(Second Neutro-Isomorphism Theorem for Singular WCNETG)*

Let X be a singular WCNETG with NT-subgroups H and K such that K is right self cancellative in H, $hK = Kh$ and $neut(h) \in Cl(H), Cl(K)$ for all $h \in H$, and $neut(k) \in Cl(K)$ for all $k \in K$. Then,

(1) $K \lhd HK \leq X$.
(2) $H \cap K, K \lhd H$.
(3) $H/H \cap K \cong HK/K$.

Proof.

(1) $hK = Kh$ for all $h \in H$ implies that $HK = KH$. So, by Theorem 2.6, HK is a NT-subgroup of X. Let $hk \in HK$, $h \in H$ and $k \in K$. Then, for any $k_1 \in K$, $(hk)k_1\ anti(hk) = h(kk_1\ anti(k))anti(h) = hk_2\ anti(h) = h\ anti(h)k_3 = neut(h)k_3 \in K$ since $neut(h) \in Cl(K)$ for all $h \in H$. So, $(hk)k_1\ anti(hk) \in K$. Also, $neut(hk)K = neut(h)neut(k)K = neut(h)K = K$. Thus, $K \lhd HK \leq X$.

(2) Let $x \in H \cap K$, then $x \in H$ and $x \in K$. So, for all $h \in H$: $hx\ anti(h) = yh\ anti(h) = y\ neut(h) = neut(h)y \in K$ and $hx\ anti(h) \in H$. Furthermore, $neut(h)(H \cap K) = neut(h)H \cap neut(h)K = H \cap K$ since $neut(h) \in Cl(H)$ for all $h \in H$. Consequently, $H \cap K \lhd H$.

For all $k \in K, h \in H$, $hk\ anti(h) = k'h\ anti(h) = k'\ neut(h) = neut(h)k' \in K$ and $neut(h)K = K$. Thence, $K \lhd H$.

(3) Let $\phi : H \to HK/K \uparrow \phi(h) = (hk)K$ for all $h \in H$ and $k \in K$. K is rsc in H implies that $k \in Cl(K)$, and so, $\phi(h) = hK$ for all $h \in H$. So, ϕ is obviously well defined. By Theorem 2.12,

$$\phi(h_1h_2) = (h_1h_2)K = h_1Kh_2K = \phi(h_1)\phi(h_2)\ \forall\ h_1, h_2 \in H.$$

Also, ϕ is onto. Thus, ϕ is a neutro-epimorphism. HK/K is neutro-right (neutro-left) cancellative by Theorem 2.25(1).

$$\ker\phi = \{h \in H|\phi(h) = neut(xK) \text{ for some } xK \in HK/K\} = \{h \in H|hK = K\} =$$

$$\{h \in H|h \in K\} = H \cap K.$$

Therefore, by Theorem 2.26(3), $H/H \cap K \cong HK/K$. □

Remark 2.29. Theorem 2.28 can be visualized as diamond lattice structure and termed the Diamond Neutro-Isomorphism Theorem for singular WCNETG.

Theorem 2.30. *(Third Neutro-Isomorphism Theorem for Singular WCNETG)*

Let X be a singular WCNETG and let $H, K \lhd X$ be right self cancellative in X such that $K \subset H$. Then, $(X/K)/(H/K) \cong X/H$.

Proof. Consider the map $\phi : X/K \to X/H \uparrow \phi(xK) = xH$ for all $x \in X$. ϕ is well defined since K is right self cancellative:

$$xK = yH \Rightarrow anti\big(xK\big)xK = anti\big(xK\big)yK \Rightarrow \big(anti(x)y\big)K = K \Rightarrow \big(anti(x)y\big) \in K \Rightarrow$$

$$\big(anti(x)y\big) \in H \Rightarrow x\big(anti(x)y\big) \in xH \Rightarrow \big(neut(x)y\big) \in xH \Rightarrow \big(neut(x)y\big)H \subseteq xHH \Rightarrow$$

$$\big(y\ neut(x)\big)H \subseteq xH \Rightarrow yH \subseteq xH.$$

Similarly, it can be shown that $xH \subseteq yH$. So, $xH = yH \Rightarrow \phi(xK) = \phi(yK)$. By Theorem 2.12,

$$\phi\big(xKyK\big) = \phi\big((xy)K\big) = (xy)H = xHyH = \phi(xK)\phi(yK)$$

and ϕ is surjective. Hence, ϕ is a neutro-homomorphism. Since H is right self cancellative, then

$$\ker\phi = \{xK \in X/K | \phi(xK) = neut(xH) \text{ for some } xH \in X/H\} = \{xK \in X/K | xH = H\} =$$

$$\{xK \in X/K | xH = H\} = \{xK \in X/K | x \in H\} = H/K.$$

For any $x \in H$ and based on the fact that K is rsc in X implies that $k \in Cl(K)$,

$$neut(xK)H/K = K \cdot H/K = K\{hK | h \in H\} = \{k(hK) | h \in H\} = \{k(Kh) | h \in H\} =$$

$$\{(kK)h | h \in H\} = \{Kh | h \in H\} = \{hK | h \in H\} = H/K.$$

Therefore, by Theorem 2.26(3), $(X/K)/(H/K) \cong X/H$. □

Remark 2.31. Theorem 2.30 is termed the double quotient Neutro-Isomorphism Theorem for singular WCNETG.

Lemma 2.32. *Let X_1 and X_2 be singular WCNETGs and let $N_1 \lhd X_1$, $N_2 \lhd X_2$ such that N_1 and N_2 are right self cancellative in X_1 and X_2 respectively. Then, $\big(X_1 \times X_2\big)/\big(N_1 \times N_2\big) \cong \big(X_1/N_1\big) \times \big(X_2/N_2\big)$.*

Proof. $X_1 \times X_2$ is a singular WCNETG since X_1 and X_2 are singular WCNETGs. Since $N_1 \lhd X_1$, $N_2 \lhd X_2$, then $N_1 \times N_2 \lhd X_1 \times X_2$. By Theorem 2.25, X_1/N_1 and X_2/N_2 are neutro-right (neutro-left) cancellative singular WCNETGs. Thus, $\big(X_1/N_1\big) \times \big(X_2/N_2\big)$ is a neutro-right (neutro-left) cancellative singular WCNETG.

Let $\phi : X_1 \times X_2 \to \big(X_1/N_1\big) \times \big(X_2/N_2\big)$. Based on Theorem 2.12, ϕ is a neutro-epimorphism and $\ker\phi = N_1 \times N_2$ using the hypothesis that N_1 and N_2 are right self cancellative in X_1 and X_2 respectively. For any $(x_1, x_2) \in X_1 \times X_2$,

$$neut((x_1, x_2))N_1 \times N_2 = \big(neut(x_1), neut(x_2)\big)N_1 \times N_2 = neut(x_1)N_1 \times neut(x_2)N_2 = N_1 \times N_2.$$

Therefore, by Theorem 2.26(3), $\big(X_1 \times X_2\big)/\big(N_1 \times N_2\big) \cong \big(X_1/N_1\big) \times \big(X_2/N_2\big)$. □

Corollary 2.33. *Let $\{X_i\}_{i=1}^n$ be a family of singular WCNETGs and let $N_i \lhd X_i$ be right self cancellative in X_i, $1 \leq i \leq n$. Then, $\prod_{i=1}^n X_i/\prod^n i = 1N_i \cong \prod^n i = 1X_i/N_i$.*

Proof. This is the generalization of Lemma 2.32. □

Theorem 2.34. *(Neutro-Correspondence Theorem for Singular WCNETGs)*

Let X and Y be singular WCNETGs and let $\phi : X \to Y$ be a neutro-epimorphism.

(1) $G \leq X$ implies $\phi(G) \leq Y$.
(2) $H \leq Y$ implies $\phi^{-1}(H) \leq X$.
(3) $G \lhd X$ implies $\phi(G) \lhd Y$.
(4) $H \lhd Y$ implies $\phi^{-1}(H) \lhd X$.
(5) $G \leq_{rsc} X$ and $\ker\phi \subset G$ implies $\phi^{-1}\Big(\phi(G)\Big) = G$.
(6) There is a 1-1 correspondence between the set of right self cancellative NT-subgroups of X that contain $\ker\phi$, and the NT-subgroups of Y.
(7) Normal NT-subgroups of X correspond to normal NT-subgroups of Y.

Proof.

(1) Let $G \leq X$. Then, for all $a, b \in G$, $\phi(a)\phi(b) = \phi(ab) \in \phi(G)$ and $anti\big(\phi(a)\big) = \phi(anti(a)) \in \phi(G)$. Thus, $\phi(G) \leq Y$.

(2) Let $H \leq Y$. Then, for all $a, b \in G$, $\phi(ab) = \phi(a)\phi(b) \in H \Rightarrow ab \in \phi^{-1}(H)$ and $\phi(anti(a)) = anti\big(\phi(a)\big) \in H \Rightarrow anti(a) \in \phi^{-1}(H)$. So, $\phi^{-1}(H) \leq X$.

(3) Let $G \lhd X$, then $neut(x)G = G$ for all $x \in X$. Thus, $\phi\big(neut(x)\big)\phi(G) = neut\big(\phi(x)\big)\phi(G) = \phi(G) \Rightarrow neut(y)\phi(G) = \phi(G)$ for all $y \in Y$, where $y = \phi(x)$.

For each $y \in Y$ there exists $x \in X$ such that $y = \phi(x)$. Let $\phi(g) \in \phi(G)$. Then, $y\phi(g)\ anti(y) = \phi(x)\phi(g)anti\big(\phi(x)\big) = \phi\big(xg\ anti(x)\big) \in \phi(G)$ since $xg\ anti(x) \in G$. From these two arguments, $\phi(G) \lhd Y$.

(4) Let $H \lhd Y$. Then, $neut(y)H = H$ for all $y \in Y$. For each $y \in Y$, there exists $x \in X$ such that $y = \phi(x)$. So,

$$neut\big(\phi(x)\big)H = H \Rightarrow \phi\big(neut(x)\big)\phi\Big(\phi^{-1}(H)\Big) = H \Rightarrow \phi\Big(neut(x)\phi^{-1}(H)\Big) = H \Rightarrow$$

$$neut(x)\phi^{-1}(H) = \phi^{-1}(H).$$

Let $g \in \phi^{-1}(H) \Rightarrow \phi(g) \in H$. Let $x \in X$, then $\phi\big(xg\ anti(x)\big) = \phi(x)\phi(g)anti\big(\phi(x)\big) \in H$ since $H \lhd Y$. Thus, $\phi\big(xg\ anti(g)\big) \in H \Rightarrow xg\ anti(x) \in \phi^{-1}(H)$. Whence, $\phi^{-1}(H) \lhd X$.

(5) Trivially, $G \subset \phi^{-1}\Big(\phi(G)\Big)$. Let $G \leq_{\text{rsc}} X$ and $\ker\phi \subset G$.

If $x \in \phi^{-1}\Big(\phi(G)\Big)$, then $\phi(x) \in \phi(G) \Rightarrow \phi(x) = \phi(g)$ for some $g \in G$. So,

$$\phi(x)anti\big(\phi(g)\big) = \phi(g)anti\big(\phi(g)\big) = neut\big(\phi(g)\big) \Rightarrow \phi\big(x\ anti(g)\big) = neut\big(\phi(g)\big) \Rightarrow$$

$$x\ anti(g) \in \ker\phi \Rightarrow x\ anti(g) \in G \Rightarrow x\ anti(g)g \in Gg \subset G \Rightarrow x\ neut(g) \in G \Rightarrow x \in G.$$

Hence, $\phi^{-1}\Big(\phi(G)\Big) \subset G$ and therefore, $\phi^{-1}\Big(\phi(G)\Big) = G$.

(6) Let $\psi : V = \{G \leq X : \ker\phi \subset G \leq_{\text{rsc}} X\} \to W = \{H \leq Y\}$ be define as $\psi(G) = \phi(G)$. Let $H \in W \Rightarrow H \leq Y$, so that $\psi(G) = H \Rightarrow G = \phi^{-1}(H) \in V$ i.e. $\ker\phi \subset \phi^{-1}(H) \leq G$. Going by (5), $\phi\Big(\phi^{-1}(H)\Big) = H$. So, ψ is surjective.

$\psi(G_1) = \psi(G_2) \Rightarrow \phi(G_1) = \phi(G_2) \Rightarrow \phi^{-1}\big(\phi(G_1)\big) = \phi^{-1}\big(\phi(G_2)\big) \Rightarrow G_1 = G_2$. So, ψ is a bijection. Therefore, there is a 1-1 correspondence between the set of right self cancellative NT-subgroups of X containing $\ker\phi$, and the NT-subgroups of Y.

(7) This follows from (3).□

Corollary 2.35. *Let X be a singular WCNETG and let $N \lhd X$. Given any $Y \leq X/N$, there exists a unique $G \leq_{rsc} X$ such that $Y = G/N$. Furthermore, $G \lhd X$ if and only if $G/N \lhd X/N$.*

Proof. By Theorem 2.25, $\phi : X \to X/N$ defined by $\phi(x) = xN$ is a neutro canonical homomorphism. By Theorem 2.34(5),(6), there is a unique $G \leq_{\text{rsc}} X$ containing

$$\ker\phi = \{x \in X | \phi(x) = neut(xN)\} = \{x \in X | xN = neut(xN)\} =$$

$$\{x \in X | xN = N\} = \{x \in X | x \in N\} = N$$

such that $Y = \phi(G) = G/N$.

Furthermore, by Theorem 2.34(3), $G \lhd X \Rightarrow \phi(G) \lhd X/N \Rightarrow G/N \lhd X/N$. Conversely, by Theorem 2.34(4), $G/N \lhd X/N \Rightarrow \phi^{-1}(G/N) = G \lhd X$. □

Definition 2.36. Let X be a NETG.

(1) The neutral of X i.e. $NEUT(X) = X^{\text{neut}}$ will be called the set of the neutrals of elements in X: $NEUT(X) = X^{\text{neut}} = \{neut(x) : x \in X\}$.

(2) The neutral set, relative to $x \in X$ i.e. $NEUT(x) = X_x^{\text{neut}}$ will be the set of the neutral of $x \in X$: $NEUT(x) = X_x^{\text{neut}} = \{neut(x)\}$. Note that $|NEUT(x)| = 1$ for al $x \in X$.

(3) A normal NT-subgroup N of X will be called a maximal normal NT-subgroup if
   (a) $N \neq X$
   (b) $Y \leq_{\text{rsc}} X$ and $Y \supset N \Rightarrow Y = N$ or $Y = X$.

(4) A singular NETG X will be said to be neutro-simple if X has no proper normal NT-subgroup; i.e. X has no normal NT-subgroup except $NEUT(x)$ for any $x \in X$ or $NEUT(X)$ and X.

Lemma 2.37.

(1) Let X be a singular NETG, then $NEUT(x) \leq X$ for each $x \in X$.

(2) Let X be a singular WCNETG.

(a) $NEUT(x) \leq NEUT(X)$ for each $x \in X$.

(b) $NEUT(X)$ is commutative and $NEUT(X) \leq_{ss} X$.

(c) $NEUT(X) \lhd X$ and $NEUT(X)$ is a NT-subgroup of any semi-strong NT-subgroup of X.

(d) $NEUT(X)$ is the smallest semi-strong NT-subgroup of X i.e. $NEUT(X) = \bigcap\limits_{H \leq_{ss} X} H$.

Proof. This is easy. □

Corollary 2.38. *Let X be a singular WCNETG and let $N \lhd X$. N is a maximal normal NT-subgroup of X if and only if X/N is neutro-simple.*

Proof. Let X be a singular WCNETG and let $N \lhd X$. If N is a maximal normal NT-subgroup of X, then $N \neq X$ and, $Y \leq_{\text{rsc}} X$ and $Y \supset N \Rightarrow Y = N$ or $Y = X$. Thus, by Corollary 2.35, $Y \lhd X \Rightarrow Y/N \lhd X/N \Rightarrow N/N \lhd X/N$ or $X/N \lhd X/N \Rightarrow \{N\} \lhd X/N$ or $X/N \lhd X/N \Rightarrow \{neut(xN)|x \in X\}$ or $X/N \lhd X/N \Rightarrow X/N$ is neutro-simple.

Conversely, if X/N is neutro-simple, then X/N has no normal NT-subgroup other than $\{N\}$ and X/N. Thus, going by Corollary 2.35, if $Y \lhd_{\text{rsc,ss}} X$ and $Y \supset N$ such that $Y/N \lhd X/N$, then $Y \lhd X$. Now, $Y/N \lhd X/N$ implies that $Y/N = \{N\} = N/N$ or $Y/N = X/N \Rightarrow Y = N$ or $Y = X$. So, N is a maximal normal NT-subgroup. □

Corollary 2.39. *Let X be a singular WCNETG and let Y, Z be maximal normal NT-subgroups of X such that $Y, Z \leq_{rsc,ss} X$. Then*

(1) $YZ \lhd_{rsc,ss} X$.

(2) $Y \cap Z$ is a maximal normal NT-subgroup of Y and of Z.

Proof.

(1) By Theorem 2.12, $yZ = Zy$ for all $y \in Y$ implies that $YZ = ZY$. Thus, by Theorem 2.6, $YZ \leq X$. Now, since $Y, Z \lhd X$, then, for all $x \in X,\ y \in Y,\ z \in Z$,

$$x(yz)\ anti(x) = x\ neut(x)yz\ anti(x) = xy\ neut(x)z\ anti(x) =$$

$$\big(xy\ anti(x)\big)\big(xz\ anti(x)\big) \in YZ \text{ and } neut(x)YZ = YZ.$$

$$\therefore\ Y, Z \lhd YZ \lhd X \tag{1}$$

Now, for any $x \in X$, $neut(x) = neut(x)neut(x) \in YZ \Rightarrow YZ \lhd_{\text{ss}} X$. For all $x, y \in X$, we already know that $xY = Y$ is equivalent to $xY = YY$ and $yZ = Z$ is equivalent to $yZ = ZZ$. So, $xYZ = YZ \Rightarrow xY = Y \Rightarrow x \in Y \subset YZ \Rightarrow x \in YZ$. So, $YZ \lhd_{\text{rsc}} X$. Therefore, $YZ \lhd_{\text{rsc,ss}} X$.

(2) Since Z is a maximal normal NT-subgroup of X, then $YZ = Z$ or $YZ = X$. But, $YZ = Z \Rightarrow Y \subset Z$, a contradiction to the fact that Y is a maximal normal NT-subgroup of X. Hence, $YZ = X$. Similarly, since Z is a maximal normal NT-subgroup of X, this also leads us to $YZ = X$.

From Theorem 2.28, $Y/Y \cap Z \cong YZ/Z$. So, $Y/Y \cap Z \cong X/Z$ and $Z/Y \cap Z \cong X/Z$. Hence, by Corollary 2.38, since Y and Z are maximal normal NT-subgroups of X, then, X/Z and X/Y are neutro-simple, whence, $Y/Y \cap Z$ and $Z/Y \cap Z$ are neutro-simple. Thus, $Y \cap Z$ is a maximal normal NT-subgroup of Y and Z. □

Definition 2.40. Let X be a singular NETG.

(1) A neutro-isomorphism $\alpha : X \to X$ will be called a neutro-automorphsim of X and the set of such mappings will be denoted by $Aut(X)$.

(2) For any fixed $g \in X$, the mapping $\alpha : X \to X$ defined by $I_g(x) = gx\ anti(g)$ for all $x \in X$ will be called a neutro-inner mapping of X at $g \in X$ and the set of such mappings will be denoted by $Inn(X)$.

Theorem 2.41.

(1) Let X be a singular NETG. Then, $Aut(X)$ is a group

(2) Let X be a singular WCNETG that is neutro-right (neutro-left) cancellative.

(a) $Inn(X) \lhd_{rsc,ss} Aut(X)$.

(b) $Inn(X)$ is a subgroup of $Aut(X)$ if and only if X is a group.

(c) If $Z(X) \subset neut(x)Z(X)$ for all $x \in X$, then $X/Z(X) \cong Inn(X)$.

Proof.

(1) This is routine.

(2) (a) For any fixed $g \in X$ and for all $x, y \in X$, the following shows that I_g is an neutro-homomorphism.

$$I_g(xy) = g(xy)\ anti(g) = g\ neut(g)xy\ anti(g) = gx\ neut(g)y\ anti(g) =$$

$$gx\ anti(g)gy\ anti(g) = I_g(x)I_g(y).$$

I_g is 1-1 based on the following arguments.

$$I_g(x) = I_g(y) \Rightarrow gx\ anti(g) = gy\ anti(g) \Rightarrow gx\ anti(g)g = gy\ anti(g)g \Rightarrow$$

$$gx\ neut(g) = gy\ neut(g) \Rightarrow g\ neut(g)x = g\ neut(g)y \Rightarrow gx = gy \Rightarrow anti(g)gx =$$

$$anti(g)gy \Rightarrow neut(g)x = neut(g)y \Rightarrow x = y.$$

Using a similar argument, it can be shown that I_g is onto. So, $Inn(X) \subseteq Aut(X)$. For any fixed $g_1, g_2 \in X$ and for all $x \in X$, the following shows that $Inn(X)$ is a groupoid.

$$I_{g_1}I_{g_2}(x) = I_{g_1}\big(g_2x\ anti(g_2)\big) = g_1g_2x\ anti(g_2)anti(g_1) = g_1g_2x\ anti\big(g_1g_2\big) =$$

$$I_{g_1g_2}(x) \Rightarrow I_{g_1}I_{g_2} = I_{g_1g_2} \in Inn(X).$$

So, $neut(I_g) = I_{neut(g)} \in Inn(X)$ for each $g \in X$. Thus, $Inn(X) \neq \emptyset$. Now,

$$I_gI_{anti(g)}(x) = g\ anti(g)x\ anti\big(anti(g)\big)anti(g) = neut(g)x\ anti\big(neut(g)\big) =$$

$$I_{neut(g)}(x) \Rightarrow I_gI_{anti(g)} = I_{neut(g)}.$$

Similarly, $I_{anti(g)}I_g = I_{neut(g)}$ and so, $anti\big(I_g\big) = I_{anti(g)} \in Inn(X)$. Hence, $Inn(X) \leq Aut(x)$.

Let $\sigma \in Aut(X)$ and let $I_g \in Inn(X)$. Then,

$$\sigma I_g\sigma(x) = \sigma\big(g\sigma^{-1}(x)anti(g)\big) = \sigma(g)x\ anti\big(\sigma(g)\big) =$$

$$I_{\sigma(g)}(x) \Rightarrow \sigma I_g\sigma = I_{\sigma(g)} \in Inn(X) \text{ and } II_g(x) = I_g(x) \Rightarrow II_g(x) = I_g \in Inn(X).$$

So, $Inn(X) \lhd_{\mathrm{rsc,ss}} Aut(X)$.

(b) $Inn(X)$ is a subgroup of $Aut(X)$ if and only if $I_{neut(g)} = I$. Now, $I_{neut(g)} = I \Rightarrow I_{neut(g)}(x) = I(x)\ \forall\ x \in X \Rightarrow neut(g)x\ anti(neut(g)) = x \Rightarrow neut(g)x = x$ and $x\ neut(g) = x \Rightarrow neut(g) = neut(x)\ \forall\, x, g \in X \Rightarrow X$ is a group. Conversely, if X is a group, then $neut(g)x\ anti(neut(g)) = x \Rightarrow I_{neut(g)} = I$. So, $Inn(X)$ is a subgroup of $Aut(X)$ if and only if X is a group.

(c) Let $\phi : X \to Aut(X)$ with $\phi(x) = I_x$. For any $x_1, x_2, x \in X$, ϕ is a neutro-homomorphism because

$$\phi(x_1x_2)(x) = I_{x_1x_2}(x) = x_1x_2x\ anti(x_1x_2) = x_1x_2x\ anti(x_2)anti(x_1) =$$

$$x_1I_{x_2}(x)anti(x_1) = I_{x_1}I_{x_2}(x) \Rightarrow \phi(x_1x_2) = I_{x_1}I_{x_2}.$$

$$\ker\phi = \{g \in X|\phi(g) = I\} = \{g \in X|\phi(g)(x) = x\ forall\ x \in X\} =$$

$$\{g \in X|\phi(g)(x) = x\ forall\ x \in X\} = \{g \in X|gx\ anti(g) = x\ forall\ x \in X\} =$$

$$\{g \in X|gx\ neut(g) = xg\ forall\ x \in X\} = \{g \in X|gx = xg\ forall\ x \in X\} = Z(X).$$

Going by Theorem 2.26(3), $X/Z(X) \cong Inn(X)$. □

Theorem 2.42. *(Neutro-Zassenhaus' Lemma for Singular WCNETG)*
Let X be a singular WCNETG such that

$B, C \leq_{rsc} X,\ B_0 \lhd B,\ C_0 \lhd C,\ B_0(B \cap C_0), C_0(C \cap B_0) \leq_{rsc} B \cap C$ *and* $B \cap C_0, C \cap B_0 \leq Cl(B \cap C)$.

If $neut(x) \in Cl(B), Cl(C)$ for all $x \in B \cap C$, then

$$\frac{B_0(B \cap C)}{B_0(B \cap C_0)} \cong \frac{C_0(C \cap B)}{C_0(C \cap B_0)}.$$

Proof. Let $K = B \cap C$ and $H = B_0(B \cap C_0)$. Since $B_0 \lhd B$, $bB_0 = B_0 b$ for all $b \in B$. So, since $K \subseteq B$, then $kB_0 = B_0 k$ for all $k \in K$. Also, $C_0 \lhd C \Rightarrow B \cap C_0 \lhd B \cap C = K$ since $neut(b)(B \cap C_0) = neut(b)B \cap neut(b)C_0 = B \cap C_0$ for all $b \in B \cap C = K$. Hence, $k(B \cap C_0) \subseteq (B \cap C_0)k$ for all $k \in K$. Thus,

$$Hk = B_0(B \cap C_0)k = B_0 k(B \cap C_0) = kB_0(B \cap C_0) = kH \Rightarrow Hk = kH\ \forall\ k \in K.$$

Let us now find HK and $H \cap K$. Thus, $HK = KH$ based on the following argument.

Since $B \cap C_0 \leq Cl(B \cap C)$, then $(B \cap C_0)(B \cap C) = B \cap C$, and so, $HK = B_0(B \cap C_0)(B \cap C) = B_0(B \cap C)$.

Let $y \in H \cap K \Rightarrow y \in H$ and $y \in K$. Now, $y \in H = B_0(B \cap C_0) \Rightarrow y = b_0 b,\ b_0 \in B_0,\ b \in B \cap C_0$. Let $b_0 b = d \in B \cap C = K$. Then, $d \in C$. Since $B \cap C_0 \subseteq C$, then $b \in C$. Now, $b_0 b = d \Rightarrow b_0\ neut(b) = d\ anti(b) \in C \Rightarrow b_0 \in C$ since $b \in B \cap C_0 \Rightarrow b \in B \Rightarrow neut(b) \in B$ and C is right self cancellative. Hence, $b_0 \in B_0 \cap C \Rightarrow b_0 b \in (B_0 \cap C)(B \cap C_0) \Rightarrow H \cap K \subseteq (B_0 \cap C)(B \cap C_0)$.

On the other hand, $B_0 \cap C \subset K,\ B \cap C_0 \subset K \Rightarrow (B_0 \cap C)(B \cap C_0) \subset K$. Since $B_0 \cap C \subseteq B_0$, then $(B_0 \cap C)(B \cap C_0) \subset H \cap K$. Thus, $H \cap K = (B_0 \cap C)(B \cap C_0)$.

Going by Theorem 2.28, if X is a singular WCNETG, with $H, K \leq X$, $H \leq_{\mathrm{rsc}} K$ and $Hk = kH$, $neut(k) \in Cl(K), Cl(H)$ for all $k \in K$, and $neut(h) \in Cl(H)$ for all $h \in H$, then

$$HK/H \cong K/H \cap K \tag{2}$$

Substituting H, K, HK and $H \cap K$ in (2) we get

$$\frac{B_0(B \cap C)}{B_0(B \cap C_0)} \cong \frac{B \cap C}{(B_0 \cap C)(B \cap C_0)} \tag{3}$$

On interchanging the roles of B and C in (3), we get

$$\frac{C_0(C \cap B)}{C_0(C \cap B_0)} \cong \frac{C \cap B}{(C_0 \cap B)(C \cap B_0)} \tag{4}$$

Since $B_0 \cap C, B \cap C_0 \lhd B \cap C$, then $(B_0 \cap C)(B \cap C_0) = (B \cap C_0)(B_0 \cap C)$. So, the right hand sides of (3) and (4) are equal. Thus, $\dfrac{B_0(B \cap C)}{B_0(B \cap C_0)} \cong \dfrac{C_0(C \cap B)}{C_0(C \cap B_0)}$. □

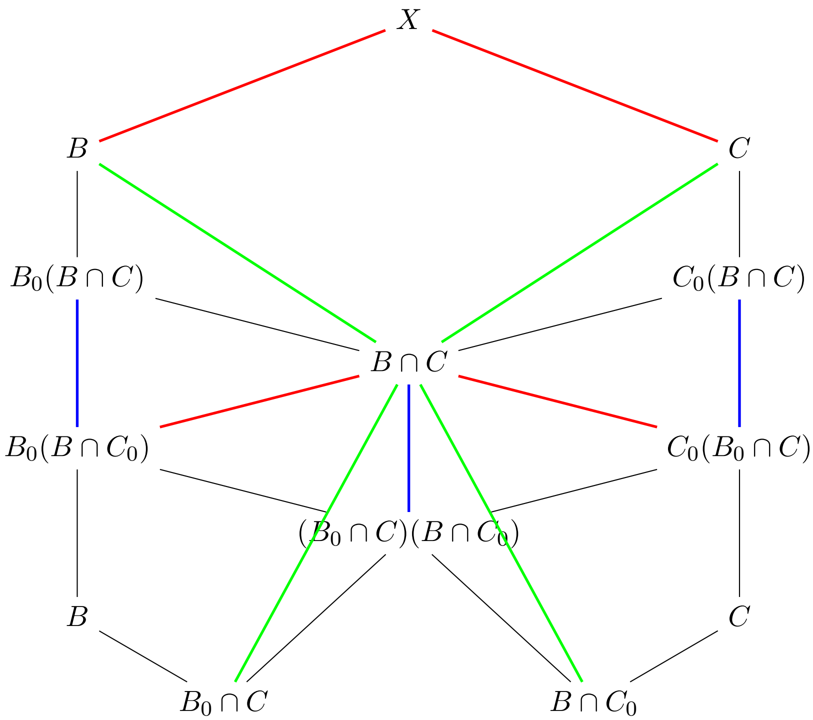

Figure 1. The Neutro-Butterfly

Remark 2.43. In Figure 1, the quotients given by the blue lines (by pairing) are neutro-isomorphic to each other based on (3) and (4), thus proving the Neutro Zassenhaus' Lemma. A black line indicates that the NT-subgroup that lie below is NT-normal in the NETG connected to it above in the plane of the figure. Also, the red (green) line indicates that the NT-subgroup that lie below is right self cancellative (closure-contained) respectively, in the NETG connected to it above in the plane of the figure. We acknowledge Kannappan Sampath [4] for adapting his LaTeX codes for Zassenhaus' Lemma for groups to generate Figure 1.

3. Conclusion

In this paper, we have been able to establish the homomorphism theorems (first, second and third neutro-isomorphism and neutro-corresponding theorems) and some other associated theorems (neutro-Zassenhaus Lemma) in singular WCNETG with the aid of newly introduced NT-subgroups such as: right cancellative, semi-strong, and maximally normal NT-subgroups. These results generalize their classical forms in group theory.

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